Runge-Kutta methods for index-2 and index-3 differential-algebraic equations arising from incompressible flow problems

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Many computational physics problems can be modelled by partial differential equations (PDEs) with constraints. In particular, we are interested in single-phase and multi-phase incompressible fluid flow problems, in which the constraint is that the velocity field is divergence-free. After discretizing the PDEs in space, a differential-algebraic equation (DAE) system is obtained. In previous work we have analyzed the accuracy of explicit Runge-Kutta methods for the single-phase incompressible Navier-Stokes equations, which form an index-2 DAE. In the current work we consider the extension to multi-phase incompressible flow problems in pipelines and channels, where a different constraint leads to a DAE with index 3. Existing time integration methods for this system lack either conservation, accuracy, or constraint-consistency.

We propose a third order half-explicit Runge-Kutta method (Hairer et al., 1989) that is consistent with the constraints of the index-3 DAE system, and with coefficients chosen such that order reduction due to the DAE nature of the equations is prevented. The method is explicit for the mass and momentum equations and implicit for the pressure. The resulting method is (i) constraint-consistent: exact conservation of the volume constraint and the incompressibility constraint; (ii) accurate: high order temporal accuracy for differential and algebraic variables; (iii) conservative: the original mass and momentum equations are solved, so that the proper shock conditions are satisfied; (iv) efficient: the only implicit part is the pressure Poisson equation, and the time step for the explicit part is restricted by a benign CFL condition based on the convective wave speeds.

Several testcases show the effectiveness of the time-integration methods: Kelvin-Helmholtz instabilities in a pipeline, liquid sloshing in a tank, and ramp-up of gas production in a multi-phase pipeline.