Outflow positivity limiting for hyperbolic systems

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Physical solutions to hyperbolic systems of conservation laws typically stay in a region of states designated as positive. Examples include the shallow water equations, which maintain positivity of the depth, and the Euler gas dynamics equations, which maintain positivity of the density and pressure. Numerical solutions that wander outside the domain of positivity are likely to become unstable due to lack of hyperbolicity for non-positive data. Finite volume methods (such as WENO or DG) are designed to exactly satisfy a discrete conservation law, but it is challenging to maintain positivity of cell average quantities while retaining high-order accuracy in space. For given solution data and numerical fluxes, one can directly calculate the largest stable time step that maintains positivity of cell averages, but this time step can become arbitrarily small, halting the simulation. The challenge is therefore to design numerical fluxes that, while preserving high order accuracy, limit the potential rate of outflow from each cell relative to the cell average, thereby guaranteeing a minimum positivity-preserving time step.

Zhang and Shu have shown how to ensure a positivity-preserving time step by linearly damping the deviation from the cell average of the high-order representation of the solution just enough to enforce positivity at a set of positivity points. We reinterpret their framework in terms of limiting outflow from each cell and thereby show how to simplify and extend their framework to work for mesh cells of arbitrary geometry while guaranteeing the same positivity-preserving time step as if the linear damping were sufficient to enforce positivity at every point in the mesh cell. High-order finite volume methods can be outfitted with outflow positivity limiters without loss of order of accuracy and with marginal additional computational expense.