Stability of plane waves in nonlinear Schrödinger equations: mathematical and numerical analysis

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The cubic nonlinear Schrödinger equation

$$i\partial_t u = -\Delta u + \lambda |u|^2 u, \qquad u = u(x,t), \tag{1}$$

with periodic boundary conditions in space ($x \in \mathbb{R}^d/(2\pi\mathbb{Z})^d$) has solutions that are plane waves:

$$u(x,t) = \rho e^{i(m\cdot x - \omega t)}$$

solves (1) for arbitrary $m \in \mathbb{Z}^d$ and $\omega = |m|^2 + \lambda\rho^2$.
In the first part of the talk the stability of these plane wave solutions under perturbations of the initial data is discussed. We show their (orbital) stability under generic perturbations that are small in a high Sobolev norm. This stability result holds over long times that extend to arbitrary negative powers of the smallness parameter. The perturbation stays small in the same Sobolev norm over such long times.
In the second part of the talk we turn to a standard numerical discretization of the cubic nonlinear Schrödinger equation, the split-step Fourier method, where a spectral collocation method in space is combined with a splitting integrator in time. Does the stability result for the exact solution extend to the fully discrete solution?