Linearly implicit methods for a class of degenerate convection-diffusion problems

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This contribution is concerned with semi-implicit numerical schemes for the discretization of strongly degenerate parabolic equations of the type

$$u_t + f(u)_x = A(u)_{xx}, \quad A(u) = \int_0^u a(s)\,\mathrm{d}s, \tag{1}$$

posed on some finite x-interval along with initial and boundary conditions, and where the function a is piecewise continuous and satisfies $a(u) \geq 0$ for all u. In particular, $a(u) = 0$ is possible on u-intervals of positive length, so (1) may turn into a first-order hyperbolic conservation law where the location of the type-change interface is unknown a priori. Thus, solutions of (1) are in general discontinuous. Consequently, the well-posedness theory and numerical analysis of (1) are based on the framework of entropy (weak) solutions.

Applications of (1) include a model of sedimentation of suspensions in mineral processing and wastewater treatment [4, 5] (after suitable simplifications). The efficient numerical solution of (1) is therefore of substantial theoretical and practical interest. Explicit monotone difference approximations to (1) go back to [7], are easily implemented, and provably converge to the entropy solution. However, the restrictive CFL condition makes explicit schemes unacceptably slow. An alternative are nonlinearly implicit semi-implicit schemes that treat the diffusive term implicitly and allow for a less restrictive CFL condition. Such methods are also supported by a convergence theory [2] and have turned out to be more efficient than their explicit counterparts in terms of error reduction versus CPU time [6]. However, their implementation requires the use of nonlinear solvers (e.g., Newton-Raphson method) which may fail to converge.

It is the purpose of this contribution to propose linearly implicit methods for the approximation of solutions of (1) as an alternative. These methods go back to Berger et al. [1], are based on a particular separate discretization of the diffusive term, enjoy a favorable CFL condition, and require the solution of a linear system at every time step, which is an advantage in practical applications. Preliminary results [3] show that these methods are competitive with the known explicit and nonlinearly implicit schemes in terms of accuracy and efficiency. It is demonstrated that these schemes are monotone, which is the key property required to demonstrate convergence to an entropy solution. The full convergence analysis is currently in preparation.

References

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