Resolvent Krylov subspace approximation to C_0-semigroups and their discretisations

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The recent success of exponential integrators and splitting methods for evolution equations triggers a very strong demand for efficient approximations of C_0-semigroups and related operator/matrix functions applied to given initial data. For fine space discretisations, this corresponds to the approximation of a matrix exponential for a huge and sparse matrix times a vector. An important finding is, that methods that work for the approximation of the semigroup in the continuous case render approximations that are independent of the space discretisation in the discretised case. These methods are therefore more appropriate for large problems resulting from a fine spatial grid. In time integration, this is nothing but the observation that certain implicit time-integration schemes are much more efficient for large stiff systems of ODEs. Despite the fact that rational approximations are known to achieve faster convergence rates in this case, the PDE community often sticks to approximations like the backward Euler and the Crank-Nicolson method that are comparatively easy to compute for fine space discretisations of evolution equations. Higher order Runge-Kutta methods and higher order rational approximations are deemed too costly and too cumbersome to implement.

A fascinating new finding is that there are methods that converge faster than the backward Euler and the Crank-Nicolson method for approximately the same computational cost. In fact, the programs computing the resolvents in the backward Euler method or the Crank-Nicolson scheme can be reused without essential changes. The idea is to approximate the semigroup with infinitesimal generator A in the resolvent Krylov subspace

$$\mathcal{K}_m((\gamma I - A)^{-1}, b) = \text{span}\{b, (\gamma I - A)^{-1}b, \ldots, (\gamma I - A)^{-m+1}b\}, \qquad \gamma > 0\,.$$

The corresponding Krylov subspace method automatically converges faster for smoother initial data b. Moreover, the convergence is independent of the grid in space for the discretised evolution equation. No method at comparable computational cost is known that shares these favourable properties. The obtained results will be illustrated by numerical experiments with different space discretisations.