Resolving singularities in parabolic initial-boundary value problems

Torsten Linß (FernUniversität in Hagen), Brice Girol

We consider a time-dependent reaction-diffusion equation with a singularity arising from incompatible initial and boundary conditions:

$$u_t - u_{xx} + b(x,t)u = f \qquad \text{in } \ (0,\ell) \times (0,T],$$

subject to boundary conditions

$$u(0,t) = \varphi_0(t), \quad u(\ell,t) = \varphi_\ell(t), \quad t \in (0,T],$$

and the initial condition

$$u(x,0) = 0, \qquad x \in (0,\ell)\,,$$

with $\varphi_0(0) \neq 0$.
The discrepancy between initial and boundary conditions causes the formation of a singularity in the vicinity of the corner $(0,0)$. This singularity s can be characterised as the solution of

$$s_t - s_{xx} + b(0,0)s = 0 \qquad \text{in } \ (0,\infty) \times (0,T],$$

subject to the boundary condition

$$s(0,t) = \varphi_0(0), \quad t \in (0,T],$$

and the initial condition

$$s(x,0) = 0, \qquad x \in (0,\infty)\,.$$

This in turn can be given analytically using the error function. Then the interesting question is:

How can the remainder $y = u - s$ be resolved numerically?

We derive bounds on the derivatives of the remainder y — under significantly less restrictive assumptions then previously assumed by other authors — and show how a numerical approximation can be obtained using an appropriately designed mesh.

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