Self-conjugate differential and difference operators in the optimal control of descriptor systems

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We analyze the structure of the differential and difference operators associated with the necessary optimality conditions of optimal control problems for descriptor systems in continuous and discrete time.
In the continuous-time case, the *linear quadratic optimal control problem* with constraints given by *differential-algebraic equations (DAEs)* is of the form

$$\begin{aligned} \text{min.} \quad & \frac{1}{2}x(\bar{t})^T M_e x(\bar{t}) + \frac{1}{2}\int_{\underline{t}}^{\bar{t}} \left(x^T W x + x^T S u + u^T S^T x + u^T R u\right)\, dt \\ \text{s. t.} \quad & E(t)\dot{x} = A(t)x + B(t)u + f(t), \quad x(\underline{t}) = \underline{x} \in \mathbb{R}^n. \end{aligned}$$

This problem has recently been discussed in several publications [1, 3, 4, 6] and it has been shown in [4] that the operator associated with the necessary optimality boundary value problem is self-conjugate. If we denote the differential-algebraic equation associated with this boundary value problem by

$$\mathcal{E}\dot{z} = \mathcal{A}z + \tilde{f},$$

then the pair $(\mathcal{E}, \mathcal{A})$ has the property that $\mathcal{E}^T = -\mathcal{E}$ and $\mathcal{A}^T = \mathcal{A} + \dot{\mathcal{E}}$.
On the other hand, we consider the *discrete-time linear-quadratic optimal control problem* given by

$$\begin{aligned} \text{min.} \quad & \frac{1}{2}x_N^T M_e x_N + \frac{1}{2}\sum_{j=0}^{N}(x_j^T W_j x_j + x_j^T S_j u_j + u_j^T S_j^T x_j + u_j^T R_j u_j), \\ \text{s. t.} \quad & E_{k+1}x_{k+1} = A_k x_k + B_k u_k + f_k, \quad x_0 = \underline{x} \in \mathbb{R}^n. \end{aligned} \tag{1}$$

The necessary optimality condition for $((x_k), (u_k))$ to be an optimal solution is the existence of a sequence of Lagrange multipliers (λ_k) such that $((x_k), (u_k), (\lambda_k))$ satisfy the discrete-time optimality system

$$\begin{bmatrix} 0 & E_{k+1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{k+1} \\ x_{k+1} \\ u_{k+1} \end{bmatrix} + \begin{bmatrix} 0 & -A_k & -B_k \\ -A_k^T & W_k & S_k \\ -B_k^T & S_k^T & R_k \end{bmatrix} \begin{bmatrix} \lambda_k \\ x_k \\ u_k \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ E_k^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_{k-1} \\ x_{k-1} \\ u_{k-1} \end{bmatrix} = \begin{bmatrix} f_k \\ 0 \\ 0 \end{bmatrix},$$

together with appropriate boundary conditions, see [5]. We will show that the special structure of the sequences of coefficient matrices corresponds

to self-conjugacy of the corresponding difference operator. The relationship between these structures is well understood in the constant coefficient case, see [2, 7]. Here, we will study the relationship between the structures in the discrete- and continuous-time case with variable coefficients and show that self-adjointness of matrix tuples (in continuous- as well as in discrete-time) is an appropriate generalization for even/palindromic, and Hamiltonian/symplectic structures in the constant coefficient case.
Discrete-time optimal control problems of the form (1) arise in the *first-discretize-then-optimize* approach for solving a linear-quadratic optimal control problem (in contrast to the approach of *first-optimize-then-discretize*). This immediately leads to the question of how to discretize the necessary optimality system in the continuous-time case so that the resulting discrete-time system has the self-adjoint structure that would have been obtained when discretizing the constraint first and then deriving the discrete-time optimality systems. In this way it can be guaranteed that the approaches first-discretize-then-optimize and first-optimize-then-discretize lead to the same structural properties of the optimality system, i.e., discretization and optimization commute, such that we can use the advantages from both approaches.

References

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