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Singularities in differential-algebraic systems with unilateral constraints

*Inequality (*unilateral*) constraints may introduce various types of singularities in differential-algebraic systems ([5]). In the present paper we consider contact conditions that are formulated as unilateral constraints for position and velocity coordinates in a mechanical multibody system. If the surfaces of the bodies being in contact are worn then the contact conditions result often in constraints that are only piecewise differentiable. We discuss a regularization that considers qualitatively the elastic deformation of the bodies.*

1. Introduction

Mechanical multibody system (MBS) models are well established in a wide area of applications (e. g. vehicle dynamics, robotics). The MBS model describes the state of the mechanical system by position coordinates $q(t) \in \mathbb{R}^{n_q}$ and velocity coordinates $v(t) = \dot{q}(t)$. The model equations

$$\begin{aligned} M(q)\ddot{q}(t) &= f(q,\dot{q},\lambda) + G^T(q)\lambda \\ 0 &= g(q) \end{aligned} \tag{1}$$

are obtained using the principles of classical mechanics. In (1) we have the symmetric positive definite mass matrix $M(q)$ and the forces $f(q,\dot{q},\lambda)$. Eqs. (1) form a differential-algebraic system of index 3 (see e. g. [3, pp. 463f]).

The quantitative consideration of the elasticity in the connections between bodies of the MBS by elastic forces results typically in solution components oscillating with high frequency and (very) small amplitude. These solution components may cause considerable difficulties in the numerical solution ([6]). Therefore it is common practice to neglect as far as possible clearance and elastic effects in the joints connecting the bodies of the MBS. These model assumptions result in (holonomic) constraints $g = 0$ and constraint forces $G^T(q)\lambda$ in (1) ($G(q) := \frac{\partial}{\partial q} g(q)$).

It is well known that the solution of higher index differential-algebraic systems is extremely sensitive to changes in the algebraic constraints. The solution of (1) does not depend continuously on small perturbations in the constraints $g(q) = 0$ ([3, Section VII.1]). Small errors in the formulation and / or in the numerical evaluation of the holonomic constraints may therefore result in large errors in the solution, it may even result in singularities.

In the present paper we discuss this problem for MBS with contact conditions and show that singularities can be avoided if the elastic deformation of the bodies being in contact is considered qualitatively.

2. Rigid body contact conditions in mechanical multibody systems

Two rigid bodies are in *contact* iff they touch but do not penetrate each other. Consider as a typical example the contact between a wheel and a rail. Exploiting the rotational symmetry of the wheel the contact condition can be formulated using the (2D) cross sections of wheel and rail (for the details we refer to [1]). The left plot of Fig. 1 shows the cross sections of **w**heel and **r**ail

$$\{ \, (\, y + \cos\varphi \cdot s - \sin\varphi \cdot F^{(w)}(s) \, , \; z + \sin\varphi \cdot s + \cos\varphi \cdot F^{(w)}(s) \,)^T \; : \; \underline{s} \le s \le \overline{s} \, \} \, ,$$

$$\{ \, (\, v \, , \; w \,)^T \; : \; w = F^{(r)}(v), \; \underline{v} \le v \le \overline{v} \, \} \, ,$$

that are defined by profile functions $F^{(w)}(s)$ and $F^{(r)}(v)$ being given by industrial standards. The cross sections are plotted in (v, w)–coordinates, the profile function $F^{(w)}(s)$ is defined for the wheel coordinate s (the s-axis is parallel to the wheel axle). y and $-z$ denote the lateral and vertical displacement of the wheel, respectively (note, that the w-axis is oriented downwards). In Fig. 1 the roll angle φ and the yaw angle ψ are fixed to $\varphi = 1.43°$, $\psi = 0$ according to the nominal position of a wheel in a rigid wheelset on a straight track with rails inclined 1/40.

Wheel and rail are in contact iff the *distance function*

$$\Delta(s; y, z) := F^{(r)}(y + \cos\varphi \cdot s - \sin\varphi \cdot F^{(w)}(s)) - (z + \sin\varphi \cdot s + \cos\varphi \cdot F^{(w)}(s))$$

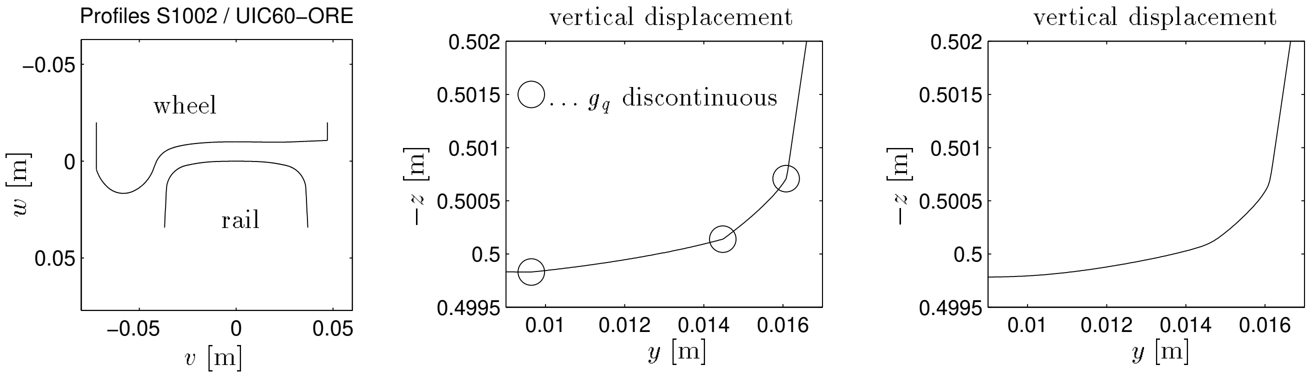

Figure 1: Left: Cross sections of wheel (profile S1002) and rail (profile UIC60–ORE).
Centre: $-z$ implicitly defined by (3). Right: $-z$ implicitly defined by (5) with $\nu = 2.0_{10}$-5.

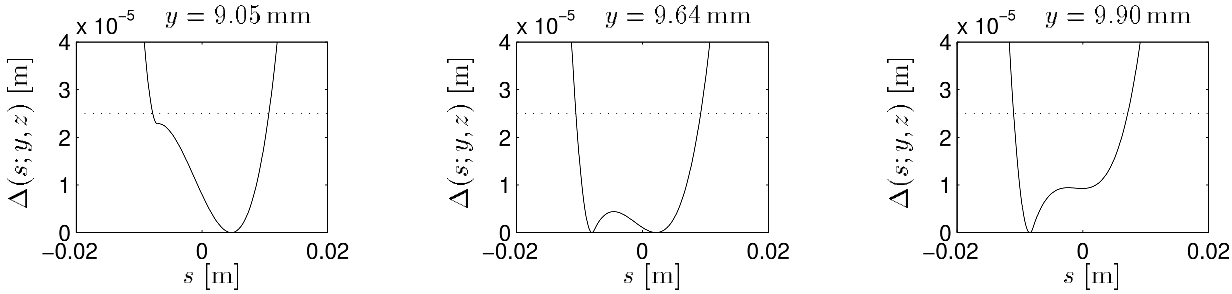

Figure 2: Distance function $\Delta(s;y,z)$ with $\varphi = 1.43°$, $\psi = 0$ and z defined by (3), profiles S1002 (wheel) and UIC60–ORE (rail).

satisfies the infinite set of unilateral constraints

$$\Delta(s;y,z) \geq 0\ , \quad (\underline{s} \leq s \leq \overline{s}) \tag{2}$$

and (at least) one of the constraints (2) is active. In a MBS model of a wheel-rail system both y and z are defined by the position coordinates q such that each wheel-rail contact results in a scalar constraint in (1):

$$0 = g_j(q) := \min_{\underline{s} \leq s \leq \overline{s}} \Delta(s; y(q), z(q))\ . \tag{3}$$

Because of (2) the bodies do not penetrate each other, because of (3) they are in contact. Eq. (3) defines implicitly the vertical displacement $-z$ as function of the lateral displacement y (see the central plot of Fig. 1).

This straightforward formulation of the contact condition yields singularities in (1) if we consider wheels with *wear profiles* that are commonly used by the Deutsche Bahn AG. The definition of wear profiles is based on measured data of wheels that are worn after frequent use. Therefore parts of the surfaces of wheel and rail are approximately parallel. This is illustrated by Fig. 2 that shows $\Delta(s;y,z)$ for all $s \in [\underline{s}, \overline{s}]$ with $\Delta(s;y,z) \leq 40\,\mu$m. For all 3 configurations we have $\Delta(s;y,z) \leq 25\,\mu$m for all $s \in [\,-8.0\,\mathrm{mm}, 7.0\,\mathrm{mm}]$.

There is an important difference between the plots: in the central plot there are two *contact points* (i. e. points s^* with $\Delta(s^*;y,z) = \min_s \Delta(s;y,z)$) but in the left and the right plot of Fig. 2 there is only one contact point. A slight change of y and z makes the contact point jump from $s^*_- = 2.4\,\mathrm{mm}$ to $s^*_+ = -8.0\,\mathrm{mm}$. If the contact point s^* changes discontinuously then the function $g_j(q)$ in (3) is continuous but in general not differentiable. In the central plot of Fig. 1 these configurations are marked by circles.

The differentiability of g is, however, essential for the existence of a classical solution of (1) ([7]). This can be seen from the *hidden constraint*

$$0 = \frac{d}{dt} g(q(t)) = g_q(q(t))\dot{q}(t) = G(q(t))v(t)$$

in (1) that has to be satisfied whenever g is differentiable. If g is not differentiable at $q^* = q(t^*) \in \mathrm{IR}^{n_q}$ and the limits $G^*_- := \lim_{t \to t^*-0} G(q(t))$ and $G^*_+ := \lim_{t \to t^*+0} G(q(t))$ exist then $v(t)$ has to be discontinuous at t^* if $\lim_{t \to t^*-0} v(t) \notin \ker G^*_+$.

In [1] we discussed the dynamical simulation of wheel-rail systems on the basis of the rigid body contact condition (3). Even in very simple benchmark problems (motion of a rigid wheelset on a straight track) we got frequent discontinuities in $v = \dot{q}$ and λ that resulted in large errors in the simulation results.

3. Regularization of contact conditions

The phenomenon that has been discussed in Section 2 for wheel-rail systems is typical of the contact between bodies having approximately parallel surfaces. This more general situation is sketched in the left plot of Fig. 3.

Up to now we have considered only *rigid* bodies. In the real mechanical system, however, the bodies being in contact are elastically deformed and there is instead of isolated contact *points* a contact *patch* Ω between the bodies. The cross-sections of the deformed bodies are sketched in the right plot of Fig. 3. We restrict ourselves to problems with plane contact patches (i. e. $\Omega \subset \mathcal{E}$ with a plane $\mathcal{E}$). The elastic deformation is caused by a force F acting in normal direction to $\mathcal{E}$. The points P_1 and P_2 in Fig. 3 denote points of Body 1 and Body 2, respectively, that are far from the contact patch, $\overrightarrow{P_1 P_2}$ is supposed to be orthogonal to $\mathcal{E}$.

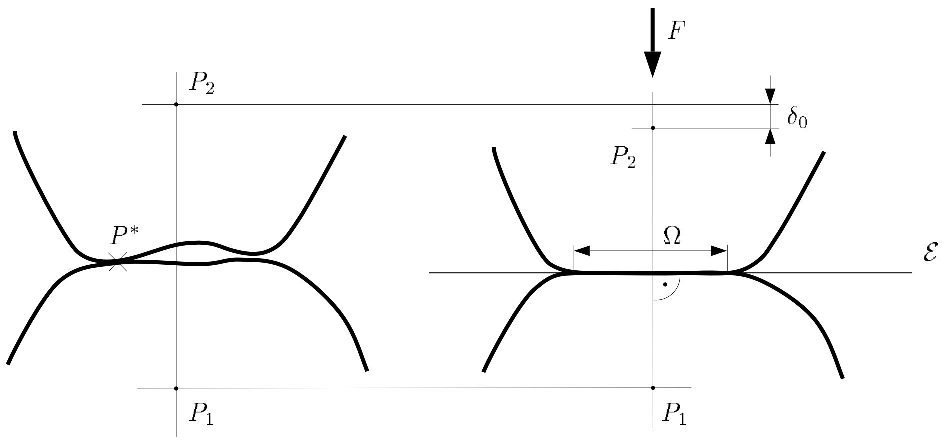

Figure 3: Elastic deformation of two bodies being in contact: undeformed (left plot) and deformed (right plot) state.

Compared with the undeformed state (left plot of Fig. 3) the distance between P_1 and P_2 is reduced by the *elastic approach* $\delta_0 > 0$ if the elastic deformation of the bodies is considered (right plot of Fig. 3). The elastic approach δ_0 depends in a complicated way on the force F, on the geometry of the surfaces being in contact, on material constants (Young moduli, Poisson number) and on other factors ([4]).

Both the rigid and the elastic contact model are essentially based on the shape of the surfaces of the undeformed bodies. The important difference is the fact that this information is used

- only *locally* (in a neighbourhood of the contact point(s) P^*) in the rigid contact model
- but *globally* (in a neighbourhood of the contact patch Ω) in the elastic model

(see Fig. 3). If there are small changes of the surfaces (caused e. g. by wear) then the elastic model is much less sensitive w. r. t. these changes than the rigid one. We want to adopt this benefit of the elastic model to avoid the singularities in the contact condition, i. e. to regularize the rigid body contact condition (3).

To discuss the details of this approach we come back to the wheel-rail contact problem. Because of the high axle loads being characteristic of railway vehicles the elastic approach

$$\delta_0 = - \min_{\underline{s} \leq s \leq \overline{s}} \Delta(s; y, z) \tag{4}$$

is in the range $10\,\mu\text{m} \ldots 100\,\mu\text{m}$ (note, that Δ is defined for the *undeformed* bodies). We do not want to consider the elastic deformation *quantitatively* because of the numerical problems mentioned in the Introduction. But *qualitatively* the vertical displacement $-z$ of the wheel depends in the elastic model on $\{ \Delta(s; y, z) \, : \, s \in I_s(\tilde{\delta}_0) \}$ with

$$I_s(\tilde{\delta}_0) := \{ s \, : \, \Delta(s; y, z) \leq \min_{\underline{s} \leq \tilde{s} \leq \overline{s}} \Delta(\tilde{s}; y, z) + \tilde{\delta}_0 \}$$

for some $\tilde{\delta}_0$ in the range $10\,\mu\text{m} \ldots 100\,\mu\text{m}$. Most of this information is lost formulating the rigid body contact condition (3). Various ideas to consider $\{ \Delta(s; y, z) \, : \, s \in I_s(\tilde{\delta}_0) \}$ have been introduced in [2]. The most promising one is the substitution of (3) by the *quasi-elastic* contact condition

$$0 = g_j^{(\nu)}(q) := \underset{\underline{s} \le s \le \overline{s}}{\mathrm{smin}^{(\nu)}} \Delta(s;y,z) := -\nu \ln \left(\int_{\underline{s}}^{\overline{s}} \exp(-\frac{1}{\nu}\Delta(s;y,z))\, ds \,/ \int_{\underline{s}}^{\overline{s}} ds \right) \tag{5}$$

with a small parameter $\nu > 0$. If $0 < \nu \ll 1$ then $\mathrm{smin}_s^{(\nu)}\Delta$ is dominated by the function values of $\Delta(s;y,z)$ for $s \in I_s(\tilde{\delta}_0)$ since $\exp(-\frac{1}{\nu}\Delta)$ is rapidly decaying for increasing Δ. Lemma 1 summarizes some properties of $\mathrm{smin}_s^{(\nu)}$ that can be proved straightforwardly.

Lemma 1. *Let a continuous function $\Delta(s;y,z)$ be given. The function $\mathrm{smin}_s^{(\nu)}\Delta(s;y,z)$ is as often differentiable w. r. t. y and z as $\Delta(s;y,z)$. If y and z are fixed then we have*

a) $\mathrm{smin}_s^{(\nu)}\Delta(s;y,z) \ge \min_s \Delta(s;y,z)$, *($\nu > 0$) and*

b) $\lim_{\nu \to 0} \mathrm{smin}_s^{(\nu)}\Delta(s;y,z) = \min_s \Delta(s;y,z)$.

Eq. (5) results in a smooth constraint in (1) that defines z implicitly. In general some of the unilateral constraints (2) are slightly violated if $\mathrm{smin}_s^{(\nu)}\Delta(s;y,z) = 0$ (see Lemma 1a). Similar to the elastic model (see (4)) we choose the parameter ν ($0 < \nu \ll 1$) such that $-\min_s \Delta(s;y,z)$ is in the range of $10\,\mu\mathrm{m} \ldots 100\,\mu\mathrm{m}$ if (5) is satisfied, typical values are $\nu \in [\,1.0_{10}\text{-}5\,,\,5.0_{10}\text{-}5\,]$. As an example the right plot of Fig. 1 shows (y,z) satisfying (5) with $\nu = 2.0_{10}\text{-}5$.

The quasi-elastic model overcomes the singularities of the rigid body contact condition (3) and avoids on the other hand the problems that are characteristic of pure elastic models. It has been implemented in the MBS simulation package SIMPACK and is used successfully in the dynamical simulation of wheel-rail systems. The details of the implementation and simulation results comparing the rigid and the quasi-elastic contact model are presented in [1].

The approach that was discussed in detail for wheel-rail systems can be carried over to the regularization of other rigid body contact conditions. It is most favorable for surfaces being approximately parallel (e. g. worn surfaces).

4. Summary

Contact conditions in mechanical multibody systems can be formulated as unilateral constraints. If the elastic deformation of the bodies being in contact is completely neglected then these contact conditions may result in singularities in the MBS model equations. To regularize the contact conditions the elastic deformation has to be considered at least qualitatively. In the dynamical simulation of wheel-rail systems a regularization that was proposed in [2] can be applied. This approach is tailored to the contact between worn surfaces with large forces acting in normal direction to the contact patch.

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