Numerical approximation of the Mittag–Leffler function and applications in fractional calculus

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The Mittag—Leffler (ML) function, introduced at the beginning of the last century by the Swedish mathematician Magnus Gösta Mittag–Leffler, is nowadays receiving renewed interest because of its applications in fractional calculus; indeed, the ML function plays for fractional differential equations (FDEs) the same key role as the exponential function does for ordinary differential equations (ODEs) of integer order.
In the last years some efforts have been dedicated in extending exponential integrators to the numerical treatment of linear and semi–linear FDEs. This approach, successfully applied to ODEs, essentially consists in solving exactly the linear (and usually stiff) term by evaluating some exponential–type function and hence applying an explicit scheme to the nonlinear (and usually non–stiff) term.
The generalization of exponential integrators to FDEs involves the evaluation of some generalized ML functions in the form

$$e_{\alpha,\beta}(t;\lambda) = t^{\beta-1}E_{\alpha,\beta}(-t^{\alpha}\lambda), \quad E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where t is an independent variable, λ is a scalar or a matrix and α and β are fixed parameters.
The numerical computation of ML functions, possibly with matrix arguments, is a challenging task. The classical definition in terms of the series is not useful for practical computation because of its slow convergence. Thus, efficient and reliable methods need to be devised.
In this talk we present and discuss some methods based on the integral representation of $e_{\alpha,\beta}(t;\lambda)$ and different approaches are compared. For some methods we present a robust error analysis allowing to select the main parameters of the method with the aim of achieving any prescribed accuracy. Some applications in the solution of FDEs are also shown.