Martin–Luther–Universität
Halle–Wittenberg
Institut für Mathematik

A Generalized Scalarization Method in Set Optimization with respect to Variable Domination Structures

E. Köbis, L. Th. Tam and Chr. Tammer

Report No. 07 (2017)

Editors:
Professors of the Institute for Mathematics, Martin Luther University Halle-Wittenberg.

Electronic version: see `http://www.mathematik.uni-halle.de/institut/reports/`

A Generalized Scalarization Method in Set Optimization with respect to Variable Domination Structures

E. Köbis, L. Th. Tam and Chr. Tammer

Report No. 07 (2017)

Elisabeth Köbis
Le Thanh Tam
Christiane Tammer

Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany

Email: elisabeth.koebis@mathematik.uni-halle.de
le.thanh-tam@mathematik.uni-halle.de
christiane.tammer@mathematik.uni-halle.de

A Generalized Scalarization Method in Set Optimization with respect to Variable Domination Structures

ELISABETH KÖBIS · THANH TAM LE · CHRISTIANE TAMMER

Abstract In this paper, we introduce new nonlinear scalarization methods to characterize minimal elements of set-valued problems equipped with variable domination structures. For certain set relations with respect to variable domination structures, we give a characterization of these relations using corresponding nonlinear scalarizing functionals and the properties of these functionals. We describe different kinds of minimal elements to a family of sets with respect to variable domination structures. Furthermore, we investigate a descent method to find approximations of minimal solutions of set-valued problems with respect to variable domination structures using nonlinear scalarization methods. Finally, we apply the results to a set-valued problem in Medical Image Registration.

Keywords nonlinear scalarization · set optimization · set relations · variable domination structures · descent method

1 Introduction

Set optimization has been studied and applied in various fields. There are three main approaches for defining solution concepts in set optimization, namely the vector approach, the set approach and the lattice approach. For more details concerning an introduction in set optimization and its applications, we refer to [14].

ELISABETH KÖBIS
Martin-Luther-University Halle-Wittenberg
Tel: +49-345-55-24682
Fax: +49-345-55-27005
E-mail: elisabeth.koebis@mathematik.uni-halle.de

THANH TAM LE
Martin-Luther-University Halle-Wittenberg
Tel: +49-345-55-24675
Fax: +49-345-55-27005
E-mail: le.thanh-tam@mathematik.uni-halle.de

CHRISTIANE TAMMER
Martin-Luther-University Halle-Wittenberg
Tel: +49-345-55-24673
Fax: +49-345-55-27005
E-mail: christiane.tammer@mathematik.uni-halle.de

It is shown in many papers that, although the solution concept based on the vector approach is of mathematical interest, it cannot be often used in practice (see, for instance, [7], [14] and [21]). Therefore, it is important to derive results and corresponding algorithms for generating solutions of set-valued problems that are equipped with natural relations for sets. Following this approach, many solution concepts have been introduced and characterized by means of different methods. A linear scalarization has been used in [8] to obtain characterizations of solutions to set-valued problems. In addition, nonlinear scalarization methods generated from the Gerstewitz scalarization are also used in many references, see e.g. [3], [4] and [17].
Moreover, it is also of interest to consider set-valued optimization problems equipped with a variable domination structure. In vector optimization, which is a special case of set-valued optimization, Eichfelder [6] has introduced minimal elements and nondominated elements of a set with respect to (w.r.t) a variable domination structure given by a cone-valued map. These elements are also characterized by scalarizing functionals when the ordering structure is given by a Bishop-Phelp cone-valued mapping. In addition, Bouza and Tammer [2] have introduced a nonlinear scalarizing functional to characterize and compute minimal points of a set with respect to a variable domination structure. Recently, Köbis [16], Eichfelder and Pilecka [7] have introduced several order relations for the case that the order is given by a cone-valued map.

Based on scalarization methods, it is of interest to derive necessary optimality conditions for different solution concepts (compare [18]) under certain regularity assumptions, see [1], [5], [13], [19] and [20].
The goal of this paper is to introduce new nonlinear scalarization methods to characterize minimal elements of a set-valued problem based on the set-approach and equipped with variable domination structures. In Section 2, we recall the well known Gerstewitz functional and some important properties that this functional satisfies and recall six certain set relations with respect to a variable domination structure, which are introduced in [18]. Section 3 introduces corresponding nonlinear scalarizing functionals for these relations and investigates useful properties of these functionals. Section 4 describes minimal elements of a family of sets and some properties of sets of these minimal elements. This section also deals with characterizations of these minimal elements by means of nonlinear functionals introduced in Section 3. In Section 5, we investigate a descent method to find approximate minimal solutions of set-valued problems w.r.t. variable domination structures using the nonlinear scalarization methods introduced in Sections 3. Finally, we apply the results of Section 4 to a set-valued problem in Medical Image Registration.

2 Preliminaries

Let Y be a linear topological space, and

$$\mathcal{P}(Y) := \{A \subseteq Y | A \neq \emptyset\}$$

denotes the power set of Y without the empty set. A subset D of Y is called proper, if D is nonempty and $D \neq Y$. Let $D \subset Y$ be a proper closed set, and $k \in Y \setminus \{0\}$ satisfying $D + [0, +\infty) \cdot k \subseteq D$. For two nonempty subsets A, B of Y, we denote the sum of sets by

$$A + B := \{a + b \mid a \in A,\ b \in B\}.$$

We introduce the functional $z^{D,k} \colon Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \overline{\mathbb{R}}$ given by

$$z^{D,k}(y) := \inf\{t \in \mathbb{R} \mid y \in tk - D\}, \tag{1}$$

where we use the convention that $\inf \emptyset = +\infty$. The functional $z^{D,k}$ assigns the smallest value t such that the property $y \in tk - D$ is fulfilled. The scalarizing functional $z^{D,k}$ was used in [11] to prove separation theorems for not necessarily convex sets. Applications of $z^{D,k}$ include coherent risk measures in financial mathematics (see, for instance, [12]). Properties of $z^{D,k}$ were studied in [10], [11] and [24]. First, let us recall the definition of C-monotonicity and properness of a functional.

Definition 1 Let Y be a linear topological space, $C \subset Y$, $C \neq \emptyset$. A functional $z\colon Y \to \overline{\mathbb{R}}$ is C-monotone, if the following implication holds

$$y_1,\ y_2 \in Y :\ y_1 \in y_2 - C \Longrightarrow z(y_1) \leq z(y_2)\,.$$

We say that z is proper if $\operatorname{dom} z \neq \emptyset$ (that is, $\{y \in Y \mid z(y) < +\infty\} \neq \emptyset$) and z does not take the value $-\infty$.

Below we provide some properties of the functional $z^{D,k}$ introduced in (1).

Theorem 1 ([10],[11]) *Let Y be a linear topological space, $C \subset Y$ a subset , $D \subset Y$ a proper closed set, and let $k \in Y \setminus \{0\}$ be such that $D + [0, +\infty) \cdot k \subseteq D$ is satisfied. Then the following properties hold for $z = z^{D,k}$:*

(a) z is lower semi-continuous.
In addition, if $k \in Y \setminus \{0\}$ such that $D + [0, +\infty) \cdot k \subseteq \operatorname{int} D$, then z is continuous.
(b) z is convex $\Longleftrightarrow$ D is convex,
(c) z is proper $\Longleftrightarrow$ D does not contain lines parallel to k, i.e., $\forall\ y \in Y\ \exists\ r \in \mathbb{R} :\ y + rk \notin D$.
(d) z is C-monotone $\Longleftrightarrow$ $D + C \subset D$.
(e) $\forall\ y \in Y,\ \forall\ r \in \mathbb{R} :\ z(y) \leq r \Longleftrightarrow y \in rk - D$.
(f) z is finite-valued $\Longleftrightarrow$ D does not contain lines parallel to k and $\mathbb{R}k - D = Y$.

For the proof of Theorem 1, see [10, Theorem 2.3.1].

Taking into account the assumptions in Theorem 1, we suppose in our whole paper the following assumption:

(H_1) Let $A, B \in \mathcal{P}(Y), \mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that $\mathcal{K}(y)$ is closed for all $y \in Y$ and

$$\forall y \in Y :\ \mathcal{K}(y) + (0, +\infty)k^0 \subseteq \mathcal{K}(y).$$

As mentioned in the previous part, we consider set-valued optimization problems with respect to variable domination structures by using the set approach. To do this, we recall in the following six relations to compare sets in a linear topological space.

Definition 2 [18] Let Y be a linear topological space, $A, B \in \mathcal{P}(Y)$ and $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map. We define binary relations on $\mathcal{P}(Y)$ w.r.t. $\mathcal{K}$ as follows:

(i) The variable generalized lower less relation ($\preccurlyeq_l^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_l^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcup_{a \in A}(a + \mathcal{K}(a)).$$

(ii) The variable generalized upper less relation ($\preccurlyeq_u^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_u^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \subseteq \bigcup_{b \in B}(b - \mathcal{K}(b)).$$

(iii) The variable generalized certainly lower less relation ($\preccurlyeq_{cl}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{cl}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \subseteq \bigcap_{a\in A}(a + \mathcal{K}(a)).$$

(iv) The variable generalized certainly upper less relation ($\preccurlyeq_{cu}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{cu}^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \subseteq \bigcap_{b\in B}(b - \mathcal{K}(b)).$$

(v) The variable generalized possible lower less relation ($\preccurlyeq_{pl}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{pl}^{\mathcal{K}} B \quad \Longleftrightarrow \quad B \cap \bigcup_{a\in A}(a + \mathcal{K}(a)) \neq \emptyset.$$

(vi) The variable generalized possible upper less relation ($\preccurlyeq_{pu}^{\mathcal{K}}$) is defined by

$$A \preccurlyeq_{pu}^{\mathcal{K}} B \quad \Longleftrightarrow \quad A \cap \bigcup_{b\in B}(b - \mathcal{K}(b)) \neq \emptyset.$$

In the following proposition, we present useful properties of the relations $\preccurlyeq_t^{\mathcal{K}}$, $t \in \{l, u, cl, cu, pl, pu\}$.

Proposition 1 *Suppose that $\mathcal{K} : Y \rightrightarrows Y$ is a set-valued map. The following statements hold true:*

(i) If $\mathcal{K}$ satisfies $0 \in \mathcal{K}(y)$ for all $y \in Y$, then the binary relations $\preccurlyeq_l^{\mathcal{K}}$, $\preccurlyeq_u^{\mathcal{K}}$, $\preccurlyeq_{pl}^{\mathcal{K}}$ and $\preccurlyeq_{pu}^{\mathcal{K}}$ are reflexive.

(ii) If $\mathcal{K}$ satisfies that

$$\forall y \in Y : \mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y)$$

and

$$\forall y \in Y, d \in \mathcal{K}(y) : \mathcal{K}(y + d) \subseteq \mathcal{K}(y),$$

then the relation $\preccurlyeq_l^{\mathcal{K}}$ and $\preccurlyeq_{cl}^{\mathcal{K}}$ are transitive.

(iii) If $\mathcal{K}$ satisfies

$$\forall y \in Y : \mathcal{K}(y) + \mathcal{K}(y) \subseteq \mathcal{K}(y)$$

and

$$\forall y \in Y, d \in \mathcal{K}(y) : \mathcal{K}(y - d) \subseteq \mathcal{K}(y),$$

then the relation $\preccurlyeq_u^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}}$ are transitive.

(iv) If for all $y, z \in Y$, $\mathcal{K}(y) \cap (-\mathcal{K}(z)) = \{0\}$, then the relation $\preccurlyeq_{cl}^{\mathcal{K}}$ and $\preccurlyeq_{cu}^{\mathcal{K}}$ are antisymmetric.

PROOF. The proof of this proposition is similar to that one in [7, Lemma 4.1] for a cone-valued map $\mathcal{K}$.

Remark 1 Let $A, B \in \mathcal{P}(Y)$. We write $A \sim B$ if $A \preccurlyeq_t^{\mathcal{K}} B$ and $B \preccurlyeq_t^{\mathcal{K}} A$, $t \in \{l, u, cl, cu, pl, pu\}$. In addition, the set of all elements $B \in \mathcal{P}(Y)$ such that $B \sim A$ is denoted by $[A]$. Obviously, if $\preccurlyeq_t^{\mathcal{K}}$ is reflexive, then $A \sim A$ and $A \in [A]$.

3 Characterizations of Variable Set Relations

Let Y be a linear topological space and $\mathcal{K} : Y \rightrightarrows Y$ be a given set-valued map. In the following, we introduce different kinds of nonlinear scalarizing functionals. These functionals are used to describe the set relations given in Definition 2 as well as to characterize minimal elements of a set defined by these relations.

3.1 Characterization of the Variable Generalized Lower Set Less Relation via Scalarization

We begin this part by introducing two kinds of scalarizing functionals for the relation $\preceq_l^{\mathcal{K}}$. Furthermore, we investigate how these functionals describe this set relation. For the following results, we define the following sets for a certain nonempty subset $A \subseteq Y$

$$\widetilde{\mathcal{K}}(A) := \bigcup_{a\in A} \mathcal{K}(a) \subset Y, \tag{2}$$

$$\text{and } \overline{\mathcal{K}}(A) := \bigcap_{a\in A} \mathcal{K}(a) \subset Y. \tag{3}$$

In addition, we will use two assumptions concerning the set-valued domination map $\mathcal{K} : Y \rightrightarrows Y$ (consequently, for $\widetilde{\mathcal{K}}(A)$ and $\overline{\mathcal{K}}(A)$) and $k^0 \in Y \setminus \{0\}$ as follows:

(H_2) Assume that for all $A \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A)$ is a proper closed set and

$$\forall A \in \mathcal{P}(Y): \ \widetilde{\mathcal{K}}(A) + (0, +\infty)k^0 \subseteq \widetilde{\mathcal{K}}(A).$$

(H_3) Assume that for all $A \in \mathcal{P}(Y)$, $\overline{\mathcal{K}}(A)$ is a proper closed set and

$$\forall A \in \mathcal{P}(Y): \ \overline{\mathcal{K}}(A) + (0, +\infty)k^0 \subseteq \overline{\mathcal{K}}(A).$$

In the next theorem, we give a characterization of $\preceq_l^{\mathcal{K}}$ by means of the functional (1) with $D = \widetilde{\mathcal{K}}(A)$ and $k = k^0$.

Theorem 2 *Let $A, B \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A) \subset Y$ be given by* (2) *and $k^0 \in Y \setminus \{0\}$ such that (H_2) holds. Assume that $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$. Then, it holds that*

$$A \preceq_l^{\mathcal{K}} B \implies \inf_{a\in A} z^{\widetilde{\mathcal{K}}(A),k^0}(a) \leq \inf_{b\in B} z^{\widetilde{\mathcal{K}}(A),k^0}(b)\,.$$

Proof. Suppose that $A \preceq_l^{\mathcal{K}} B$ holds, i.e.,

$$\forall\, b \in B \ \exists\, a_b \in A: \ b \in a_b + \mathcal{K}(a_b).$$

This implies that

$$\forall\, b \in B \ \exists\, a_b \in A: \ b \in a_b + \widetilde{\mathcal{K}}(A).$$

Under the assumption $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$, the functional $z^{\widetilde{\mathcal{K}}(A),k^0}$ is $\widetilde{\mathcal{K}}(A)$-monotone because of Theorem 1 (d). Therefore,

$$\forall\, b \in B \ \exists\, a_b \in A: \ z^{\widetilde{\mathcal{K}}(A),k^0}(a_b) \leq z^{\widetilde{\mathcal{K}}(A),k^0}(b),$$

which yields the assertion

$$\inf_{a\in A} z^{\widetilde{\mathcal{K}}(A),k^0}(a) \leq \inf_{b\in B} z^{\widetilde{\mathcal{K}}(A),k^0}(b).$$

Remark 2 Note that the property $\widetilde{\mathcal{K}}(A)+\widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ does not imply that $\widetilde{\mathcal{K}}(A)$ is a convex cone. Consider, for instance, the case $\widetilde{\mathcal{K}}(A) := \mathbb{N}$, the set of natural numbers. Then $\widetilde{\mathcal{K}}(A)+\widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ is fulfilled, but $\widetilde{\mathcal{K}}(A)$ is neither a cone nor a convex set.

Now, we give a characterization of $\preceq_l^{\mathcal{K}}$ by means of the functional (1) with $D = a + \widetilde{\mathcal{K}}(A)$ or $D = a + \overline{\mathcal{K}}(A)$, where $A, B \in \mathcal{P}(Y)$ and $a \in A$ are given and $k = k^0$.

Theorem 3 *Let $A, B \in \mathcal{P}(Y)$, $\widetilde{\mathcal{K}}(A)$ and $\overline{\mathcal{K}}(A)$ are given by* (2) *and* (3)*, respectively. In addition, let $k^0 \in Y \setminus \{0\}$ such that* (H_2) *and* (H_3) *are fulfilled. Then, it holds that* *(i)* $A \preceq_l^{\mathcal{K}} B \implies \sup_{b\in B} \inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(A),k^0}(-b) \le 0.$

(ii) Suppose that $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b)$ *is attained for all* $b \in B$. *Then,*

$$\sup_{b\in B} \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \le 0 \quad \implies \quad A \preceq_l^{\mathcal{K}} B.$$

PROOF.

(i) Suppose that $A \preceq_l^{\mathcal{K}} B$, i.e.,

$$\forall\, b \in B\ \exists\, a_b \in A:\ b \in a_b + \mathcal{K}(a_b).$$

This leads to

$$\forall\, b \in B\ \ \exists\, a_b \in A:\ a_b - b \in -\widetilde{\mathcal{K}}(A)\,.$$

Because of Theorem 1 (e), we get

$$\forall\, b \in B\ \ \exists\, a_b \in A:\quad z^{a_b+\widetilde{\mathcal{K}}(A),k^0}(-b) \le 0$$

and this implies

$$\sup_{b\in B}\ \inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(A),k^0}(-b) \le 0\,.$$

(ii) Now, let $k^0 \in \overline{\mathcal{K}}(A) \setminus \{0\}$ be given such that for all $b \in B$ the infimum $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b)$ is attained. Let

$$\sup_{b\in B} \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \le 0.$$

That is,

$$\forall\, b \in B:\ \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \le 0\,.$$

Because for all $b \in B$ the infimum $\inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b)$ is attained, we obtain

$$\forall\, b \in B\ \exists\, \overline{a}_b \in A:\ z^{a_b+\overline{\mathcal{K}}(A),k^0}(-b) = \inf_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \le 0\,.$$

By Theorem 1 (e), we conclude

$$\forall\, b \in B\ \exists\, \overline{a}_b \in A:\ \overline{a}_b - b \in -\overline{\mathcal{K}}(A) \subseteq -\mathcal{K}(\bar{a}_b),$$

which implies that

$$\forall\, b \in B\ \exists\, \bar{a}_b \in A:\ b \in \bar{a}_b + \mathcal{K}(\bar{a}_b),$$

and this means that $A \preceq_l^{\mathcal{K}} B$.

Remark 3 Let us note that the property $\widetilde{\mathcal{K}}(A) + \widetilde{\mathcal{K}}(A) \subseteq \widetilde{\mathcal{K}}(A)$ is not needed in Theorem 3.

Remark 4 It is obvious that

$$\forall B \in \mathcal{P}(Y): \widetilde{\mathcal{K}}(B) \subseteq \widetilde{\mathcal{K}}(Y) \text{ and } \overline{\mathcal{K}}(Y) \subseteq \overline{\mathcal{K}}(B).$$

Then, under the assumptions given by Theorem 3, we have that

(i)

$$A \preceq_l^{\mathcal{K}} B \Longrightarrow \sup_{b\in B} \inf_{a\in A} z^{a+\widetilde{\mathcal{K}}(Y),k^0}(-b) \le 0.$$

(ii) Suppose that $\inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(Y),k^0}(-b)$ is attained for all $b \in B$. Then

$$\sup_{b\in B}\inf_{a\in A} z^{a+\overline{\mathcal{K}}(Y),k^0}(-b) \leq 0 \quad \Longrightarrow \quad A \preceq_l^{\mathcal{K}} B.$$

The following example shows that the attainment property in Theorem 3 (ii) is indeed necessary for the implication to hold true. Moreover, it is shown that the attainment property may not hold for all $k \in Y \setminus \{0\}$ that satisfy (H_3).

Example 1 1. Consider the open set $A := \{(y_1, y_2)^T \in \mathbb{R}^2 \mid 0 < y_1 < 1,\ 0 < y_2 < 1\}$, the vector $b = (0, 0)^T$ and the single-valued set $B = \{b\}$. Let $\overline{\mathcal{K}}(A) := \mathbb{R}^2$. We choose any $k^0 \in \operatorname{int} \overline{\mathcal{K}}(A)$. Then $\sup\limits_{b\in B}\inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) = \inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(0) = 0$, but this value is not attained due to the openness of A. This means that $\sup\limits_{b\in B}\inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(-b) \leq 0$ is fulfilled; however, $A \preceq_l^{\mathcal{K}} B$ does not hold true. Hence, the implication given in Theorem 3 (ii) does not hold true.

2. It is sufficient in Theorem 3 that the attainment property is fulfilled by some $k \in \overline{\mathcal{K}}(A)$, and not by *all* $k \in Y \setminus \{0\}$ such that (H_3) holds. For instance, let $Y := \mathbb{R}^2$, $\overline{\mathcal{K}}(A) := \mathbb{R}^2_+$, $b := (0, 0)^T$, $B := \{b\}$, $A := \{(y_1, y_2)^T \in \mathbb{R}^2_+ \mid y_1^2 + y_2^2 < 1\} \cup \{(y_1, y_2)^T \in \mathbb{R}^2_+ \mid y_1^2 + y_2^2 = -1 \text{ for } y_1 \in [-\frac{1}{2}, \frac{1}{2}]\}$. Then we have $A = -A$, and A is neither closed nor open. Now choose $k := (0.7, \sqrt{0.51})^T$. Then $\inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k}(0) = 1$, but this value is never attained. For $k^0 := (0.1, \sqrt{0.99})^T$, we get $\inf\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(0) = \min\limits_{a\in A} z^{a+\overline{\mathcal{K}}(A),k^0}(0) = -1$.

Observe that we cannot get an equivalent assertion in Theorem 3. Therefore, we introduce in the following a different scalarizing functional to obtain an equivalence between the set relation $\preceq_l^{\mathcal{K}}$ and properties of this functional. Now, we consider $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. We introduce a new scalarizing functional defined as follows

$$\begin{aligned} g^{\preceq_l^{\mathcal{K}}} &: \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}}, \\ g^{\preceq_l^{\mathcal{K}}}(A, B) &:= \sup_{b\in B}\inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b). \end{aligned} \tag{4}$$

In (4), we are using the functional (1) with $D = a + \mathcal{K}(a)$, $a \in A$ fixed, $k = k^0$, such that, for $b \in B$, the functional (1) has the form $z^{a+\mathcal{K}(a),k^0}(-b) = \inf\{t \in \mathbb{R} \mid -b \in tk^0 - (a + \mathcal{K}(a))\}$. In the following theorem, we show the relationships between the value of this new functional (4) for $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved with respect to the relation $\preceq_l^{\mathcal{K}}$.

Theorem 4 *Consider $A, B \in \mathcal{P}(Y)$, a set-valued map $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true*

(a) $g^{\preceq_l^{\mathcal{K}}}(A, B) \leq r \Longrightarrow \bigcup\limits_{t>r}(tk^0 + B) \subseteq \bigcup\limits_{a\in A}(a + \mathcal{K}(a))$, *i.e.,* $A \preceq_l^{\mathcal{K}} \bigcup\limits_{t>r}(tk^0 + B)$.

(b) $rk^0 + B \subseteq \bigcup\limits_{a\in A}(a + \mathcal{K}(a)) \Longrightarrow g^{\preceq_l^{\mathcal{K}}}(A, B) \leq r$.

Proof.

(a) Suppose that $g^{\preceq_l^{\mathcal{K}}}(A,B) \leq r$ holds true. Consider $\varepsilon > 0$, arbitrarily, but fixed. We are using the functional $g^{\preceq_l^{\mathcal{K}}}(\cdot,\cdot)$ given by (4). Then, we have that

$$\begin{aligned}
&\forall\, b \in B : \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) < r + \varepsilon \\
\Longleftrightarrow &\forall\, b \in B : \inf_{a\in A} \inf\{t \in \mathbb{R} : b + tk^0 \in a + \mathcal{K}(a)\} < r + \varepsilon \\
\Longleftrightarrow &\forall\, b \in B : \exists\, a \in A, \exists l < r + \varepsilon : b + lk^0 \in a + \mathcal{K}(a).
\end{aligned}$$

This means that for all $b \in B$ there exist an element $a \in A$ and an element $l < r + \varepsilon$ such that

$$b + (r+\varepsilon)k^0 = b + lk^0 + (r + \varepsilon - l)k^0 \in a + \mathcal{K}(a) + (r + \varepsilon - l)k^0.$$

Taking into account the implication

$$r + \varepsilon - l > 0 \Rightarrow (r + \varepsilon - l)k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a),$$

we get that

$$b + (r+\varepsilon)k^0 \in a + \mathcal{K}(a).$$

Thus, $\forall\, \varepsilon > 0, \forall\, b \in B,$ there is $a \in A$ such that $b + (r+\varepsilon)k^0 \in a + \mathcal{K}(a), i.e.,$

$$\bigcup_{t>r}(tk^0 + B) \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

(b) Let $rk^0 + B \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a))$. It holds

$$\begin{aligned}
&\forall\, b \in B : rk^0 + b \in \bigcup_{a\in A}(a + \mathcal{K}(a)) \\
\Longleftrightarrow\ &\forall\, b \in B, \exists\, a_b \in A : rk^0 + b \in a_b + \mathcal{K}(a_b) \\
\Longrightarrow\ &\forall\, b \in B : \ \inf_{a\in A} \inf\{t \in \mathbb{R} : tk^0 + b \in a + \mathcal{K}(a)\} \leq r \\
\Longrightarrow\ &\sup_{b\in B} \inf_{a\in A} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(a)\} \leq r \\
\Longleftrightarrow\ &\sup_{b\in B} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) \leq r \\
\Longrightarrow\ &g^{\preceq_l^{\mathcal{K}}}(A,B) \leq r.
\end{aligned}$$

Example 2 Let us consider two sets in $\mathbb{R}^2$ given by

$$A := \{(y_1,y_2) \in \mathbb{R}^2 |\ 1 \leq y_1, y_2 \leq 2\} \text{ and } B := \left\{(y_1,y_2) \in \mathbb{R}^2 |\ 0 \leq y_1 \leq 1, 0 \leq y_2 \leq \frac{1}{2}\right\}.$$

Furthermore, a set-valued map $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ is given by

$$\mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } y \in \mathbb{R}^2 \setminus \{(2,2)\}, \\ \{(d_1,d_2) |\ d_1 \in \mathbb{R},\ d_2 \geq 0\} & \text{if } y = (2,2). \end{cases}$$

We choose $k^0 := (1,1)$ and it satisfies that $\forall\, t \in [0,+\infty) : tk^0 + \mathcal{K}(y) \in \mathcal{K}(y)$. Obviously, we have that

$$1 \cdot k^0 + B = \left\{(y_1,y_2) \in \mathbb{R}^2 |\ 1 \leq y_1 \leq 2,\ 1 \leq y_2 \leq \frac{3}{2}\right\} \subseteq \bigcup_{a\in A}(a + \mathcal{K}(a)).$$

Taking into account Theorem 4 (b), it holds that

$$g^{\preceq_l^{\mathcal{K}}}(A,B) \leq 1.$$

In addition, from Theorem 4 (a) we get $\underset{t>1}{\cup}(tk^0+B) \subseteq \underset{a\in A}{\cup}(a+\mathcal{K}(a))$. See Figure 1 for an illustration of this example.

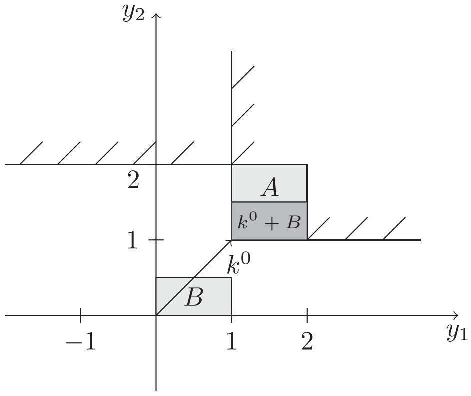

Figure 1: Illustration for Example 2

Remark 5 Theorem 4 states that, if the functional $g^{\preceq_l^{\mathcal{K}}}$ given by (4) takes values that do not exceed r at (A,B), then the set A is smaller (w.r.t. $\preceq_l^{\mathcal{K}}$) than the union of all sets which are the movements of B along the direction tk^0, where $t > r$. However, the conversion is not always true.

In the following, we give an equivalence of the comparison $A \preceq_l^{\mathcal{K}} B$, where $A,B \in \mathcal{P}(Y)$ by means of the functional $g^{\preceq_l^{\mathcal{K}}}$ given by (4).

Theorem 5 *Let $A,B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. Suppose that $\underset{a\in A}{\cup}(a+\mathcal{K}(a))$ is closed. Then*

$$A \preceq_l^{\mathcal{K}} B \iff g^{\preceq_l^{\mathcal{K}}}(A,B) \leq 0.$$

Proof. Obviously, $A \preceq_l^{\mathcal{K}} B \iff 0k^0 + B \subseteq \underset{a\in A}{\cup}(a+\mathcal{K}(a))$. By Theorem 4 (b), it holds that

$$g^{\preceq_l^{\mathcal{K}}}(A,B) \leq 0.$$

Now, we prove the sufficient condition. By Theorem 4 (a), we get that

$$\underset{t>0}{\cup}(tk^0+B) \subseteq \underset{a\in A}{\cup}(a+\mathcal{K}(a)).$$

This means for all $n > 0$, $\frac{1}{n}k^0 + B \subseteq \underset{a\in A}{\cup}(a+\mathcal{K}(a))$. Taking the limit for $n \to +\infty$, we obtain

$$B \subseteq \mathrm{cl}(\underset{a\in A}{\cup}(a+\mathcal{K}(a))) = \underset{a\in A}{\cup}(a+\mathcal{K}(a)).$$

Therefore, $B \subseteq \underset{a\in A}{\cup}(a+\mathcal{K}(a))$, i.e., $A \preceq_l^{\mathcal{K}} B$. The proof is complete.

The following examples indicates that even if $B \sim A$ or $B = A$, we can get $g^{\preceq_l^{\mathcal{K}}}(A, B) < 0$.

Let $B := \{(x_1, x_2) \in \mathbb{R}^2 | \ -1 \leq x_1, x_2 \leq 0 \text{ and } x_1 + x_2 \geq -1\}$ and a set-valued mapping $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined as

$$\forall y \in \mathbb{R}^2 : \ \mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } (y_1, y_2) \in \mathbb{R}^2 \setminus B, \\ \{(d_1, d_2) \in \mathbb{R}^2 | 0 \leq d_1\} & \text{if } (y_1, y_2) \in B. \end{cases}$$

Choose $k^0 := (0, 1)$. Then for all $b \in B$, $b - tk^0 \in b + \mathcal{K}(b)$ holds true for all $t \in \mathbb{R}$. Therefore, $g^{\preceq_l^{\mathcal{K}}}(B, B) = -\infty$ since

$$g^{\preceq_l^{\mathcal{K}}}(B, B) = \sup_{b \in B} \inf_{a \in B} \inf\{t \in \mathbb{R} | a - b \in tk^0 - \mathcal{K}(a)\} = -\infty.$$

See Figure 2 for an illustration of this example.

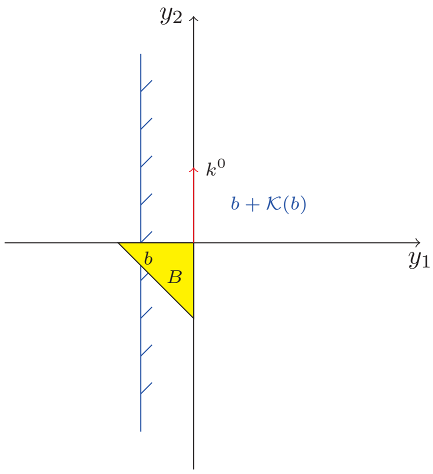

Figure 2: Illustration for Example **??**

Example 3 Let $A := \{(x_1, x_2) \in \mathbb{R}^2 | \ 1 \leq x_1, x_2 \leq 2 \text{ and } x_1 + x_2 \leq 3\}$ and $B := \{(2, d) | \ 2 \leq d \leq 3\} \cup \{(d, 2) | \ 2 \leq d \leq 3\}$. Let $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be determined as

$$\forall y \in \mathbb{R}^2 : \ \mathcal{K}(y) := \begin{cases} \mathbb{R}^2_+ & \text{if } (y_1, y_2) \in \mathbb{R}^2 \setminus B, \\ \{(d_1, d_2) \in \mathbb{R}^2 | -1 \leq d_1, -1 \leq d_2\} & \text{if } (y_1, y_2) \in B. \end{cases} \tag{5}$$

It is clear that $A \sim B$, since A and B are both subsets of the following set

$$\bigcup_{a \in A} (a + \mathcal{K}(a)) = \bigcup_{b \in B} (b + \mathcal{K}(b)) = \{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 1, d_2 \geq 1\}.$$

Let $k^0 := (1, 1)$. Then,

$$g^{\preceq_l^{\mathcal{K}}}(A, B) = g^{\preceq_l^{\mathcal{K}}}(B, B) = -1 < 0.$$

For an illustration, see Figure 3.

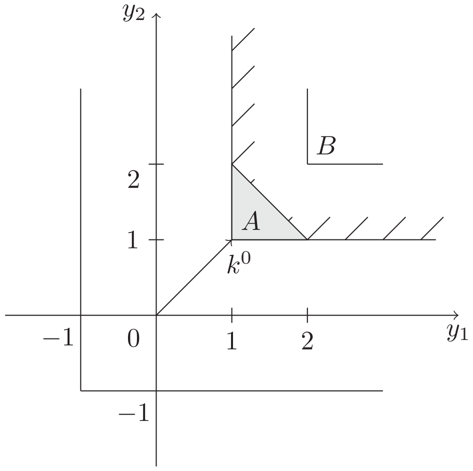

Figure 3: Illustration for Example 3

3.2 Characterization of the Variable Generalized Upper Set Less Relation via Scalarization

In the following theorem, we are using the functional (1) with $D = -b + \widetilde{\mathcal{K}}(B)$ or $D = -b + \overline{\mathcal{K}}(B)$, where $A, B \in \mathcal{P}(Y)$ and $b \in B$ are given and $k = k^0$. The proof is similar to the one given for Theorem 3 and is therefore skipped.

Theorem 6 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_2) and (H_3) hold true. Then*

(i) $A \preceq_u^{\mathcal{K}} B \implies \sup_{a \in A} \inf_{b \in B} z^{-b+\widetilde{\mathcal{K}}(B),k^0}(a) \leq 0.$

(ii) Suppose that $\inf_{b \in B} z^{-b+\overline{\mathcal{K}}(B),k^0}(a)$ *is attained for all $a \in A$. Then,*

$$\sup_{a \in A} \inf_{b \in B} z^{-b+\overline{\mathcal{K}}(B),k^0}(a) \leq 0 \quad \implies \quad A \preceq_u^{\mathcal{K}} B.$$

Remark 6 It is obvious that

$$\forall B \in \mathcal{P}(Y) : \widetilde{\mathcal{K}}(B) \subseteq \widetilde{\mathcal{K}}(Y) \text{ and } \overline{\mathcal{K}}(Y) \subseteq \overline{\mathcal{K}}(B).$$

Then, under the assumptions given by Theorem 6, we have that

(i)
$$A \preceq_u^{\mathcal{K}} B \implies \sup_{a \in A} \inf_{b \in B} z^{-b+\tilde{\mathcal{K}}(Y),k^0}(a) \leq 0.$$

(ii) If $\inf_{b \in B} z^{-b+\overline{\mathcal{K}}(Y),k^0}(a)$ is attained for all $a \in A$ then,

$$\sup_{a \in A} \inf_{b \in B} z^{-b+\overline{\mathcal{K}}(Y),k^0}(a) \leq 0 \implies A \preceq_u^{\mathcal{K}} B.$$

Observe that we cannot get an equivalent assertion in Theorem 6. Therefore, we introduce in the following a different scalarizing functional to obtain an equivalence between the set relation $\preceq_u^{\mathcal{K}}$ and properties of this functional.

Now, let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. We consider the following functional:

$$g^{\preceq_u^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq_u^{\mathcal{K}}}(A, B) := \sup_{a \in A} \inf_{b \in B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{6}$$

In (6), we are using the functional (1) with $D = -b + \mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$ such that, for $a \in A$, (1) has the form $z^{-b+\mathcal{K}(b),k^0}(a) = \inf\{t \in \mathbb{R} | \ a \in tk^0 - (\mathcal{K}(b) - b)\}$.

Similarly to Theorem 4, we have the following relationships between the value of the function $g^{\preceq_u^{\mathcal{K}}}$ given by (6) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved with respect to the relation $\preceq_u^{\mathcal{K}}$. :

Theorem 7 *Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following characterizations of the relation $\preceq_u^{\mathcal{K}}$ by means of the functional $g^{\preceq_u^{\mathcal{K}}}$ given by (6) hold true*

(a) $g^{\preceq_u^{\mathcal{K}}}(A, B) \le r \Longrightarrow \bigcup_{t>r}(A - tk^0) \subseteq \bigcup_{b \in B}(b - \mathcal{K}(b))$, *i.e.,* $\bigcup_{t>r}(A - tk^0) \preceq_u^{\mathcal{K}} B$.

(b) $A - rk^0 \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)) \Longrightarrow g^{\preceq_u^{\mathcal{K}}}(A, B) \le r$.

Proof.

(a) Let $g^{\preceq_u^{\mathcal{K}}}(A, B) \le r$ and $\varepsilon > 0$, arbitrary. We are using the functional $g^{\preceq_u^{\mathcal{K}}}(\cdot, \cdot)$ given by (6). It holds that

$$\begin{aligned} &\forall\, a \in A : \inf_{b\in B} z^{-b+\mathcal{K}(b),k^0}(a) < r + \varepsilon \\ \Leftrightarrow &\forall\, a \in A : \inf_{b\in B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(b)\} < r + \varepsilon \\ \Leftrightarrow &\forall\, a \in A, \exists b \in B, \exists l < r + \varepsilon : a - lk^0 \in b - \mathcal{K}(b). \end{aligned}$$

This means that for all $a \in A$ there exist elements $b \in B$ and $l < r + \varepsilon$ such that

$$a - (r + \varepsilon)k^0 = a - lk^0 - (r + \varepsilon - l)k^0 \in b - (\mathcal{K}(b) + (r + \varepsilon - l)k^0) \subseteq b - \mathcal{K}(b).$$

Therefore,

$$\begin{aligned} &\forall\, \varepsilon > 0, \forall\, a \in A, \exists b \in B : a - (r + \varepsilon)k^0 \in b - \mathcal{K}(b) \\ \Leftrightarrow &\bigcup_{t>r}(A - tk^0) \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b)). \end{aligned}$$

(b) Let $A - rk^0 \subseteq \bigcup_{b\in B}(b - \mathcal{K}(b))$. It holds that

$$\begin{aligned} &\forall\, a \in A : a - rk^0 \in \bigcup_{b\in B}(b - \mathcal{K}(b)) \\ \Leftrightarrow &\forall\, a \in A, \exists b_a \in B : a - rk^0 \in b_a - \mathcal{K}(b_a) \\ \Rightarrow &\forall\, a \in A, \inf_{b\in B} \inf\{t \in \mathbb{R} : a - tk^0 \in b - \mathcal{K}(b)\} \le r \\ \Rightarrow &\sup_{a\in A} \inf_{b\in B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(b)\} \le r \\ \Leftrightarrow &g^{\preceq_u^{\mathcal{K}}}(A, B) \le r. \end{aligned}$$

The proof is complete.

Example 4 Let $A, B \in \mathcal{P}(\mathbb{R}^2)$ be determined by

$$A := \{(y_1, y_2) \in \mathbb{R}^2 | (y_1 - 6)^2 + (y_2 - 1)^2 \leq 1\},$$

and

$$B := \{(y_1, y_2) \in \mathbb{R}^2 | 3 \leq y_1 \leq 5, 3 \leq y_2 \leq 4\}.$$

Let a set-valued mapping $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ be given by

$$\mathcal{K}(y) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 | 0 \leq d_1 \leq d_2\} & \text{if } y \neq (5, 3), \\ \mathbb{R}^2_+ & \text{if } y = (5, 3). \end{cases}$$

Choose $k^0 = (1, 1)$. Obviously, $\forall y \in \mathbb{R}^2$, $\forall t \in [0, +\infty)$: $tk^0 + \mathcal{K}(y) \subseteq \mathcal{K}(y)$. We have that $A - 2k^0 \subseteq \bigcup_{b \in B} (b - \mathcal{K}(b))$. By Theorem 7(b), $g^{\preceq^{\mathcal{K}}_u}(A, B) \leq 2$. In addition, the following assertion also holds true

$$\bigcup_{t > 2} (A - tk^0) \subseteq \bigcup_{b \in B} (b - \mathcal{K}(b)).$$

For an illustration of this example, see Figure 4.

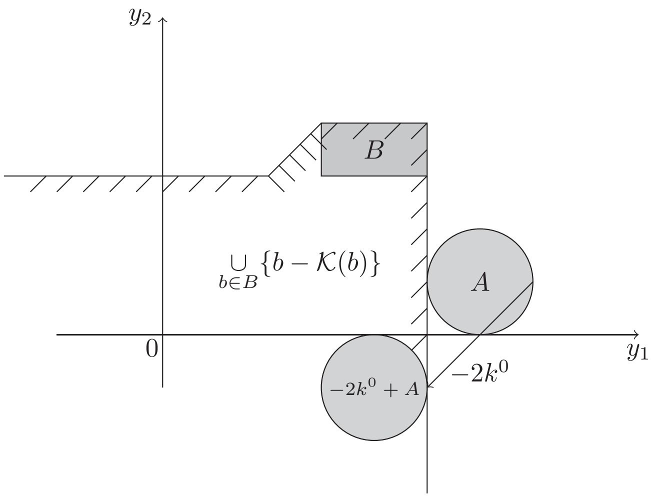

Figure 4: Illustration for Example 4

The comparison $A \preceq^{\mathcal{K}}_u B$ can be described by an equivalent assertion by means of the nonlinear scalarizing functional $g^{\preceq^{\mathcal{K}}_u}$ as follow.

Theorem 8 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Suppose that $\bigcup_{b \in B} (b - \mathcal{K}(b))$ is closed. Then, it holds that*

$$A \preceq^{\mathcal{K}}_u B \Leftrightarrow g^{\preceq^{\mathcal{K}}_u}(A, B) \leq 0.$$

Proof. The necessary condition is a consequence of Theorem 7(b) with $r := 0$. Now we prove the sufficient condition. Let $g^{\preceq^{\mathcal{K}}_u}(A, B) \leq 0$. By Theorem 7(a), it holds that

$$\bigcup_{t > 0} (A - tk^0) \subseteq \bigcup_{b \in B} (b - \mathcal{K}(b)).$$

This means that for all $n > 0$, we have $(A - \frac{1}{n}k^0) \subseteq \underset{b\in B}{\cup}(b - \mathcal{K}(b))$. Taking the limit when $n \to +\infty$ we obtain

$$A \subseteq \mathrm{cl}(\underset{b\in B}{\cup}(b - \mathcal{K}(b))) = \underset{b\in B}{\cup}(b - \mathcal{K}(b)).$$

Therefore, $A \subseteq \underset{b\in A}{\cup}(b - \mathcal{K}(b))$, i.e., $A \preceq_u^{\mathcal{K}} B$. The proof is complete.

3.3 Characterization of the Variable Generalized Certainly Lower Less Order Relation via Scalarization

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. We consider a scalarizing functional

$$g^{\preceq_{cl}^{\mathcal{K}}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq_{cl}^{\mathcal{K}}}(A, B) := \sup_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b). \tag{7}$$

In (7), we are using the functional (1) with $D = a + \mathcal{K}(a)$, $a \in A$ fixed, and $k = k^0$.
The following result describes the relationships between the value of the functional $g^{\preceq_{cl}^{\mathcal{K}}}$ given by (7) at (A, B), where $A, B \in \mathcal{P}(Y)$ and the comparison of sets where A, B are involved with respect to the relation $\preceq_{cl}^{\mathcal{K}}$

Theorem 9 *Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following characterizations of the relation $\preceq_{cl}^{\mathcal{K}}$ by means of the functional $g^{\preceq_{cl}^{\mathcal{K}}}$ given by (7) hold true*

(a) $g^{\preceq_{cl}^{\mathcal{K}}}(A, B) \leq r \Longrightarrow \underset{t>r}{\cup}(tk^0 + B) \subseteq \underset{a\in A}{\cap}(a + \mathcal{K}(a))$, *i.e.,* $A \preceq_{cl}^{\mathcal{K}} \underset{t>r}{\cup}(tk^0 + B)$

(b) $rk^0 + B \subseteq \underset{a\in A}{\cap}(a + \mathcal{K}(a)) \Longrightarrow g^{\preceq_{cl}^{\mathcal{K}}}(A, B) \leq r$.

Proof.

(a) Let $g^{\preceq_{cl}^{\mathcal{K}}}(A, B) \leq r$ and $\varepsilon > 0$, arbitrary. It holds thats

$$\begin{aligned} & \forall\, b \in B : \sup_{a\in A} z^{a+\mathcal{K}(a),k^0}(-b) < r + \varepsilon \\ \Leftrightarrow\quad & \forall\, b \in B : \sup_{a\in A} \inf\{t \in \mathbb{R} : b + tk^0 \in a + \mathcal{K}(a)\} < r + \varepsilon \\ \Leftrightarrow\quad & \forall\, b \in B : \forall a \in A, \exists l < r + \varepsilon : b + lk^0 \in a + \mathcal{K}(a). \end{aligned}$$

We have that

$$b + (r + \varepsilon)k^0 = b + lk^0 + (r + \varepsilon - l)k^0 \in a + \mathcal{K}(a) + (r + \varepsilon - l)k^0.$$

Taking into account the implication

$$r + \varepsilon - l > 0 \Rightarrow (r + \varepsilon - l)k^0) + \mathcal{K}(a) \subseteq \mathcal{K}(a),$$

we get that

$$b + (r + \varepsilon)k^0 \in a + \mathcal{K}(a).$$

Thus, $\forall\, \varepsilon > 0, \forall\, b \in B, \forall a \in A$ it holds that $b + (r + \varepsilon)k^0 \in a + \mathcal{K}(a)$, *i.e.*,

$$\bigcup_{t>r} (tk^0 + B) \subseteq \bigcap_{a\in A} (a + \mathcal{K}(a)).$$

(b) Let $rk^0 + B \subseteq \bigcap_{a\in A} (a + \mathcal{K}(a))$. We have that

$$\begin{aligned} & \forall\, b \in B : rk^0 + b \in \bigcap_{a\in A} (a + \mathcal{K}(a)) \\ \Leftrightarrow\quad & \forall\, b \in B, \forall a \in A : rk^0 + b \in a + \mathcal{K}(a) \\ \Rightarrow\quad & \forall\, b \in B, \sup_{a\in A} \inf\{t \in \mathbb{R} : tk^0 + b \in a + \mathcal{K}(a)\} \leq r \\ \Rightarrow\quad & \sup_{(a,b)\in A\times B} \inf\{t \in \mathbb{R} : a - b \in tk^0 - \mathcal{K}(a)\} \leq r \\ \Leftrightarrow\quad & \sup_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b) \leq r \\ \Leftrightarrow\quad & g^{\preceq^{\mathcal{K}}_{cl}}(A, B) \leq r. \end{aligned}$$

We also get an equivalent result for the comparison $A \preceq^{\mathcal{K}}_{cl} B$ by means of the functional $g^{\preceq^{\mathcal{K}}_{cl}}$ given by (7) as follow.

Theorem 10 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, it holds that*

$$A \preceq^{\mathcal{K}}_{cl} B \Leftrightarrow g^{\preceq^{\mathcal{K}}_{cl}}(A, B) \leq 0.$$

Proof. Since for all $a \in A$, $\mathcal{K}(a))$ is closed, $\bigcap_{a\in A} (a+\mathcal{K}(a))$ is closed. Therefore, we apply Theorem 9 and use the same arguments as in the proof of Theorem 5 to get the desired conclusion.

3.4 Characterization of the Variable Generalized Certainly Upper Less Relation via Scalarization

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. We consider a scalarizing functional

$$g^{\preceq^{\mathcal{K}}_{cu}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{cu}}(A, B) := \sup_{(a,b)\in A\times B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{8}$$

In (8), we are using the functional (1) with $D = -b + \mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$. Similary to Theorem 9 and Theorem 10, we get the following relationships between the value of the function $g^{\preceq^{\mathcal{K}}_{cu}}$ given by (8) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved with respect to the relation $\preceq^{\mathcal{K}}_{cu}$.

Theorem 11 *Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y \setminus \{0\}$ be given such that (H_1) holds. Then, the following characterizations of the relation $\preceq^{\mathcal{K}}_{cu}$ by means of the functional $g^{\preceq^{\mathcal{K}}_{cu}}$ given by* (8) *hold true*

(a) $g^{\preceq^{\mathcal{K}}_{cu}}(A, B) \leq r \Longrightarrow \bigcup_{t>r} (A - tk^0) \subseteq \bigcap_{b\in B} (b - \mathcal{K}(b))$, *i.e.*, $\bigcup_{t>r} (A - tk^0) \preceq^{\mathcal{K}}_{cu} B$.

(b) $A - rk^0 \subseteq \bigcap_{b\in B} (b - \mathcal{K}(b)) \Longrightarrow g^{\preceq^{\mathcal{K}}_{cu}}(A, B) \leq r$.

Theorem 12 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, it holds that*

$$A \preceq^{\mathcal{K}}_{cu} B \Longleftrightarrow g^{\preceq^{\mathcal{K}}_{cu}}(A, B) \leq 0.$$

3.5 Characterization of the Variable Generalized Possible Lower Less Relation via Scalarization

Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) is fulfilled. In this section, we consider the following scalarizing functional

$$g^{\preceq^{\mathcal{K}}_{pl}} : \mathcal{P}(Y) \times \mathcal{P}(Y) \to \overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{pl}}(A, B) := \inf_{(a,b) \in A \times B} z^{a+\mathcal{K}(a),k^0}(-b). \tag{9}$$

In (9), we are using the functional (1) with $D = a + \mathcal{K}(a)$, $a \in A$ fixed, and $k = k^0$.
The following theorem illustrates the relationships between the value of the functional $g^{\preceq^{\mathcal{K}}_{pl}}$ given by (9) at (A, B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved with respect to the relation $\preceq^{\mathcal{K}}_{pl}$.

Theorem 13 *Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true*

(a) $g^{\preceq^{\mathcal{K}}_{pl}}(A, B) \leq r \Longrightarrow \bigcup_{t>r} \{tk^0\} \subseteq \bigcup_{(a,b) \in A \times B} (a + \mathcal{K}(a) - b)$.

(b) $rk^0 \in \bigcup_{(a,b) \in A \times B} (a + \mathcal{K}(a) - b) \Longrightarrow g^{\preceq^{\mathcal{K}}_{pl}}(A, B) \leq r$.

Proof.

(a) Let $g^{\preceq^{\mathcal{K}}_{pl}}(A, B) \leq r$ and $\varepsilon > 0$, arbitrary. It yields

$$\begin{aligned} &\exists\, b \in B : \inf_{a \in A} z^{a+\mathcal{K}(a),k^0}(-b) < r + \varepsilon \\ \Leftrightarrow &\exists\, b \in B : \inf_{a \in A} \inf\{t \in \mathbb{R} : b + tk^0 \in a + \mathcal{K}(a)\} < r + \varepsilon \\ \Leftrightarrow &\exists\, b \in B, \exists a \in A, \exists l < r + \varepsilon : b + lk^0 \in a + \mathcal{K}(a). \end{aligned}$$

Taking into account the implication $r + \varepsilon - l > 0 \Rightarrow (r + \varepsilon - l)k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a)$, we get

$$\begin{aligned} &b + (r + \varepsilon)k^0 = b + lk^0 + (r + \varepsilon - l)k^0 \in a + \mathcal{K}(a) + (r + \varepsilon - l)k^0 \\ \Rightarrow\;& b + (r + \varepsilon)k^0 \in a + \mathcal{K}(a). \end{aligned}$$

Thus, $\forall\, \varepsilon > 0, \exists\, b \in B, \exists a \in A$ it holds that $b + (r + \varepsilon)k^0 \in a + \mathcal{K}(a)$, *i.e.*,

$$\bigcup_{t>r} \{tk^0\} \subseteq \bigcup_{(a,b) \in A \times B} (a + \mathcal{K}(a) - b).$$

(b) Let $rk^0 \in \bigcup\limits_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b)$. It holds that

$$\begin{aligned}
&\exists\ (a,b)\in A\times B : rk^0 \in (a+\mathcal{K}(a)-b)\\
\Leftrightarrow\ &\exists\ (a,b)\in A\times B : tk^0 + b \in a+\mathcal{K}(a)\\
\Rightarrow\ &\inf_{(a,b)\in A\times B} \inf\{t\in\mathbb{R} : a-b\in tk^0-\mathcal{K}(a)\}\le r\\
\Leftrightarrow\ &\inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b)\le r\\
\Leftrightarrow\ &g^{\preceq^{\mathcal{K}}_{pl}}(A,B)\le r
\end{aligned}$$

Furthermore, the comparison $A \preceq^{\mathcal{K}}_{pl} B$, where $A, B \in \mathcal{P}(Y)$ can be described by an equivalent assertion as follows.

Theorem 14 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y\setminus\{0\}$ such that (H_1) holds. Suppose that for all $A, B \in \mathcal{P}(Y)$, $\bigcup\limits_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b)$ is closed. Then, we have the following characterization of the relation $\preceq^{\mathcal{K}}_{pl}$ be means of the functional $g^{\preceq^{\mathcal{K}}_{pl}}$ given by* (9)

$$A \preceq^{\mathcal{K}}_{pl} B \Longleftrightarrow g^{\preceq^{\mathcal{K}}_{pl}}(A,B)\le 0.$$

Proof. " $\Longrightarrow$ " : Suppose that $A \preceq^{\mathcal{K}}_{pl} B$. Then, we get that

$$\exists (a,b)\in A\times B : b\in a+\mathcal{K}(a).$$

This is equivalent to $0k^0 \in \bigcup\limits_{a\in A\times B}(a+\mathcal{K}(a)-b)$. Applying Theorem 13 (b), we have that

$$\inf_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b)\le 0.$$

" $\Longleftarrow$ " : Suppose that $\inf\limits_{(a,b)\in A\times B} z^{a+\mathcal{K}(a),k^0}(-b)\le 0$. Taking into account Theorem 13 (a), we get that

$$\bigcup_{t>0}\{tk^0\}\subseteq \bigcup_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b).$$

Therefore, for all $t=\frac{1}{n}, n>0$ it holds that $\frac{1}{n}k^0 \in \bigcup\limits_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b)$. Let $n\to+\infty$, we have that

$$0\in \mathrm{cl}(\bigcup_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b)).$$

Taking into account $\bigcup\limits_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b)$ is closed, it holds that

$$0\in \bigcup_{(a,b)\in A\times B}(a+\mathcal{K}(a)-b).,\ \text{i.e., } A \preceq^{\mathcal{K}}_{pl} B.$$

The proof is complete.

3.6 Characterization of the Variable Generalized Possible Upper Less Relation via Scalarization

Consiser $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$, $k^0 \in Y\setminus\{0\}$ such that (H_1) holds and set

$$g^{\preceq^{\mathcal{K}}_{pu}} : \mathcal{P}(Y)\times\mathcal{P}(Y)\to\overline{\mathbb{R}},$$

defined as

$$g^{\preceq^{\mathcal{K}}_{pu}}(A,B) := \inf_{(a,b)\in A\times B} z^{-b+\mathcal{K}(b),k^0}(a). \tag{10}$$

In (10), we are using the functional (1) with $D = -b + \mathcal{K}(b)$, $b \in B$ fixed, and $k = k^0$. The following theorems illustrate some relationships between the value of the function $g^{\preceq^{\mathcal{K}}_{pu}}$ given by (10) at (A,B), where $A, B \in \mathcal{P}(Y)$ and a comparison of sets where A and B are involved with respect to the relation $\preceq^{\mathcal{K}}_{pu}$. Since their proofs are similar to that of Theorem 13 and Theorem 14, we skip them in this paper.

Theorem 15 *Consider $A, B \in \mathcal{P}(Y)$ and let $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Then, the following assertions hold true*

(a) $g^{\preceq^{\mathcal{K}}_{pu}}(A,B) \le r$ *implies* $\bigcup_{t>r} \{tk^0\} \subseteq \bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b))$.

(b) $rk^0 \in \bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b))$ *implies* $g^{\preceq^{\mathcal{K}}_{pu}}(A,B) \le r$.

Theorem 16 *Let $A, B \in \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds. Suppose $\bigcup_{(a,b)\in A\times B} (a - b + \mathcal{K}(b))$ is closed. Then, we have the following characterization of the relation $\preceq^{\mathcal{K}}_{pu}$ be means of the functional $g^{\preceq^{\mathcal{K}}_{pu}}$ given by (10):*

$$A \preceq^{\mathcal{K}}_{pu} B \Longleftrightarrow g^{\preceq^{\mathcal{K}}_{pu}}(A,B) \le 0.$$

4 Characterizations of Minimal Elements Defined by Set Relations

Let us define minimal elements and strict minimal elements of $\mathcal{A} \in \mathcal{P}(Y)$ w.r.t. $\preceq^{\mathcal{K}}_t$ where $t \in \{l, u, cl, cu, pl, pu\}$. From now on, we assume that the relations $\preceq^{\mathcal{K}}_t$ are reflexive and transitive for $t \in \{l, u, cl, cu, pl, pu\}$.

Definition 3 Let Y be real linear topological space, $\mathcal{A} \in \mathcal{P}(Y)$. Let $\mathcal{K} : Y \rightrightarrows Y$ be a set-valued map.

(a) A set $\bar{A} \in \mathcal{A}$ is called a minimal element of $\mathcal{A}$ w.r.t. $\preceq^{\mathcal{K}}_t$ if

$$A \in \mathcal{A}, A \preceq^{\mathcal{K}}_t \bar{A} \text{ implies } \bar{A} \preceq^{\mathcal{K}}_t A.$$

(b) A set $\bar{A} \in \mathcal{A}$ is called a strong minimal element of $\mathcal{A}$ w.r.t. $\preceq^{\mathcal{K}}_t$ if

$$\forall\ A \in \mathcal{A}, \bar{A} \preceq^{\mathcal{K}}_t A.$$

(c) A set $\bar{A} \in \mathcal{A}$ is called a strict minimal element of $\mathcal{A}$ w.r.t. $\preceq^{\mathcal{K}}_t$ if

$$A \in \mathcal{A}, A \preceq^{\mathcal{K}}_t \bar{A} \text{ implies } \bar{A} = A.$$

The sets of all minimal, strong minimal and strict minimal elements of $\mathcal{A}$ w.r.t. $\preceq^{\mathcal{K}}_t$ are denoted by $\mathrm{Min}(\mathcal{A}, \preceq^{\mathcal{K}}_t)$, $\mathrm{SoMin}(\mathcal{A}, \preceq^{\mathcal{K}}_t))$ and $\mathrm{SiMin}(\mathcal{A}, \preceq^{\mathcal{K}}_t)$, respectively.

Remark 7 When $\mathcal{A}$ is a family of singleton sets and $\mathcal{K}(y)$ is a closed, convex, pointed cone for each $y \in Y$, the definition of strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq^{\mathcal{K}}_l$ reduces to the definition of nondominated elements of $\mathcal{A}$ w.r.t. $\mathcal{K}$ (see [6, Definition 2.7]), that is: An element $\bar{z} \in \mathcal{A}$ is a nondominated point of $\mathcal{A}$ with respect to $\mathcal{K}$, if there is no point $z \in \mathcal{A} \setminus \{\bar{z}\}$ such that $\bar{z} \in z + \mathcal{K}(z)$. On the other hand, the definition of strictly minimal elements of $\mathcal{A}$ w.r.t. $\preceq_u^{\mathcal{K}}$ reduces to the definition of minimal elements of $\mathcal{A}$ w.r.t. $\mathcal{K}$ (see [6, Definition 2.7]), that is: An element $\bar{z} \in \mathcal{A}$ is a nondominated point of $\mathcal{A}$ with respect to $\mathcal{K}$, if there is no point $z \in \mathcal{A} \setminus \{\bar{z}\}$ such that $\bar{z} \in z + \mathcal{K}(\bar{z})$.

Remark 8 If $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ then for all $B \sim \bar{A}$ it holds that $B \in \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. In addition, from the Definition 3 we get that

$$\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \text{ and } \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}}).$$

However, neither $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ nor $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \subseteq \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ always holds true. This is illustrated by the following example.

Example 5 Consider the four following sets:
$A_1 := \{(y_1, y_2) \in \mathbb{R}^2 | \ 2 \le y_1, y_2 \le 3, y_1 + y_2 \le 5\}$.
$A_2 := \{(2, y_2) \in \mathbb{R}^2 | \ 2 \le y_2 \le 3\} \cup \{(y_1, 2) \in \mathbb{R}^2 | \ 2 \le y_1 \le 3\}$.
$A_3 := \{(5, 5)\}$.
$A_4 := \{(y_1, y_2) \in \mathbb{R}^2 | \ 3 \le y_1 \le 5, 0 \le y_2 \le 1\}$.
We define a set-valued mapping $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$\mathcal{K}(y) := \begin{cases} \{(d_1, d_2) | \ 0 \le d_1 \le 2\ d_2\} & \text{if } y \in \mathbb{R}^2 \setminus \{(2,3)\}, \\ \mathbb{R}^2_+ & \text{if } y = (2,3). \end{cases}$$

It follows from the definition of $\preceq_l^{\mathcal{K}}$ given in Definition 2 that:

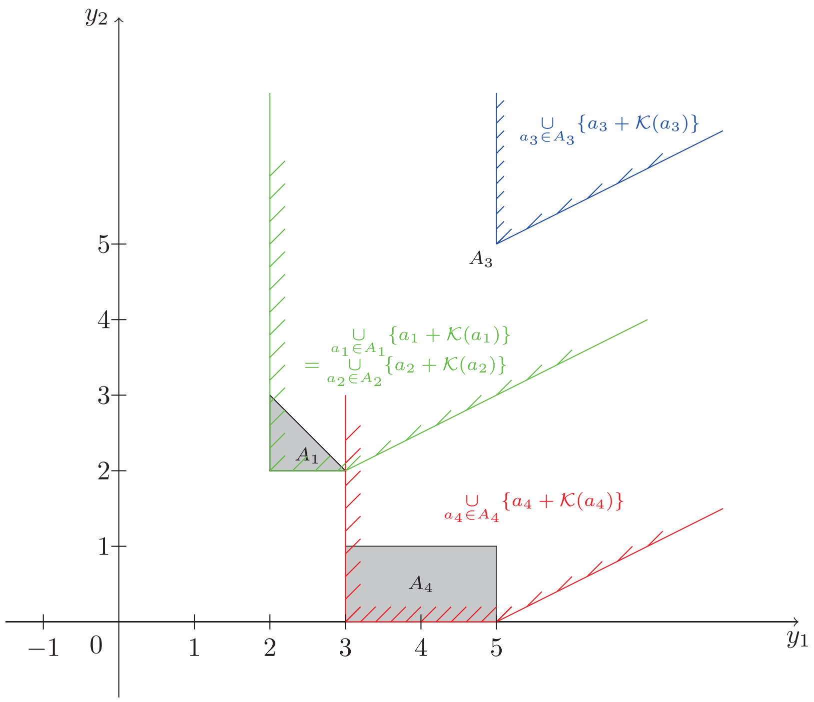

Figure 5: Illustration for Example 5

$$\begin{cases} A_1 \preceq_l^{\mathcal{K}} A_2, A_1 \preceq_l^{\mathcal{K}} A_3, A_1 \not\preceq_l^{\mathcal{K}} A_4, \\ A_2 \preceq_l^{\mathcal{K}} A_1, A_2 \preceq_l^{\mathcal{K}} A_3, A_2 \not\preceq_l^{\mathcal{K}} A_4, \\ A_3 \not\preceq_l^{\mathcal{K}} A_1, A_3 \not\preceq^{\mathcal{K}} A_2, A_3 \not\preceq_l^{\mathcal{K}} A_4, \\ A_4 \not\preceq_l^{\mathcal{K}} A_1, A_4 \not\preceq_l^{\mathcal{K}} A_2, A_4 \preceq_l^{\mathcal{K}} A_3. \end{cases}$$

Let $\mathcal{A} := \{A_1, A_2, A_3\}$. We have that

$$\mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \ \ \mathrm{SoMin}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \ \ \mathrm{SiMin}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \emptyset.$$

Let $\mathcal{A}' := \{A_1, A_2, A_3, A_4\}$. We get that

$$\mathrm{Min}(\mathcal{A}', \preceq_l^{\mathcal{K}}) = \{A_1, A_2\}, \ \ \mathrm{SoMin}(\mathcal{A}', \preceq_l^{\mathcal{K}}) = \emptyset, \ \ \mathrm{SiMin}(\mathcal{A}, \preceq_l^{\mathcal{K}}) = \{A_4\}.$$

Let $\mathcal{A}'' := \{A_3, A_4\}$. It holds that

$$\mathrm{Min}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \mathrm{SoMin}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \mathrm{SiMin}(\mathcal{A}'', \preceq_l^{\mathcal{K}}) = \{A_4\}.$$

For an illustration of this example, see Figure 5.

In addition, the sets $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ and $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ have following properties.

Proposition 2 *Let $\mathcal{A}$ be a family of sets in $\mathcal{P}(Y)$, $S \in \mathcal{P}(Y)$ and let $|S|$ denote the number of elements of S. Then, for $t \in \{l, u, cl, cu, pl, pu\}$, it holds that*

(a) If $|\,\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$*, then* $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.
(b) If $|\,\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| > 1$*, then* $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \emptyset$.
(c) If $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$*, then* $\begin{cases} |\,\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = |\,\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = 1 \\ \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) = \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}). \end{cases}$

PROOF.

(a) Suppose that $A_1, A_2 \in \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $A_1 \neq A_2$, and $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$. Let $B \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. It holds that

$$\begin{cases} A_1 \preceq_t^{\mathcal{K}} B, B \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A_1 = B, \\ A_2 \preceq_t^{\mathcal{K}} B, B \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A_2 = B. \end{cases}$$

Therefore $A_1 = A_2$, that is a contradiction.

(b) Let $B_1, B_2 \in \mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}), B_1 \neq B_2$ and suppose that $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$. Let $A \in \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$. We have that

$$\begin{cases} A \in \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A \preceq_t^{\mathcal{K}} B_1 \Rightarrow A = B_1, \\ A \in \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \Rightarrow A \preceq_t^{\mathcal{K}} B_2 \Rightarrow A = B_2. \end{cases}$$

Thus, $B_1 = B_2$, a contradiction.

(c) Part (a) and part (b) yield $|\,\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| \leq 1$. Therefore, if

$$\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \cap \mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}}) \neq \emptyset$$

then

$$|\,\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = |\,\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})| = 1.$$

The proof is complete.

The preceding section shows how we present set order relations using different nonlinear functionals of type (1). Since these relations are used to define minimal elements of a family of sets, in the following we characterize these elements by means of the corresponding nonlinear functionals.

Theorem 17 *Let $\mathcal{A} \subseteq \mathcal{P}(Y)$, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that (H_1) holds true. Assume that for $A \in \mathcal{A}$, $\bigcup_{a\in A}(a + \mathcal{K}(a))$ is closed. Then*

(a) $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0.$$

(b) $\bar{A} \in \mathrm{SoMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A} : \ g^{\preceq_l^{\mathcal{K}}}(\bar{A}, A) \leq 0.$$

(c) $\bar{A} \in \mathrm{SiMin}(\mathcal{A}, \preceq_l^{\mathcal{K}})$ *if and only if*

$$\forall A \in \mathcal{A}, A \in \mathcal{A} \setminus \bar{A} : \ g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0.$$

Proof.

(a) Let $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$. We are using the definition of $g^{\preceq_l^{\mathcal{K}}}$ given by (4). Suppose by contradiction that

$$\exists A \in \mathcal{A}, A \not\sim \bar{A} : \ g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) \leq 0 \text{ i.e.,}$$
$$\exists A \in \mathcal{A}, A \not\sim \bar{A} : \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d) \leq 0.$$

By Theorem 5, it holds that

$$\sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d) \leq 0 \Rightarrow A \preceq_l^{\mathcal{K}} \bar{A}.$$

Taking into account $\bar{A} \in \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}})$, we get that $\bar{A} \preceq_l^{\mathcal{K}} A$. Thus, $A \sim \bar{A}$, a contradiction. Conversely, assume that

$$\forall A \in \mathcal{A}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0 \text{ and } \bar{A} \notin \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}}), \text{ i.e.,}$$

$$\forall A \in \mathcal{A}, A \not\sim \bar{A} : \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d) > 0 \text{ and } \bar{A} \notin \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}}).$$

We have the following implication

$$\bar{A} \notin \mathrm{Min}(\mathcal{A}, \preceq_l^{\mathcal{K}}) \Longrightarrow \exists A \in \mathcal{A}, A \preceq_l^{\mathcal{K}} \bar{A} \text{ and } \bar{A} \not\preceq_l^{\mathcal{K}} A.$$

By Theorem 5, it holds that

$$A \preceq_l^{\mathcal{K}} \bar{A} \Rightarrow \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d) \leq 0, \text{ a contradiction.}$$

(b) It implies directly from the definition of $\preceq_l^{\mathcal{K}}$ and Theorem 5.
(c) Analogously to part (a).

As for $\preceq_t^{\mathcal{K}}$, where $t \in \{u, cl, cu, pl, pu\}$ we can also obtain similar results as Theorem 17. We illustrate them by the following table, which describes characterizations for elements in $\mathrm{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $\mathrm{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ and $\mathrm{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$ by means of the corresponding scalarizing functionals.

	$A \in \text{Min}(\mathcal{A}, \preceq_t^{\mathcal{K}})$	$A \in \text{SoMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$	$A \in \text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$
$t = l$ $\bigcup_{a\in A}(a + \mathcal{K}(a))$ is closed	$\forall A \not\sim \bar{A}$: $g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0$	$\forall A \in \mathcal{A}$: $g^{\preceq_t^{\mathcal{K}}}(\bar{A}, A) \leq 0$	$\forall A \in \mathcal{A} \setminus \{\bar{A}\}$: $g^{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0$
$t = u$ $\bigcup_{b\in B}(b - \mathcal{K}(b))$ is closed			
$t = cl$			
$t = cu$			
$t = pl$ $\bigcup_{a\in A}(a + \mathcal{K}(a) - b)$ is closed			
$t = pu$ $\bigcup_{(a,b)\in A\times B}(a - b + \mathcal{K}(b))$ is closed			

Table 1: Characterizations of minimal elements defined by set relations w.r.t. variable domination structures

Proposition 3 *Let Y be a linear topological space and $\mathcal{A}$ be a nonempty subset of $\mathcal{P}(Y)$. Suppose that $\mathcal{K} : Y \rightrightarrows Y$ and k^0 are given such that (H_1) is fulfilled. Assume $\bar{A} \in \text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $t \in \{l, u, cl, cu, pl, pu\}$ and the corresponding closedness assumptions given in Table 1 are satisfied, then*

$$g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) = \operatorname*{Min}_{V\in\mathcal{A}} g^{\preceq_t^{\mathcal{K}}}(V, \bar{A}).$$

Proof. Since $\bar{A} \in \text{SiMin}(\mathcal{A}, \preceq_t^{\mathcal{K}})$, $t \in \{l, u, cl, cu, pl, pu\}$, we have that for all $A \neq \bar{A}$ the following assertion holds true:

$$g_{\preceq_t^{\mathcal{K}}}(A, \bar{A}) > 0.$$

On the other hand, because $\preceq_t^{\mathcal{K}}$ is reflexive, we have that $\bar{A} \preceq_t^{\mathcal{K}} \bar{A}$ and therefore $g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) \leq 0$. Thus,

$$g^{\preceq_t^{\mathcal{K}}}(\bar{A}, \bar{A}) = \operatorname*{Min}_{V\in\mathcal{A}} g^{\preceq_t^{\mathcal{K}}}(V, \bar{A}).$$

Note that in vector optimization w.r.t. variable domination structure, Tammer and Bouza in [2] have introduced a nonlinear scalarization method which gets the value zero at the minimal points of a given set. We also obtain this result by using our scalarizing functional for the relations $\preceq_l^{\mathcal{K}}$ and $\preceq_u^{\mathcal{K}}$ as follows:

Proposition 4 *Let Y be a linear topological space, $\mathcal{A} \subset Y$ be a nonempty set, $\mathcal{K} : Y \rightrightarrows Y$ and $k^0 \in Y \setminus \{0\}$ such that for all $y \in Y$, $\mathcal{K}(y)$ is a closed convex pointed cone and (H_1) holds. Then*

(a) If $\bar{y}$ is a minimal element of $\mathcal{A}$ and $k^0 \in \mathcal{K}(\bar{y}) \setminus \{0\}$ then

$$g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) = 0.$$

(b) If $\bar{y}$ is a nondominated element of $\mathcal{A}$ and $k^0 \in \mathcal{K}(\bar{y}) \setminus \{0\}$ then

$$g^{\preceq_l^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) = 0.$$

Proof. We consider $\mathcal{A}$ is a family of singleton sets and we only prove part (a) since part (b) can be done by similar lines. By Remark 8, $\{\bar{y}\} \in \text{SiMin}(\mathcal{A}, \preceq_u^{\mathcal{K}})$. Taking into account Theorem 8 and $\{\bar{y}\} \preceq_u^{\mathcal{K}} \{\bar{y}\}$, it hold that $g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) \leq 0$. Now, we assume that $g^{\preceq_u^{\mathcal{K}}}(\{\bar{y}\}, \{\bar{y}\}) < 0$. Then, there exists $t < 0$ such that $\bar{y} - \bar{y} \in tk^0 - \mathcal{K}(\bar{y})$. This implies that $tk^0 \in \mathcal{K}(\bar{y})$. On the other hand, since

$\mathcal{K}(\bar{y})$ is a cone, $k^0 \in \mathcal{K}(\bar{y})$ and $-t > 0$, we get that $-tk^0 \in \mathcal{K}(\bar{y})$. Therefore, $tk^0 \in \mathcal{K}(\bar{y}) \cap (-\mathcal{K}(\bar{y}))$. Taking into account $\mathcal{K}(\bar{y})$ is pointed, it holds that $k^0 = 0$, a contradiction.

5 Descent Method

This section is devoted developing a numerical method for finding approximations of minimal solutions of a set-valued optimization problem equipped with a variable domination structure without any convexity assumptions. In addition to providing a numerical method, we present a convergence result. In the literature, there already exist some algorithms for solving set-valued optimization problems. Jahn [22] proposes a descent method that generates approximations of minimal elements of set-valued optimization problems under convexity assumptions on the considered sets. In [22], the set less order relation is characterized by means of linear functionals. More recently, in [15], the authors propose a similar descent method for obtaining approximations of minimal elements of set-valued optimization problems with a fixed domination structure. The approaches in [15,22] all rely on set order relations where the involved domination structure is given by fixed cones. We relax this assumption and consider a domination structure which is variable.

Here we consider a set-valued optimization problem in the following setting: Let $Y = \mathbb{R}^m$, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map, $S \subset \mathbb{R}^n$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ such that for each $y \in Y$, $\mathcal{K}(y)$ is a proper closed set.

Let $k^0 \in Y \setminus \{0\}$ such that

$$\forall y \in Y : \; y+\mathcal{K}(y) + (0, +\infty)k^0 \subset \operatorname{int}(y + \mathcal{K}(y)). \tag{11}$$

This section provides a descent method to find approximations of minimal solutions of the following problem

$$\operatorname*{Min}_{x \in S} F(x) \tag{P_1}$$

equipped with the reflexive and transitive relations $\preceq_t^{\mathcal{K}}, t \in \{u, l, cu, cl, pu, pl\}$ given in Definition 2. The minimality notion of (P_1) is defined as follows:

Definition 4 A point $\bar{x} \in S$ is called a minimal solution of problem (P_1) w.r.t. $\preceq_t^{\mathcal{K}}$, where $t \in \{u, l, cu, cl, pu, pl\}$, if

$$x \in S, \; F(x) \preceq_t^{\mathcal{K}} F(\bar{x}) \Longrightarrow F(\bar{x}) \preceq_t^{\mathcal{K}} F(x).$$

Definition 4 yields that $\bar{x}$ is a minimal solution of (P_1) w.r.t. $\preceq_t^{\mathcal{K}}$ if and only if $F(\bar{x}) \in \operatorname{Min}(F(S), \preceq_t^{\mathcal{K}})$. From now on, let $k^0 \in Y \setminus \{0\}$ satisfy (11).

In the following, we present a descent method for (P_1) w.r.t. $\preceq_l^{\mathcal{K}}$. Note that we can modify this method in order to obtain approximate minimal solutions of (P_1) w.r.t. $\preceq_t^{\mathcal{K}}$, where $t \in \{u, cl, cu, pl, pu\}$. However, we restrict ourselves to the relation $\preceq_l^{\mathcal{K}}$ for the sake of brevity. Again, we assume in the following that the relation $\preceq_l^{\mathcal{K}}$ is reflexive and transitive. In addition, suppose that for all $x \in S$, $\bigcup_{y \in F(x)} (y + \mathcal{K}(y))$ is closed. Now, we define a functional $p : S \times S \rightrightarrows \mathbb{R}$ as

$$p(z, x) := \sup_{b \in F(x)} \inf_{a \in F(z)} z^{a+\mathcal{K}(a), k^0}(-b) = g^{\preceq_l^{\mathcal{K}}}(F(z), F(x)),$$

where again $k^0 \in Y \setminus \{0\}$ such that (11) holds true.

Notice that the functional $z^{a+\mathcal{K}(a), k^0}(\cdot)$ is well-defined, as $a+\mathcal{K}(a)$ is a closed set. In addition, since

$k^0 \in Y \setminus \{0\}$ and the condition (11) holds true, for each $a \in Y$, $z^{a+\mathcal{K}(a),k^0}(\cdot)$ is continuous (Theorem 1 (a)).

Obviously, it follows from Theorem 5 that

$$p(z,x) \leq 0 \Leftrightarrow F(z) \preceq_l^{\mathcal{K}} F(x).$$

In the following, we present a descent method for computing approximate minimal solutions of (P_1) w.r.t. $\preceq_l^{\mathcal{K}}$, where $S = \mathbb{R}^n$. For one given starting point x^0, Algorithm 1 approximates one minimal solution of problem (P_1) w.r.t. $\preceq_l^{\mathcal{K}}$. To find more than one approximation of minimal solutions, one needs to vary the input parameters, such as choosing a different starting point $x^0 \in \mathbb{R}^n$, or modifying the vector k^0.

Algorithm 1 (A descent method for finding an approximation of a minimal solution of the set-valued problem (P_1))

Input: $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, $S = \mathbb{R}^n$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$, order relation $\preceq_l^{\mathcal{K}}$,
starting point $x^0 \in \mathbb{R}^n$, $k \in \mathbb{R}^m \setminus \{0\}$) satisfies (11), maximal number i_{max} of iterations, number of search
directions n_s, maximal number j_{max} of iterations for the determination of the step size,
initial step size h_0 and minimum step size $h_{\min}$, $\{\lambda_1, \dots, \lambda_N\} \subset [0,1]$
for $p = 1 : 1 : N$ **do**
    % initialization for the descent method
    $i := 0$, $h := h_0$
    choose n_s points $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^{n_s}$ on the unit sphere around $0_{\mathbb{R}^n}$
    % iteration loop
    while $i \leq i_{max}$ **do**
        check $F(x^i + h\tilde{x}^j) \preceq_l^{\mathcal{K}} F(x^i)$ for every $j \in \{1, \dots, n_s\}$ by evaluating the extremal
        term (e. g. $p(x^i + h\tilde{x}^j, x^i) = g^{\preceq_l^{\mathcal{K}}}(F(x^i + h\tilde{x}^j), F(x^i))$.
        Choose the index $n_0 := j$ with the smallest function value extremal$_{term}$.
        if extremal$_{term} \leq 0$ **then**
            $x^{i+1} := x^i + h\tilde{x}^{n_0}$ % new iteration point
            $j := 1$
            while $F(x^i + (j+1)h\tilde{x}^{n_0}) \preceq_l^{\mathcal{K}} F(x^i + jh\tilde{x}^{n_0})$ and $j \leq j_{max}$ **do**
                $j := j + 1$
                $x^{i+1} := x^{i+1} + h\tilde{x}^{n_0}$ % new iteration point
            end while
        else
            $h := h/2$
            if $h \leq h_{\min}$ **then**
                STOP. Output: $x := x^i$
            end if
        end if
        $i := i + 1$
    end while
end for
Output: A set of approximations x of minimal solutions of the set-valued problem (P_1) w.r.t. $\preceq_l^{\mathcal{K}}$.

We now set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In order to show a convergence result for Algorithm 1, we need the following modifications in the algorithm:

(A) Assume that the pattern contains at least one direction of descent whenever a set $F(x^i)$ ($i \in \mathbb{N}_0$) can be improved.

(B) Let some $\beta \in (0,1)$ and an arbitrary null sequence $(\varepsilon^i)_{i\in\mathbb{N}_0}$ with $\varepsilon^i < 0$ for all $i \in \mathbb{N}_0$ be given. While $p(x^{i+1}, x^i) \geq \varepsilon^i$, set $h := \beta^q h$ for $q := 0, 1, 2, \ldots$ after line 27 of Algorithm 1.

Remark 9 It is shown in [15] and [22] that the continuity of the function $p(\cdot,\cdot)$ is a sufficient condition such that assumption (A) holds true. In addition, we can use a weaker condition that ensures the fulfillment of the assumption (A), that is: If for every given element $a \in \mathbb{R}^n$, $p(\cdot, a)$ is continuous then the assumption (A) also holds true. Indeed, when x^i is not the final iteration point, then there is a descent direction and a point $\bar{x} \in \mathbb{R}^n$ such that $p(\bar{x}, x^i) < 0$. By the continuity of $p(\cdot, x^i)$, it follows that there is some ball $B(\bar{x}, \delta)$ around $\bar{x}$ with radius δ such that for all $x \in B(\bar{x}, \delta)$, $p(x, x^i) < 0$. Therefore, we can get a descent direction by refining the grid, and the assumption (A) is fulfilled. It is also an interesting topic for further research to find other sufficient conditions for this assumption.

Theorem 18 *Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a nonempty and compact-valued map. Let Algorithm 1 with the additional specifications (A) and (B) generate an iteration sequence $(x^i)_{i\in\mathbb{N}}$, where $x^0 \in \mathbb{R}^n$ denotes the initial iteration point. In addition, assume that $z^{y+\mathcal{K}(y),k^0}$ is finite-valued for all $y \in Y$ and k^0 satisfying* (11). *Then*

$$\limsup_{i\to+\infty} p(x^{i+1}, x^i) = 0.$$

Proof. Observe that for all $i \in \mathbb{N}, F(x^i) \preceq_l^{\mathcal{K}} F(x^0)$ since the relation $\preceq_l^{\mathcal{K}}$ is transitive. Because $F(x^i)$ is compact for all $x^i \in \mathbb{R}^n$, there is $a^{i+1} \in F(x^{i+1})$, $b^i \in F(x^i)$ such that $p(x^{i+1}, x^i) = z^{a^{i+1}+\mathcal{K}(a^{i+1}),k^0}(-b^i)$. Thus, $\{p(x^{i+1}, x^i)\}$, $i \in \mathbb{N}_0$ is bounded. Consequently, there exists $\alpha := \limsup_{i\to+\infty} p(x^{i+1}, x^i)$. We assume now that $\alpha \neq 0$. By specification (A), it holds that $p(x^{i+1}, x^i) \leq 0$. Therefore,

$$\forall i \in \mathbb{N}_0 : p(x^{i+1}, x^i) \leq 0.$$

Taking into account $\limsup_{i\to+\infty} p(x^{i+1}, x^i) = \alpha \neq 0$, we get that $\alpha < 0$ and

$$\exists N_1 \in \mathbb{N} : \forall r > N_1,\ p(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} < 0.$$

Let $\{\varepsilon^i\}_{i\in\mathbb{N}_0}$ be a null sequence. Then, there is $N_2 \in \mathbb{N}$ such that

$$\forall r \geq N_2 :\ \frac{\alpha}{2} \leq \varepsilon^{i_r} < 0.$$

Therefore, let $N = \max\{N_1, N_2\}$ it holds that

$$\forall r \geq N :\ p(x^{i_r+1}, x^{i_r}) \leq \frac{\alpha}{2} \leq \varepsilon^{i_r}.$$

This is a contradiction to specification (B).

6 An Application in Medical Image Registration

Medical image registration has been used widely in medical treatment, for instance in radiotherapy (treatment verification, treatment planning, treatment guidance), Orthopaedic surgery and surgical microscope. The problem of image registration is finding a transformation matching two given sets of data (images). Suppose that T is a subset of transformations. H and K are two images obtained by X-Ray or Angiography (2D image) or Computed Tomography (CT), Magnetic Resonance Tomography (MRT) and PET (Positron Emission Tomography)(3D image). Suppose that $H, K \subseteq Y$ where $Y = \mathbb{R}^2$ or $Y = \mathbb{R}^3$. For each $t \in T$, a set of comparison mappings $\{f_i(t, H, K) \subset \mathbb{R}, i = 1, ..., m\}$ is calculated, where $f_i : (T, H, K) \rightrightarrows \mathbb{R}$. Note that f_i can be different distance functions used to compare H and K, $i = 1, .., m$, see [9]. However, we choose the set-valued mappings $f_i : (T, H, K) \rightrightarrows \mathbb{R}$, $i = 1, ..., m$ since the fact that there may exist some movements of the patient during the time his images are taken, which can lead to perturbed data. For each point $y \in \mathbb{R}^m$, we attach a weight $\omega(y) := (\omega_1(y), \omega_2(y), ..., \omega_m(y)) \in \mathbb{R}^m_+$, which is chosen by the desision maker, see [9] and [23] for more detail. To formulate our Image registration problem w.r.t. a variable domination structure, we use a set-valued map $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ introduced in [9] and given as

$$\begin{aligned} \mathcal{K} :\ & \mathbb{R}^m \rightrightarrows \mathbb{R}^m \\ \forall y \in \mathbb{R}^m, \mathcal{K}(y) :&= \{d \in \mathbb{R}^m | \sum_{i=1}^{m} \operatorname{sign}(d_i)\omega_i(y) \geq 0\}, \end{aligned} \tag{12}$$

where $\operatorname{sign}(d_i) := \begin{cases} 1 & \text{if } d_i > 0, \\ 0 & \text{if } d_i = 0, \\ -1 & \text{if } d_i < 0. \end{cases}$

Obviously, for each $y \in \mathbb{R}^m$, $\mathcal{K}(y)$ is a cone satisfying $\mathbb{R}^m_+ \subseteq \mathcal{K}(y)$. We introduce an Image registration problem by

$$\mathcal{K} - \operatorname*{Min}_{t \in T}(f_1(t, H, K), ..., f_m(t, H, K)). \tag{P_2}$$

Here, we use the set relation $\preceq^{\mathcal{K}}_l$ since this relation is often used when the decision maker concerns the best cases. Observe that if we set

$$\hat{\mathcal{A}} := \{(f_1(t, H, K), ..., f_m(t, H, K)), t \in T\}$$

then each element of $\hat{\mathcal{A}}$ is a set $A \subseteq \mathbb{R}^m$. We are looking for a transformation $\bar{t} \in T$ such that

$$\bar{A} := (f_1(\bar{t}, H, K), ..., f_m(\bar{t}, H, K)) \in \operatorname{Min}(\hat{\mathcal{A}}, \preceq^{\mathcal{K}}_l).$$

Motivated by Theorem 17, we obtain the following corollary. Since the proof of it is similar to that of Theorem 17, we omit it in this paper. As in previous sections, from now on let $k^0 \in Y \setminus \{0\}$ satisfying (H_1).

Corollary 1 *Let $\hat{A} \subset \mathcal{P}(\mathbb{R}^m)$, $\mathcal{K} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ and $k^0 \in \mathbb{R}^m \setminus \{0\}$ such that (H_1) is satisfied. Assume that for all $A \in \hat{\mathcal{A}}$, $\bigcup_{a \in A}(a + \mathcal{K}(a))$ is closed. Then $\bar{t} \in T$ is a solution of the problem of the image registration problem (P_2) if*
$\bar{A} := (f_1(\bar{t}, H, K), ..., f_m(\bar{t}, H, K)))$ is a minimal element of $\hat{\mathcal{A}}$ w.r.t. $\preceq^{\mathcal{K}}_l$., i.e

$$\forall A \in \hat{\mathcal{A}}, A \not\sim \bar{A} : g^{\preceq^{\mathcal{K}}_l}(A, \bar{A}) > 0. \tag{13}$$

Observe that $g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) = \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d)$, where

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} : a - d \in rk^0 - \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} : d + rk^0 - a \in \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} : \sum_{i=1}^{m} \operatorname{sign}(d + rk^0 - a)_i \omega_i(a) \geq 0\}, \qquad (14) \\ &\text{for all } a \in A, d \in \bar{A}. \end{aligned}$$

Now, we apply the characterization of minimal elements of $\hat{\mathcal{A}}$ which is derived in Corollary 1 to the image registration problem (P_2) when $m = 2$. We present in the following a proposition in which $z^{a+\mathcal{K}(a),k^0}(-d)$ is calculated in detail.

Proposition 5 *Let $\hat{\mathcal{A}} \subset \mathcal{P}(\mathbb{R}^2)$, $\mathcal{K} : \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ and $k^0 \in \mathbb{R}^2 \setminus \{0\}$ such that (H_1) is satisfied. Assume that for all $A \in \hat{\mathcal{A}}$, $\bigcup_{a\in A}(a + \mathcal{K}(a))$ is closed. Then $\bar{t} \in T$ is a solution of the problem (P_2) if $\bar{A} := (f_1(\bar{t}, H, K), f_2(\bar{t}, H, K)))$ is a minimal element of $\hat{\mathcal{A}}$ w.r.t. $\preceq_l^{\mathcal{K}}$, i.e.,*

$$\forall A \in \hat{\mathcal{A}}, A \not\sim \bar{A} : g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) > 0,$$

where $g^{\preceq_l^{\mathcal{K}}}(A, \bar{A}) = \sup_{d\in\bar{A}} \inf_{a\in A} z^{a+\mathcal{K}(a),k^0}(-d)$ and $z^{a+\mathcal{K}(a),k^0}(-d)$, $a \in A, \bar{d} \in \bar{A}$ is determined by

$$z^{a+\mathcal{K}(a),k^0}(-d) = \begin{cases} \frac{a_1-d_1}{k_1^0} & \text{if } \omega_1(a) > 0, \omega_2(a) = 0 \text{ or } \omega_1(a) \geq \omega_2(a) > 0, \\ \frac{a_2-d_2}{k_2^0} & \text{if } \omega_1(a) = 0, \omega_2(a) > 0 \text{ or } \omega_2(a) \geq \omega_1(a) > 0. \end{cases}$$

Proof. As shown in [9], for each $\omega = (\omega_1, \omega_2) \in \mathbb{R}^m_+$ one has for each $a \in A$,

$$\mathcal{K}(a) = \begin{cases} \{d \in \mathbb{R}^2 | d_1 \geq 0, d_2 \in \mathbb{R}\} & \text{if } \omega_1(a) > 0, \omega_2(a) = 0, \\ \{d \in \mathbb{R}^2 | d_1 \in \mathbb{R}, d_2 \geq 0\} & \text{if } \omega_1(a) = 0, \omega_2(a) > 0, \\ \{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \text{ or} (d_1 < 0, d_2 > 0)\} & \text{if } \omega_2(a) \geq \omega_1(a) > 0, \\ \{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \text{ or} (d_1 > 0, d_2 < 0)\} & \text{if } \omega_1(a) \geq \omega_2(a) > 0. \end{cases}$$

In the following, we illustrate the formulation of $z^{a+\mathcal{K}(a),k^0}(-d)$, $a \in A, d \in \bar{A}$ in case m=2 for all possible weight $\omega(a) := (\omega_1(a), \omega_2(a))$.

(a) $\omega_1(a) > 0, \omega_2(a) = 0$. We have that $\mathcal{K}(a) = \{d \in \mathbb{R}^2 | d_1 \geq 0, d_2 \in \mathbb{R}\}$. Now, (14) becomes

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} : rk^0 - (a - d) \in \mathcal{K}(a)\} \\ &= \inf\{r \in \mathbb{R} : (d_1 + rk_1^0 - a_1) \geq 0\}. \qquad (15) \end{aligned}$$

Since $\forall \gamma > 0 : \gamma k^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a)$ and $0 \in \mathcal{K}(a)$, we get that

$$\forall \gamma > 0 : \gamma k^0 \in \mathcal{K}(a) \Leftrightarrow \gamma k_1^0 \geq 0, \forall \gamma > 0.$$

Therefore,

$$k_1^0 \geq 0.$$

We choose k^0 such that $k_1^0 > 0$. By (15), it holds that

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} : r \geq \frac{a_1 - d_1}{k_1^0}\} = \frac{a_1 - d_1}{k_1^0}.$$

(b) $\omega_1 = 0, \omega_2(a) > 0$. We can prove this part analogously to part (a) and obtain:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_2 - d_2}{k_2^0}.$$

(c) $\omega_2(a) \geq \omega_1(a) > 0$. Then

$$\mathcal{K}(a) = \{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \,\text{or}\, (d_1 < 0, d_2 > 0)\}.$$

Now, (14) becomes

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} :& ((d_1 + rk_1^0 - a_1) \geq 0, (d_2 + rk_2^0 - a_2) \geq 0) \\ & \text{or}((d_1 + rk_1^0 - a_1) < 0, (d_2 + rk_2^0 - a_2) > 0)\}. \end{aligned} \quad (16)$$

Since

$$\forall t > 0 : tk^0 + \mathcal{K}(a) \subseteq \mathcal{K}(a) \text{ and } 0 \in \mathcal{K}(a),$$

it holds that

$$\begin{aligned} &(tk_1^0 \geq 0, tk_2^0 \geq 0) \text{ or } (tk_1^0 < 0, tk_2^0 > 0) \\ \Longleftrightarrow &(k_1^0 \geq 0, k_2^0 \geq 0) \text{ or } (k_1^0 < 0, k_2^0 > 0). \end{aligned}$$

Now, we consider two cases:
Case 1: $(k_1^0 \geq 0, k_2^0 \geq 0)$, we choose $k_1^0 > 0$ and $k_2^0 > 0$, then (16) becomes:

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} :\ & (r \geq \frac{a_1 - d_1}{k_1^0}, r \geq \frac{a_2 - d_2}{k_2^0}) \\ & \text{or } (r < \frac{a_1 - d_1}{k_1^0}, r > \frac{a_2 - d_2}{k_2^0})\}. \end{aligned}$$

If $\frac{a_1 - d_1}{k_1^0} \leq \frac{a_2 - d_2}{k_2^0}$ then

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} :\ r \geq \frac{a_2 - d_2}{k_2^0}\} = \frac{a_2 - d_2}{k_2^0}.$$

If $\frac{a_1 - d_1}{k_1^0} > \frac{a_2 - d_2}{k_2^0}$ then

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} :\ r \geq \frac{a_1 - d_1}{k_1^0} \text{ or } r > \frac{a_2 - d_2}{k_2^0}\} \\ &= \frac{a_2 - d_2}{k_2^0}. \end{aligned}$$

Case 2: $(k_1^0 < 0, k_2^0 > 0)$, then (16) becomes:

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} :\ & (r \leq \frac{a_1 - d_1}{k_1^0}, r \geq \frac{a_2 - d_2}{k_2^0}) \\ & \text{or } (r > \frac{a_1 - d_1}{k_1^0}, r > \frac{a_2 - d_2}{k_2^0})\}. \end{aligned}$$

If $\frac{a_1-d_1}{k_1^0} < \frac{a_2-d_2}{k_2^0}$ then

$$z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} : \; r > \frac{a_2 - d_2}{k_2^0}\} = \frac{a_2 - d_2}{k_2^0}.$$

If $\frac{a_1-d_1}{k_1^0} \geq \frac{a_2-d_2}{k_2^0}$ then,

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) &= \inf\{r \in \mathbb{R} : \; \frac{a_2 - d_2}{k_2^0} \leq r \leq \frac{a_1 - d_1}{k_1^0} \text{ or } r > \frac{a_1 - d_1}{k_1^0}\} \\ &= \frac{a_2 - d_2}{k_2^0}. \end{aligned}$$

From the two above cases, we conclude that

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_2 - d_2}{k_2^0}.$$

(d) $\omega_1(a) \geq \omega_2(a) > 0$. Then,

$$\mathcal{K}(a) = \{d \in \mathbb{R}^2 | (d_1 \geq 0, d_2 \geq 0) \,\text{or}\, (d_1 > 0, d_2 < 0)\}.$$

Now, (14) becomes

$$\begin{aligned} z^{a+\mathcal{K}(a),k^0}(-d) = \inf\{r \in \mathbb{R} : &((d_1 + rk_1^0 - a_1) \geq 0, (d_2 + rk_2^0 - a_2) \geq 0) \\ &\text{or}((d_1 + rk_1^0 - a_1) > 0, (d_2 + rk_2^0 - a_2) < 0)\}. \end{aligned} \tag{17}$$

By using the same arguments like in part (c), we obtain:

$$z^{a+\mathcal{K}(a),k^0}(-d) = \frac{a_1 - d_1}{k_1^0}.$$

The proof is complete.

7 Conclusion

This paper develops novel nonlinear scalarization methods for the characterization of minimal elements defined by several set relations of a set-valued problem equipped with variable domination structures. The assumptions that we consider on the domination mapping are very broad and therefore widely applicable; especially, our results do not rely on any convexity assumptions. Our results leave various avenues for future research. First, it is our goal to apply the proposed descent method using our theoretical scalarization results for real-world medical image registration problems as well as other practical problems. Second, we would like to develop existence results for set-valued optimization problems with variable domination structures. Moreover, it is important to find further sufficient conditions for assumption (A) in the convergence result in Theorem 18.

References

1. Ait Mansour, M., Durea, M. and Théra, M.: A lower semicontinuous regularization for set-valued mappings and its applications. *J. Convex Anal.*, 15(3), 473–484 (2008).
2. Bouza, G. and Tammer, Chr.: *Nonlinear scalarizing functions for computing minimal points under variable ordering structures,* Appl. Anal. Optimization **1**(1), 67–97 (2017).

3. Gutiérrez, C. and Jiménez, B. and Miglierina, E. and Molho, E.: *Scalarization in set optimization with solid and nonsolid ordering cones*. J. Global Optim., 61(3), 525–552 (2015).
4. Gutiérrez, C. and Miglierina, E. and Molho, E. and Novo, V.: *Pointwise well-posedness in set optimization with cone proper sets*. Nonlinear Anal., 75(4), 1822-1833 (2012).
5. Ernst, E. and Théra, M.: On the necessity of the Moreau-Rockafellar-Robinson qualification condition in Banach spaces. *Math. Program.*, 117(1-2, Ser. B), 149–161 (2009).
6. Eichfelder, G.: *Variable ordering structures in vector optimization*. Springer, Heidelberg (2014).
7. Eichfelder, G. and Pilecka, M.: *Set Approach for Set Optimization with Variable Ordering Structures Part I: Set Relations and Relationship to Vector Approach*, J. Optim. Theory Appl. **171**(3), 931–946 (2016).
8. Eichfelder, G. and Pilecka, M.: *Set Approach for Set Optimization with Variable Ordering Structures Part II: Scalarization Approaches,* J. Optim. Theory Appl. **171**(3), 947–963 (2016).
9. Eichfelder, G.: *Optimal elements in vector optimization with a variable order- ing structure*. J. Optim. Theory Appl., 151(2), 217–240 (2011).
10. A. Göpfert, Chr. Tammer, H. Riahi and C. Zălinescu: *Variational Methods in Partially Ordered Spaces*, Springer-Verlag, Berlin (2003).
11. Gerth (Tammer), Chr. and Weidner, P.: *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl. **67**, 297–320 (1990).
12. Heyde, F.: *Coherent risk measures and vector optimization*, In: Multicriteria decision making and fuzzy systems. Theory, methods and applications, Editor: K.-H. Küfer, Shaker-Verlag, 3–12 (2006).
13. Huynh, V.N. and Théra, M.: Directional metric regularity of multifunctions. *Math. Oper. Res.*, 40(4), 969–991 (2015).
14. Khan, A. A., Tammer, Chr. and Zălinescu, C.: *Set-valued optimization: An introduction with applications*, Springer, Berlin (2015).
15. Köbis, E. and Köbis, M. A.: Treatment of set order relations by means of a nonlinear scalarization functional: A full characterization. *Optim.*, 65(10), 1805–1827 (2016).
16. Köbis, E.: *Variable ordering structures in set optimization*. Preprint 378, Univ. Erlangen-Nürnberg (2014)
17. Köbis, E. and Tammer, Chr.: *Characterization of Set Relations by Means of a Nonlinear Scalarization Functional*, Modelling, Computation and Optimization in Information Systems and Management Sciences Advances in Intelligent Systems and Computing **359**, 491-503 (2015).
18. Köbis, E., Le, Th. T. and Tammer, Chr.: *Necessary Necessary optimality conditions for solutions of set optimization with respect to variable domination structures*, Reports on Optimization and Stochastics 04-2017 (2017).
19. Ngai, H.V., Tron, N.H. and Théra, M.: Metric regularity of the sum of multifunctions and applications. *J. Optim. Theory Appl.*, 160(2), 355–390 (2014).
20. Ngai, H.V., Kruger, A.Y. and Théra, M.: Slopes of multifunctions and extensions of metric regularity. *Vietnam J. Math.*, 40(2-3), 355–369 (2012).
21. Jahn, J.: *Vectorization in set optimization*, J. Optim. Theory Appl. **167**(3), 783–795 (2015).
22. Jahn, J.: *A derivative-free descent method in set optimization*, Computational Optimization and Applications **60**(2), 393–411 (2015).
23. Wacker, M.: *Multikriterielle Optimierung bei Registrierung medizinischer Daten.* Master thesis, Friedrich-Alexander-University Erlangen-Nuremberg (2008).
24. Weidner, P.: *Ein Trennungskonzept und seine Anwendung auf Vektoroptimierungsverfahren*, Dissertation B, Martin-Luther-Universität Halle-Wittenberg (1990).

Reports of the Institutes 2017

01-17. M. Klinge and R. Weiner, *Strong stability preserving explicit peer methods for discontinuous Galerkin discretizations*

02-17. Chr. Günther and Chr. Tammer, *On generalized-convex constrained multi-objective optimization*

03-17. E. Köbis, L. Th. Tam, Chr. Tammer and J.-Ch. Yao, *Necessary conditions for solutions of set optimization problems with respect to variable domination structures*

04-17. E. Köbis, Chr. Tammer and J.-Ch. Yao, *Optimality conditions for set-valued optimization problems based on set approach and applications in uncertain optimization*

05-17. L. Th. Tam, *Multiobjective approaches based on variable ordering structures for intensity problems in radiotherapy treatment*

06-17. Chr. Günter and N. Popovici, *Jahn-Graef-Younes type algorithms for discrete vector optimization based on cone-monotone sorting functions*

Reports are available via WWW: http://www.mathematik.uni-halle.de/institut/reports/