Index Reduction in Industrial Multibody System Simulation

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1 Introduction

The *multibody system* (MBS) approach is frequently used in industrial simulation packages in robotics, vehicle system dynamics, and biomechanics. A multibody system model consists of a finite number of rigid or elastic bodies and their interconnections like, e.g., joints, springs, dampers, and actuators. The equations of motion may be generated in a systematic way by *multibody formalisms* that are based on the principles of classical mechanics [24].
The efficient and robust numerical integration of these equations is a very challenging problem in the development of simulation packages, since dynamical simulation is a frequently used and one of the most time consuming analysis methods for MBS models. The equations of motion form a nonlinear system of differential-algebraic equations (DAEs) of index 3 with a very special structure that should be exploited in the numerical solution [8].
In the present paper we study the time integration of MBS model equations following a novel approach that has originally been developed for the time integration of

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linear DAEs of arbitrary high index [17], [19]. In contrast to standard textbook presentations like [8] we do not restrict ourselves to classical conservative constrained mechanical systems but consider the more complex model equations that are actually used in state-of-the-art MBS simulation packages [9].

2 Multibody System Models in Industrial Applications

2.1 Constrained Mechanical Systems

A multibody system consists of a finite number of bodies and their interconnections, e.g. joints and springs. In contrast to springs and other force elements a joint restricts the relative motion of bodies in the system. The multibody system performs motions that are compatible with all joints of the system. Multibody systems can be represented by a graph, see Figure 1. Here, the knots represent the bodies and the edges represent the joints.

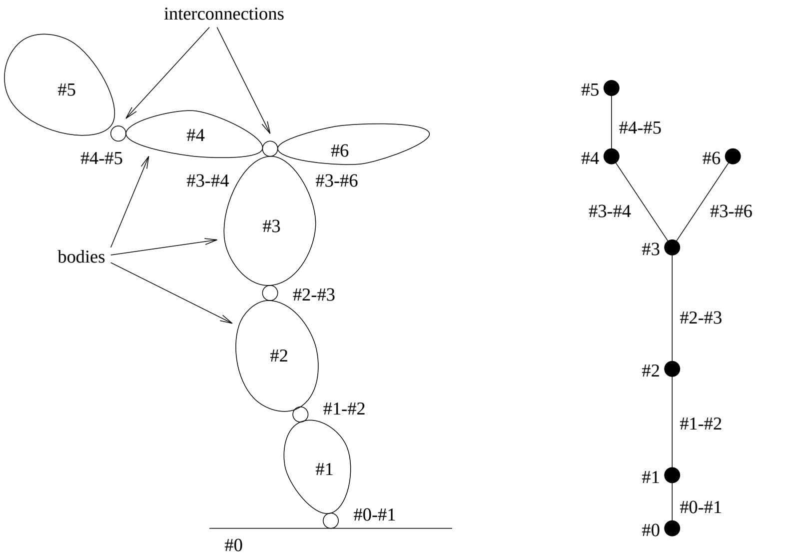

Figure 1: Multibody system in tree structure with 6 bodies and the corresponding graph

One may obtain the equations of motion for instance by using the following multibody formalism.

Step 1: If the graph of a multibody system contains loops, then cut edges such that the resulting graph is in tree structure. These cut joints are called *loop-closing joints*.

Step 2: Define *joint coordinates* $\mathbf{q} = [q_i]_{i=1..n_\mathbf{q}}$ [9],[21] to describe the dynamical behavior of the multibody system, for instance the distances and angles between different bodies.

Step 3: Define *absolute coordinates* $\mathbf{p} = [p_i]_{i=1..n_\mathbf{p}}$ [9],[21] with respect to a fixed inertial coordinate system to describe the dynamical behavior of the multibody system. Usually the set of the coordinates $\mathbf{p}$ is not minimal. Between the absolute and relative coordinates there are $n_\mathbf{p}$ kinematic relations

$0 = \tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) = [\tilde{k}_i(\mathbf{p},\mathbf{q},t)]_{i=1..n_\mathbf{p}}$. The Jacobian $\tilde{\mathbf{K}}(\mathbf{p},\mathbf{q},t) = \partial\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t)/\partial\mathbf{p}$ is assumed to be nonsingular, i.e. the relations are not redundant or contradictory with respect to $\mathbf{p}$. Redundant means that at least one equation can be expressed by other equations. If the multibody system has tree structure then the absolute coordinates $\mathbf{p}$ can be computed recursively from the relative coordinates $\mathbf{q}$.

Step 4: If there exist n_λ *constraints* which describe those kinematic loops that were cut in Step 1 then we denote them by $0 = \tilde{\mathbf{g}}(\mathbf{p},t) = [\tilde{g}_i(\mathbf{p},t)]_{i=1..n_\lambda}$. We assume that the constraints $0 = \tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t)$ and $0 = \tilde{\mathbf{g}}(\mathbf{p},t)$ together are not redundant or contradictory with respect to $\mathbf{p}$ and $\mathbf{q}$, i.e. the Jacobian

$$\begin{bmatrix} \frac{\partial}{\partial\mathbf{p}}\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) & \frac{\partial}{\partial\mathbf{q}}\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) \\ \frac{\partial}{\partial\mathbf{p}}\tilde{\mathbf{g}}(\mathbf{p},t) & 0 \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{K}} & \tilde{\mathbf{J}} \\ \tilde{\mathbf{G}} & 0 \end{bmatrix} \quad \text{of} \quad \begin{bmatrix} \tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) \\ \tilde{\mathbf{g}}(\mathbf{p},t) \end{bmatrix}$$

has full rank. If the original MBS has tree structure, then there are no loops to be cut in Step 1 such that $n_\lambda = 0$ and the joint coordinates q are minimal. Theoretically, any MBS could be described by minimal coordinates $\mathbf{q}$ (resulting in $n_\lambda = 0$) but this approach from classical mechanics may result in very complicated kinematics for MBS with closed loops and is therefore in general less efficient than a formulation that includes constraints.

Step 5: Determine the *kinetic energy* $T(\dot{\mathbf{p}},\mathbf{p})$ and the *potential energy* $U(\mathbf{p})$ of the multibody system.

Step 6: The *Lagrange-function* has the form

$$L = T(\dot{\mathbf{p}},\mathbf{p}) - U(\mathbf{p}) - \sum_{j=1}^{n_\mathbf{p}} \gamma_j \tilde{k}_j(\mathbf{p},\mathbf{q},t) - \sum_{j=1}^{n_\lambda} \lambda_j \tilde{g}_j(\mathbf{p},t),$$

where $\gamma = [\gamma_j]_{j=1..n_\mathbf{p}}$ and $\lambda = [\lambda_j]_{j=1..n_\lambda}$ denote the Lagrange multipliers.

Step 7: Use the *Euler-Lagrange equations*

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial(\dot{\mathbf{p}},\dot{\mathbf{q}},\dot{\lambda},\dot{\gamma})}\right) - \frac{\partial L}{\partial(\mathbf{p},\mathbf{q},\lambda,\gamma)}$$

to obtain the following derivatives:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial(\dot{\mathbf{p}})}\right) &= \frac{\mathrm{d}}{\mathrm{d}t}\left(T_{,\dot{\mathbf{p}}}(\dot{\mathbf{p}},\mathbf{p})\right) = T_{,\dot{\mathbf{p}}\mathbf{p}}(\dot{\mathbf{p}},\mathbf{p})\dot{\mathbf{p}} + T_{,\dot{\mathbf{p}}\dot{\mathbf{p}}}(\dot{\mathbf{p}},\mathbf{p})\ddot{\mathbf{p}}, \\
\frac{\partial L}{\partial\mathbf{p}} &= T_{,\mathbf{p}}(\dot{\mathbf{p}},\mathbf{p}) - U_{,\mathbf{p}}(\mathbf{p}) - \sum_{j=1}^{n_\mathbf{p}} \gamma_j \tilde{k}_{j,\mathbf{p}}(\mathbf{p},\mathbf{q},t) - \sum_{j=1}^{n_\lambda} \lambda_j \tilde{g}_{j,\mathbf{p}}(\mathbf{p},t), \\
\Rightarrow 0 &= \underbrace{T_{,\dot{\mathbf{p}}\dot{\mathbf{p}}}(\dot{\mathbf{p}},\mathbf{p})}_{\tilde{\mathbf{M}}(\mathbf{p})}\ddot{\mathbf{p}} + \underbrace{T_{,\dot{\mathbf{p}}\mathbf{p}}(\dot{\mathbf{p}},\mathbf{p})\dot{\mathbf{p}} - T_{\mathbf{p}}(\dot{\mathbf{p}},\mathbf{p}) + U_{\mathbf{p}}(\mathbf{p})}_{-\tilde{\mathbf{f}}(\dot{\mathbf{p}},\mathbf{p})} \\
&\quad + \underbrace{\sum_{j=1}^{n_\mathbf{p}} \gamma_j \tilde{k}_{j,\mathbf{p}}(\mathbf{p},\mathbf{q},t)}_{\tilde{\mathbf{K}}^T(\mathbf{p},\mathbf{q},t)\gamma} + \underbrace{\sum_{j=1}^{n_\lambda} \lambda_j \tilde{g}_{j,\mathbf{p}}(\mathbf{p},t)}_{\tilde{\mathbf{G}}^T(\mathbf{p},t)\lambda}, \\
\Rightarrow 0 &= \tilde{\mathbf{M}}(\mathbf{p})\ddot{\mathbf{p}} - \tilde{\mathbf{f}}(\dot{\mathbf{p}},\mathbf{p}) + \tilde{\mathbf{K}}^T(\mathbf{p},\mathbf{q},t)\gamma + \tilde{\mathbf{G}}^T(\mathbf{p},t)\lambda, \qquad (1)
\end{aligned}$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial(\dot{\mathbf{q}})}\right) - \frac{\partial L}{\partial \mathbf{q}} = 0 - \Bigg(\underbrace{-\sum_{j=1}^{n_{\mathbf{p}}} \gamma_j \tilde{k}_{j,\mathbf{q}}(\mathbf{p},\mathbf{q},t)}_{\tilde{\mathbf{J}}^T(\mathbf{p},\mathbf{q},t)\gamma} \Bigg), \\
\Rightarrow 0 &= \tilde{\mathbf{J}}^T(\mathbf{p},\mathbf{q},t)\gamma, \qquad (2)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\lambda}}\right) - \frac{\partial L}{\partial \lambda} = 0 - \Big(-\tilde{\mathbf{g}}(\mathbf{p},t) \Big), \\
\Rightarrow 0 &= \tilde{\mathbf{g}}(\mathbf{p},t), \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\gamma}}\right) - \frac{\partial L}{\partial \gamma} = 0 - \Big(-\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) \Big), \\
\Rightarrow 0 &= \tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) \qquad (4)
\end{aligned}
$$

Then, the equations of motion (1)-(4) of a multibody system in *mixed form* are given by

$$
\begin{aligned}
\tilde{\mathbf{M}}(\mathbf{p})\ddot{\mathbf{p}} &= \tilde{\mathbf{f}}(\dot{\mathbf{p}},\mathbf{p}) - \tilde{\mathbf{K}}^T(\mathbf{p},\mathbf{q},t)\gamma - \tilde{\mathbf{G}}^T(\mathbf{p},t)\lambda, && (5)\\
0 &= \tilde{\mathbf{J}}^T(\mathbf{p},\mathbf{q},t)\gamma, && (6)\\
0 &= \tilde{\mathbf{g}}(\mathbf{p},t), && (7)\\
0 &= \tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t). && (8)
\end{aligned}
$$

Here $\mathbf{p} \in \mathbb{R}^{n_{\mathbf{p}}}$ and $\mathbf{q} \in \mathbb{R}^{n_{\mathbf{q}}}$ are the absolute and relative coordinates of the multibody system and $\tilde{\mathbf{G}} = \tilde{\mathbf{g}}_{,\mathbf{p}}(\mathbf{p},t)$, $\tilde{\mathbf{K}} = \tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t)$ and $\tilde{\mathbf{J}} = \tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t)$. The matrix $\tilde{\mathbf{K}}$ is nonsingular and $\tilde{\mathbf{M}}$ is the symmetric, positive definite mass matrix. In the 2D case and a model in mixed (5-8) form the mass matrix is constant. The vector $\tilde{\mathbf{f}}(\dot{\mathbf{p}},\mathbf{p}) \in \mathbb{R}^{n_{\mathbf{p}}}$ represents the vector of applied and gyroscopic forces, $\tilde{\mathbf{G}}(\mathbf{p},t)$ contains column-wise the so called inaccessible directions of motion.

Step 8: Differentiate (8) twice with respect to the time t to obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) &= \tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t)\dot{\mathbf{p}} + \tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t)\dot{\mathbf{q}} + \tilde{\mathbf{k}}_{,t}(\mathbf{p},\mathbf{q},t), \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\tilde{\mathbf{k}}(\mathbf{p},\mathbf{q},t) &= \tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t)\ddot{\mathbf{p}} + \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t))\dot{\mathbf{p}} + \tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t)\ddot{\mathbf{q}} \\
&\quad + \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t))\dot{\mathbf{q}} + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{k}}_{,t}(\mathbf{p},\mathbf{q},t) \\
&= \underbrace{\tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t)}_{\tilde{\mathbf{K}}(\mathbf{p},\mathbf{q},t)}\ddot{\mathbf{p}} + \underbrace{\tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t)}_{\tilde{\mathbf{J}}(\mathbf{p},\mathbf{q},t)}\ddot{\mathbf{q}} \\
&\quad + \underbrace{\frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\mathbf{k}}_{,\mathbf{p}}(\mathbf{p},\mathbf{q},t))\dot{\mathbf{p}} + \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{\mathbf{k}}_{,\mathbf{q}}(\mathbf{p},\mathbf{q},t))\dot{\mathbf{q}} + \frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathbf{k}}_{,t}(\mathbf{p},\mathbf{q},t)}_{\tilde{\mathbf{k}}^{(II)}(\mathbf{p},\mathbf{q},t)}, \\
\Rightarrow 0 &= \tilde{\mathbf{K}}(\mathbf{p},\mathbf{q},t)\ddot{\mathbf{p}} + \tilde{\mathbf{J}}(\mathbf{p},\mathbf{q},t)\ddot{\mathbf{q}} + \tilde{\mathbf{k}}^{(II)}(\mathbf{p},\mathbf{q},t). \qquad (9)
\end{aligned}
$$

From (5),(6), (7) and (9) we obtain the equations of motion in the form

$$
\left[\begin{array}{ccc|c}
\tilde{\mathbf{M}} & \tilde{\mathbf{K}}^T & 0 & \tilde{\mathbf{G}}^T \\
\tilde{\mathbf{K}} & 0 & \tilde{\mathbf{J}} & 0 \\
0 & \tilde{\mathbf{J}}^T & 0 & 0 \\
\hline
0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} \ddot{\mathbf{p}} \\ \gamma \\ \ddot{\mathbf{q}} \\ \lambda \end{array}\right]
=
\left[\begin{array}{c} \tilde{\mathbf{f}} \\ -\tilde{\mathbf{k}}^{(II)} \\ 0 \\ \tilde{\mathbf{g}} \end{array}\right]. \qquad (10)
$$

With row manipulations using the Block-Gauss algorithm we obtain

$$\begin{bmatrix} \tilde{\mathbf{M}} & \tilde{\mathbf{K}}^T & 0 & \tilde{\mathbf{G}}^T \\ 0 & -\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}^T & \tilde{\mathbf{J}} & -\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{G}}^T \\ 0 & \tilde{\mathbf{J}}^T & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ \gamma \\ \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{f}} \\ -\tilde{\mathbf{k}}^{(II)} - \tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{f}} \\ 0 \\ \tilde{\mathbf{g}} \end{bmatrix},$$

$$\begin{bmatrix} \tilde{\mathbf{M}} & \tilde{\mathbf{K}}^T & 0 & \tilde{\mathbf{G}}^T \\ 0 & -\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}^T & \tilde{\mathbf{J}} & -\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{G}}^T \\ 0 & 0 & \tilde{\mathbf{J}}^T\tilde{\mathbf{K}}^{-T}\tilde{\mathbf{M}}\tilde{\mathbf{K}}^{-1}\tilde{\mathbf{J}} & -\tilde{\mathbf{J}}^T\tilde{\mathbf{K}}^{-T}\tilde{\mathbf{G}}^T \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{p}} \\ \gamma \\ \ddot{\mathbf{q}} \\ \lambda \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{f}} \\ -\tilde{\mathbf{k}}^{(II)} - \tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{f}} \\ -\tilde{\mathbf{J}}^T(\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}^T)^{-1}(\tilde{\mathbf{k}}^{(II)} + \tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{f}}) \\ \tilde{\mathbf{g}} \end{bmatrix}. \tag{11}$$

If we are only interested in the relative coordinates $\mathbf{q}$ and the Lagrange multipliers λ, which describe the dynamical behavior completely, we have to consider the last two equations

$$\begin{aligned} \underbrace{[\tilde{\mathbf{J}}^T\tilde{\mathbf{K}}^{-T}\tilde{\mathbf{M}}\tilde{\mathbf{K}}^{-1}\tilde{\mathbf{J}}]}_{=\mathbf{M}(\mathbf{q},t)}\ddot{\mathbf{q}} &= \underbrace{[-\tilde{\mathbf{J}}^T(\tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{K}}^T)^{-1}(\tilde{\mathbf{k}}^{(II)} + \tilde{\mathbf{K}}\tilde{\mathbf{M}}^{-1}\tilde{\mathbf{f}})]}_{=\mathbf{f}(\dot{\mathbf{q}},\mathbf{q},t)} + \underbrace{[\tilde{\mathbf{J}}^T\tilde{\mathbf{K}}^{-T}\tilde{\mathbf{G}}^T]}_{=-\mathbf{G}^T(\mathbf{q},t)}\lambda, \\ 0 &= \underbrace{\tilde{\mathbf{g}}(\mathbf{p}(\mathbf{q},t),t)}_{=\mathbf{g}(\mathbf{q},t)}. \end{aligned}$$

This corresponds to the *descriptor form* [15],[23]

$$\begin{aligned} \mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} &= \mathbf{f}(\dot{\mathbf{q}},\mathbf{q},t) - \mathbf{G}^T(\mathbf{q},t)\lambda, \\ 0 &= \mathbf{g}(\mathbf{q},t). \end{aligned} \tag{12}$$

As already mentioned before, the set of coordinates $\mathbf{p} \in \mathbb{R}^{n_\mathbf{p}}$ is not minimal and therefore redundant. The redundancies are described by holonomic constraints, i.e. the redundancies are described by a finite number of equations only with respect to the coordinates $\mathbf{p}$ itself. In (12) the matrix $\mathbf{M}(\mathbf{q},t) \in \mathbb{R}^{n_\mathbf{q},n_\mathbf{q}}$ is the symmetric and positive definite mass matrix, $\mathbf{f}(\dot{\mathbf{q}},\mathbf{q},t) \in \mathbb{R}^{n_\mathbf{q}}$ represents the vector of applied and gyroscopic forces and $\mathbf{G}(\mathbf{q},t) = \partial\mathbf{g}(\mathbf{q},t)/\partial\mathbf{q} \in \mathbb{R}^{n_\lambda,n_\mathbf{q}}$, since

$$\begin{aligned} \mathbf{G}(\mathbf{q},t) &:= \frac{\partial\mathbf{g}(\mathbf{q},t)}{\partial\mathbf{q}} \\ &= \frac{\partial\tilde{\mathbf{g}}(\mathbf{p}(\mathbf{q},t),t)}{\partial\mathbf{q}} \\ &= \frac{\partial\tilde{\mathbf{g}}(\mathbf{p}(\mathbf{q},t),t)}{\partial\mathbf{p}}\,\frac{\partial\mathbf{p}(\mathbf{q},t)}{\partial\mathbf{q}} \\ &= -\tilde{\mathbf{G}}(\mathbf{p},t)\mathbf{K}^{-1}(\mathbf{p},\mathbf{q},t)\mathbf{J}(\mathbf{p},\mathbf{q},t). \end{aligned}$$

Furthermore, $\mathbf{p} \in \mathbb{R}^{n_\mathbf{p}}$, $\mathbf{q} \in \mathbb{R}^{n_\mathbf{q}}$ and $\lambda \in \mathbb{R}^{n_\lambda}$.
If the set of relative coordinates $\mathbf{q} \in \mathbb{R}^{n_\mathbf{q}}$ is minimal, then there are no constraints $\mathbf{g}(\mathbf{q},t)$ and no constraint forces $\mathbf{G}^T\lambda$, and therefore the *state space form*

$$\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}} = \mathbf{f}(\dot{\mathbf{q}},\mathbf{q},t) \tag{13}$$

is obtained. Here $n_\mathbf{q}$ is the number of degrees of freedom of the system.

Remark 2.1.1 The Block-Gauss algorithm used in (11) is inefficient for multibody systems with a large number of bodies. For practical computations the left upper block of the equations of motion (10) is transformed to an *augmented block tridiagonal form* by row and column manipulations, see Example 2.1.2. Then one can use specially adapted solvers for the computation of $\mathbf{M}(\mathbf{q},t)\ddot{\mathbf{q}}$ and $\mathbf{f}(\dot{\mathbf{q}},\mathbf{q},t)$ ("$\mathcal{O}(N)$-formalisms").

Example 2.1.2 The left upper block of the matrix in (10) may be reordered with respect to the coordinates of each body of the system:

$$\begin{bmatrix} \tilde{\mathbf{M}} & \tilde{\mathbf{K}}^T & 0 \\ \tilde{\mathbf{K}} & 0 & \tilde{\mathbf{J}} \\ 0 & \tilde{\mathbf{J}}^T & 0 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \boxed{\#6} & 0 & 0 & \boxed{\#3\text{-}\#6} & 0 & 0 \\ 0 & \boxed{\#5} & \boxed{\#4\text{-}\#5} & 0 & 0 & 0 \\ 0 & \boxed{\#4\text{-}\#5} & \boxed{\#4} & \boxed{\#3\text{-}\#4} & 0 & 0 \\ \boxed{\#3\text{-}\#6} & 0 & \boxed{\#3\text{-}\#4} & \boxed{\#3} & \boxed{\#2\text{-}\#3} & 0 \\ 0 & 0 & 0 & \boxed{\#2\text{-}\#3} & \boxed{\#2} & \boxed{\#1\text{-}\#2} \\ 0 & 0 & 0 & 0 & \boxed{\#1\text{-}\#2} & \boxed{\#1} \end{bmatrix},$$

where $\#*$ describes the number of a body and $\# * -\#*$ describes the number of a joint between two bodies. The above structure refers to the topology of the multibody system in Figure 1. Each block of the matrix above has again the same structure as the left upper block in (10). This special structure can be exploited, in the solution process, when using the block-Cholesky or the block-Gauss algorithm [20].

2.2 Equations of Motion for Complex Multibody System Models

In the present paper we consider equations of motion in the following form. Here, the coordinate vector $\mathbf{p}$ contains arbitrary, not only absolute coordinates of the multibody system.

$$\begin{aligned} \dot{\mathbf{p}} &= \mathbf{Z}(\mathbf{p})\mathbf{v}, && (14)\\ \mathbf{M}(\mathbf{p},\mathbf{u})\dot{\mathbf{v}} &= \mathbf{f}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) - \mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},\mathbf{s},\mathbf{u})\lambda, && (15)\\ \dot{\mathbf{r}} &= \mathbf{b}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}), && (16)\\ 0 &= \mathbf{d}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}), && (17)\\ 0 &= \mathbf{c}(\mathbf{p},\mathbf{s},\mathbf{u}), && (18)\\ 0 &= \mathbf{g}(\mathbf{p},\mathbf{s},\mathbf{u}). && (19) \end{aligned}$$

For several reasons we prefer this formulation of the equations of motion that include more information and more details than the classical equations of motions (12) and (13).

- The equations of motion for spatial multibody systems with constraints were discussed in [9]. If the differential equation of second order (like (5), (12) or (13)) is reduced to first order one has to observe the relation between the *generalized velocity* $\dot{\mathbf{p}}$ and the velocities $\mathbf{v}$. To transform the second order system (5)-(8) to an equivalent first order system we introduce a velocity vector $\mathbf{v}$ and get

  $$\dot{\mathbf{p}} = \mathbf{Z}(\mathbf{p})\mathbf{v}, \qquad (20)$$

  with a nonsingular matrix $\mathbf{Z}(\mathbf{p})$, that determines the angular velocities. The transformation matrix $\mathbf{Z}(\mathbf{p})$ occurs only if there are rotations in three dimensional space, it may be determined by Poisson's kinematical equations [1],[9]. In the two dimensional case we have $\mathbf{Z}(\mathbf{p}) = \mathbf{I},\ \dot{\mathbf{p}} = \mathbf{v}$.

- In a real multibody system, there are dynamic force elements which are described by the state vector $\mathbf{r} \in \mathbb{R}^{n_\mathbf{r}}$ and determined by equation (16), see [9, Section 1.3.4.].

- Sometimes force laws and constraints may be formulated more conveniently using auxilary variables $\mathbf{w}$ that are implicitly defined by the nonlinear equation (17) with nonsingular Jacobian $\partial\mathbf{d}/\partial\mathbf{w}$.

- Furthermore, not all holonomic constraints of a multibody system are directly described by the state variables $\mathbf{p}$, but depend on certain contact points with coordinates $\mathbf{s}$ on the surface of some bodies. The relationship between these contact point coordinates $\mathbf{s}$ and the state variables $\mathbf{p}$ are given by (18) with nonsingular Jacobian $\partial\mathbf{c}/\partial\mathbf{s}$, see Figure 2.

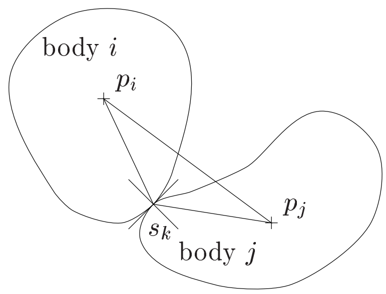

Figure 2: Contact point between two bodies

- The variable $\mathbf{u}$ describes the control. If the equations of motion are non-autonomous, then there exists an explicit dependence on time t. In this case the time will be modelled as a component of the control $\mathbf{u}$, for example $u_1(t) = t$. In the following the control $\mathbf{u}$ is assumed to be given.

Equations (14)-(19) form the equations of motion of an arbitrary spatial multibody system with holonomic constraints.
The $n_\mathbf{p}$ equations (14) are called kinematic equations of motion. They accomplish the order reduction of the equations of motion (5-8), (12) and (13) from order two to order one.
The $n_\mathbf{p}$ equations (15) are called dynamic equations of motion. They follow from the equilibrium of forces and momenta and include the mass matrix $\mathbf{M}(\mathbf{p},\mathbf{u})$, the vector $\mathbf{f}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u})$ of the applied and gyroscopic forces, the constraint matrix $\mathbf{G}(\mathbf{p},\mathbf{s},\mathbf{u})$ of the holonomic constraints, the associated constraint forces $\mathbf{G}^T(\mathbf{p},\mathbf{s},\mathbf{u})\lambda$ and the Lagrange multipliers λ. The mass matrix $\mathbf{M}(\mathbf{p},\mathbf{u})$ is nonsingular and positive definite, since the kinetic energy is a positive quadratic form. The mass matrix includes the inertia properties of the multibody system.
The Jacobians

$$\frac{\partial\mathbf{d}}{\partial\mathbf{w}} \text{ and } \frac{\partial\mathbf{c}}{\partial\mathbf{s}} \tag{21}$$

are assumed to be nonsingular, such that (17) and (18) determine $\mathbf{w} = \mathbf{w}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{s},\lambda,\mathbf{u})$ and $\mathbf{s} = \mathbf{s}(\mathbf{p},\mathbf{u})$.
Furthermore, the equations of motion are affected by the n_λ holonomic constraints (19). From the constraints $\mathbf{g}(\mathbf{p},\mathbf{s},\mathbf{u}) = 0$ one obtains the constraint matrix $\mathbf{G}(\mathbf{p},\mathbf{s},\mathbf{u})$ which contains column-wise the inaccessible directions of motion.

Lemma 2.2.1 *The constraint matrix* $\mathbf{G}$ *in (15) is given by*

$$\mathbf{G}(\mathbf{p},\mathbf{s},\mathbf{u}) = \left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}} - \frac{\partial \mathbf{g}}{\partial \mathbf{s}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{s}}\right)^{-1}\frac{\partial \mathbf{c}}{\partial \mathbf{p}}\right](\mathbf{p},\mathbf{s},\mathbf{u}). \tag{22}$$

Proof: We have

$$\begin{aligned}
\mathbf{G}(\mathbf{p},\mathbf{s},\mathbf{u}) &:= \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\mathbf{p}}(\mathbf{p},\mathbf{s}(\mathbf{p},\mathbf{u}),\mathbf{u}) \\
&= \left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}} + \frac{\partial \mathbf{g}}{\partial \mathbf{s}}\frac{\partial \mathbf{s}}{\partial \mathbf{p}}\right](\mathbf{p},\mathbf{s}(\mathbf{p},\mathbf{u}),\mathbf{u}) \\
&= \left[\frac{\partial \mathbf{g}}{\partial \mathbf{p}} - \frac{\partial \mathbf{g}}{\partial \mathbf{s}}\left(\frac{\partial \mathbf{c}}{\partial \mathbf{s}}\right)^{-1}\frac{\partial \mathbf{c}}{\partial \mathbf{p}}\right](\mathbf{p},\mathbf{s},\mathbf{u}).
\end{aligned}$$

The Jacobian $\frac{\partial \mathbf{s}}{\partial \mathbf{p}}$ has been obtained from (18) by implicit differentiation with respect to $\mathbf{p}$. ■

To avoid redundant constraints we assume that the so called *Grübler condition* is satisfied [11], i.e. that $\mathbf{G}(\mathbf{p},\mathbf{s},\mathbf{u})$ has full row rank.

In general one obtains a nonlinear differential-algebraic equation of the form

$$0 = \mathbf{F}(\dot{\mathbf{x}},\mathbf{x},\mathbf{u}), \tag{23}$$

where

$$\mathbf{F}(\dot{\mathbf{x}},\mathbf{x},\mathbf{u}) = \begin{bmatrix} \dot{\mathbf{p}} - \mathbf{Z}(\mathbf{p})\mathbf{v} \\ \mathbf{M}(\mathbf{p},\mathbf{u})\dot{\mathbf{v}} - \mathbf{f}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) \\ \qquad +\mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},\mathbf{s},\mathbf{u})\lambda \\ \dot{\mathbf{r}} - \mathbf{b}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) \\ \mathbf{d}(\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) \\ \mathbf{c}(\mathbf{p},\mathbf{s},\mathbf{u}) \\ \mathbf{g}(\mathbf{p},\mathbf{s},\mathbf{u}) \end{bmatrix}, \tag{24}$$

$$\begin{aligned}
\mathbf{x}(t) &= \left[\mathbf{v}^T,\mathbf{p}^T,\mathbf{r}^T,\mathbf{w}^T,\mathbf{s}^T,\lambda^T\right]^T \\
&\in \mathbb{R}^{n_\mathbf{p}+n_\mathbf{p}+n_\mathbf{r}+n_\mathbf{w}+n_\mathbf{s}+n_\lambda}.
\end{aligned} \tag{25}$$

2.3 Linearized Equations of Motion

In this section we linearize the equations of motion (14) - (19) in function space along reference functions $\tilde{\mathbf{v}}(t)$, $\tilde{\mathbf{p}}(t)$, ... respectively. The linearization in function space is called quasi-linearization. The reference function does not have to be a solution of the equations of motion, but for a good approximating linearization, it should lie in a small neighbourhood of a solution.

$$\begin{aligned}
\mathbf{v} &= \tilde{\mathbf{v}} + \Delta\mathbf{v}, & \mathbf{p} &= \tilde{\mathbf{p}} + \Delta\mathbf{p}, & \mathbf{r} &= \tilde{\mathbf{r}} + \Delta\mathbf{r}, \\
\mathbf{w} &= \tilde{\mathbf{w}} + \Delta\mathbf{w}, & \mathbf{s} &= \tilde{\mathbf{s}} + \Delta\mathbf{s}, & \lambda &= \tilde{\lambda} + \Delta\lambda, \\
\mathbf{u} &= \tilde{\mathbf{u}} + \Delta\mathbf{u}.
\end{aligned} \tag{26}$$

With this approach the kinematic (14) and dynamic (15) equations of motion have the following form.

$$\begin{aligned}
\dot{\tilde{\mathbf{p}}} + \dot{\Delta\mathbf{p}} &= \mathbf{Z}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\tilde{\mathbf{v}} + \Delta\mathbf{v}), \\
\mathbf{M}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})&(\dot{\tilde{\mathbf{v}}} + \dot{\Delta\mathbf{v}}) \\
&= \mathbf{f}(\tilde{\mathbf{v}} + \Delta\mathbf{v}, \tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{r}} + \Delta\mathbf{r}, \tilde{\mathbf{w}} + \Delta\mathbf{w}, \tilde{\mathbf{s}} + \Delta\mathbf{s}, \tilde{\lambda} + \Delta\lambda, \tilde{\mathbf{u}} + \Delta\mathbf{u}) \\
&\quad -\mathbf{Z}^T(\tilde{\mathbf{p}} + \Delta\mathbf{p})\mathbf{G}^T(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{s}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\tilde{\lambda} + \Delta\lambda)
\end{aligned}$$

Expanding every function by Taylor's theorem to first order we obtain from Definition A.0.15 and Lemma A.0.16

$$\begin{aligned}
\dot{\tilde{\mathbf{p}}} + \dot{\Delta\mathbf{p}} &= \mathbf{Z}(\tilde{\mathbf{p}})\tilde{\mathbf{v}} + (\mathbf{Z}(\tilde{\mathbf{p}})\tilde{\mathbf{v}})_{,\mathbf{p}}\Delta\mathbf{p} + \mathbf{Z}(\tilde{\mathbf{p}})\Delta\mathbf{v} + (\mathbf{Z}(\tilde{\mathbf{p}})\Delta\mathbf{v})_{,\mathbf{p}}\Delta\mathbf{p} + \text{h. o. t.}\\
\mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\tilde{\mathbf{v}}} &+ (\mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}}\Delta\mathbf{p} + (\mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\tilde{\mathbf{v}}})_{,\mathbf{u}}\Delta\mathbf{u} + \mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\Delta\mathbf{v}}\\
&+(\mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\Delta\mathbf{v}})_{,\mathbf{p}}\Delta\mathbf{p} + (\mathbf{M}(\tilde{\mathbf{p}},\tilde{\mathbf{u}})\dot{\Delta\mathbf{v}})_{,\mathbf{u}}\Delta\mathbf{u}\\
&= \mathbf{f}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}}) + \mathbf{f}_{,\mathbf{v}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{v}\\
&+ \mathbf{f}_{,\mathbf{p}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{p} + \mathbf{f}_{,\mathbf{r}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{r}\\
&+ \mathbf{f}_{,\mathbf{w}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{w} + \mathbf{f}_{,\mathbf{s}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{s}\\
&+ \mathbf{f}_{,\lambda}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\lambda + \mathbf{f}_{,\mathbf{u}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{u}\\
&- \mathbf{Z}^T(\tilde{\mathbf{p}})\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\tilde{\lambda} - \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\tilde{\lambda})_{,\mathbf{p}}\Delta\mathbf{p}\\
&- \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\tilde{\lambda})_{,\mathbf{s}}\Delta\mathbf{s} - \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\tilde{\lambda})_{,\mathbf{u}}\Delta\mathbf{u}\\
&- \mathbf{Z}^T(\tilde{\mathbf{p}})\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\lambda - \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\lambda)_{,\mathbf{p}}\Delta\mathbf{p}\\
&- \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\lambda)_{,\mathbf{s}}\Delta\mathbf{s} - \mathbf{Z}^T(\tilde{\mathbf{p}})(\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\lambda)_{,\mathbf{u}}\Delta\mathbf{u}\\
&- \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\tilde{\lambda} - \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{p})\tilde{\lambda}\\
&- \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{s})\tilde{\lambda} - \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{u})\tilde{\lambda}\\
&- \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\lambda - \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{p})\Delta\lambda\\
&- \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{s})\Delta\lambda - \mathbf{Z}^T_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\mathbf{G}^T_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\Delta\mathbf{u})\Delta\lambda\\
&+ \text{h. o. t.}
\end{aligned}$$

If we neglect the quadratic and higher order terms, reorder and omit the arguments, using Definition A.0.8 and Remark A.0.9, it follows that

$$\begin{aligned}
\dot{\Delta\mathbf{p}} &= \mathbf{Z}\Delta\mathbf{v} + (\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}}\Delta\mathbf{p} + (-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}),\\
\mathbf{M}\dot{\Delta\mathbf{v}} &= \mathbf{f}_{,\mathbf{v}}\Delta\mathbf{v} + (-(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}} + \mathbf{f}_{,\mathbf{p}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{p}} - \mathbf{Z}^T_{,\mathbf{p}}()(\cdot)\mathbf{G}^T\tilde{\lambda})\Delta\mathbf{p} + \mathbf{f}_{,\mathbf{r}}\Delta\mathbf{r}\\
&\quad+\mathbf{f}_{,\mathbf{w}}\Delta\mathbf{w} + (\mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{s}})\Delta\mathbf{s} + (\mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T)\Delta\lambda\\
&\quad+(-(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{u}} + \mathbf{f}_{,\mathbf{u}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{u}})\Delta\mathbf{u} + (-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}).
\end{aligned}$$

Finally we consider equations (16)-(19) in the same way.

$$\begin{aligned}
(\dot{\tilde{\mathbf{r}}} + \dot{\Delta\mathbf{r}}) &= \mathbf{b}(\tilde{\mathbf{v}}+\Delta\mathbf{v},\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{r}}+\Delta\mathbf{r},\tilde{\mathbf{w}}+\Delta\mathbf{w},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\lambda}+\Delta\lambda,\tilde{\mathbf{u}}+\Delta\mathbf{u}),\\
0 &= \mathbf{d}(\tilde{\mathbf{v}}+\Delta\mathbf{v},\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{r}}+\Delta\mathbf{r},\tilde{\mathbf{w}}+\Delta\mathbf{w},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\lambda}+\Delta\lambda,\tilde{\mathbf{u}}+\Delta\mathbf{u}),\\
0 &= \mathbf{c}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s}),\\
0 &= \mathbf{g}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u}).
\end{aligned}$$

Taylor series expansion of the first order yields

$$\begin{aligned}
(\dot{\tilde{\mathbf{r}}} + \dot{\Delta\mathbf{r}}) &= \mathbf{b}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}}) + \mathbf{b}_{,\mathbf{v}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{v}\\
&+ \mathbf{b}_{,\mathbf{p}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{p} + \mathbf{b}_{,\mathbf{r}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{r}\\
&+ \mathbf{b}_{,\mathbf{w}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{w} + \mathbf{b}_{,\mathbf{s}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{s}\\
&+ \mathbf{b}_{,\lambda}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\lambda + \mathbf{b}_{,\mathbf{u}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{u},\\
0 &= \mathbf{d}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}}) + \mathbf{d}_{,\mathbf{v}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{v}\\
&+ \mathbf{d}_{,\mathbf{p}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{p} + \mathbf{d}_{,\mathbf{r}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{r}\\
&+ \mathbf{d}_{,\mathbf{w}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{w} + \mathbf{d}_{,\mathbf{s}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{s}\\
&+ \mathbf{d}_{,\lambda}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\lambda + \mathbf{d}_{,\mathbf{u}}(\tilde{\mathbf{v}},\tilde{\mathbf{p}},\tilde{\mathbf{r}},\tilde{\mathbf{w}},\tilde{\mathbf{s}},\tilde{\lambda},\tilde{\mathbf{u}})\Delta\mathbf{u},\\
0 &= \mathbf{c}(\tilde{\mathbf{p}},\tilde{\mathbf{s}}) + \mathbf{c}_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}})\Delta\mathbf{p} + \mathbf{c}_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}})\Delta\mathbf{s},\\
0 &= \mathbf{g}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}}) + \mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\mathbf{p} + \mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\mathbf{s} + \mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})\Delta\mathbf{u}.
\end{aligned}$$

Neglecting quadratic and higher order terms, reordering and omitting the dependences we get

$$\begin{aligned}
\dot{\Delta\mathbf{r}} &= \mathbf{b}_{,\mathbf{v}}\Delta\mathbf{v} + \mathbf{b}_{,\mathbf{p}}\Delta\mathbf{p} + \mathbf{b}_{,\mathbf{r}}\Delta\mathbf{r} + \mathbf{b}_{,\mathbf{w}}\Delta\mathbf{w} + \mathbf{b}_{,\mathbf{s}}\Delta\mathbf{s} + \mathbf{b}_{,\lambda}\Delta\lambda + \mathbf{b}_{,\mathbf{u}}\Delta\mathbf{u} + (-\dot{\tilde{\mathbf{r}}} + \mathbf{b}),\\
0 &= \mathbf{d}_{,\mathbf{v}}\Delta\mathbf{v} + \mathbf{d}_{,\mathbf{p}}\Delta\mathbf{p} + \mathbf{d}_{,\mathbf{r}}\Delta\mathbf{r} + \mathbf{d}_{,\mathbf{w}}\Delta\mathbf{w} + \mathbf{d}_{,\mathbf{s}}\Delta\mathbf{s} + \mathbf{d}_{,\lambda}\Delta\lambda + \mathbf{d}_{,\mathbf{u}}\Delta\mathbf{u} + \mathbf{d},\\
0 &= \mathbf{c}_{,\mathbf{p}}\Delta\mathbf{p} + \mathbf{c}_{,\mathbf{s}}\Delta\mathbf{s} + \mathbf{c},\\
0 &= \mathbf{g}_{,\mathbf{p}}\Delta\mathbf{p} + \mathbf{g}_{,\mathbf{s}}\Delta\mathbf{s} + \mathbf{g}_{,\mathbf{u}}\Delta\mathbf{u} + \mathbf{g}.
\end{aligned}$$

In this way we obtain a linear inhomogeneous differential-algebraic equation with time dependent coefficients of the form

$$\mathbf{E}(t)\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{h}(t). \tag{27}$$

Here the matrices $\mathbf{E}(t), \mathbf{A}(t)$, the inhomogeneity $\mathbf{h}(t)$ and the vector of the independent variables $\mathbf{x}(t)$ are given by:

$$\mathbf{E}(t) = \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{M} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{h}(t) = \begin{bmatrix} -\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}\\ -\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}\\ -\dot{\tilde{\mathbf{r}}} + \mathbf{b}\\ \mathbf{d}\\ \mathbf{c}\\ \mathbf{g} \end{bmatrix},$$

$$\mathbf{A}(t) = \begin{bmatrix} (\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & \mathbf{Z} & 0 & 0 & 0 & 0\\ A_{\mathbf{pp}} & \mathbf{f}_{,\mathbf{v}} & \mathbf{f}_{,\mathbf{r}} & \mathbf{f}_{,\mathbf{w}} & A_{\mathbf{ps}} & A_{\mathbf{p}\lambda}\\ \mathbf{b}_{,\mathbf{p}} & \mathbf{b}_{,\mathbf{v}} & \mathbf{b}_{,\mathbf{r}} & \mathbf{b}_{,\mathbf{w}} & \mathbf{b}_{,\mathbf{s}} & \mathbf{b}_{,\lambda}\\ \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda}\\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0\\ \mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}, \mathbf{x}(t) = \begin{bmatrix} \Delta\mathbf{p}\\ \Delta\mathbf{v}\\ \Delta\mathbf{r}\\ \Delta\mathbf{w}\\ \Delta\mathbf{s}\\ \Delta\lambda \end{bmatrix}, \tag{28}$$

with

$$\begin{aligned}
A_{\mathbf{pp}} &= -(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}} + \mathbf{f}_{,\mathbf{p}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{p}} - \mathbf{Z}^T_{,\mathbf{p}}()(\cdot)(\mathbf{G}^T\tilde{\lambda}),\\
A_{\mathbf{ps}} &= \mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T\mathbf{G}^T_{,\mathbf{s}}\tilde{\lambda},\\
A_{\mathbf{p}\lambda} &= \mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T.
\end{aligned}$$

Recall that the control $\mathbf{u}$ is assumed to be given, it remains as nonlinear function in the matrices $\mathbf{E}$, $\mathbf{A}$ and the vector $\mathbf{h}$.
The matrix entries are evaluated at the reference functions $\tilde{\mathbf{v}}(t)$, $\tilde{\mathbf{p}}(t)$, ... respectively. For this reason the matrices $\mathbf{E}(t)$, $\mathbf{A}(t)$ and the vector $\mathbf{h}(t)$ only depend on time, since the reference function and the control are given.

2.4 The Index of the Equations of Motion

If (23) is differentiated l times with respect to t, we get the $(l+1)n$ *derivative array equations*

$$\tilde{\mathbf{F}}_l(\mathbf{z}, \dot{\mathbf{x}}, \mathbf{x}, t) = \begin{bmatrix} \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t)\\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t)\\ \vdots\\ \frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \end{bmatrix},$$

where

$$\mathbf{z} = [\mathbf{x}^{(k+1)}, ..., \ddot{\mathbf{x}}].$$

This notation was introduced by Campbell [5].

A definition of the *differentiation index* of a nonlinear differential-algebraic equation (23) is given in [6] as

Definition 2.4.1 *Suppose that (23) is a solvable DAE on an open set Ω. If $\mathbf{y}$ is uniquely determined by $\mathbf{x}$, t and $\tilde{\mathbf{F}}_l(\mathbf{z}, \mathbf{y}, \mathbf{x}, t) = 0$ for all consistent values and ν_d is the least such integer l that holds for, we call ν_d the differention index of the DAE.*

See also [12] and [13].
It is shown in [18] that a nonlinear differential-algebraic equation (23) has the same differentiation index as its linearization (27). The differentiation index of the linear differential-algebraic equation (27) is given in [16] by the following Lemma.

Lemma 2.4.2 (Differentiation index of a linear DAE) *Let $\mathbf{E}(t)$ and $\mathbf{A}(t)$ in (27) be sufficiently smooth and let*

$$\mathbf{E}_l(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \mathbf{E}(t),\ \mathbf{A}_l(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \mathbf{A}(t),\ \mathbf{x}_l(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \mathbf{x}(t),\ \mathbf{h}_l(t) = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^l \mathbf{h}(t),$$

for $l = 0, 1, \ldots$. Formal differentiation of (27) then leads to linear equations of the form

$$\mathcal{M}_l(t)\dot{\mathbf{z}}_l(t) = \mathcal{N}_l(t)\mathbf{z}_l(t) + \mathcal{H}_l(t)$$

for any l, where

$$\begin{aligned}
(\mathcal{M}_l(t))_{ij} &= \binom{i}{j}\mathbf{E}_{i-j}(t) - \binom{i}{j+1}\mathbf{A}_{i-j-1}(t),\\
(\mathcal{N}_l(t))_{ij} &= \begin{cases} \mathbf{A}_i(t) & \text{for } i = 0, ..., l,\ j = 0,\\ 0 & \text{else},\end{cases}\\
(\mathcal{H}_l(t))_i &= \mathbf{h}_i(t),\\
(\mathbf{z}_l(t))_i &= \mathbf{x}_i(t),
\end{aligned}$$

$i, j = 0, ..., l$. We use the convention that $\binom{i}{j} = 0$ for $i < 0$, $j < 0$ or $j > i$.
The differentiation index ν of (27) is then the smallest value $l \in \mathbb{N}_0$ for which $\mathcal{M}_l(t)$ is smoothly 1-full, i.e., for which there exists a continuous and nonsingular matrix function $\mathbf{R}(t)$ with

$$\mathbf{R}(t)\mathcal{M}_l(t) = \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & \times \end{bmatrix}, \tag{29}$$

for all t.

Remark 2.4.3 **1)** If the differentiation index of a differential-algebraic equation is ν, then the derivatives up to the order ν of the original differential-algebraic equation will be needed to create the underlying ordinary differential equation.
2) If the equations of the original differential-algebraic equation are perturbed then the derivatives up to order $\nu-1$ of these perturbations have influence on the solution of this system [8]. Perturbations with small amplitudes but high frequencies can lead to large oscillations of the solution.

Lemma 2.4.4 *The equations of motion (24) form a system of differentiation index 3 as long as*

$$\mathbf{GZM}^{-1}\left[-\frac{\partial \mathbf{f}}{\partial \lambda}+\frac{\partial \mathbf{f}}{\partial \mathbf{w}}\left(\frac{\partial \mathbf{d}}{\partial \mathbf{w}}\right)^{-1}\frac{\partial \mathbf{d}}{\partial \lambda}+\mathbf{Z}^T\mathbf{G}^T\right](\mathbf{v},\mathbf{p},\mathbf{r},\mathbf{s},\lambda,\mathbf{u}) \tag{30}$$

is non-singular.

Proof: It is possible to prove the Lemma directly by using Definition 2.4.1 or by using Lemma 2.4.2. We will use Lemma 2.4.2, since the used transformations are similar to the transformations which one needs for the index reduction later. Therefore, we will determine the differentiation index of the linearized equations of motion (28) checking successively if the matrix $\mathcal{M}_l(t)$ is 1-full for $l = 0, 1, \dots$. We have

$$\mathcal{M}_0 = \mathbf{E} = \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\mathbf{E}$ does not have full rank, there does not exist a matrix $\mathbf{R}_0$ such that $\mathbf{R}_0\mathcal{M}_0$ has the form (29). Consider

$$\mathcal{M}_1 = \begin{bmatrix} \mathbf{E} & 0 \\ \dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} \end{bmatrix},$$

where

$$\dot{\mathbf{E}} - \mathbf{A} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & \dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ -\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}.$$

Since $\mathbf{M}, \mathbf{d}_{,\mathbf{w}}$ and $\mathbf{c}_{,\mathbf{s}}$ are nonsingular, there exists a matrix $\mathbf{R}_1$ such that

$$\mathbf{R}_1\mathcal{M}_1 = \tilde{\mathbf{R}}_1 \begin{bmatrix} \mathbf{E} & 0 \\ \tilde{\mathcal{M}}_{21} & \mathbf{E} \end{bmatrix},$$

where

$$\tilde{\mathcal{M}}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_{\mathbf{p}\lambda} + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{b}_{,\lambda} + \mathbf{b}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & \mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

With $A_{\mathbf{p}\lambda} = \mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T$, it follows that

$$\mathbf{R}_1\mathcal{M}_1 = \begin{bmatrix} \mathcal{M}_{11} & 0 \\ \mathcal{M}_{21} & \mathbf{E} \end{bmatrix},$$

where

$$\mathcal{M}_{11} = \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & \mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{b}_{,\lambda} + \mathbf{b}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Again it is not possible to find a matrix $\mathbf{R}_1$ such that

$$\mathbf{R}_1\mathcal{M}_1 = \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & \times \end{bmatrix},$$

since any linear combination of the second and third row of $\mathcal{M}_{21}$ added to the fourth row of $\mathcal{M}_{11}$ leads to entries at the upper right block.
Now consider

$$\mathcal{M}_2 = \begin{bmatrix} \mathbf{E} & 0 & 0 \\ \dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} & 0 \\ \ddot{\mathbf{E}} - 2\dot{\mathbf{A}} & 2\dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} \end{bmatrix},$$

where

$$\dot{\mathbf{E}} - \mathbf{A} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & \dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ -\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$\ddot{\mathbf{E}} - 2\dot{\mathbf{A}} = \begin{bmatrix} -2(\mathbf{Z}\tilde{\mathbf{v}})^{\cdot}_{,\mathbf{p}} & -2\dot{\mathbf{Z}} & 0 & 0 & 0 & 0 \\ -2\dot{A}_{\mathbf{pp}} & \ddot{\mathbf{M}} - 2\dot{\mathbf{f}}_{,\mathbf{v}} & -2\dot{\mathbf{f}}_{,\mathbf{r}} & -2\dot{\mathbf{f}}_{,\mathbf{w}} & -2\dot{A}_{\mathbf{ps}} & -2\dot{A}_{\mathbf{p}\lambda} \\ -2\dot{\mathbf{b}}_{,\mathbf{p}} & -2\dot{\mathbf{b}}_{,\mathbf{v}} & -2\dot{\mathbf{b}}_{,\mathbf{r}} & -2\dot{\mathbf{b}}_{,\mathbf{w}} & -2\dot{\mathbf{b}}_{,\mathbf{s}} & -2\dot{\mathbf{b}}_{,\lambda} \\ -2\dot{\mathbf{d}}_{,\mathbf{p}} & -2\dot{\mathbf{d}}_{,\mathbf{v}} & -2\dot{\mathbf{d}}_{,\mathbf{r}} & -2\dot{\mathbf{d}}_{,\mathbf{w}} & -2\dot{\mathbf{d}}_{,\mathbf{s}} & -2\dot{\mathbf{d}}_{,\lambda} \\ -2\dot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & -2\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ -2\dot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & -2\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$2\dot{\mathbf{E}} - \mathbf{A} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & 2\dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ -\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}.$$

We can transform the derivative array $\mathcal{M}_2$ in the following way. Compare also with the transformations of $\mathcal{M}_1$.

$$\mathbf{R}_2\mathcal{M}_2 = \bar{\mathbf{R}}_2 \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 \\ \ddot{\mathbf{E}} - 2\dot{\mathbf{A}} & 2\dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} \end{bmatrix}$$

$$= \tilde{\mathbf{R}}_2 \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 \\ \tilde{\mathcal{M}}_{31} & \tilde{\mathcal{M}}_{32} & \mathbf{E} \end{bmatrix},$$

where

$$\tilde{\mathcal{M}}_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\dot{\mathbf{f}}_{,\mathbf{w}} & 0 & -2\dot{A}_{\mathbf{p}\lambda} \\ 0 & 0 & 0 & -2\dot{\mathbf{b}}_{,\mathbf{w}} & 0 & -2\dot{\mathbf{b}}_{,\lambda} \\ 0 & 0 & 0 & -2\dot{\mathbf{d}}_{,\mathbf{w}} & 0 & -2\dot{\mathbf{d}}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{32} = \begin{bmatrix} 0 & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ 0 & 2\dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & 0 & -A_{\mathbf{p}\lambda} \\ 0 & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & 0 & -\mathbf{b}_{,\lambda} \\ 0 & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & 0 & -\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Furthermore, it follows that

$$\mathbf{R}_2 \mathcal{M}_2 = \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 \\ \mathcal{M}_{31} & \mathcal{M}_{21} & \mathbf{E} \end{bmatrix},$$

where

$$\mathcal{M}_{11} = \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & \mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{b}_{,\lambda} + \mathbf{b}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{31} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{Z}\mathbf{M}^{-1}(-\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda}) \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

In the following we consider $\mathcal{M}_3$, where the first three block rows will be transformed like $\mathcal{M}_2$.

$$\mathcal{M}_3 = \begin{bmatrix} \mathbf{E} & 0 & 0 & 0 \\ \dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} & 0 & 0 \\ \ddot{\mathbf{E}} - 2\dot{\mathbf{A}} & 2\dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} & 0 \\ \mathbf{E}^{(3)} - 3\ddot{\mathbf{A}} & 3\ddot{\mathbf{E}} - 3\dot{\mathbf{A}} & 3\dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} \end{bmatrix}$$

and

$$\mathbf{R}_3\mathcal{M}_3 = \hat{\mathbf{R}}_3 \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 & 0 \\ \mathcal{M}_{31} & \mathcal{M}_{21} & \mathbf{E} & 0 \\ \mathbf{E}^{(3)} - 3\ddot{\mathbf{A}} & 3\ddot{\mathbf{E}} - 3\dot{\mathbf{A}} & 3\dot{\mathbf{E}} - \mathbf{A} & \mathbf{E} \end{bmatrix},$$

where

$$\mathbf{E}^{(3)} - 3\ddot{\mathbf{A}} = \begin{bmatrix} -3(\mathbf{Z}\tilde{\mathbf{v}})\ddot{}_{,\mathbf{p}} & -3\ddot{\mathbf{Z}} & 0 & 0 & 0 & 0 \\ -3\ddot{A}_{\mathbf{pp}} & \mathbf{M}^{(3)} - 3\ddot{\mathbf{f}}_{,\mathbf{v}} & -3\ddot{\mathbf{f}}_{,\mathbf{r}} & -3\ddot{\mathbf{f}}_{,\mathbf{w}} & -3\ddot{A}_{\mathbf{ps}} & -3\ddot{A}_{\mathbf{p}\lambda} \\ -3\ddot{\mathbf{b}}_{,\mathbf{p}} & -3\ddot{\mathbf{b}}_{,\mathbf{v}} & -3\ddot{\mathbf{b}}_{,\mathbf{r}} & -3\ddot{\mathbf{b}}_{,\mathbf{w}} & -3\ddot{\mathbf{b}}_{,\mathbf{s}} & -3\ddot{\mathbf{b}}_{,\lambda} \\ -3\ddot{\mathbf{d}}_{,\mathbf{p}} & -3\ddot{\mathbf{d}}_{,\mathbf{v}} & -3\ddot{\mathbf{d}}_{,\mathbf{r}} & -3\ddot{\mathbf{d}}_{,\mathbf{w}} & -3\ddot{\mathbf{d}}_{,\mathbf{s}} & -3\ddot{\mathbf{d}}_{,\lambda} \\ -3\ddot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & -3\ddot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ -3\ddot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & -3\ddot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$3\ddot{\mathbf{E}} - 3\dot{\mathbf{A}} = \begin{bmatrix} -3(\mathbf{Z}\tilde{\mathbf{v}})\dot{}_{,\mathbf{p}} & -3\dot{\mathbf{Z}} & 0 & 0 & 0 & 0 \\ -3\dot{A}_{\mathbf{pp}} & 3\ddot{\mathbf{M}} - 3\dot{\mathbf{f}}_{,\mathbf{v}} & -3\dot{\mathbf{f}}_{,\mathbf{r}} & -3\dot{\mathbf{f}}_{,\mathbf{w}} & -3\dot{A}_{\mathbf{ps}} & -3\dot{A}_{\mathbf{p}\lambda} \\ -3\dot{\mathbf{b}}_{,\mathbf{p}} & -3\dot{\mathbf{b}}_{,\mathbf{v}} & -3\dot{\mathbf{b}}_{,\mathbf{r}} & -3\dot{\mathbf{b}}_{,\mathbf{w}} & -3\dot{\mathbf{b}}_{,\mathbf{s}} & -3\dot{\mathbf{b}}_{,\lambda} \\ -3\dot{\mathbf{d}}_{,\mathbf{p}} & -3\dot{\mathbf{d}}_{,\mathbf{v}} & -3\dot{\mathbf{d}}_{,\mathbf{r}} & -3\dot{\mathbf{d}}_{,\mathbf{w}} & -3\dot{\mathbf{d}}_{,\mathbf{s}} & -3\dot{\mathbf{d}}_{,\lambda} \\ -3\dot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & -3\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ -3\dot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & -3\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$3\dot{\mathbf{E}} - \mathbf{A} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & 3\dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ -\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}.$$

Hence

$$\mathbf{R}_3\mathcal{M}_3 = \bar{\mathbf{R}}_3 \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 & 0 \\ \mathcal{M}_{31} & \mathcal{M}_{21} & \mathbf{E} & 0 \\ \bar{\mathcal{M}}_{41} & \bar{\mathcal{M}}_{42} & \bar{\mathcal{M}}_{43} & \mathbf{E} \end{bmatrix},$$

where

$$\bar{\mathcal{M}}_{41} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\ddot{\mathbf{f}}_{,\mathbf{w}} & 0 & -3\ddot{A}_{\mathbf{p}\lambda} \\ 0 & 0 & 0 & -3\ddot{\mathbf{b}}_{,\mathbf{w}} & 0 & -3\ddot{\mathbf{b}}_{,\lambda} \\ 0 & 0 & 0 & -3\ddot{\mathbf{d}}_{,\mathbf{w}} & 0 & -3\ddot{\mathbf{d}}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\mathcal{M}}_{42} = \begin{bmatrix} 0 & -3\dot{\mathbf{Z}} & 0 & 0 & 0 & 0 \\ 0 & 3\ddot{\mathbf{M}} - 3\dot{\mathbf{f}}_{,\mathbf{v}} & -3\dot{\mathbf{f}}_{,\mathbf{r}} & -3\dot{\mathbf{f}}_{,\mathbf{w}} & 0 & -3\dot{A}_{\mathbf{p}\lambda} \\ 0 & -3\dot{\mathbf{b}}_{,\mathbf{v}} & -3\dot{\mathbf{b}}_{,\mathbf{r}} & -3\dot{\mathbf{b}}_{,\mathbf{w}} & 0 & -3\dot{\mathbf{b}}_{,\lambda} \\ 0 & -3\dot{\mathbf{d}}_{,\mathbf{v}} & -3\dot{\mathbf{d}}_{,\mathbf{r}} & -3\dot{\mathbf{d}}_{,\mathbf{w}} & 0 & -3\dot{\mathbf{d}}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\mathcal{M}}_{43} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & 3\dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ -\mathbf{g}_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From (22) it follows that $-\mathbf{g}_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} = -\mathbf{G}$ and therefore

$$\mathbf{R}_3\mathcal{M}_3 = \tilde{\mathbf{R}}_3 \begin{bmatrix} \mathcal{M}_{11} & 0 & 0 & 0 \\ \mathcal{M}_{21} & \mathbf{E} & 0 & 0 \\ \mathcal{M}_{31} & \mathcal{M}_{21} & \mathbf{E} & 0 \\ \tilde{\mathcal{M}}_{41} & \tilde{\mathcal{M}}_{42} & \tilde{\mathcal{M}}_{43} & \mathbf{E} \end{bmatrix},$$

where

$$\mathcal{M}_{\mathbf{R}11} = \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & \mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{\mathbf{R}21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & -\mathbf{b}_{,\lambda} + \mathbf{b}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathcal{M}_{\mathbf{R}31} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \mathbf{Z}\mathbf{M}^{-1}(-\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda}) \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{41} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3\ddot{\mathbf{f}}_{,\mathbf{w}} & 0 & -3\ddot{A}_{\mathbf{p}\lambda} \\ 0 & 0 & 0 & -3\ddot{\mathbf{b}}_{,\mathbf{w}} & 0 & -3\ddot{\mathbf{b}}_{,\lambda} \\ 0 & 0 & 0 & -3\ddot{\mathbf{d}}_{,\mathbf{w}} & 0 & -3\ddot{\mathbf{d}}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(-\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda}) \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{42} = \begin{bmatrix} 0 & -3\dot{\mathbf{Z}} & 0 & 0 & 0 & 0 \\ 0 & 3\ddot{\mathbf{M}} - 3\dot{\mathbf{f}}_{,\mathbf{v}} & -3\dot{\mathbf{f}}_{,\mathbf{r}} & -3\dot{\mathbf{f}}_{,\mathbf{w}} & 0 & -3\dot{A}_{\mathbf{p}\lambda} \\ 0 & -3\dot{\mathbf{b}}_{,\mathbf{v}} & -3\dot{\mathbf{b}}_{,\mathbf{r}} & -3\dot{\mathbf{b}}_{,\mathbf{w}} & 0 & -3\dot{\mathbf{b}}_{,\lambda} \\ 0 & -3\dot{\mathbf{d}}_{,\mathbf{v}} & -3\dot{\mathbf{d}}_{,\mathbf{r}} & -3\dot{\mathbf{d}}_{,\mathbf{w}} & 0 & -3\dot{\mathbf{d}}_{,\lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\tilde{\mathcal{M}}_{43} = \begin{bmatrix} -(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ -A_{\mathbf{pp}} & 3\dot{\mathbf{M}} - \mathbf{f}_{,\mathbf{v}} & -\mathbf{f}_{,\mathbf{r}} & -\mathbf{f}_{,\mathbf{w}} & -A_{\mathbf{ps}} & -A_{\mathbf{p}\lambda} \\ -\mathbf{b}_{,\mathbf{p}} & -\mathbf{b}_{,\mathbf{v}} & -\mathbf{b}_{,\mathbf{r}} & -\mathbf{b}_{,\mathbf{w}} & -\mathbf{b}_{,\mathbf{s}} & -\mathbf{b}_{,\lambda} \\ -\mathbf{d}_{,\mathbf{p}} & -\mathbf{d}_{,\mathbf{v}} & -\mathbf{d}_{,\mathbf{r}} & -\mathbf{d}_{,\mathbf{w}} & -\mathbf{d}_{,\mathbf{s}} & -\mathbf{d}_{,\lambda} \\ -\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & -\mathbf{c}_{,\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

If $\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(-\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda})$ is non-singular then it is possible to transform the matrix $\mathcal{M}_3$ to the form (29) using the 1st, 2nd, 3rd, 4th, 5th and the last row.
Hence, if $\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(-\mathbf{f}_{,\lambda} + \mathbf{Z}^T\mathbf{G}^T + \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda})$ is nonsingular, then the matrix $\mathcal{M}_3$ is 1-full and the equations of motion (24) have differentiation index three. ■

In the following considerations we assume that this condition is satisfied, i.e. that (24) has differentiation index three.

Remark 2.4.5 1) Redundant constraints or other redundant equations in the equations of motion are excluded by conditions (21) and (30).
2) The equations of motion for elastic multibody systems have the same structure as for rigid multibody systems. For this reason they can be treated analogously, e.g. [26].

3 The Projected Linear Index-One Formulation of the Equations of Motion

3.1 Derivative Array and Linearization

In the following we consider the equations of motion in the form of the nonlinear differential-algebraic equation (24) using general coordinates. We omit for simplicity the arguments of all included functions, see Remark A.0.6, i.e. we have

$$
\begin{aligned}
0 = \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) &= \begin{bmatrix} \dot{\mathbf{p}} - \mathbf{Z}(\mathbf{p})\mathbf{v} \\ \mathbf{M}(\mathbf{p},\mathbf{u})\dot{\mathbf{v}} - \mathbf{f}(\mathbf{p},\mathbf{v},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) + \mathbf{Z}^T(\mathbf{p})\mathbf{G}^T(\mathbf{p},\mathbf{s},\mathbf{u})\lambda \\ \dot{\mathbf{r}} - \mathbf{b}(\mathbf{p},\mathbf{v},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) \\ \mathbf{d}(\mathbf{p},\mathbf{v},\mathbf{r},\mathbf{w},\mathbf{s},\lambda,\mathbf{u}) \\ \mathbf{c}(\mathbf{p},\mathbf{s},\mathbf{u}) \\ \mathbf{g}(\mathbf{p},\mathbf{s},\mathbf{u}) \end{bmatrix} \\
&= \begin{bmatrix} \dot{\mathbf{p}} - \mathbf{Z}\mathbf{v} \\ \mathbf{M}\dot{\mathbf{v}} - \mathbf{f} + \mathbf{Z}^T\mathbf{G}^T\lambda \\ \dot{\mathbf{r}} - \mathbf{b} \\ \mathbf{d} \\ \mathbf{c} \\ \mathbf{g} \end{bmatrix}. \qquad (31)
\end{aligned}
$$

We obtain the first derivative $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t)$ of this differential-algebraic equation with respect to the time t by using Lemma A.0.20, i.e.

$$
0 = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) = \begin{bmatrix} \ddot{\mathbf{p}} - \mathbf{Z}_{,\mathbf{p}}()(\dot{\mathbf{p}})\mathbf{v} - \mathbf{Z}\dot{\mathbf{v}} \\ \mathbf{M}_{,\mathbf{p}}()(\dot{\mathbf{p}})\dot{\mathbf{v}} + \mathbf{M}_{,\mathbf{u}}()(\dot{\mathbf{u}})\dot{\mathbf{v}} + \mathbf{M}\ddot{\mathbf{v}} - \mathbf{f}_{,\mathbf{p}}\dot{\mathbf{p}} - \mathbf{f}_{,\mathbf{v}}\dot{\mathbf{v}} - \mathbf{f}_{,\mathbf{r}}\dot{\mathbf{r}} - \mathbf{f}_{,\mathbf{w}}\dot{\mathbf{w}} \\ \quad -\mathbf{f}_{,\mathbf{s}}\dot{\mathbf{s}} - \mathbf{f}_{,\lambda}\dot{\lambda} - \mathbf{f}_{,\mathbf{u}}\dot{\mathbf{u}} + \mathbf{Z}^T_{,\mathbf{p}}()(\dot{\mathbf{p}})\mathbf{G}^T\lambda + \mathbf{Z}^T(\mathbf{G}^T_{,\mathbf{p}}()(\dot{\mathbf{p}}) \\ \quad +\mathbf{G}^T_{,\mathbf{s}}()(\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{u}}()(\dot{\mathbf{u}}))\lambda + \mathbf{Z}^T\mathbf{G}^T\dot{\lambda} \\ \ddot{\mathbf{r}} - \mathbf{b}_{,\mathbf{p}}\dot{\mathbf{p}} - \mathbf{b}_{,\mathbf{v}}\dot{\mathbf{v}} - \mathbf{b}_{,\mathbf{r}}\dot{\mathbf{r}} - \mathbf{b}_{,\mathbf{w}}\dot{\mathbf{w}} - \mathbf{b}_{,\mathbf{s}}\dot{\mathbf{s}} - \mathbf{b}_{,\lambda}\dot{\lambda} - \mathbf{b}_{,\mathbf{u}}\dot{\mathbf{u}} \\ \mathbf{d}_{,\mathbf{p}}\dot{\mathbf{p}} + \mathbf{d}_{,\mathbf{v}}\dot{\mathbf{v}} + \mathbf{d}_{,\mathbf{r}}\dot{\mathbf{r}} + \mathbf{d}_{,\mathbf{w}}\dot{\mathbf{w}} + \mathbf{d}_{,\mathbf{s}}\dot{\mathbf{s}} + \mathbf{d}_{,\lambda}\dot{\lambda} + \mathbf{d}_{,\mathbf{u}}\dot{\mathbf{u}} \\ \mathbf{c}_{,\mathbf{p}}\dot{\mathbf{p}} + \mathbf{c}_{,\mathbf{s}}\dot{\mathbf{s}} + \mathbf{c}_{,\mathbf{u}}\dot{\mathbf{u}} \\ \mathbf{g}_{,\mathbf{p}}\dot{\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\dot{\mathbf{s}} + \mathbf{g}_{,\mathbf{u}}\dot{\mathbf{u}} \end{bmatrix}.
$$

Using Lemma A.0.20 and using Lemma A.0.10 we get the second derivative $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t)$ with respect to the time t as

$$
0 = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t)
$$

$$
= \left[\begin{array}{l}
\dddot{\mathbf{p}} - \mathbf{Z}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}})\mathbf{v} - \mathbf{Z}_{,\mathbf{p}}()(\ddot{\mathbf{p}})\mathbf{v} - 2\mathbf{Z}_{,\mathbf{p}}()(\dot{\mathbf{p}})\dot{\mathbf{v}} - \mathbf{Z}\ddot{\mathbf{v}} \\
\mathbf{M}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}})\dot{\mathbf{v}} + 2\mathbf{M}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}})\dot{\mathbf{v}} + \mathbf{M}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}})\dot{\mathbf{v}} + \mathbf{M}_{,\mathbf{p}}()(\ddot{\mathbf{p}})\dot{\mathbf{v}} \\
\qquad +\mathbf{M}_{,\mathbf{u}}()(\ddot{\mathbf{u}})\dot{\mathbf{v}} + 2\mathbf{M}_{,\mathbf{p}}()(\dot{\mathbf{p}})\ddot{\mathbf{v}} + 2\mathbf{M}_{,\mathbf{u}}()(\dot{\mathbf{u}})\ddot{\mathbf{v}} + \mathbf{M}\dddot{\mathbf{v}} - \mathbf{f}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) \\
\qquad -\mathbf{f}_{,\mathbf{vv}}()(\dot{\mathbf{v}},\dot{\mathbf{v}}) - \mathbf{f}_{,\mathbf{rr}}()(\dot{\mathbf{r}},\dot{\mathbf{r}}) - \mathbf{f}_{,\mathbf{ww}}()(\dot{\mathbf{w}},\dot{\mathbf{w}}) - \mathbf{f}_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) \\
\qquad -\mathbf{f}_{,\lambda\lambda}()(\dot{\lambda},\dot{\lambda}) - \mathbf{f}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) - 2\mathbf{f}_{,\mathbf{pv}}()(\dot{\mathbf{p}},\dot{\mathbf{v}}) - 2\mathbf{f}_{,\mathbf{pr}}()(\dot{\mathbf{p}},\dot{\mathbf{r}}) \\
\qquad -2\mathbf{f}_{,\mathbf{pw}}()(\dot{\mathbf{p}},\dot{\mathbf{w}}) - 2\mathbf{f}_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) - 2\mathbf{f}_{,\mathbf{p}\lambda}()(\dot{\mathbf{p}},\dot{\lambda}) - 2\mathbf{f}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) \\
\qquad -2\mathbf{f}_{,\mathbf{vr}}()(\dot{\mathbf{v}},\dot{\mathbf{r}}) - 2\mathbf{f}_{,\mathbf{vw}}()(\dot{\mathbf{v}},\dot{\mathbf{w}}) - 2\mathbf{f}_{,\mathbf{vs}}()(\dot{\mathbf{v}},\dot{\mathbf{s}}) - 2\mathbf{f}_{,\mathbf{v}\lambda}()(\dot{\mathbf{v}},\dot{\lambda}) \\
\qquad -2\mathbf{f}_{,\mathbf{vu}}()(\dot{\mathbf{v}},\dot{\mathbf{u}}) - 2\mathbf{f}_{,\mathbf{rw}}()(\dot{\mathbf{r}},\dot{\mathbf{w}}) - 2\mathbf{f}_{,\mathbf{rs}}()(\dot{\mathbf{r}},\dot{\mathbf{s}}) - 2\mathbf{f}_{,\mathbf{r}\lambda}()(\dot{\mathbf{r}},\dot{\lambda}) \\
\qquad -2\mathbf{f}_{,\mathbf{ru}}()(\dot{\mathbf{r}},\dot{\mathbf{u}}) - 2\mathbf{f}_{,\mathbf{ws}}()(\dot{\mathbf{w}},\dot{\mathbf{s}}) - 2\mathbf{f}_{,\mathbf{w}\lambda}()(\dot{\mathbf{w}},\dot{\lambda}) - 2\mathbf{f}_{,\mathbf{wu}}()(\dot{\mathbf{w}},\dot{\mathbf{u}}) \\
\qquad -2\mathbf{f}_{,\mathbf{s}\lambda}()(\dot{\mathbf{s}},\dot{\lambda}) - 2\mathbf{f}_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) - 2\mathbf{f}_{,\lambda\mathbf{u}}()(\dot{\lambda},\dot{\mathbf{u}}) - \mathbf{f}_{,\mathbf{p}}\ddot{\mathbf{p}} - \mathbf{f}_{,\mathbf{v}}\ddot{\mathbf{v}} - \mathbf{f}_{,\mathbf{r}}\ddot{\mathbf{r}} \\
\qquad -\mathbf{f}_{,\mathbf{w}}\ddot{\mathbf{w}} - \mathbf{f}_{,\mathbf{s}}\ddot{\mathbf{s}} - \mathbf{f}_{,\lambda}\ddot{\lambda} - \mathbf{f}_{,\mathbf{u}}\ddot{\mathbf{u}} + \mathbf{Z}^T_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}})\mathbf{G}^T\lambda + \mathbf{Z}^T_{,\mathbf{p}}()(\ddot{\mathbf{p}})\mathbf{G}^T\lambda \\
\qquad +\mathbf{Z}^T(\mathbf{G}^T_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) + \mathbf{G}^T_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) + \mathbf{G}^T_{,\mathbf{p}}()(\ddot{\mathbf{p}}) \\
\qquad\qquad +\mathbf{G}^T_{,\mathbf{sp}}()(\dot{\mathbf{s}},\dot{\mathbf{p}}) + \mathbf{G}^T_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) + \mathbf{G}^T_{,\mathbf{s}}()(\ddot{\mathbf{s}}) \\
\qquad\qquad +\mathbf{G}^T_{,\mathbf{up}}()(\dot{\mathbf{u}},\dot{\mathbf{p}}) + \mathbf{G}^T_{,\mathbf{us}}()(\dot{\mathbf{u}},\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) + \mathbf{G}^T_{,\mathbf{u}}()(\ddot{\mathbf{u}}))\lambda \\
\qquad +2\mathbf{Z}^T_{,\mathbf{p}}()(\dot{\mathbf{p}})(\mathbf{G}^T_{,\mathbf{p}}()(\dot{\mathbf{p}}) + \mathbf{G}^T_{,\mathbf{s}}()(\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{u}}()(\dot{\mathbf{u}}))\lambda + 2\mathbf{Z}^T_{,\mathbf{p}}()(\dot{\mathbf{p}})\mathbf{G}^T\dot{\lambda} \\
\qquad +2\mathbf{Z}^T(\mathbf{G}^T_{,\mathbf{p}}()(\dot{\mathbf{p}}) + \mathbf{G}^T_{,\mathbf{s}}()(\dot{\mathbf{s}}) + \mathbf{G}^T_{,\mathbf{u}}()(\dot{\mathbf{u}}))\dot{\lambda} + \mathbf{Z}^T\mathbf{G}^T\ddot{\lambda} \\
\ddot{\mathbf{r}} - \mathbf{b}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) - \mathbf{b}_{,\mathbf{vv}}()(\dot{\mathbf{v}},\dot{\mathbf{v}}) - \mathbf{b}_{,\mathbf{rr}}()(\dot{\mathbf{r}},\dot{\mathbf{r}}) - \mathbf{b}_{,\mathbf{ww}}()(\dot{\mathbf{w}},\dot{\mathbf{w}}) \\
\qquad -\mathbf{b}_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) - \mathbf{b}_{,\lambda\lambda}()(\dot{\lambda},\dot{\lambda}) - \mathbf{b}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\mathbf{pv}}()(\dot{\mathbf{p}},\dot{\mathbf{v}}) \\
\qquad -2\mathbf{b}_{,\mathbf{pr}}()(\dot{\mathbf{p}},\dot{\mathbf{r}}) - 2\mathbf{b}_{,\mathbf{pw}}()(\dot{\mathbf{p}},\dot{\mathbf{w}}) - 2\mathbf{b}_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) - 2\mathbf{b}_{,\mathbf{p}\lambda}()(\dot{\mathbf{p}},\dot{\lambda}) \\
\qquad -2\mathbf{b}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\mathbf{vr}}()(\dot{\mathbf{v}},\dot{\mathbf{r}}) - 2\mathbf{b}_{,\mathbf{vw}}()(\dot{\mathbf{v}},\dot{\mathbf{w}}) - 2\mathbf{b}_{,\mathbf{vs}}()(\dot{\mathbf{v}},\dot{\mathbf{s}}) \\
\qquad -2\mathbf{b}_{,\mathbf{v}\lambda}()(\dot{\mathbf{v}},\dot{\lambda}) - 2\mathbf{b}_{,\mathbf{vu}}()(\dot{\mathbf{v}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\mathbf{rw}}()(\dot{\mathbf{r}},\dot{\mathbf{w}}) - 2\mathbf{b}_{,\mathbf{rs}}()(\dot{\mathbf{r}},\dot{\mathbf{s}}) \\
\qquad -2\mathbf{b}_{,\mathbf{r}\lambda}()(\dot{\mathbf{r}},\dot{\lambda}) - 2\mathbf{b}_{,\mathbf{ru}}()(\dot{\mathbf{r}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\mathbf{ws}}()(\dot{\mathbf{w}},\dot{\mathbf{s}}) - 2\mathbf{b}_{,\mathbf{w}\lambda}()(\dot{\mathbf{w}},\dot{\lambda}) \\
\qquad -2\mathbf{b}_{,\mathbf{wu}}()(\dot{\mathbf{w}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\mathbf{s}\lambda}()(\dot{\mathbf{s}},\dot{\lambda}) - 2\mathbf{b}_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) - 2\mathbf{b}_{,\lambda\mathbf{u}}()(\dot{\lambda},\dot{\mathbf{u}}) \\
\qquad -\mathbf{b}_{,\mathbf{p}}\ddot{\mathbf{p}} - \mathbf{b}_{,\mathbf{v}}\ddot{\mathbf{v}} - \mathbf{b}_{,\mathbf{r}}\ddot{\mathbf{r}} - \mathbf{b}_{,\mathbf{w}}\ddot{\mathbf{w}} - \mathbf{b}_{,\mathbf{s}}\ddot{\mathbf{s}} - \mathbf{b}_{,\lambda}\ddot{\lambda} - \mathbf{b}_{,\mathbf{u}}\ddot{\mathbf{u}} \\
\mathbf{d}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) + \mathbf{d}_{,\mathbf{vv}}()(\dot{\mathbf{v}},\dot{\mathbf{v}}) + \mathbf{d}_{,\mathbf{rr}}()(\dot{\mathbf{r}},\dot{\mathbf{r}}) + \mathbf{d}_{,\mathbf{ww}}()(\dot{\mathbf{w}},\dot{\mathbf{w}}) + \mathbf{d}_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) \\
\qquad +\mathbf{d}_{,\lambda\lambda}()(\dot{\lambda},\dot{\lambda}) + \mathbf{d}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) + 2\mathbf{d}_{,\mathbf{pv}}()(\dot{\mathbf{p}},\dot{\mathbf{v}}) + 2\mathbf{d}_{,\mathbf{pr}}()(\dot{\mathbf{p}},\dot{\mathbf{r}}) \\
\qquad +2\mathbf{d}_{,\mathbf{pw}}()(\dot{\mathbf{p}},\dot{\mathbf{w}}) + 2\mathbf{d}_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) + 2\mathbf{d}_{,\mathbf{p}\lambda}()(\dot{\mathbf{p}},\dot{\lambda}) + 2\mathbf{d}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) \\
\qquad +2\mathbf{d}_{,\mathbf{vr}}()(\dot{\mathbf{v}},\dot{\mathbf{r}}) + 2\mathbf{d}_{,\mathbf{vw}}()(\dot{\mathbf{v}},\dot{\mathbf{w}}) + 2\mathbf{d}_{,\mathbf{vs}}()(\dot{\mathbf{v}},\dot{\mathbf{s}}) + 2\mathbf{d}_{,\mathbf{v}\lambda}()(\dot{\mathbf{v}},\dot{\lambda}) \\
\qquad +2\mathbf{d}_{,\mathbf{vu}}()(\dot{\mathbf{v}},\dot{\mathbf{u}}) + 2\mathbf{d}_{,\mathbf{rw}}()(\dot{\mathbf{r}},\dot{\mathbf{w}}) + 2\mathbf{d}_{,\mathbf{rs}}()(\dot{\mathbf{r}},\dot{\mathbf{s}}) + 2\mathbf{d}_{,\mathbf{r}\lambda}()(\dot{\mathbf{r}},\dot{\lambda}) \\
\qquad +2\mathbf{d}_{,\mathbf{ru}}()(\dot{\mathbf{r}},\dot{\mathbf{u}}) + 2\mathbf{d}_{,\mathbf{ws}}()(\dot{\mathbf{w}},\dot{\mathbf{s}}) + 2\mathbf{d}_{,\mathbf{w}\lambda}()(\dot{\mathbf{w}},\dot{\lambda}) + 2\mathbf{d}_{,\mathbf{wu}}()(\dot{\mathbf{w}},\dot{\mathbf{u}}) \\
\qquad +2\mathbf{d}_{,\mathbf{s}\lambda}()(\dot{\mathbf{s}},\dot{\lambda}) + 2\mathbf{d}_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) + 2\mathbf{d}_{,\lambda\mathbf{u}}()(\dot{\lambda},\dot{\mathbf{u}}) \\
\qquad +\mathbf{d}_{,\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{d}_{,\mathbf{v}}\ddot{\mathbf{v}} + \mathbf{d}_{,\mathbf{r}}\ddot{\mathbf{r}} + \mathbf{d}_{,\mathbf{w}}\ddot{\mathbf{w}} + \mathbf{d}_{,\mathbf{s}}\ddot{\mathbf{s}} + \mathbf{d}_{,\lambda}\ddot{\lambda} + \mathbf{d}_{,\mathbf{u}}\ddot{\mathbf{u}} \\
\mathbf{c}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) + \mathbf{c}_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) + \mathbf{c}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) + 2\mathbf{c}_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) + 2\mathbf{c}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) \\
\qquad +2\mathbf{c}_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) + \mathbf{c}_{,\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{c}_{,\mathbf{s}}\ddot{\mathbf{s}} + \mathbf{c}_{,\mathbf{u}}\ddot{\mathbf{u}} \\
\mathbf{g}_{,\mathbf{pp}}()(\dot{\mathbf{p}},\dot{\mathbf{p}}) + \mathbf{g}_{,\mathbf{ss}}()(\dot{\mathbf{s}},\dot{\mathbf{s}}) + \mathbf{g}_{,\mathbf{uu}}()(\dot{\mathbf{u}},\dot{\mathbf{u}}) + 2\mathbf{g}_{,\mathbf{ps}}()(\dot{\mathbf{p}},\dot{\mathbf{s}}) + 2\mathbf{g}_{,\mathbf{pu}}()(\dot{\mathbf{p}},\dot{\mathbf{u}}) \\
\qquad +2\mathbf{g}_{,\mathbf{su}}()(\dot{\mathbf{s}},\dot{\mathbf{u}}) + \mathbf{g}_{,\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\ddot{\mathbf{s}} + \mathbf{g}_{,\mathbf{u}}\ddot{\mathbf{u}}
\end{array}\right].
$$

Now we can determine the derivative array see [5],[7] and [18], of the form

$$\tilde{\mathbf{F}}_l(\mathbf{x}^{(l+1)}, ..., \dot{\mathbf{x}}, \mathbf{x}, t) = 0,$$

which summarizes the original equation and all its derivatives up to order l in one large system, i. e.

$$\tilde{\mathbf{F}}_l(\mathbf{x}^{(l+1)}, ..., \dot{\mathbf{x}}, \mathbf{x}, t) = \begin{bmatrix} \mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \\ \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \\ \vdots \\ \frac{\mathrm{d}^l}{\mathrm{d}t^l}\mathbf{F}(\dot{\mathbf{x}}, \mathbf{x}, t) \end{bmatrix}.$$

With the first and the second derivative of the differential-algebraic system we would obtain $\tilde{\mathbf{F}}_2(\dddot{\mathbf{x}}, \ddot{\mathbf{x}}, \dot{\mathbf{x}}, \mathbf{x}, t)$. But since the derivative array would be very large and since we know that only the derivatives of the kinematical equation (first equation of (31)) of the contact point equation (before the last equation of (31)) and of the holonomic constraints (last equation of (31)) play an important role, we will omit the derivatives of the other equations and obtain

$$
\begin{array}{rcl}
0 & = & \mathbf{F}_2(\ddot{\mathbf{x}}, \ddot{\mathbf{x}}, \dot{\mathbf{x}}, \mathbf{x}, \mathbf{u}) \\
& = & \left[\begin{array}{l}
\dot{\mathbf{p}} - \mathbf{Z}\mathbf{v} \\
\mathbf{M}\dot{\mathbf{v}} - \mathbf{f} + \mathbf{Z}^T\mathbf{G}^T\lambda \\
\dot{\mathbf{r}} - \mathbf{b} \\
\mathbf{d} \\
\mathbf{c} \\
\mathbf{g} \\
\hline
\ddot{\mathbf{p}} - \mathbf{Z}_{,\mathbf{p}}()(\dot{\mathbf{p}})\mathbf{v} - \mathbf{Z}\dot{\mathbf{v}} \\
\mathbf{c}_{,\mathbf{p}}\dot{\mathbf{p}} + \mathbf{c}_{,\mathbf{s}}\dot{\mathbf{s}} + \mathbf{c}_{,\mathbf{u}}\dot{\mathbf{u}} \\
\mathbf{g}_{,\mathbf{p}}\dot{\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\dot{\mathbf{s}} + \mathbf{g}_{,\mathbf{u}}\dot{\mathbf{u}} \\
\hline
\dddot{\mathbf{p}} - \mathbf{Z}_{,\mathbf{pp}}()(\dot{\mathbf{p}}, \dot{\mathbf{p}})\mathbf{v} - \mathbf{Z}_{,\mathbf{p}}()(\ddot{\mathbf{p}})\mathbf{v} - 2\mathbf{Z}_{,\mathbf{p}}()(\dot{\mathbf{p}})\dot{\mathbf{v}} - \mathbf{Z}\ddot{\mathbf{v}} \\
\mathbf{c}_{,\mathbf{pp}}()(\dot{\mathbf{p}}, \dot{\mathbf{p}}) + \mathbf{c}_{,\mathbf{ss}}()(\dot{\mathbf{s}}, \dot{\mathbf{s}}) + \mathbf{c}_{,\mathbf{uu}}()(\dot{\mathbf{u}}, \dot{\mathbf{u}}) + 2\mathbf{c}_{,\mathbf{ps}}()(\dot{\mathbf{p}}, \dot{\mathbf{s}}) \\
\qquad +2\mathbf{c}_{,\mathbf{pu}}()(\dot{\mathbf{p}}, \dot{\mathbf{u}}) + 2\mathbf{c}_{,\mathbf{su}}()(\dot{\mathbf{s}}, \dot{\mathbf{u}}) + \mathbf{c}_{,\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{c}_{,\mathbf{s}}\ddot{\mathbf{s}} + \mathbf{c}_{,\mathbf{u}}\ddot{\mathbf{u}} \\
\mathbf{g}_{,\mathbf{pp}}()(\dot{\mathbf{p}}, \dot{\mathbf{p}}) + \mathbf{g}_{,\mathbf{ss}}()(\dot{\mathbf{s}}, \dot{\mathbf{s}}) + \mathbf{g}_{,\mathbf{uu}}()(\dot{\mathbf{u}}, \dot{\mathbf{u}}) + 2\mathbf{g}_{,\mathbf{ps}}()(\dot{\mathbf{p}}, \dot{\mathbf{s}}) \\
\qquad +2\mathbf{g}_{,\mathbf{pu}}()(\dot{\mathbf{p}}, \dot{\mathbf{u}}) + 2\mathbf{g}_{,\mathbf{su}}()(\dot{\mathbf{s}}, \dot{\mathbf{u}}) + \mathbf{g}_{,\mathbf{p}}\ddot{\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\ddot{\mathbf{s}} + \mathbf{g}_{,\mathbf{u}}\ddot{\mathbf{u}}
\end{array}\right].
\end{array}
\tag{32}
$$

In this paper we assume that the control $\mathbf{u}$ is given, but nevertheless to keep generality also for control applications, we will linearize the derivative array also along a reference control function. The first part of the derivative array (32) corresponds to the equations of motion (14)-(19) and its linearization is given by (28). The linearization of the remaining equations about a reference function and its derivatives will be carried out with the same strategy as in the Chapter 2.3.
For the first derivative of the kinematical equation of motion we have

$$
\begin{array}{rcl}
0 & = & (\ddot{\tilde{\mathbf{p}}} + \ddot{\Delta}\mathbf{p}) - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\dot{\tilde{\mathbf{p}}} + \dot{\Delta}\mathbf{p})(\tilde{\mathbf{v}} + \Delta\mathbf{v}) - \mathbf{Z}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\dot{\tilde{\mathbf{v}}} + \dot{\Delta}\mathbf{v}) \\
& \approx & \ddot{\Delta}\mathbf{p} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\Delta}\mathbf{p})\tilde{\mathbf{v}} - \mathbf{Z}(\tilde{\mathbf{p}})\dot{\Delta}\mathbf{v} - \mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}}, \Delta\mathbf{p})\tilde{\mathbf{v}} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\dot{\tilde{\mathbf{v}}} \\
& & -\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}})\Delta\mathbf{v} + (\ddot{\tilde{\mathbf{p}}} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}})\tilde{\mathbf{v}} - \mathbf{Z}(\tilde{\mathbf{p}})\dot{\tilde{\mathbf{v}}}),
\end{array}
$$

and for the second derivative we obtain

$$
\begin{array}{rcl}
0 & = & (\dddot{\tilde{\mathbf{p}}} + \Delta\dddot{\mathbf{p}}) - \mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\dot{\tilde{\mathbf{p}}} + \dot{\Delta}\mathbf{p}, \dot{\tilde{\mathbf{p}}} + \dot{\Delta}\mathbf{p})(\tilde{\mathbf{v}} + \Delta\mathbf{v}) \\
& & -\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\ddot{\tilde{\mathbf{p}}} + \ddot{\Delta}\mathbf{p})(\tilde{\mathbf{v}} + \Delta\mathbf{v}) - 2\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\dot{\tilde{\mathbf{p}}} + \dot{\Delta}\mathbf{p})(\dot{\tilde{\mathbf{v}}} + \dot{\Delta}\mathbf{v}) \\
& & -\mathbf{Z}(\tilde{\mathbf{p}} + \Delta\mathbf{p})(\ddot{\tilde{\mathbf{p}}} + \ddot{\Delta}\mathbf{p}) \\
& = & \Delta\dddot{\mathbf{p}} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\ddot{\Delta}\mathbf{p})\tilde{\mathbf{v}} - \mathbf{Z}(\tilde{\mathbf{p}})\ddot{\Delta}\mathbf{v} - 2\mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\dot{\Delta}\mathbf{p}, \dot{\tilde{\mathbf{p}}})\tilde{\mathbf{v}} \\
& & -2\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\Delta}\mathbf{p})\dot{\tilde{\mathbf{v}}} - 2\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}})\dot{\Delta}\mathbf{v} - \mathbf{Z}_{,\mathbf{ppp}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}}, \dot{\tilde{\mathbf{p}}}, \Delta\mathbf{p})\tilde{\mathbf{v}} - \mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\ddot{\tilde{\mathbf{p}}}, \Delta\mathbf{p})\tilde{\mathbf{v}} \\
& & -2\mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}}, \Delta\mathbf{p})\dot{\tilde{\mathbf{v}}} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\Delta\mathbf{p})\ddot{\tilde{\mathbf{v}}} - \mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}}, \dot{\tilde{\mathbf{p}}})\Delta\mathbf{v} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\ddot{\tilde{\mathbf{p}}})\Delta\mathbf{v} \\
& & +(\dddot{\tilde{\mathbf{p}}} - \mathbf{Z}_{,\mathbf{pp}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}}, \dot{\tilde{\mathbf{p}}})\tilde{\mathbf{v}} - \mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\ddot{\tilde{\mathbf{p}}})\tilde{\mathbf{v}} - 2\mathbf{Z}_{,\mathbf{p}}(\tilde{\mathbf{p}})(\dot{\tilde{\mathbf{p}}})\dot{\tilde{\mathbf{v}}} - \mathbf{Z}(\tilde{\mathbf{p}})\ddot{\tilde{\mathbf{v}}}).
\end{array}
$$

For the derivatives of the holonomic constraints we obtain

$$
\begin{array}{rcl}
0 & = & \mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{s}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{p}}} + \dot{\Delta}\mathbf{p}) + \mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{s}}} + \dot{\Delta}\mathbf{s}) \\
& & +\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{u}}} + \dot{\Delta}\mathbf{u}) \\
& = & \mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{s}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{p}}}) + \mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{s}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\Delta}\mathbf{p}) \\
& & +\mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{s}}}) + \mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\Delta}\mathbf{s}) \\
& & +\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\tilde{\mathbf{u}}}) + \mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}} + \Delta\mathbf{p}, \tilde{\mathbf{s}} + \Delta\mathbf{p}, \tilde{\mathbf{u}} + \Delta\mathbf{u})(\dot{\Delta}\mathbf{u}) \\
& \approx & \mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\Delta}\mathbf{p}) + \mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\Delta}\mathbf{s}) + \mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\Delta}\mathbf{u}) \\
& & +\mathbf{g}_{,\mathbf{pp}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}}, \Delta\mathbf{p}) + \mathbf{g}_{,\mathbf{sp}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}}, \Delta\mathbf{p}) + \mathbf{g}_{,\mathbf{up}}(\tilde{\mathbf{p}}, \tilde{\mathbf{s}}, \tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}}, \Delta\mathbf{p})
\end{array}
$$

$$
\begin{aligned}
&+\mathbf{g}_{,\mathbf{ps}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{ss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{us}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\Delta\mathbf{s})\\
&+\mathbf{g}_{,\mathbf{pu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{su}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{uu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\Delta\mathbf{u})\\
&+\mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}})+\mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}})+\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}}).
\end{aligned}
$$

Finally the second derivative of the holonomic constraints is

$$
\begin{aligned}
0 \;=\;& \mathbf{g}_{,\mathbf{pp}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{p}}}+\dot{\Delta\mathbf{p}},\dot{\tilde{\mathbf{p}}}+\dot{\Delta\mathbf{p}})\\
&+\mathbf{g}_{,\mathbf{ss}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{s}}}+\dot{\Delta\mathbf{s}},\dot{\tilde{\mathbf{s}}}+\dot{\Delta\mathbf{s}})\\
&+\mathbf{g}_{,\mathbf{uu}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{u}}}+\dot{\Delta\mathbf{u}},\dot{\tilde{\mathbf{u}}}+\dot{\Delta\mathbf{u}})\\
&+2\mathbf{g}_{,\mathbf{ps}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{p}}}+\dot{\Delta\mathbf{p}},\dot{\tilde{\mathbf{s}}}+\dot{\Delta\mathbf{s}})\\
&+2\mathbf{g}_{,\mathbf{pu}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{p}}}+\dot{\Delta\mathbf{p}},\dot{\tilde{\mathbf{u}}}+\dot{\Delta\mathbf{u}})\\
&+2\mathbf{g}_{,\mathbf{su}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\dot{\tilde{\mathbf{s}}}+\dot{\Delta\mathbf{s}},\dot{\tilde{\mathbf{u}}}+\dot{\Delta\mathbf{u}})\\
&+\mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\ddot{\tilde{\mathbf{p}}}+\ddot{\Delta\mathbf{p}})+\mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\ddot{\tilde{\mathbf{s}}}+\ddot{\Delta\mathbf{s}})\\
&+\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}}+\Delta\mathbf{p},\tilde{\mathbf{s}}+\Delta\mathbf{s},\tilde{\mathbf{u}}+\Delta\mathbf{u})(\ddot{\tilde{\mathbf{u}}}+\ddot{\Delta\mathbf{u}})\\
\approx\;& +\mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\Delta\mathbf{p}})+\mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\Delta\mathbf{s}})+\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\Delta\mathbf{u}})\\
&+2\mathbf{g}_{,\mathbf{pp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\Delta\mathbf{p}})+2\mathbf{g}_{,\mathbf{sp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\Delta\mathbf{p}})+2\mathbf{g}_{,\mathbf{up}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\Delta\mathbf{p}})\\
&+2\mathbf{g}_{,\mathbf{ss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\Delta\mathbf{s}})+2\mathbf{g}_{,\mathbf{ps}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\Delta\mathbf{s}})+2\mathbf{g}_{,\mathbf{us}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\Delta\mathbf{s}})\\
&+2\mathbf{g}_{,\mathbf{uu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\Delta\mathbf{u}})+2\mathbf{g}_{,\mathbf{pu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\Delta\mathbf{u}})+2\mathbf{g}_{,\mathbf{su}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\Delta\mathbf{u}})\\
&+\mathbf{g}_{,\mathbf{ppp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{p}}},\Delta\mathbf{p})+\mathbf{g}_{,\mathbf{ssp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{p})+\mathbf{g}_{,\mathbf{uup}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{p})\\
&\quad+2\mathbf{g}_{,\mathbf{psp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{p})+2\mathbf{g}_{,\mathbf{pup}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{p})+2\mathbf{g}_{,\mathbf{sup}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{p})\\
&\quad+\mathbf{g}_{,\mathbf{pp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{p}}},\Delta\mathbf{p})+\mathbf{g}_{,\mathbf{sp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{s}}},\Delta\mathbf{p})+\mathbf{g}_{,\mathbf{up}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{u}}},\Delta\mathbf{p})\\
&+\mathbf{g}_{,\mathbf{pps}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{p}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{sss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{uus}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{s})\\
&\quad+2\mathbf{g}_{,\mathbf{pss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{s})+2\mathbf{g}_{,\mathbf{pus}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{s})+2\mathbf{g}_{,\mathbf{sus}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{s})\\
&\quad+\mathbf{g}_{,\mathbf{ps}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{p}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{ss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{s}}},\Delta\mathbf{s})+\mathbf{g}_{,\mathbf{us}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{u}}},\Delta\mathbf{s})\\
&+\mathbf{g}_{,\mathbf{ppu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{p}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{ssu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{uuu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{u})\\
&\quad+2\mathbf{g}_{,\mathbf{psu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{s}}},\Delta\mathbf{u})+2\mathbf{g}_{,\mathbf{puu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{u})+2\mathbf{g}_{,\mathbf{suu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{u}}},\Delta\mathbf{u})\\
&\quad+\mathbf{g}_{,\mathbf{pu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{p}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{su}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{s}}},\Delta\mathbf{u})+\mathbf{g}_{,\mathbf{uu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{u}}},\Delta\mathbf{u})\\
&+(\mathbf{g}_{,\mathbf{pp}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{p}}})+\mathbf{g}_{,\mathbf{ss}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{s}}})\\
&\quad+\mathbf{g}_{,\mathbf{uu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{u}}},\dot{\tilde{\mathbf{u}}})+2\mathbf{g}_{,\mathbf{ps}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{s}}})+2\mathbf{g}_{,\mathbf{pu}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{p}}},\dot{\tilde{\mathbf{u}}})\\
&\quad+2\mathbf{g}_{,\mathbf{su}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\dot{\tilde{\mathbf{s}}},\dot{\tilde{\mathbf{u}}})+\mathbf{g}_{,\mathbf{p}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{p}}})+\mathbf{g}_{,\mathbf{s}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{s}}})+\mathbf{g}_{,\mathbf{u}}(\tilde{\mathbf{p}},\tilde{\mathbf{s}},\tilde{\mathbf{u}})(\ddot{\tilde{\mathbf{u}}})).
\end{aligned}
$$

The linearization of the derivatives of the contact point equations follows analogously to the linearization of the derivatives of the holonomic constraints above, therefore we will not describe it explicitly.
We obtain the linearization of the derivative array (32) in the form of a linear differential-algebraic equation.

$$
\mathcal{M}(t)\dot{\mathbf{z}} = \mathcal{N}(t)\mathbf{z} + \mathcal{H}(t) \quad \text{with a matrix pencil} \quad (\mathcal{M}(t), \mathcal{N}(t)) \tag{33}
$$

The linearization with respect to the control $\mathbf{u}$ will be useful for the control of MBS but for our work the control $\mathbf{u}$ is known and therefore it will be kept as nonlinear function in the matrices, i.e. all terms with derivatives with respect to $\mathbf{u}$ are dropped. One has to note the relation

$$
\begin{aligned}
\dot{\mathbf{g}}_{,\mathbf{y}}(\mathbf{p},\mathbf{s},\mathbf{u})(\cdot) &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{g}_{,\mathbf{y}}(\mathbf{p},\mathbf{s},\mathbf{u})(\cdot)\\
&= \mathbf{g}_{,\mathbf{yp}}(\mathbf{p},\mathbf{s},\mathbf{u})(\cdot,\dot{\mathbf{p}})+\mathbf{g}_{,\mathbf{ys}}(\mathbf{p},\mathbf{s},\mathbf{u})(\cdot,\dot{\mathbf{s}})+\mathbf{g}_{,\mathbf{yu}}(\mathbf{p},\mathbf{s},\mathbf{u})(\cdot,\dot{\mathbf{u}})\\
&= \mathbf{g}_{,\mathbf{py}}(\mathbf{p},\mathbf{s},\mathbf{u})(\dot{\mathbf{p}},\cdot)+\mathbf{g}_{,\mathbf{sy}}(\mathbf{p},\mathbf{s},\mathbf{u})(\dot{\mathbf{s}},\cdot)+\mathbf{g}_{,\mathbf{uy}}(\mathbf{p},\mathbf{s},\mathbf{u})(\dot{\mathbf{u}},\cdot),
\end{aligned}
$$

which follows from Lemma A.0.10 and where $\mathbf{y}$ can correspond to the vectors $\mathbf{p}, \mathbf{s}$ or $\mathbf{u}$. This holds analogously holds for $\mathbf{c}(\mathbf{p}, \mathbf{s}, \mathbf{u})$. The matrices $\mathcal{M}(t)$ and $\mathcal{N}(t)$ of the linearized derivative array (33) are given by

$$\mathcal{M}(t) = \begin{bmatrix} \mathbf{E} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{21} & M_{22} \end{bmatrix}, \tag{34}$$

where
$$M_{21} = \begin{bmatrix} -\mathbf{Z}_{,\mathbf{p}}()(\cdot)\tilde{\mathbf{v}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\ \mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$M_{22} = \begin{bmatrix} \mathbf{I}_{,n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$M_{31} = \begin{bmatrix} -2\mathbf{Z}_{,\mathbf{pp}}()(\cdot, \dot{\tilde{\mathbf{p}}})\tilde{\mathbf{v}} - 2\mathbf{Z}_{,\mathbf{p}}()(\cdot)\dot{\tilde{\mathbf{v}}} & -2\mathbf{Z}_{,\mathbf{p}}()(\dot{\tilde{\mathbf{p}}}) & 0 & 0 & 0 & 0 \\ 2\dot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & 2\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ 2\dot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & 2\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix},$$

and

$$\mathcal{N}(t) = \begin{bmatrix} \mathbf{A} & 0 & 0 \\ N_{21} & 0 & 0 \\ N_{31} & 0 & 0 \end{bmatrix}, \tag{35}$$

where
$$N_{21} = \begin{bmatrix} \mathbf{Z}_{,\mathbf{pp}}()(\dot{\tilde{\mathbf{p}}}, \cdot)\tilde{\mathbf{v}} + \mathbf{Z}_{,\mathbf{p}}()(\cdot)\dot{\tilde{\mathbf{v}}} & \mathbf{Z}_{,\mathbf{p}}()(\dot{\tilde{\mathbf{p}}}) & 0 & 0 & 0 & 0 \\ -\dot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & -\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ -\dot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & -\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix},$$

$$N_{31} = \begin{bmatrix} N_{31}^{11} & N_{31}^{12} & 0 & 0 & 0 & 0 \\ -\ddot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & -\ddot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ -\ddot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & -\ddot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix}$$

and $N_{31}^{11} = \mathbf{Z}_{,\mathbf{ppp}}()(\dot{\tilde{\mathbf{p}}}, \dot{\tilde{\mathbf{p}}}, \cdot)\tilde{\mathbf{v}} + \mathbf{Z}_{,\mathbf{pp}}()(\ddot{\tilde{\mathbf{p}}}, \cdot)\tilde{\mathbf{v}} + 2\mathbf{Z}_{,\mathbf{pp}}()(\dot{\tilde{\mathbf{p}}}, \cdot)\dot{\tilde{\mathbf{v}}} + \mathbf{Z}_{,\mathbf{p}}()(\cdot)\ddot{\tilde{\mathbf{v}}}$,
$N_{31}^{12} = \mathbf{Z}_{,\mathbf{pp}}()(\dot{\tilde{\mathbf{p}}}, \dot{\tilde{\mathbf{p}}}) + \mathbf{Z}_{,\mathbf{p}}()(\ddot{\tilde{\mathbf{p}}})$.

Furthermore, we have

$$\mathcal{H}(t) = \left[\begin{array}{c} -\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}} \\ -\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda} \\ -\dot{\tilde{\mathbf{r}}} + \mathbf{b} \\ \mathbf{d} \\ \mathbf{c} \\ \mathbf{g} \\ \hline -(\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot} \\ -\dot{\mathbf{c}} \\ -\dot{\mathbf{g}} \\ \hline -(\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot\cdot} \\ -\ddot{\mathbf{c}} \\ -\ddot{\mathbf{g}} \end{array}\right],$$

$$\mathbf{z}(t) = [\mathbf{x}^T, \dot{\mathbf{x}}^T, \ddot{\mathbf{x}}^T]^T,$$
$$\mathbf{x}(t) = [\Delta\mathbf{p}^T, \Delta\mathbf{v}^T, \Delta\mathbf{w}^T, \Delta\mathbf{r}^T, \Delta\mathbf{s}^T, \Delta\lambda^T]^T.$$

The blocks of the matrices $\mathcal{M}(t), \mathcal{N}(t)$ and of the inhomogenity $\mathcal{H}(t)$ depend only on the reference functions $\tilde{\mathbf{v}}(t)$, $\tilde{\mathbf{q}}(t)$, ... Since the reference functions are given, the matrices $\mathcal{M}(t), \mathcal{N}(t)$ and the inhomogeneity $\mathcal{H}(t)$ actually depend only on the time t.

$$\begin{aligned}\mathcal{M}(t), \mathcal{N}(t) &\in \mathbb{R}^{(2n_{\mathbf{p}}+n_{\mathbf{r}}+n_{\mathbf{w}}+n_{\mathbf{s}}+n_{\lambda}+n_{\mathbf{p}}+n_{\mathbf{s}}+n_{\lambda}+n_{\mathbf{p}}+n_{\mathbf{s}}+n_{\lambda},3(2n_{\mathbf{p}}+n_{\mathbf{r}}+n_{\mathbf{w}}+n_{\mathbf{s}}+n_{\lambda}))}\\ \mathcal{H}(t) &\in \mathbb{R}^{(2n_{\mathbf{p}}+n_{\mathbf{r}}+n_{\mathbf{w}}+n_{\mathbf{s}}+n_{\lambda}+n_{\mathbf{p}}+n_{\mathbf{s}}+n_{\lambda}+n_{\mathbf{p}}+n_{\mathbf{s}}+n_{\lambda})}\end{aligned}$$

Remark 3.1.1 Note that due to Taylor's theorem the derivative array of the linearized equations of motion (27) is the same as the linearized derivative array (33) of the original, nonlinear equations of motion (24).

3.2 Reduction to a Linear Index-One-System

In this section we determine an equivalent linear index-one-formulation of the linearized equations of motion in general coordinates (28). We will use the theory developed in [17] and [18].
In [17] the so called characteristic quantities of a matrix pencil were introduced. We assume that all characteristic quantities of $(\mathcal{M}_l(t), \mathcal{N}_l(t))$ (see Definition 2.4.2) are independent of the time t for all $l = 0, .., 2$.

First we have to determine the projector $\mathbf{Z}_2 \in \mathbb{R}^{(n+\mu(n_{\mathbf{p}}+n_{\mathbf{s}}+n_{\lambda}), a_\mu)}$, where a_μ corresponds to the number of all constraints, including the so called hidden constraints. This projector has to satisfy

$$\mathbf{Z}_2^T \mathcal{M}(t) = 0,$$

i.e. the columns of $\mathbf{Z}_2$ are a basis of kernel$(\mathcal{M}^T(t))$ = corange$(\mathcal{M}(t))$. In the following for simplicity we omit the dependency on the time t. In a first step rank$(\mathcal{M})$ is determined analyzing the rank deficiency with respect to the rows. We apply row transformations to $\mathcal{M}$ as in the proof of Lemma 2.4.4. In the following the sign "$\overset{\text{new}}{\sim}$" means that the directly following notation is adapted to the new block structure such that same symbols can represent another entry. With (34)

$$\begin{aligned}
\mathbf{E} &= \begin{bmatrix} \mathbf{I}_{n_{\mathbf{p}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{21} &= \begin{bmatrix} -\mathbf{Z}_{,\mathbf{p}}()(\cdot)\tilde{\mathbf{v}} & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\ \mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}, \\
M_{31} &= \begin{bmatrix} -2\mathbf{Z}_{,\mathbf{pp}}()(\cdot, \dot{\mathbf{p}})\tilde{\mathbf{v}} - 2\mathbf{Z}_{,\mathbf{p}}()(\cdot)\dot{\tilde{\mathbf{v}}} & -2\mathbf{Z}_{,\mathbf{p}}()(\dot{\mathbf{p}}) & 0 & 0 & 0 & 0 \\ 2\dot{\mathbf{c}}_{,\mathbf{p}} & 0 & 0 & 0 & 2\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ 2\dot{\mathbf{g}}_{,\mathbf{p}} & 0 & 0 & 0 & 2\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix}, \\
M_{22} &= \begin{bmatrix} \mathbf{I}_{,n_{\mathbf{p}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{M}(t) &= \begin{bmatrix} \mathbf{E} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{21} & M_{22} \end{bmatrix} \\
&\overset{\text{new}}{\sim} \begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{22} \end{bmatrix},
\end{aligned}$$

where

$$
\begin{aligned}
M_{11} &= \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{21} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}, \\
M_{31} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\dot{\mathbf{c}}_{,\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 2\dot{\mathbf{g}}_{,\mathbf{s}} & 0 \end{bmatrix}, \\
M_{22} &= \begin{bmatrix} \mathbf{I}_{,n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{32} &= \begin{bmatrix} 0 & -\mathbf{Z} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix}.
\end{aligned}
$$

Removing blocks via block-Gauss elimination we obtain

$$
\mathcal{M}(t) \overset{\text{new}}{\sim} \begin{bmatrix} M_{11} & 0 & 0 \\ M_{21} & M_{22} & 0 \\ M_{31} & M_{32} & M_{22} \end{bmatrix},
$$

where

$$
\begin{aligned}
M_{11} &= \begin{bmatrix} \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{21} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix}, \\
M_{31} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{22} &= \begin{bmatrix} \mathbf{I}_{,n_\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\
M_{32} &= \begin{bmatrix} 0 & \mathbf{I}_{n_\mathbf{p}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Therefore, $\operatorname{rank}(\mathcal{M}) = 4n_\mathbf{p} + n_\mathbf{r} + 2n_\mathbf{s}$ and the rank deficiency with respect to the rows is $\dim(\operatorname{kernel}(\mathcal{M}^T)) = 3n_\lambda + n_\mathbf{w} + n_\mathbf{s}$. Hence

$$
\mathbf{Z}_2^T \in \mathbb{R}^{(n_\mathbf{w}+n_\mathbf{s}+n_\lambda+n_\lambda+n_\lambda,\, n+n_\mathbf{p}+n_\mathbf{s}+n_\lambda+n_\mathbf{p}+n_\mathbf{s}+n_\lambda)}
$$

is given by

$$
\mathbf{Z}_2^T = \begin{bmatrix} \mathbf{Z}_{2(1)}^T & \mathbf{Z}_{2(2)}^T & \mathbf{Z}_{2(3)}^T \end{bmatrix},
$$

where

$$
\begin{aligned}
\mathbf{Z}_{2(1)}^T &= \left[\begin{array}{cccccc}
0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{I}_{n_\lambda} \\
-(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}) & 0 & 0 & 0 & 0 & 0 \\
\left(\begin{array}{c}
-2\dot{\mathbf{g}}_{,\mathbf{p}} + 2\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} \\
-2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}})\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} \\
-(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}})(\mathbf{Z}\mathbf{v})_{,\mathbf{p}}
\end{array}\right) & -(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}})\mathbf{Z}\mathbf{M}^{-1} & 0 & 0 & 0 & 0
\end{array}\right], \\
\mathbf{Z}_{2(2)}^T &= \left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda} \\
-(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}) & 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}})\mathbf{c}_{,\mathbf{s}}^{-1} & 0
\end{array}\right], \\
\mathbf{Z}_{2(3)}^T &= \left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda}
\end{array}\right].
\end{aligned}
$$

Here $n = n_\mathbf{p} + n_\mathbf{p} + n_\mathbf{r} + n_\mathbf{w} + n_\mathbf{s} + n_\lambda$ is the dimension of the original DAE (31). With the constraint matrix $\mathbf{G}$ as in (22) and the time derivatives

$$
\begin{aligned}
\mathbf{G} &= \mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}, \\
\dot{\mathbf{G}} &= \dot{\mathbf{g}}_{,\mathbf{p}} - \dot{\mathbf{g}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}(\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\mathbf{c}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}},
\end{aligned}
\tag{36}
$$

and using the identity

$$
\begin{aligned}
\mathbf{I} = \mathbf{c}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} \;\Rightarrow\; 0 = \dot{\mathbf{c}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} + \mathbf{c}_{,\mathbf{s}}(\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot} \;\Rightarrow\; 0 = \mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} + (\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot} \\
\Rightarrow\; \mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} = -(\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot},
\end{aligned}
$$

it follows that

$$
\begin{aligned}
\mathbf{Z}_2^T &= \left[\begin{array}{ccc} \mathbf{Z}_{2(1)}^T & \mathbf{Z}_{2(2)}^T & \mathbf{Z}_{2(3)}^T \end{array}\right], \\
& \mathbf{Z}_{2(1)}^T = \left[\begin{array}{cccccc}
0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{I}_{n_\lambda} \\
-\mathbf{G} & 0 & 0 & 0 & 0 & 0 \\
-2\dot{\mathbf{G}} - \mathbf{G}(\mathbf{Z}\mathbf{v})_{,\mathbf{p}} & -\mathbf{G}\mathbf{Z}\mathbf{M}^{-1} & 0 & 0 & 0 & 0
\end{array}\right], \\
& \mathbf{Z}_{2(2)}^T = \left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda} \\
-\mathbf{G} & -2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot} & 0
\end{array}\right], \\
& \mathbf{Z}_{2(3)}^T = \left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda}
\end{array}\right].
\end{aligned}
$$

Therefore,

$$\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n\\0_n\\0_n\end{bmatrix} = \begin{bmatrix}\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda}\\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0\\ \mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0\\ N_{z41} & -\mathbf{GZ} & 0 & 0 & N_{z45} & 0\\ N_{z51} & N_{z52} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & N_{z55} & N_{z56}\end{bmatrix}$$

with $\mathbf{I}_n \in \mathbb{R}^{n,n}$ is the identity matrix, $0_n \in \mathbb{R}^{n,n}$ is the zero matrix and

$$\begin{aligned}
N_{z41} &= -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} - \dot{\mathbf{g}}_{,\mathbf{p}}\\
N_{z45} &= \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}}\\
N_{z51} &= -\mathbf{GZM}^{-1}N_{21} - (2\dot{\mathbf{G}} + \mathbf{G}(\mathbf{Zv})_{,\mathbf{p}})(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} - \mathbf{G}N_{71}\\
&\quad +2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\,\dot{\mathbf{c}}_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{p}} - \ddot{\mathbf{g}}_{,\mathbf{p}}\\
&= -\mathbf{GZM}^{-1}(-(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}} + \mathbf{f}_{,\mathbf{p}} - \mathbf{Z}^T(\mathbf{G}\tilde{\lambda})_{,\mathbf{p}}^T - \mathbf{Z}_{,\mathbf{p}}^T()(\cdot)(\mathbf{G}^T\tilde{\lambda}))\\
&\quad -(2\dot{\mathbf{G}} + \mathbf{G}(\mathbf{Zv})_{,\mathbf{p}})(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} - \mathbf{G}(\mathbf{Z}_{,\mathbf{pp}}()(\dot{\tilde{\mathbf{p}}},\cdot)\tilde{\mathbf{v}} + \mathbf{Z}_{,\mathbf{p}}()(\cdot)\dot{\tilde{\mathbf{v}}})\\
&\quad +2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}}_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{p}} - \ddot{\mathbf{g}}_{,\mathbf{p}}\\
N_{z52} &= -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{v}} - (2\dot{\mathbf{G}} + \mathbf{G}(\mathbf{Zv})_{,\mathbf{p}})\mathbf{Z} - \mathbf{GZ}_{,\mathbf{p}}()(\dot{\tilde{\mathbf{p}}})\\
N_{z55} &= -\mathbf{GZM}^{-1}N_{25} + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}}_{,\mathbf{s}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{s}} - \ddot{\mathbf{g}}_{,\mathbf{s}}\\
&= -\mathbf{GZM}^{-1}(\mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{s}}) + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}}_{,\mathbf{s}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{s}} - \ddot{\mathbf{g}}_{,\mathbf{s}}\\
N_{z56} &= -\mathbf{GZM}^{-1}N_{26}\\
&= -\mathbf{GZM}^{-1}(\mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T).
\end{aligned}$$

We determine the rank of

$$\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n\\0_n\\0_n\end{bmatrix} \tag{37}$$

by a sequence of block-Gauss-eliminations.

$$\begin{aligned}
\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n\\0_n\\0_n\end{bmatrix} &\sim \begin{bmatrix}\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda}\\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0\\ \mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & 0 & 0\\ N_{z41} & -\mathbf{GZ} & 0 & 0 & N_{z45} & 0\\ N_{z51} & N_{z52} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & N_{z55} & N_{z56}\end{bmatrix}\\
&= \begin{bmatrix}\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda}\\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0\\ \mathbf{G} & 0 & 0 & 0 & 0 & 0\\ N_{z41} & -\mathbf{GZ} & 0 & 0 & N_{z45} & 0\\ N_{z51} & N_{z52} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & N_{z55} & N_{z56}\end{bmatrix}\\
&\sim \begin{bmatrix}\mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\lambda}\\ 0 & \mathbf{c}_{,\mathbf{s}} & \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0\\ 0 & 0 & \mathbf{G} & 0 & 0 & 0\\ 0 & N_{z45} & N_{z41} & -\mathbf{GZ} & 0 & 0\\ -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & N_{z55} & N_{z51} & N_{z52} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & N_{z56}\end{bmatrix}\\
&\sim \begin{bmatrix}\mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\lambda}\\ 0 & \mathbf{c}_{,\mathbf{s}} & \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0\\ 0 & 0 & \mathbf{G} & 0 & 0 & 0\\ 0 & N_{z42}^1 & N_{z43}^1 & -\mathbf{GZ} & 0 & 0\\ 0 & N_{z52}^1 & N_{z53}^1 & N_{z54}^1 & N_{z55}^1 & N_{z56}^1\end{bmatrix}
\end{aligned}$$

where

$$
\begin{aligned}
N^1_{z42} &= N_{z45} = \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}} \\
N^1_{z43} &= N_{z41} = -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} - \dot{\mathbf{g}}_{,\mathbf{p}} \\
N^1_{z52} &= N_{z55} + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\mathbf{s}} \\
&= (-\mathbf{GZM}^{-1}(\mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{s}}) + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})\dot{}\dot{\mathbf{c}}_{,\mathbf{s}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{s}} - \ddot{\mathbf{g}}_{,\mathbf{s}}) \\
&\quad + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\mathbf{s}} \\
N^1_{z53} &= N_{z51} + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\mathbf{p}} \\
N^1_{z54} &= N_{z52} + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\mathbf{v}} \\
N^1_{z55} &= -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\mathbf{r}} \\
N^1_{z56} &= N_{z56} + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\
&= -\mathbf{GZM}^{-1}(\mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T) + \mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} \\
&= -\mathbf{GZM}^{-1}(\mathbf{f}_{,\lambda} - \mathbf{f}_{,\mathbf{w}}\mathbf{d}_{,\mathbf{w}}^{-1}\mathbf{d}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T)
\end{aligned}
$$

is non-singular, see (30). Then

$$
\begin{aligned}
\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n \\ 0_n \\ 0_n\end{bmatrix} &\sim \begin{bmatrix}
\mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{r}} \\
0 & \mathbf{c}_{,\mathbf{s}} & 0 & 0 & \mathbf{c}_{,\mathbf{p}} & 0 \\
0 & N^1_{z52} & N^1_{z56} & N^1_{z54} & N^1_{z53} & N^1_{z55} \\
0 & N^1_{z42} & 0 & -\mathbf{GZ} & N^1_{z43} & 0 \\
0 & 0 & 0 & 0 & \mathbf{G} & 0
\end{bmatrix} \\
&\sim \begin{bmatrix}
\mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{r}} \\
0 & \mathbf{c}_{,\mathbf{s}} & 0 & 0 & \mathbf{c}_{,\mathbf{p}} & 0 \\
0 & 0 & N^1_{z56} & N^1_{z54} & N^1_{z53} - N^1_{z52}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} & N^1_{z55} \\
0 & 0 & 0 & -\mathbf{GZ} & N^2_{z45} & 0 \\
0 & 0 & 0 & 0 & \mathbf{G} & 0
\end{bmatrix}
\end{aligned}
$$

where

$$
\begin{aligned}
N^2_{z45} &= N^1_{z43} - N^1_{z42}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} \\
&= (-\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} - \dot{\mathbf{g}}_{,\mathbf{p}}) - (\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}})\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} \\
&= -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + (-\dot{\mathbf{g}}_{,\mathbf{p}} + \dot{\mathbf{g}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}}) \\
&= -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} - \dot{\mathbf{G}} \qquad \text{see (36).}
\end{aligned}
$$

Since $\mathbf{d}_{,\mathbf{w}} \in \mathbb{R}^{n_\mathbf{w},n_\mathbf{w}}$, $\mathbf{c}_{,\mathbf{s}} \in \mathbb{R}^{n_\mathbf{s},n_\mathbf{s}}$ and $N^1_{z56} \in \mathbb{R}^{n_\lambda,n_\lambda}$ are non-singular and $\mathbf{G}, \mathbf{GZ} \in \mathbb{R}^{n_\lambda,n_\mathbf{p}}$ have full rank we have

$$
\operatorname{rank}(\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n \\ 0_n \\ 0_n\end{bmatrix}) = n_\mathbf{w} + n_\mathbf{s} + 3n_\lambda = a_\mu.
$$

Furthermore, we have to determine the projector $\mathbf{T}_2 \in \mathbb{R}^{n,2n_f+n_\mathbf{r}}$ such that

$$
0 = \mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n \\ 0_n \\ 0_n\end{bmatrix}\mathbf{T}_2.
$$

The rank of the matrix (37) is $n_\mathbf{w} + n_\mathbf{s} + 3n_\lambda$ and therefore the deficiency with respect to the columns, i.e. the dimension of the kernel of (37), is $2n_f + n_\mathbf{r}$. Here, $n_f = n_\mathbf{p} - n_\lambda$ is the number of degrees of freedom of the considered multibody

system.
If we use the same transformations as for the determination of the rank of (37), except the row manipulations, we get

$$\mathbf{Z}_2^T\mathcal{N}\begin{bmatrix}\mathbf{I}_n\\0_n\\0_n\end{bmatrix}\sim\mathcal{N}_{\mathbf{Z}}=\begin{bmatrix}\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda}\\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0\\ N_{z53}^1-N_{z52}^1\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} & N_{z54}^1 & N_{z55}^1 & 0 & 0 & N_{z56}^1\\ N_{z45}^2 & -\mathbf{GZ} & 0 & 0 & 0 & 0\\ \mathbf{G} & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

compare with (38). Therefore, the columns of $\mathbf{T}_2$ span also kernel($\mathcal{N}_{\mathbf{Z}}$), i.e.

$$\begin{aligned}\mathbf{T}_2 &= \begin{bmatrix}\mathbf{G}^- & 0 & 0\\ \mathbf{G}_{\mathbf{Z}}^+N_{z45}^2\mathbf{G}^- & \mathbf{G}_{\mathbf{Z}}^- & 0\\ 0 & 0 & \mathbf{I}\\ T_{2(41)} & T_{2(42)} & T_{2(43)}\\ -\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}\mathbf{G}^- & 0 & 0\\ T_{2(61)} & -(N_{z56}^1)^{-1}N_{z54}^1\mathbf{G}_{\mathbf{Z}}^- & -(N_{z56}^1)^{-1}N_{z55}^1\end{bmatrix}\\ &\in \mathbb{R}^{(n,n_f+n_f+n_{\mathbf{r}})},\end{aligned}$$

with

$$\begin{aligned}T_{2(41)} &= -\mathbf{d}_{,\mathbf{w}}^{-1}[\mathbf{d}_{,\mathbf{p}}\mathbf{G}^-+\mathbf{d}_{,\mathbf{v}}\mathbf{G}_{\mathbf{Z}}^+N_{z45}^2\mathbf{G}^--\mathbf{d}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}\mathbf{G}^-+\mathbf{d}_{,\lambda}T_{2(61)}],\\ T_{2(42)} &= -\mathbf{d}_{,\mathbf{w}}^{-1}[\mathbf{d}_{,\mathbf{v}}\mathbf{G}_{\mathbf{Z}}^--\mathbf{d}_{,\lambda}(N_{z56}^1)^{-1}N_{z54}^1\mathbf{G}_{\mathbf{Z}}^-],\\ T_{2(43)} &= -\mathbf{d}_{,\mathbf{w}}^{-1}[\mathbf{d}_{,\mathbf{r}}-(N_{z56}^1)^{-1}N_{z55}^1\mathbf{d}_{\lambda}],\\ T_{2(61)} &= -(N_{z56}^1)^{-1}[(N_{z53}^1-N_{z52}^1\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}})\mathbf{G}^-+N_{z54}^1\mathbf{G}_{\mathbf{Z}}^+N_{z45}^2\mathbf{G}^-],\\ N_{z45}^2 &= -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}}-\dot{\mathbf{G}}, \qquad \text{see (36).}\end{aligned}$$

Here, the columns of $\mathbf{G}^-\in\mathbb{R}^{n_{\mathbf{p}},n_f}$ span kern($\mathbf{G}$) and the columns of $\mathbf{G}_{\mathbf{Z}}^-\in\mathbb{R}^{n_{\mathbf{p}},n_f}$ span the kernel of $\mathbf{GZ}$, i.e. $\mathbf{GG}^-=0\in\mathbb{R}^{n_\lambda,n_f}$ and $\mathbf{GZG}_{\mathbf{Z}}^-=0\in\mathbb{R}^{n_\lambda,n_f}$. Furthermore, $\mathbf{G}_{\mathbf{Z}}^+\in\mathbb{R}^{n_{\mathbf{p}},n_\lambda}$ is a pseudo inverse of $\mathbf{GZ}$, i.e. $\mathbf{GZG}_{\mathbf{Z}}^+=\mathbf{I}\in\mathbb{R}^{n_\lambda,n_\lambda}$. It follows that

$$\begin{aligned}\mathbf{ET}_2 &= \begin{bmatrix}\mathbf{I}_{n_{\mathbf{p}}} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{M} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\mathbf{T}_2\\ &= \begin{bmatrix}\mathbf{G}^- & 0 & 0\\ -\mathbf{MG}_{\mathbf{Z}}^+(\mathbf{G}(\tilde{\mathbf{Z}}\mathbf{v})_{,\mathbf{p}}+\dot{\mathbf{G}})\mathbf{G}^- & \mathbf{MG}_{\mathbf{Z}}^- & 0\\ 0 & 0 & \mathbf{I}\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\end{aligned}$$

and thus

$$\mathrm{rank}(\mathbf{ET}_2)=2n_f+n_{\mathbf{r}}=d_\mu.$$

Last but not least, we have to determine a third projector $\mathbf{Z}_1^T\in\mathbb{R}^{(d_\mu,n)}$ such that

$$\mathrm{rank}(\mathbf{Z}_1^T\mathbf{ET}_2)=d_\mu=2n_f+n_{\mathbf{r}},$$

$$\Leftrightarrow \quad \text{rank}\left(\mathbf{Z}_1^T \begin{bmatrix} \mathbf{G}^- & 0 & 0 \\ -\mathbf{M}\mathbf{G}_{\mathbf{Z}}^+(\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + \dot{\mathbf{G}})\mathbf{G}^- & \mathbf{M}\mathbf{G}_{\mathbf{Z}}^- & 0 \\ 0 & 0 & \mathbf{I} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right) = d_\mu = 2n_f + n_{\mathbf{r}}.$$

We choose

$$\mathbf{Z}_1^T = \begin{bmatrix} \mathbf{P}_{\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{\mathbf{v}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \end{bmatrix},$$

such that $\mathbf{P}_{\mathbf{p}}$ and $\mathbf{P}_{\mathbf{v}}$ are parts of permutation matrices and $\mathbf{P}_{\mathbf{p}}\mathbf{G}^- \in \mathbb{R}^{(n_f, n_f)}$ and $\mathbf{P}_{\mathbf{v}}\mathbf{M}\mathbf{G}_{\mathbf{Z}}^- \in \mathbb{R}^{(n_f, n_f)}$ are non-singular. With the projectors $\mathbf{Z}_1^T$ and $\mathbf{Z}_2^T$ we obtain a linear index-one system $\hat{\mathbf{E}}\dot{\mathbf{x}} = \hat{\mathbf{A}}\mathbf{x} + \hat{\mathbf{h}}$ of the form

$$\left[\begin{array}{c} \hat{\mathbf{E}}_1 \\ \hline 0 \end{array}\right]\dot{\mathbf{x}} = \left[\begin{array}{c} \hat{\mathbf{A}}_1 \\ \hline \hat{\mathbf{A}}_2 \end{array}\right]\mathbf{x} + \left[\begin{array}{c} \hat{\mathbf{h}}_1 \\ \hline \hat{\mathbf{h}}_2 \end{array}\right] \tag{38}$$

with

$$\begin{aligned}
\hat{\mathbf{E}}_1 &= \mathbf{Z}_1^T\mathbf{E} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{\mathbf{v}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{I}_{n_{\mathbf{p}}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{M} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{\mathbf{v}}\mathbf{M} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{A}}_1 &= \mathbf{Z}_1^T\mathbf{A} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{\mathbf{v}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} (\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & \mathbf{Z} & 0 & 0 & 0 & 0 \\ A_{\mathbf{pp}} & \mathbf{f}_{,\mathbf{v}} & \mathbf{f}_{,\mathbf{r}} & \mathbf{f}_{,\mathbf{w}} & A_{\mathbf{ps}} & A_{\mathbf{p}\lambda} \\ \mathbf{b}_{,\mathbf{p}} & \mathbf{b}_{,\mathbf{v}} & \mathbf{b}_{,\mathbf{r}} & \mathbf{b}_{,\mathbf{w}} & \mathbf{b}_{,\mathbf{s}} & \mathbf{b}_{,\lambda} \\ \mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda} \\ \mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\ \mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & \mathbf{P}_{\mathbf{p}}\mathbf{Z} & 0 & 0 & 0 & 0 \\ \mathbf{P}_{\mathbf{v}}A_{\mathbf{pp}} & \mathbf{P}_{\mathbf{v}}\mathbf{f}_{,\mathbf{v}} & \mathbf{P}_{\mathbf{v}}\mathbf{f}_{,\mathbf{r}} & \mathbf{P}_{\mathbf{v}}\mathbf{f}_{,\mathbf{w}} & \mathbf{P}_{\mathbf{v}}A_{\mathbf{ps}} & \mathbf{P}_{\mathbf{v}}A_{\mathbf{p}\lambda} \\ \mathbf{b}_{,\mathbf{p}} & \mathbf{b}_{,\mathbf{v}} & \mathbf{b}_{,\mathbf{r}} & \mathbf{b}_{,\mathbf{w}} & \mathbf{b}_{,\mathbf{s}} & \mathbf{b}_{,\lambda} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{h}}_1 &= \mathbf{Z}_1^T\mathbf{h} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{P}_{\mathbf{v}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{I}_{n_{\mathbf{r}}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}} \\ -\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda} \\ -\dot{\tilde{\mathbf{r}}} + \mathbf{b} \\ \mathbf{d} \\ \mathbf{c} \\ \mathbf{g} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{P}_{\mathbf{p}}(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) \\ \mathbf{P}_{\mathbf{v}}(-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}) \\ -\dot{\tilde{\mathbf{r}}} + \mathbf{b} \end{bmatrix},
\end{aligned}$$

$$
\begin{aligned}
\hat{\mathbf{A}}_2 &= \mathbf{Z}_2^T \mathcal{N} \begin{bmatrix} \mathbf{I}_n \\ 0_n \\ 0_n \end{bmatrix} \\
&= \begin{bmatrix}
\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda} \\
\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\
\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \\
N_{z41} & -\mathbf{GZ} & 0 & 0 & N_{z45} & 0 \\
N_{z51} & N_{z52} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & N_{z55} & N_{z56}
\end{bmatrix}, \\
\hat{\mathbf{h}}_2 &= \mathbf{Z}_2^T \mathcal{H} \\
&= \begin{bmatrix} \mathbf{Z}_{2(1)}^T & \mathbf{Z}_{2(2)}^T & \mathbf{Z}_{2(3)}^T \end{bmatrix}
\left[\begin{array}{c}
-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}} \\
-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda} \\
-\dot{\tilde{\mathbf{r}}} + \mathbf{b} \\
\mathbf{d} \\
\mathbf{c} \\
\mathbf{g} \\ \hline
-(\tilde{\mathbf{p}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot} \\
-\dot{\mathbf{c}} \\
-\dot{\mathbf{g}} \\ \hline
-(\tilde{\mathbf{p}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot\cdot} \\
-\ddot{\mathbf{c}} \\
-\ddot{\mathbf{g}}
\end{array}\right],
\end{aligned}
$$

$$
\mathbf{Z}_{2(1)}^T = \begin{bmatrix}
0 & 0 & 0 & \mathbf{I}_{n_\mathbf{w}} & 0 & 0 \\
0 & 0 & 0 & 0 & \mathbf{I}_{n_\mathbf{s}} & 0 \\
0 & 0 & 0 & 0 & 0 & \mathbf{I}_{n_\lambda} \\
-(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}) & 0 & 0 & 0 & 0 & 0 \\
\begin{pmatrix} -2\dot{\mathbf{g}}_{,\mathbf{p}} + 2\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} \\ -2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}})\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}} \\ -(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}})(\mathbf{Zv})_{,\mathbf{p}} \end{pmatrix} & -(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}})\mathbf{ZM}^{-1} & 0 & 0 & 0 & 0
\end{bmatrix},
$$

$$
\mathbf{Z}_{2(2)}^T = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda} \\
-(\mathbf{g}_{,\mathbf{p}} - \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\mathbf{c}_{,\mathbf{p}}) & 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}})\mathbf{c}_{,\mathbf{s}}^{-1} & 0
\end{bmatrix},
$$

$$
\mathbf{Z}_{2(3)}^T = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & -\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1} & \mathbf{I}_{n_\lambda}
\end{bmatrix},
$$

$$
= \begin{bmatrix}
\mathbf{d} \\
\mathbf{c} \\
\mathbf{g} \\
-\mathbf{G}(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}} - \dot{\mathbf{g}} \\
\begin{pmatrix} (-2\dot{\mathbf{G}} - \mathbf{G}(\mathbf{Zv})_{,\mathbf{p}})(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) - \mathbf{GZM}^{-1}(-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}) \\ +\mathbf{G}(\tilde{\mathbf{p}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot} + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}} - \ddot{\mathbf{g}} \end{pmatrix}
\end{bmatrix}
$$

$$= \left[\begin{array}{c} \mathbf{d} \\ \mathbf{c} \\ \mathbf{g} \\ -\mathbf{G}(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}} - \dot{\mathbf{g}} \\ \left(\begin{array}{c} 2\dot{\mathbf{G}}\dot{\tilde{\mathbf{p}}} - 2\dot{\mathbf{G}}\mathbf{Z}\tilde{\mathbf{v}} + \mathbf{G}(\mathbf{Z}\mathbf{v})_{,\mathbf{p}}\dot{\tilde{\mathbf{p}}} - \mathbf{G}(\mathbf{Z}\mathbf{v})_{,\mathbf{p}}\mathbf{Z}\tilde{\mathbf{v}} + \mathbf{G}\mathbf{Z}\dot{\tilde{\mathbf{v}}} - \mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{f} \\ +\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}\mathbf{Z}^T\mathbf{G}^T\tilde{\lambda} + \mathbf{G}(\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot} + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}} - \ddot{\mathbf{g}} \end{array} \right) \end{array} \right].$$

With

$$\begin{aligned}
\hat{A}_{71} &= -\mathbf{G}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{p}} - \dot{\mathbf{g}}_{,\mathbf{p}} \\
\hat{A}_{75} &= \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}}_{,\mathbf{s}} - \dot{\mathbf{g}}_{,\mathbf{s}} \\
\hat{A}_{21} &= \mathbf{P}_{\mathbf{v}}A_{\mathbf{pp}} \\
&= \mathbf{P}_{\mathbf{v}}(-(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}} + \mathbf{f}_{,\mathbf{p}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{p}} - \mathbf{Z}_{,\mathbf{p}}^T()(\cdot)(\mathbf{G}^T\tilde{\lambda})) \\
\hat{A}_{25} &= \mathbf{P}_{\mathbf{v}}A_{\mathbf{ps}} \\
&= \mathbf{P}_{\mathbf{v}}(\mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T\mathbf{G}_{,\mathbf{s}}^T\tilde{\lambda}) \\
\hat{A}_{26} &= \mathbf{P}_{\mathbf{v}}A_{\mathbf{p}\lambda} \\
&= \mathbf{P}_{\mathbf{v}}(\mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T) \\
\hat{A}_{81} &= N_{z51} \\
&= -\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(-(\mathbf{M}\dot{\tilde{\mathbf{v}}})_{,\mathbf{p}} + \mathbf{f}_{,\mathbf{p}} - \mathbf{Z}^T(\mathbf{G}\tilde{\lambda})_{,\mathbf{p}}^T - \mathbf{Z}_{,\mathbf{p}}^T()(\cdot)(\mathbf{G}^T\tilde{\lambda})) \\
&\quad -(2\dot{\mathbf{G}} + \mathbf{G}(\mathbf{Z}\mathbf{v})_{,\mathbf{p}})(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} - \mathbf{G}(\mathbf{Z}_{,\mathbf{pp}}()(\dot{\tilde{\mathbf{p}}},\cdot)\tilde{\mathbf{v}} + \mathbf{Z}_{,\mathbf{p}}()(\cdot)\dot{\tilde{\mathbf{v}}}) + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}}_{,\mathbf{p}} \\
&\quad +\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{p}} - \ddot{\mathbf{g}}_{,\mathbf{p}} \\
\hat{A}_{85} &= N_{z55} \\
&= -\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(\mathbf{f}_{,\mathbf{s}} - \mathbf{Z}^T(\mathbf{G}^T\tilde{\lambda})_{,\mathbf{s}}) + 2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\dot{\mathbf{c}}_{,\mathbf{s}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}}_{,\mathbf{s}} - \ddot{\mathbf{g}}_{,\mathbf{s}} \\
\hat{A}_{86} &= N_{z56} \\
&= -\mathbf{G}\mathbf{Z}\mathbf{M}^{-1}(\mathbf{f}_{,\lambda} - \mathbf{Z}^T\mathbf{G}^T)
\end{aligned}$$

the equivalent linear index-one-formulation or strangeness free formulation [18] (i.e. without hidden constraints) has the following explicit form.

$$
\left[\begin{array}{cccccc}
\mathbf{P_p} & 0 & 0 & 0 & 0 & 0 \\
0 & \mathbf{P_v M} & 0 & 0 & 0 & 0 \\
0 & 0 & \mathbf{I}_{n_\mathbf{r}} & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\left[\begin{array}{c}
\Delta \mathbf{p} \\ \Delta \mathbf{v} \\ \Delta \mathbf{r} \\ \Delta \mathbf{w} \\ \Delta \mathbf{s} \\ \Delta \lambda
\end{array}\right]^{\cdot}
=
\left[\begin{array}{cccccc}
\mathbf{P_p}(\mathbf{Z}\tilde{\mathbf{v}})_{,\mathbf{p}} & \mathbf{P_p Z} & 0 & 0 & 0 & 0 \\
\hat{A}_{21} & \mathbf{P_v f}_{,\mathbf{v}} & \mathbf{P_v f}_{,\mathbf{r}} & \mathbf{P_v f}_{,\mathbf{w}} & \hat{A}_{25} & \hat{A}_{26} \\
\mathbf{b}_{,\mathbf{p}} & \mathbf{b}_{,\mathbf{v}} & \mathbf{b}_{,\mathbf{r}} & \mathbf{b}_{,\mathbf{w}} & \mathbf{b}_{,\mathbf{s}} & \mathbf{b}_{,\lambda} \\
\hline
\mathbf{d}_{,\mathbf{p}} & \mathbf{d}_{,\mathbf{v}} & \mathbf{d}_{,\mathbf{r}} & \mathbf{d}_{,\mathbf{w}} & \mathbf{d}_{,\mathbf{s}} & \mathbf{d}_{,\lambda} \\
\mathbf{c}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{c}_{,\mathbf{s}} & 0 \\
\mathbf{g}_{,\mathbf{p}} & 0 & 0 & 0 & \mathbf{g}_{,\mathbf{s}} & 0 \\
\hat{A}_{71} & -\mathbf{GZ} & 0 & 0 & \hat{A}_{75} & 0 \\
\hat{A}_{81} & \hat{A}_{82} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{r}} & -\mathbf{GZM}^{-1}\mathbf{f}_{,\mathbf{w}} & \hat{A}_{85} & \hat{A}_{86}
\end{array}\right]
\left[\begin{array}{c}
\Delta \mathbf{p} \\ \Delta \mathbf{v} \\ \Delta \mathbf{r} \\ \Delta \mathbf{w} \\ \Delta \mathbf{s} \\ \Delta \lambda
\end{array}\right]
$$

$$
+\left[\begin{array}{c}
\mathbf{P_p}(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) \\
\mathbf{P_v}(-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}) \\
-\dot{\tilde{\mathbf{r}}} + \mathbf{b} \\
\hline
\mathbf{d} \\
\mathbf{c} \\
\mathbf{g} \\
-\mathbf{G}(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\dot{\mathbf{c}} - \dot{\mathbf{g}} \\
\left(\begin{array}{c}
(-2\dot{\mathbf{G}} - \mathbf{G}(\mathbf{Z}\mathbf{v})_{,\mathbf{p}})(-\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}}) - \mathbf{GZM}^{-1}(-\mathbf{M}\dot{\tilde{\mathbf{v}}} + \mathbf{f} - \mathbf{Z}^T\mathbf{G}^T\tilde{\lambda}) + \mathbf{G}(\dot{\tilde{\mathbf{p}}} + \mathbf{Z}\tilde{\mathbf{v}})^{\cdot} \\
+2(\mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1})^{\cdot}\,\dot{\mathbf{c}} + \mathbf{g}_{,\mathbf{s}}\mathbf{c}_{,\mathbf{s}}^{-1}\ddot{\mathbf{c}} - \ddot{\mathbf{g}}
\end{array}\right)
\end{array}\right]
$$

4 Summary

In the present paper we have examined the index reduction of the linearized equations of motion of a multibody system as used in industrial applications. The original equations of motions have differentiation index three and can in generally not be solved straightforwardly by standard ODE techniques. Up to now standard techniques in industrial applications are based on a combination of index reduction and projection steps such as BDF applied to the Gear-Gupta-Leimkuhler formulation [14].
In [17] and [19] a formalism has been developed for general linear differential-algebraic equations of arbitrary index and without special structure to determine an equivalent index-one-formulation. Based on this general formalism we have given an adapted formalism to determine a linear differentiation index-one-fromulation for a very general class of equations of motion. This index-one-formulation is analytically equivalent to the linearization of the original index-three system.
The linear index-one-formulation can be used for the piecewise approximation and integration of the originally nonlinear equations of motion using stiff ODE solvers like Runge-Kutta algorithms or BDF methods.

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A Notation and Definitions

We denote by $\mathbb{R}^{n,m}$ and $\mathbb{C}^{n,m}$ the set of real and complex $m \times n$-matrices. Column vectors belong to $\mathbb{R}^{n,1}$ respectively $\mathbb{C}^{n,1}$ and row vectors accordingly to $\mathbb{R}^{1,n}$ respectively $\mathbb{C}^{1,n}$. As special cases one obtains $\mathbb{R} = \mathbb{R}^{1,1}$ respectively $\mathbb{C} = \mathbb{C}^{1,1}$.

Notation: The following notation is used, compare [3, Section 8.3].

row vector	$\mathbf{x}$	$= [x_j]_{j=1\ldots n}$	$\in \mathbb{R}^{1,n}$
column vector	$\mathbf{x}$	$= [x^i]^{i=1\ldots n}$	$\in \mathbb{R}^{n,1}$
matrix	$\mathbf{A}$	$= [A^i_j]^{i=1\ldots m}_{j=1\ldots n}$	$\in \mathbb{R}^{m,n}$
tensors of higher order	$\mathbf{A}$	$= [A^{i_1\ldots i_l}_{j_1\ldots j_k}]^{i_s=1\ldots m_s}_{j_t=1\ldots n_t}$	$\in \mathbb{R}^{m_1,\ldots,m_l,n_1,\ldots,n_k}$ $s = 1,\ldots,l;\ t = 1,\ldots,k.$

Here, i denotes a row index and j a column index.

Remark A.0.1 We will only use small Latin letters as index of tensors. For this reason the summation convention holds only for these small Latin indices. For instance capital Latin indices denote the tensor itself.

Definition A.0.2 (Derivative of a scalar function) *Let $f(\mathbf{x}) : \mathbb{R}^{n,1} \to \mathbb{R}$ be a scalar function and $f(\mathbf{x})$ at $\mathbf{x}_0 \in \mathbb{R}^{n,1}$ differentiable. Then the vector*

$$f_{,\mathbf{x}}(\mathbf{x}_0) = \frac{\partial}{\partial \mathbf{x}} f(\mathbf{x}_0) = \left[\frac{\partial}{\partial x^1} f(\mathbf{x}_0) \ \ldots \ \frac{\partial}{\partial x^n} f(\mathbf{x}_0) \right] = \left[\frac{\partial}{\partial x^i} f(\mathbf{x}_0) \right]_{i=1,\ldots,n} \in \mathbb{R}^{1,n}$$

is called the derivative of $f(\mathbf{x})$ w.r.t. $\mathbf{x}$ at $\mathbf{x}_0$ [4].

Definition A.0.3 (Jacobian of vector-valued function) *If $\mathbf{f}(\mathbf{x}) : \mathbb{R}^{n,1} \to \mathbb{R}^{m,1}$ is a vector-valued function and if each component of $\mathbf{f}(\mathbf{x})$ is at $\mathbf{x}_0 \in \mathbb{R}^{n,1}$ differentiable, then the matrix*

$$\mathbf{f}_{,\mathbf{x}}(\mathbf{x}_0) = \frac{\partial}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}_0) = \begin{bmatrix} \frac{\partial}{\partial x^1} f_1(\mathbf{x}_0) & \ldots & \frac{\partial}{\partial x^n} f_1(\mathbf{x}_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial x^1} f_m(\mathbf{x}_0) & \ldots & \frac{\partial}{\partial x^n} f_m(\mathbf{x}_0) \end{bmatrix} = \left[\frac{\partial}{\partial x^j} f^i(\mathbf{x}_0) \right]_{j=1,\ldots,n}^{i=1,\ldots,m} \in \mathbb{R}^{m,n}$$

is called the Jacobian of $\mathbf{f}(\mathbf{x})$ regarding to $\mathbf{x}$ at $\mathbf{x}_0$.

Derivatives of tensor products may be evaluated using the product rule, see [3, Section 8.3]. For any $\mathbf{A}(\mathbf{x}) : \mathbb{R}^{l,1} \to \mathbb{R}^{m,n}$ and $\mathbf{y} \in \mathbb{R}^{n,1}$ we have

$$\begin{aligned} (\mathbf{A}(\mathbf{x})\mathbf{y})_{,\mathbf{x}} &= \mathbf{A}_{,\mathbf{x}}(\mathbf{x})\mathbf{y} + \mathbf{A}(\mathbf{x})\mathbf{y}_{,\mathbf{x}} \\ &= \mathbf{A}_{,\mathbf{x}}(\mathbf{x})\mathbf{y}. \end{aligned}$$

These considerations motivate the following Definition.

Definition A.0.4 (Derivative of a matrix-valued mapping) *If $\mathbf{A}(\mathbf{x}) : \mathbb{R}^{l,1} \to \mathbb{R}^{m,n}$, then the first partial derivative of $\mathbf{A}(\mathbf{x})\mathbf{y}$ and of $\mathbf{A}(\mathbf{x})\mathbf{B}$ with respect to the vector $\mathbf{x} \in \mathbb{R}^{l,1}$ is defined by*

$$\mathbf{A}_{,\mathbf{x}}(\mathbf{x})\mathbf{y} = (\mathbf{A}(\mathbf{x})\mathbf{y})_{,\mathbf{x}} = \left[\sum_{k=1}^{n} \frac{\partial A^i_k(\mathbf{x})}{\partial x^j} y^k \right]_{j=1,\ldots,l}^{i=1,\ldots,m} \quad \textit{with} \quad \mathbf{y} \in \mathbb{R}^{n,1}, \ \mathbf{A}_{,\mathbf{x}}\mathbf{y} \in \mathbb{R}^{m,l},$$

$$\mathbf{A}_{,\mathbf{x}}(\mathbf{x})\mathbf{B} = (\mathbf{A}(\mathbf{x})\mathbf{B})_{,\mathbf{x}} = \left[\sum_{k=1}^{n} \frac{\partial A^i_k(\mathbf{x})}{\partial x^j} B^k_s \right]_{\substack{j=1,\ldots,l \\ s=1,\ldots,t}}^{i=1,\ldots,m} \quad \textit{with} \quad \mathbf{B} \in \mathbb{R}^{n,t}, \ \mathbf{A}_{,\mathbf{x}}\mathbf{B} \in \mathbb{R}^{m,l,t}.$$

Definition A.0.5 (Higher order derivative of a matrix-valued mapping) *If $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$, then the higher order partial derivative is defined by*

$$\mathbf{A}_{,\mathbf{y}_1 \ldots \mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) = \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\mathbf{z}_1, \ldots, \mathbf{z}_K)}{\partial y_1^{r_1} \ldots \partial y_L^{r_L}} x_1^{r_1} \cdot \ldots \cdot x_L^{r_L} \right]_{j=1,\ldots,n}^{i=1,\ldots,m}$$

$$\textit{with } \mathbf{A}_{,\mathbf{y}_1 \ldots \mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) \in \mathbb{R}^{m,n}.$$

Here, $\mathbf{y}_i, \mathbf{x}_i \in \mathbb{R}^{n_I,1}$, $n_I \in \{m_1, ..., m_K\}$, $\forall I = 1, ..., L$.

Remark A.0.6 **1)** For convenience we will use the simplified notation

$$\mathbf{A}_{,\mathbf{y}_1 \ldots \mathbf{y}_L}()(\mathbf{x}_1, ..., \mathbf{x}_l) := \mathbf{A}_{,\mathbf{y}_1 \ldots \mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_l).$$

2) For $L = 0$ we get the matrix $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K)$ itself and for $L = 1$ we will also use the notation of Definition A.0.4.

Lemma A.0.7 (Linearity) *If* $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$. *and if* $\mathbf{z}_I \in \mathbb{R}^{m_I,1}$, $\forall I = 1, ..., K$ *and* $\mathbf{y}_I \in \mathbb{R}^{n_I,1}$, $\forall I = 1, ..., L$ *are given, then*

$$\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) : (\mathbb{R}^{n_1,1} \times ... \times \mathbb{R}^{n_L,1}) \to \mathbb{R}^{m,n} \tag{39}$$

defines a linear mapping with respect to $(\mathbf{x}_1, ..., \mathbf{x}_L)$.

Proof: The proof follows directly from Definition A.0.5. ■

Since for any fixed given $\mathbf{y}_I$ and $\mathbf{z}_J$ the mapping (39) is linear, it is also a linear mapping if additional $\mathbf{x}_1, ..., \mathbf{x}_{L-1}$are kept fixed.
If the matrix $\mathbf{A} = \mathbf{a}$ contains only one column vector $\mathbf{a}$, then the linear mapping $\mathbf{a}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_{L-1}, \cdot) : \mathbb{R}^{n_L,1} \to \mathbb{R}^{m,1}$ may be written as follows:

Definition A.0.8 (Matrix representation of higher order derivatives) *Let* $\mathbf{a}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,1}$, *then the matrix representation of the linear mapping* $\mathbf{a}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_{L-1}, \cdot) : \mathbb{R}^{n_L,1} \to \mathbb{R}^{m,1}$ *is defined by*

$$\begin{aligned}
&\mathbf{a}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_{L-1}, \cdot) \\
&:= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_{L-1}=1}^{n_{L-1}} \frac{\partial^L a^i(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_{L-1}^{r_{L-1}} \partial y_L^j} x_1^{r_1} \cdot ... \cdot x_{L-1}^{r_{L-1}} \right]_{j=1,...,n_L}^{i=1,...,m} \in \mathbb{R}^{m,n_L}.
\end{aligned}$$

If $\mathbf{a}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,1}$ then

$$\begin{aligned}
&\mathbf{a}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A_j^i(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L} \right]^{i=1,...,m} \\
&= \left[\sum_{r_L=1}^{n_L} \left(\sum_{r_1=1}^{n_1} \cdots \sum_{r_{L-1}=1}^{n_{L-1}} \frac{\partial^L A_j^i(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_{L-1}^{r_{L-1}} \right) x_L^{r_L} \right]^{i=1,...,m} \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_{L-1}=1}^{n_{L-1}} \frac{\partial^L A_j^i(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_{L-1}^{r_{L-1}} \right]_{r_L=1,...,n_L}^{i=1,...,m} \mathbf{x}_L \\
&= \mathbf{a}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_{L-1}, \cdot)\mathbf{x}_L.
\end{aligned}$$

Remark A.0.9 1) The vector-valued function $\mathbf{a}(\mathbf{z}_1, ..., \mathbf{z}_K)$ arises in our treatments often as $\mathbf{a}(\mathbf{z}_1, ..., \mathbf{z}_K) = \mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K)\mathbf{v}$.
2) Definition A.0.8 can be generalized for arbitrary vector-valued functions. For instance we use

$$\mathbf{A}_{,\mathbf{y}}(\mathbf{y})(\mathbf{x})\mathbf{B}(\mathbf{z})\mathbf{v} = (\mathbf{A}_{,\mathbf{y}}(\mathbf{y})(\cdot)\mathbf{B}(\mathbf{z})\mathbf{v})\mathbf{x} = \left[\sum_{j=1}^{n} \frac{\partial A_j^i(\mathbf{y})}{\partial y^r} \sum_{k=1}^{s} B_k^j(\mathbf{z}) v^k \right]_{r=1,...,l}^{i=1,...,m} \mathbf{x}$$

$$(\mathbf{A}_{,\mathbf{y}}(\mathbf{y})(\cdot)\mathbf{B}(\mathbf{z})\mathbf{v}) := \left[\sum_{j=1}^{n} \frac{\partial A_j^i(\mathbf{y})}{\partial y^r} \sum_{k=1}^{s} B_k^j(\mathbf{z}) v^k \right]_{r=1,...,l}^{i=1,...,m}$$

with $\mathbf{A}(\mathbf{y}) : \mathbb{R}^{l,1} \to \mathbb{R}^{m,n}$, $\mathbf{B}(\mathbf{z}) : \mathbb{R}^{k,1} \to \mathbb{R}^{n,s}$, $\mathbf{v} \in \mathbb{R}^{s,1}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{l,1}$ in the linearization for the nonlinear equations of motion.

Lemma A.0.10 (Sequence of derivatives) *If* $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$ *is sufficiently smoothly differentiable with respect to* $\mathbf{z}_i$, $\forall i = 1, ..., K$, *then*

$$\begin{aligned}
&\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_S...\mathbf{y}_T...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_S, ..., \mathbf{x}_T, ...\mathbf{x}_L) \\
&= \mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_T...\mathbf{y}_S...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_T, ..., \mathbf{x}_S, ...\mathbf{x}_L).
\end{aligned}$$

Proof: The proof follows directly from Definition A.0.5 and the Schwarz's Theorem, e.g. [10]. ∎

Remark A.0.11 Because of the symmetry with respect to the sequence of derivatives Definition A.0.4 may be applied not only for the last argument $\mathbf{x}_L$. It may be generalized to an arbitrary argument $\mathbf{x}_I,\ \forall I = 1, ..., L$.

Lemma A.0.12 *Let* $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$ *and* $\mathbf{v} \in \mathbb{R}^{n,1}$*, then*

$$\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L)\mathbf{v} = (\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K)\mathbf{v})_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{x}_1, ..., \mathbf{x}_L).$$

Proof:

$$\begin{aligned}
&\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{z}_1, ..., \mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L)\mathbf{v} \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L}\right]_{j=1,...,n}^{i=1,...,m} \mathbf{v} \\
&= \left[\sum_{j=1}^{n}\left(\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\mathbf{z}_1, ..., \mathbf{z}_K)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L}\right) v^j\right]^{i=1,...,m} \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L \sum_{j=1}^{n} A^i_j(\mathbf{z}_1, ..., \mathbf{z}_K) v^j}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L}\right]^{i=1,...,m} \\
&= (\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K)\mathbf{v})_{,\mathbf{y}_1...\mathbf{y}_L}(\mathbf{x}_1, ..., \mathbf{x}_l).
\end{aligned}$$

∎

Lemma A.0.13 *Let* $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$*, then*

$$(\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_L}()(\mathbf{x}_1, ..., \mathbf{x}_L))_{,\mathbf{y}_{L+1}}()(\mathbf{x}_{L+1}) = \mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_{L+1}}()(\mathbf{x}_1, ..., \mathbf{x}_{L+1}).$$

Proof:

$$\begin{aligned}
&(\mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_L}()(\mathbf{x}_1, ..., \mathbf{x}_L))_{,\mathbf{y}_{L+1}}()(\mathbf{x}_{L+1}) \\
&= \left(\left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\cdot)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L}\right]_{j=1,...,n}^{i=1,...,m}\right)_{,\mathbf{y}_{L+1}} ()(\mathbf{x}_{L+1}) \\
&= \sum_{r_{L+1}=1}^{m_{L+1}} \frac{\partial}{\partial y_{L+1}^{r_{L+1}}} \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\cdot)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L}\right]_{j=1,...,n}^{i=1,...,m} x_{L+1}^{r_{L+1}} \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_L=1}^{n_L} \sum_{r_{L+1}=1}^{n_{L+1}} \frac{\partial}{\partial y_{L+1}^{r_{L+1}}} \frac{\partial^L A^i_j(\cdot)}{\partial y_1^{r_1} \cdots \partial y_L^{r_L}} x_1^{r_1} \cdot ... \cdot x_L^{r_L} \cdot x_{L+1}^{r_{L+1}}\right]_{j=1,...,n}^{i=1,...,m} \\
&= \left[\sum_{r_1=1}^{n_1} \cdots \sum_{r_{L+1}=1}^{n_{L+1}} \frac{\partial^{L+1} A^i_j(\cdot)}{\partial y_1^{r_1} \cdots \partial y_{L+1}^{r_{L+1}}} x_1^{r_1} \cdot ... \cdot x_{L+1}^{r_{L+1}} \cdot\right]_{j=1,...,n}^{i=1,...,m} \\
&= \mathbf{A}_{,\mathbf{y}_1...\mathbf{y}_{L+1}}()(\mathbf{x}_1, ..., \mathbf{x}_{L+1})
\end{aligned}$$

∎

Lemma A.0.14 *Let* $\mathbf{A} : \mathbb{R}^{l,1} \to \mathbb{R}^{m,n};\ \mathbf{A} : \mathbf{x} \to \mathbf{A}(\mathbf{x})$*, then*

$$\left[\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k} x^k\right]_{j=1,\ldots,n}^{i=1,\ldots,m} \mathbf{y} = (\mathbf{A}(\mathbf{x})\mathbf{y})_{,\mathbf{x}}\mathbf{x}.$$

Proof:

$$\begin{aligned}
\left[\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k} x^k\right]_{j=1,\ldots,n}^{i=1,\ldots,m} \mathbf{y} &= \left[\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k} x^k\right]_{j=1,\ldots,n}^{i=1,\ldots,m} \left[y^j\right]^{j=1,\ldots,n} \\
&= \left[\sum_{j=1}^{n}\left(\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k} x^k\right) y^j\right]^{i=1,\ldots,m} \\
&= \left[\sum_{k=1}^{n}\sum_{j=1}^{n} \frac{\partial A^i_j(\mathbf{x}) y^j}{\partial x^k} x^k\right]^{i=1,\ldots,m} \\
&= \left[\sum_{k=1}^{n} \frac{\partial \sum\limits_{j=1}^{n} A^i_j(\mathbf{x}) y^j}{\partial x^k} x^k\right]^{i=1,\ldots,m} \\
&= \left[\frac{\partial \sum\limits_{j=1}^{n} A^i_j(\mathbf{x}) y^j}{\partial x^k}\right]_{k=1,\ldots,n}^{i=1,\ldots,m} \left[x^k\right]^{k=1,\ldots,n} \\
&= (\mathbf{A}(\mathbf{x})\mathbf{y})_{,\mathbf{x}}\mathbf{x}.
\end{aligned}$$

■

Definition A.0.15 (First order Taylor series expansion of a vector-valued mapping) *Let* $\mathbf{f}(\mathbf{x}) : \mathbb{R}^{l,1} \to \mathbb{R}^{m,1}$*, then the first order Taylor series has the following form*

$$\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}_{,\mathbf{x}}(\mathbf{x})\Delta\mathbf{x} + \mathcal{O}(||\Delta\mathbf{x}||^2).$$

Lemma A.0.16 (First order Taylor series expansion of a matrix-valued mapping) *Let* $\mathbf{A} : \mathbb{R}^{l,1} \to \mathbb{R}^{m,n};\ \mathbf{A} : \mathbf{x} \to \mathbf{A}(\mathbf{x})$*, then the first order Taylor series expansion has the following form*

$$\begin{aligned}
\mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) &= \mathbf{A}(\mathbf{x}) + \mathbf{A}_{,\mathbf{x}}(\mathbf{x})(\Delta\mathbf{x}) + \mathcal{O}(||\Delta\mathbf{x}||^2) \\
&= \mathbf{A}(\mathbf{x}) + \left[\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k} \Delta x^k\right]_{j=1,\ldots,n}^{i=1,\ldots,m} + \mathcal{O}(||\Delta\mathbf{x}||^2).
\end{aligned}$$

Furthermore, for $\mathbf{y} \in \mathbb{R}^{n,1}$ *it follows that*

$$\begin{aligned}
\mathbf{A}(\mathbf{x} + \Delta\mathbf{x})\mathbf{y} &= \mathbf{A}(\mathbf{x})\mathbf{y} + \mathbf{A}_{,\mathbf{x}}(\mathbf{x})(\Delta\mathbf{x})\mathbf{y} + \mathcal{O}(||\Delta\mathbf{x}||^2) \\
&= \mathbf{A}(\mathbf{x})\mathbf{y} + (\mathbf{A}_{,\mathbf{x}}(\mathbf{x})\mathbf{y})\Delta\mathbf{x} + \mathcal{O}(||\Delta\mathbf{x}||^2). \qquad (40)
\end{aligned}$$

Proof:

$$\mathbf{A}(\mathbf{x} + \Delta\mathbf{x}) = [\underbrace{A^i_j(\mathbf{x} + \Delta\mathbf{x})}_{\mathbb{R}}]_{j=1,\ldots,n}^{i=1,\ldots,m}$$

$$
\begin{aligned}
&= [\underbrace{A^i_j(\mathbf{x})}_{\mathbb{R}} + \underbrace{A^i_{j,\mathbf{x}}(\mathbf{x})}_{\mathbb{R}^{1,l}}\underbrace{\Delta\mathbf{x}}_{\mathbb{R}^{l,1}} + \underbrace{\mathcal{O}(||\Delta\mathbf{x}||^2)}_{\mathbb{R}}]^{i=1,\ldots,m}_{j=1,\ldots,n} \\
&= [A^i_j(\mathbf{x})]^{i=1,\ldots,m}_{j=1,\ldots,n} + [A^i_{j,\mathbf{x}}(\mathbf{x})\Delta\mathbf{x}]^{i=1,\ldots,m}_{j=1,\ldots,n} + \mathcal{O}(||\Delta\mathbf{x}||^2) \\
&= \mathbf{A}(\mathbf{x}) + [A^i_{j,\mathbf{x}}(\mathbf{x})\Delta\mathbf{x}]^{i=1,\ldots,m}_{j=1,\ldots,n} + \mathcal{O}(||\Delta\mathbf{x}||^2) \\
&= \mathbf{A}(\mathbf{x}) + \left[\sum_{k=1}^{n} \frac{\partial A^i_j(\mathbf{x})}{\partial x^k}\Delta x^k\right]^{i=1,\ldots,m}_{j=1,\ldots,n} + \mathcal{O}(||\Delta\mathbf{x}||^2).
\end{aligned}
$$

The second statement may be proven considering the Taylor series expansion of $\mathbf{f}(\mathbf{x} + \Delta\mathbf{x}) = \mathbf{A}(\mathbf{x} + \Delta\mathbf{x})\mathbf{y}$, or using Lemma A.0.12. ■

Remark A.0.17 Note that in contrast to standard matrix notation the parantheses in the notation of 3rd order tensors in (40) are significant and should not be left out.

Lemma A.0.18 (First order Taylor series expansion of higher derivatives of matrix-valued mappings) *Let* $\mathbf{A}(\mathbf{z}_1, ..., \mathbf{z}_K) : (\mathbb{R}^{m_1,1} \times ... \times \mathbb{R}^{m_K,1}) \to \mathbb{R}^{m,n}$, *then the Taylor series expansion has the following form*

$$
\begin{aligned}
&\mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L}(\mathbf{z}_{1N} + \Delta\mathbf{z}_1, ..., \mathbf{z}_{KN} + \Delta\mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&\quad = \mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L}(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&\quad + \sum_{I=1}^{K} \mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L\mathbf{z}_I}(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})(\mathbf{x}_1, ..., \mathbf{x}_L, \Delta\mathbf{z}_I) + \mathcal{O}(||\Delta\mathbf{z}||^2).
\end{aligned}
$$

Proof:

$$
\begin{aligned}
&\mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L}(\mathbf{z}_{1N} + \Delta\mathbf{z}_1, ..., \mathbf{z}_{KN} + \Delta\mathbf{z}_K)(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&\quad = \left[\sum_{r_1=1}^{n_1}\cdots\sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\mathbf{z}_{1N} + \Delta\mathbf{z}_1, ..., \mathbf{z}_{KN} + \Delta\mathbf{z}_K)}{\partial y_1^{r_1}\cdots\partial y_L^{r_L}} x_1^{r_1}\cdot ...\cdot x_L^{r_L}\right]^{i=1,\ldots,m}_{j=1,\ldots,n} \\
&\quad = \left[\sum_{r_1=1}^{n_1}\cdots\sum_{r_L=1}^{n_L} \frac{\partial^L A^i_j(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})}{\partial y_1^{r_1}\cdots\partial y_L^{r_L}} x_1^{r_1}\cdot ...\cdot x_L^{r_L}\right]^{i=1,\ldots,m}_{j=1,\ldots,n} \\
&\qquad + \left[\sum_{r_1=1}^{n_1}\cdots\sum_{r_L=1}^{n_L} \frac{\partial^L \sum\limits_{I=1}^{K} A^i_{j,\mathbf{z}_I}(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})\Delta\mathbf{z}_I}{\partial y_1^{r_1}\cdots\partial y_L^{r_L}} x_1^{r_1}\cdot ...\cdot x_L^{r_L}\right]^{i=1,\ldots,m}_{j=1,\ldots,n} \\
&\qquad + \mathcal{O}(||\Delta\mathbf{z}||^2) \\
&\quad = \mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L}(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&\qquad + \left[\sum_{r_1=1}^{n_1}\cdots\sum_{r_L=1}^{n_L} \frac{\partial^L \sum\limits_{I=1}^{K}\sum\limits_{s_I=1}^{m_I} \frac{\partial A^i_j(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})}{\partial z_I^{s_I}}\Delta z_I^{s_I}}{\partial y_1^{r_1}\cdots\partial y_L^{r_L}} x_1^{r_1}\cdot ...\cdot x_L^{r_L}\right]^{i=1,\ldots,m}_{j=1,\ldots,n} \\
&\qquad + \mathcal{O}(||\Delta\mathbf{z}||^2) \\
&\quad = \mathbf{A}_{,\mathbf{y}_1\ldots\mathbf{y}_L}(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})(\mathbf{x}_1, ..., \mathbf{x}_L) \\
&\qquad + \sum_{I=1}^{K}\left[\sum_{r_1=1}^{n_1}\cdots\sum_{r_L=1}^{n_L}\sum_{s_I=1}^{m_I} \frac{\partial^L A^i_j(\mathbf{z}_{1N}, ..., \mathbf{z}_{KN})}{\partial y_1^{r_1}\cdots\partial y_L^{r_L}\partial z_I^{s_I}} x_1^{r_1}\cdot ...\cdot x_L^{r_L}\cdot\Delta z_I^{s_I}\right]^{i=1,\ldots,m}_{j=1,\ldots,n}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{O}(||\Delta\mathbf{z}||^2)\\
&= \mathbf{A}_{,\mathbf{y}_1\dots\mathbf{y}_L}(\mathbf{z}_{1N},...,\mathbf{z}_{KN})(\mathbf{x}_1,...,\mathbf{x}_L)\\
&\quad+\sum_{I=1}^{K}\mathbf{A}_{,\mathbf{y}_1\dots\mathbf{y}_L\mathbf{z}_I}(\mathbf{z}_{1N},...,\mathbf{z}_{KN})(\mathbf{x}_1,...,\mathbf{x}_L,\Delta\mathbf{z}_I)+\mathcal{O}(||\Delta\mathbf{z}||^2).
\end{aligned}
$$

■

Lemma A.0.19 (Total time derivative of a scalar function) *If* $f(\mathbf{x}):\mathbb{R}^{l,1}\to\mathbb{R}$ *and* $\mathbf{x}(t):\mathbb{R}\to\mathbb{R}^{l,1}$*, then the total time derivative is defined by*

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{x}) = f_{,\mathbf{x}}(\mathbf{x})\dot{\mathbf{x}}.$$

Lemma A.0.20 (Total time derivative of a matrix valued function) *Let* $\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K):(\mathbb{R}^{l_1,1}\times...\times\mathbb{R}^{l_K,1})\to\mathbb{R}^{m,n}$ *and* $\mathbf{x}_I(t):\mathbb{R}\to\mathbb{R}^{l_I,1}$*,* $\forall I=1,...,K$*, then it follows that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K) = \mathbf{A}_{,\mathbf{x}_1}()(\dot{\mathbf{x}}_1)+...+\mathbf{A}_{,\mathbf{x}_K}()(\dot{\mathbf{x}}_K)=\sum_{I=1}^{K}\mathbf{A}_{,\mathbf{x}_I}()(\dot{\mathbf{x}}_I), \tag{41}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K) = \sum_{I=1}^{K}\sum_{J=1}^{K}\mathbf{A}_{,\mathbf{x}_I\mathbf{x}_J}()(\dot{\mathbf{x}}_I,\dot{\mathbf{x}}_J)+\sum_{I=1}^{K}\mathbf{A}_{,\mathbf{x}_I}()(\ddot{\mathbf{x}}_I). \tag{42}$$

Proof:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K)\\
&= \left[\frac{\mathrm{d}}{\mathrm{d}t}A^i_j(\mathbf{x}_1,...,\mathbf{x}_K)\right]^{i=1,\dots,m}_{j=1,\dots,n}\\
&= \left[A^i_{j,\mathbf{x}_1}()\dot{\mathbf{x}}_1+...+A^i_{j,\mathbf{x}_K}()\dot{\mathbf{x}}_K\right]^{i=1,\dots,m}_{j=1,\dots,n}\\
&= \left[A^i_{j,\mathbf{x}_1}()\dot{\mathbf{x}}_1\right]^{i=1,\dots,m}_{j=1,\dots,n}+...+\left[A^i_{j,\mathbf{x}_K}()\dot{\mathbf{x}}_K\right]^{i=1,\dots,m}_{j=1,\dots,n}\\
&= \mathbf{A}_{,\mathbf{x}_1}()\dot{\mathbf{x}}_1+...+\mathbf{A}_{,\mathbf{x}_K}()\dot{\mathbf{x}}_K
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x}_1,...,\mathbf{x}_K)\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{A}_{,\mathbf{x}_1}()\dot{\mathbf{x}}_1+...+\mathbf{A}_{,\mathbf{x}_K}()\dot{\mathbf{x}}_K\right)\\
&= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{I=1}^{K}\mathbf{A}_{,\mathbf{x}_I}()\dot{\mathbf{x}}_I\\
&= \sum_{I=1}^{K}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{A}_{,\mathbf{x}_I}()\dot{\mathbf{x}}_I)\\
&= \sum_{I=1}^{K}\frac{\mathrm{d}}{\mathrm{d}t}\left[\sum_{r=1}^{l}\frac{\partial}{\partial x^r_I}A^i_j()\dot{x}^r_I\right]^{i=1,\dots,m}_{j=1,\dots,n}\\
&= \sum_{I=1}^{K}\left[\sum_{r_I=1}^{l}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial x^{r_I}_I}A^i_j()\dot{x}^{r_I}_I\right)\right]^{i=1,\dots,m}_{j=1,\dots,n}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{I=1}^{K}\left[\sum_{r_I=1}^{l}\left(\frac{\mathrm{d}}{\mathrm{dt}}\frac{\partial}{\partial x_I^r}A_j^i()\dot{x}_I^r + \frac{\partial}{\partial x_I^{r_I}}A_j^i()\frac{\mathrm{d}}{\mathrm{dt}}\dot{x}_I^{r_I}\right)\right]_{j=1,\dots,n}^{i=1,\dots,m} \\
&= \sum_{I=1}^{K}\left[\sum_{r_I=1}^{l}\left(\sum_{J=1}^{K}\sum_{r_J=1}^{l}\frac{\partial}{\partial x_I^{r_I}\partial x_J^{r_J}}A_j^i()\dot{x}_J^{r_J}\dot{x}_I^{r_I} + \frac{\partial}{\partial x_I^{r_I}}A_j^i()\ddot{x}_I^{r_I}\right)\right]_{j=1,\dots,n}^{i=1,\dots,m} \\
&= \sum_{I=1}^{K}\sum_{J=1}^{K}\left[\sum_{r_I=1}^{l}\sum_{r_J=1}^{l}\frac{\partial}{\partial x_I^r \partial x_J^r}A_j^i()\dot{x}_I^r\dot{x}_J^r\right]_{j=1,\dots,n}^{i=1,\dots,m} \\
&\quad + \sum_{I=1}^{K}\left[\sum_{r_I=1}^{l}\frac{\partial}{\partial x_I^{r_I}}A_j^i()\ddot{x}_I^{r_I}\right]_{j=1,\dots,n}^{i=1,\dots,m} \\
&= \sum_{I=1}^{K}\sum_{J=1}^{K}\mathbf{A}_{,\mathbf{x}_I\mathbf{x}_J}()(\dot{\mathbf{x}}_I,\dot{\mathbf{x}}_J) + \sum_{I=1}^{K}\mathbf{A}_{,\mathbf{x}_I}()(\ddot{\mathbf{x}}_I)
\end{aligned}
$$

■

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