Impasse points and descriptor systems with unilateral constraints

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ABSTRACT

Descriptor systems with inequality (*unilateral*) constraints may have singularities (*impasse points*) if the set of active constraints changes in the time interval of interest. A typical example are contact conditions in constrained mechanical systems that can be formulated as unilateral constraints for the position and velocity coordinates. We discuss these singularities in detail for a simple model problem and use a method proposed in [2] to regularize the contact conditions. The theoretical results are applied to the contact between a wheel and a rail, they can be used in the dynamical simulation of wheel-rail systems.

1. INTRODUCTION

The existence and uniqueness of the solution of an initial value problem

$$F(t, x(t), \dot{x}(t)) = 0 \ , \quad (\, t \in (0, T)\,) \ , \quad x(0) = x_0 \ , \quad \dot{x}(0) = \dot{x}_0$$

for a differential-algebraic equation (DAE) can in general be guaranteed only, if the initial values x_0, $\dot{x}_0$ are consistent and if F is sufficiently differentiable (see e. g. [5]). In the present paper we are concerned with DAEs violating these smoothness assumptions. The practical background are mechanical multibody systems (MBS) with contact conditions that result in constraints being only piecewise differentiable.

The descriptor form approach is well established in the dynamical simulation of MBS. The state of the MBS is described by position coordinates $q(t) \in \mathbb{R}^{n_q}$ and velocity coordinates $v(t) = \dot{q}(t)$. Based on the principles of classical mechanics we obtain the model equations

$$\begin{aligned} M(q)\ddot{q}(t) &= f(q, \dot{q}, \lambda) + G^T(q)\lambda \ , \quad (\, t \in (0, T)\,) \ , \\ 0 &= g(q) \ , \qquad q(0) = q_0 \ , \quad \dot{q}(0) = v_0 \ , \quad \lambda(0) = \lambda_0 \end{aligned} \tag{1}$$

with the generalized mass matrix $M(q) \in \mathbb{R}^{n_q \times n_q}$, the force term $f(q, \dot{q}, \lambda)$ and n_λ (holonomic) constraints $g = 0$. The constraint forces $G^T(q)\lambda$ with $G(q) := \frac{\partial}{\partial q} g(q)$ and $\lambda(t) \in \mathbb{R}^{n_\lambda}$ have to guarantee that the constraints are always satisfied. We suppose that $M(q)$ is non-singular and that $M(q)$ and $f(q, \dot{q}, \lambda)$ are Lipschitzian.

Definition 1 *The functions $q \in C^2[0, \hat{T}]$, $\lambda \in C[0, \hat{T}]$ are a* classical solution *of (1) on the time interval $[0, \hat{T}] \subset [0, T]$ iff they satisfy both the initial conditions and the DAE in (1) (for all $t \in (0, \hat{T})$).*

Remark 1 a) Because of $g(q(t)) \equiv 0$ a classical solution of (1) has to satisfy the *hidden constraints*

$$0 = \frac{d}{dt} g(q(t)) = g_q(q(t))\dot{q}(t) = G(q(t))v(t) =: \gamma^{(v)}(q(t), v(t)) \ , \tag{2}$$

$$0 = \frac{d^2}{dt^2} g(q(t)) = g_{qq}(q)(v,v) + G(q)M^{-1}(q)(f(q,v,\lambda) - G^T(q)\lambda) =: \gamma^{(\lambda)}(q,v,\lambda) \qquad (3)$$

(if $g_q(q)$ and $g_{qq}(q)$ do exist).
b) The existence and uniqueness of a classical solution of (1) can be guaranteed ([7]) if g is two times continuously differentiable, $G(q)$ has full row rank and the initial values are consistent, i. e.

$$g(q_0) = \gamma^{(v)}(q_0, v_0) = \gamma^{(\lambda)}(q_0, v_0, \lambda_0) = 0 \ .$$

Eqs. (1) and (2) show that we have to expect singularities if g is not differentiable in a point $q^* \in \mathbb{R}^{n_q}$, i. e. if $G(q^*) = g_q(q)|_{q=q^*}$ does not exist.

Definition 2 *Denote by $\mathbb{T}(q_0, v_0, \lambda_0) \subset (0,T]$ the set of all $\hat{T} \in (0,T]$ such that the initial value problem (1) has a classical solution on $[0,\hat{T}]$. The point $(q^*, v^*, \lambda^*) \in \mathbb{R}^{n_q} \times \mathbb{R}^{n_q} \times \mathbb{R}^{n_\lambda}$ is an* impasse point *of the DAE in (1) iff there are initial values q_0, v_0, λ_0 such that*

$$t^* := \sup(\{0\} \cup \{\, t \ : \ t \in \mathbb{T}(q_0, v_0, \lambda_0) \,\}) \in (0,T)$$

and the (classical) solution of (1) satisfies $q(t^*) = q^*$, $v(t^*) = v^*$, and $\lambda(t^*) = \lambda^*$.

Impasse points in the sense of Definition 2 are points that can be reached but not passed by a classical solution (see also [6]).
In Section 2 we illustrate with a very simple model problem that contact conditions in MBS models may result in impasse points in the model equations. In practical applications impasse points result typically from simplifications in the modelling process. In Section 3 we discuss how the impasse points in the MBS model equations could be avoided considering the elastic deformation of the bodies in the contact area.

2. CONTACT CONDITIONS AND IMPASSE POINTS: A MODEL PROBLEM

Contact conditions are in general formulated analytically using parametrizations of the surfaces being in contact. In this section we describe for a model problem the basic steps to get from the geometrical contact condition to a constraint $g(q) = 0$ in the model equations (1).
Consider as a very simple example a mass point in $\mathbb{R}^2$ that is attached to the centre of a massless plane disk with radius $\varrho = 1$. This disk (with the mass point) moves on the "floor" (see Fig. 1)

$$\{\, (x,y)^T \ : \ y \leq u(x) \,\}$$

that is defined by a convex function $u \in C^\infty(\mathbb{R})$ with

$$u(x) = |x| \ , \ \text{ if } \ |x| \geq \sqrt{2} - 1 \ . \qquad (4)$$

If the position coordinates $(x,y)^T$ of the mass point are denoted by $q = (q_1, q_2)^T$ then the disk and the floor do not penetrate each other iff all points of the disk surface

$$\{\, (x,y)^T \ : \ (x-q_1)^2 + (y-q_2)^2 = 1 \,\}$$

are "above" the floor, i. e. iff

$$\Delta(x;q) := q_2 - \sqrt{1-(x-q_1)^2} - u(x) \geq 0 \ , \ \ (q_1 - 1 \leq x \leq q_1 + 1) \ . \qquad (5)$$

It is verified straightforwardly that (5) for $x^{(1)} = q_1 + 1/\sqrt{2}$ and $x^{(2)} = q_1 - 1/\sqrt{2}$ implies $q_2 \geq \sqrt{2}$ and $q_2 - \sqrt{1-(x-q_1)^2} \geq \sqrt{2} - 1$ such that the region of the floor with $|x| < \sqrt{2} - 1$ can never get in contact with the disk since

$$u(x) < \sqrt{2} - 1 \ , \ \ (\,|x| < \sqrt{2} - 1\,) \ .$$

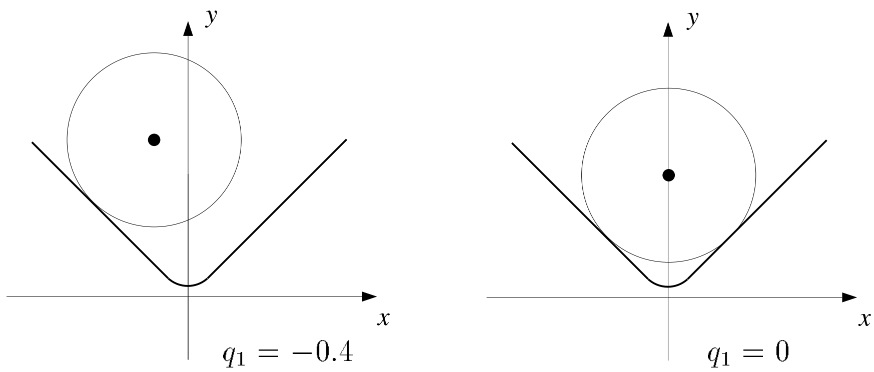

Figure 1: *Geometry of the model problem.*

Out of this region $u(x)$ coincides with $|x| = \max\{x, -x\}$ (see (4)). Therefore (5) is equivalent to

$$\left.\begin{array}{rcccl} \tilde{\Delta}^{(1)}(x;q) & := & q_2 - \sqrt{1-(x-q_1)^2} - x & \geq & 0 \\ \tilde{\Delta}^{(2)}(x;q) & := & q_2 - \sqrt{1-(x-q_1)^2} + x & \geq & 0 \end{array}\right\} \ , \quad (q_1 - 1 \leq x \leq q_1 + 1) \ . \tag{6}$$

If q is fixed then $\tilde{\Delta}^{(1)}(x;q)$ reaches its minimum in $x^{(1)} = q_1 + 1/\sqrt{2}$ and $\tilde{\Delta}^{(2)}(x;q)$ reaches its minimum in $x^{(2)} = q_1 - 1/\sqrt{2}$. Instead of the infinite set of constraints (6) we may consider the equivalent set of 2 constraints

$$g^{(1)}(q) \geq 0 \quad \text{with} \quad g^{(1)}(q) := \left(\begin{array}{c} \tilde{\Delta}^{(1)}(x^{(1)};q) \\ \tilde{\Delta}^{(2)}(x^{(2)};q) \end{array} \right) = \left(\begin{array}{c} q_2 - q_1 - \sqrt{2} \\ q_2 + q_1 - \sqrt{2} \end{array} \right) \tag{7}$$

(the inequality has to be read componentwise) or the equivalent scalar constraint

$$g^{(2)}(q) := \min_{\{x\,:\,|x-q_1|\leq 1\}} \Delta(x;q) = \min\{\, g_1^{(1)}(q), g_2^{(1)}(q) \,\} = q_2 - |q_1| + \sqrt{2} \geq 0 \ . \tag{8}$$

Disk and floor are in contact iff this unilateral constraint (8) is active. Equivalent conditions are that the infinite set of constraints (5) and the finite set of constraints (7) contain at least one active constraint.

The sets of unilateral constraints (5), (7), and (8) are equivalent but the infinite set (5) is unsuitable for practical computations. Using either (7) or (8) we end up with a complementarity problem

$$\begin{array}{rcl} M(q)\ddot{q}(t) & = & f(q,\dot{q},\lambda) + (G^{(i)}(q))^T \lambda \\ g^{(i)}(q) & \geq & 0 \ , \ \ \lambda \geq 0 \ , \ \ \lambda^T(t) g^{(i)}(q(t)) = 0 \end{array} \tag{9}$$

to describe the motion of the mass point ([4]). In (9) we have either $i = 1$ or $i = 2$. In contrast to (1) the Lagrangian multipliers λ in (9) are restricted to non-negative values. $\lambda_j > 0$ implies that the corresponding unilateral constraint $g_j^{(i)}(q) \geq 0$ is active.

At first sight there are no singularities in (9) with $i = 1$ since the surfaces of disk and floor are described by smooth functions and $g^{(1)}(q)$ depends linearly on q. However, it is well known ([4]) that changes of the set of active constraints in (9) may result in discontinuities in $\dot{q}$ and λ. For simplicity we restrict ourselves in the following to the special case that disk and floor remain in contact for all $t \in [0, T]$. Then there may be singularities in trajectories passing states $(\tilde{q}, \tilde{v}, \tilde{\lambda})$ with $\tilde{q}_1 = 0$ since the set of active constraints (7) changes from $\{\, g_2^{(1)}(q) \geq 0 \,\}$ (if $q_1 < 0$) via $\{\, g_1^{(1)}(q) \geq 0,\, g_2^{(1)}(q) \geq 0 \,\}$ (if $q_1 = 0$) to $\{\, g_1^{(1)}(q) \geq 0 \,\}$ (if $q_1 > 0$). If there is a $t^* \in (0, T)$ such that $q_1(t) < 0$, $(t < t^*)$ and $q_1(t) > 0$, $(t > t^*)$ then we have because of (2)

$$\dot{q}_2(t) = \left\{ \begin{array}{rl} -\dot{q}_1(t) & \text{if} \ \ t < t^* \, , \\ \dot{q}_1(t) & \text{if} \ \ t > t^* \, , \end{array} \right.$$

i. e. $\dot{q}$ has to be discontinuous at $t = t^*$ if $\lim_{t \to t^*-0} \dot{q}_1(t) \neq 0$.
In the case of permanent contact between disk and floor the scalar constraint (8) is always active and (9) with $i = 2$ is equivalent to the model equations (1) with $g(q) := g^{(2)}(q)$. Since $g^{(2)}(q)$ is only piecewise differentiable the hidden constraint (2) results in

$$\dot{q}_2(t) = \begin{cases} -\dot{q}_1(t) & \text{if } q_1(t) < 0\,, \\ \dot{q}_1(t) & \text{if } q_1(t) > 0 \end{cases}$$

such that all points (q^*, v^*, λ^*) with

$$g^{(2)}(q) = 0\ ,\quad v_1^* \neq 0\ ,\quad |v_2^*| = |v_1^*|$$

and a consistent $\lambda^* \in \mathbb{R}$ may be impasse points of the model equations (1) for this test example.
The basic ideas of this analysis for the model problem can be carried over to the contact between bodies with more complicated geometry. In general we get again the following results:

a) Contact conditions can be expressed analytically by the condition that a suitably defined *distance function* $\Delta(x; q)$ is *non-negative* for all x and q (see (5)).

b) This infinite set of unilateral constraints can be reduced to a *finite* set using (e. g.) the local minima of $\Delta(.; q)$ (see (7)).

c) Using the global minimum of $\Delta(.; q)$ the contact condition can be expressed by a *scalar* unilateral constraint (see (8)). In contrast to the constraints in a) and b) this scalar constraint may be only *piecewise differentiable* w. r. t. q even if the surfaces of the bodies are given by smooth functions.

d) Contact conditions result in complementarity problems that can be reduced to model equations (1) if the bodies are permanent in contact. The resulting constraints $g(q) = 0$ in (1) are in general only piecewise differentiable if the set of active constraints in the corresponding complementarity problem changes in the time interval of interest. The changes in the set of active constraints may introduce singularities in $\dot{q}$ and λ and may cause impasse points in (1).

3. REGULARIZATION OF CONTACT CONDITIONS

Contact conditions for rigid bodies may result in impasse points in the model equations. In this section we discuss a regularization of these contact conditions and sketch the application to the dynamical simulation of wheel-rail systems.
The singularities in the solution of the complementarity problem (9) that correspond to the impasse points in (1) may be interpreted as *impacts* in the mechanical system ([4]). From a theoretical point of view (9) has to be completed by a physical model describing the changes of $\dot{q}$ and λ during the impacts. This combination of the complementarity problem with an impact model guarantees for many problems of practical interest the existence and uniqueness of a (generalized) solution $q \in C[0, T]$ ([4]).
From a practical point of view, however, this approach is not very attractive if there are frequent changes of the set of active constraints in (9) since the time integration has to be interrupted whenever this set changes and each of these interruptions causes a considerable numerical effort. Furthermore, the model error of the impact model may introduce large errors in the solution ([1]). Instead of computing a generalized solution of (1) with frequent discontinuities in $\dot{q}$ and λ we therefore proposed in [2] a regularization of the contact conditions. This regularization is tailored to the contact between a wheel with wear profile and a rail (see the right plot of Fig. 2).

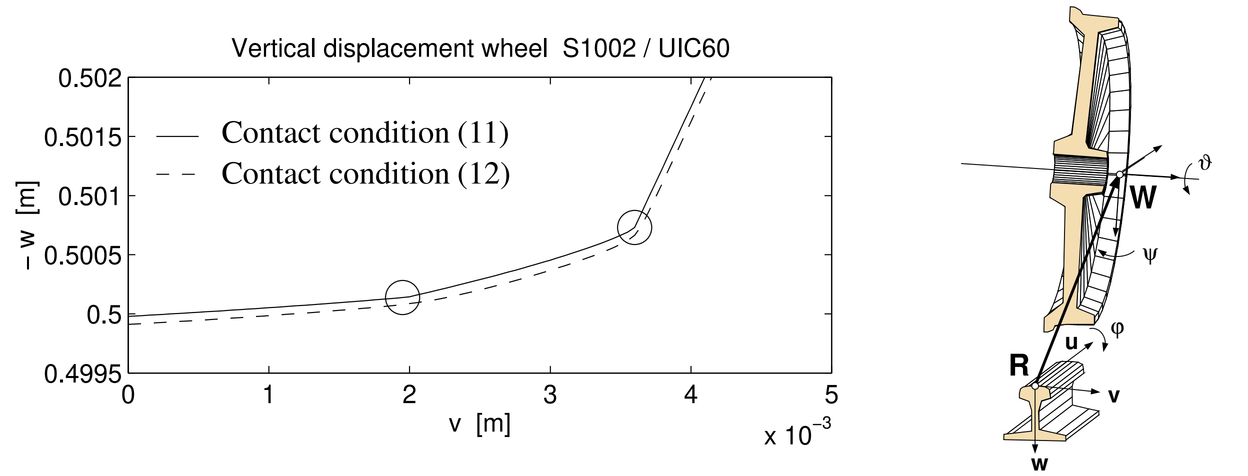

Figure 2: Left plot: *Vertical displacement of a wheel vs. its lateral displacement (profiles S1002 (wheel), UIC60–ORE (rail)).* Right plot: *The geometry of wheel and rail.*

Because of the complex geometry of wheel and rail the formulation of the rigid body contact condition is more complicated than for the model problem but it follows the basic steps discussed in Section 2 (for the details we refer to [1]). There is a distance function $\Delta(s;q)$ depending on the wheel coordinate s (the s-axis of the local system of coordinates $\boldsymbol{W}$ is parallel to the wheel axle). Wheel and rail do not penetrate each other iff

$$\Delta(s;q) \geq 0\ ,\quad (\,\underline{s} \leq s \leq \overline{s}\,)\ , \tag{10}$$

they are in contact iff (at least) one of the unilateral constraints (10) is active:

$$g(q) = \min_{\underline{s} \leq s \leq \overline{s}} \Delta(s;q) = 0\ . \tag{11}$$

As in the model problem of Section 2 the constraint (11) is continuous but only piecewise differentiable. This is illustrated by the left plot of Fig. 2 that shows for fixed roll angle φ and fixed yaw angle ψ the vertical displacement $-w$ of the wheel as function of the lateral displacement v. Here w is defined by the constraint (11) with the coordinates $q = (u, v, w, \varphi, \vartheta, \psi)^T$ defining the relative position of the wheel to the rail (both $\Delta(s;q)$ and $g(q)$ are independent of x and ϑ [1]). The circles in Fig. 2 mark the discontinuities of $\frac{\partial}{\partial q} g(q)$.

Up to now we have considered wheel and rail as idealized rigid bodies. In reality wheel and rail are elastically deformed because of the high axle loads being characteristic of railway vehicles. The relative position of the deformed wheel to the deformed rail is such that rigid bodies in the same position would penetrate each other and (10) is violated:

$$\delta_0 := -\min_{\underline{s} \leq s \leq \overline{s}} \Delta(s;q) > 0$$

(note, that Δ is defined for the *undeformed* surfaces). This *elastic approach* δ_0 depends on the axle load, on the material of wheel and rail (Young moduli, Poisson number), on the local geometry of the surfaces of wheel and rail in the contact patch and on other factors ([3]). Typical values of δ_0 are in the range $10\,\mu$m $\ldots$ $100\,\mu$m.

The key observation is that the relative position of the wheel to the rail actually depends on the distance $\Delta(s;q)$ for all

$$s \in \{\, s\ :\ \Delta(s;q) \leq \min_{\underline{s} \leq \tilde{s} \leq \overline{s}} \Delta(\tilde{s};q) + \tilde{\delta}_0 \,\}$$

with some $\tilde{\delta}_0$ in the range $10\,\mu$m $\ldots$ $100\,\mu$m. Most of this information is lost in the rigid contact condition (11). We therefore substitute (11) by

$$g(q) := -\nu\ \ln \left(\int_{\underline{s}}^{\overline{s}} \exp(-\frac{1}{\nu}\Delta(s;q))\,ds \,/\, \int_{\underline{s}}^{\overline{s}} ds \right) \tag{12}$$

with a small parameter $\nu > 0$ ([2]). For fixed q the constraint (12) tends to (11) if $\nu \to 0$. In contrast to (11) this *regularized* contact condition (12) defines, however, a function g that is as often differentiable w. r. t. q as $\Delta(s;q)$. This is illustrated by the dashed line in the left plot of Fig. 2 that was obtained from (12) with $\nu = 2.0_{10}\text{-}5$.
Using the regularized contact condition (12) we get model equations (1) with smooth constraints such that the existence and uniqueness of a classical solution is guaranteed. This approach has been implemented in the MBS simulation package SIMPACK, it is successfully used in the dynamical simulation of wheel-rail systems. We refer to [1] for details of the implementation and for tests comparing simulations on the basis of the rigid body contact condition (11) with simulations using the regularized contact condition (12).

SUMMARY

Contact conditions in MBS can be formulated as infinite sets of unilateral constraints. They may result in impasse points in the MBS model equations. In the real mechanical system a part of the constraints is slightly violated because of the elastic deformation of the bodies being in contact. This effect is considered qualitatively in an alternative (regularized) approach to the formulation of contact conditions that has been applied successfully to the dynamical simulation of wheel-rail systems.

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