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OPTIMALITY CONDITIONS FOR SET-VALUED OPTIMIZATION PROBLEMS BASED ON SET APPROACH AND APPLICATIONS IN UNCERTAIN OPTIMIZATION

ELISABETH KÖBIS, CHRISTIANE TAMMER, AND JEN-CHIH-YAO

Abstract. In this paper, we derive necessary optimality conditions for optimal solutions of set-valued optimization problems based on a set approach. These necessary optimality conditions are derived in terms of Bouligand and Mordukhovich generalized differentiation objects for optimal solutions of set-valued problems with respect to the possibly and lower set less order relation. Finally, we apply our results to vector optimization problems that are contaminated by uncertain data.

1. Introduction

In this paper we investigate optimality conditions for set-valued optimization problems based on a set approach using the possibly and lower set less order relation for the comparison of sets. In order to prove the optimality conditions for set-valued optimization problems based on the set approach we will use corresponding results (see Durea [4] and Durea, Strugariu [6]) shown for set-valued optimization problems based on the vector approach, i.e., for solutions defined on the graph of the set-valued objective map. Furthermore, extending ideas by Eichfelder and Pilecka [8, 9], we show relationships between solutions of set-valued optimization problems based on set approach and solutions based on vector approach that are important in our context. We structure our line of study into two directions: On the one hand, we work on primal spaces, where derivative concepts for set-valued maps are used and, on the other hand, we consider generalized differentiation objects lying in the dual spaces. Some of these auxiliary results are taken from the literature (mostly from [4], [6]), while the presented optimality conditions for set-valued optimization problems based on a set approach are obtained here for the first time, in this respect being of interest on their own.

Generally speaking, in set-valued optimization, there are (at least) three approaches for defining solution concepts: the vector approach (defined for elements belonging to the graph of the objective set-valued map), the set approach ([13]), and the lattice approach ([15], [17], [18]). In the last two

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cases, the generalized differentiation techniques developed so far have not been very effective because of the structure of the involved spaces. In this paper, having in mind the aim to formulate optimality conditions in terms of derivatives and coderivatives of set-valued maps, we develop some necessary optimality conditions for solution concepts based on the set approach.

We propose the following structure for the paper. Section 2 prepares the notions and preliminary results used in the sequel. Section 3 proposes some optimality notions for set-valued optimization problems, which are either based on a set approach or on the graph of the set-valued mapping, and their relationships. Necessary optimality conditions are derived in Section 4, where we divide our analysis into two parts, namely into approaches in primal spaces and approaches in dual spaces. In the primal space approach, we distinguish between the solid case, where the interior of the ordering cone is assumed to be nonempty, and the general case, which works for ordering cones with empty interior. We apply our results to uncertain vector optimization problems, i.e., vector optimization problems which are perturbed by uncertain data in the objective map, in Section 5.

2. Preliminaries

In this section, we recall the notation and preliminary results used in this paper. Throughout this manuscript, let X, Y, Z be Banach spaces over the real field $\mathbb{R}$. $B_X(x, \varepsilon)$ and $D_X(x, \varepsilon)$ denote the open ball and the closed ball, respectively, with center at $x \in X$ and radius $\varepsilon > 0$. B_X, D_X, S_X are the open unit ball, the closed unit ball and the unit sphere of X, respectively. On a product space we consider the sum norm, unless otherwise stated. The topological dual space of X is denoted by X^* and by w^* we mean the weak star topology on X^*. For given $x \in X$, $\mathcal{V}(x)$ is the system of the neighborhoods of x. For $x \in X$ and $S \subset X$, we denote the distance from x to S by $d(x, S) := \inf_{y \in X} \|x - y\|$ (by convention, we set $d(x, \emptyset) = \infty$). As usual, $\operatorname{cl} S$, $\operatorname{int} S$ denote the topological closure and the topological interior of S, respectively. In the sequel, we suppose that K is a closed, convex, proper (i.e., $K \neq \{0\}$ and $K \neq Y$) and pointed (i.e., $K \cap (-K) = \{0\}$) cone in Y. Under these assumptions, the cone K induces an order relation on Y by the equivalence $y_1 \leq_K y_2$ if and only if $y_2 - y_1 \in K$. We call $K^+ := \{y^* \in Y^* \mid \forall y \in K : y^*(y) \geq 0\}$ the positive dual cone of K.

In this paper, we consider set-valued maps $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$. As usual, the graph of F is $\operatorname{Gr} F := \{(x, y) \in X \times Y \mid y \in F(x)\}$. If $A \subset X$, we denote the image of A under F by $F(A) := \bigcup\limits_{x \in A} F(x)$ and the inverse set-valued map of F is $F^{-1} : Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} F^{-1}$ iff $(x, y) \in \operatorname{Gr} F$. We write $\operatorname{cl} F$ for the multifunction whose graph is $\operatorname{cl} \operatorname{Gr} F$. (F, G) is the set-valued map defined on X, which takes values on $Y \times Z$ and is defined by

$$(F, G)(x) := \{(y, z) \mid y \in F(x), z \in G(x)\}.$$

The aim of this paper is to derive optimality conditions in terms of the Bouligand Derivative. We now introduce the main objects we use in the sequel. The definitions are standard, but they are used under different names in literature (see, for instance, [1]).

Definition 2.1 (Bouligand Tangent Cone)**.** Let S be a nonempty subset of X and $\overline{x} \in X$. The Bouligand tangent cone to S at $\overline{x}$ is the set

$$T_B(S, \overline{x}) = \{u \in X \mid \exists (t_n) \downarrow 0, \exists (u_n) \to u, \exists n_0 \in \mathbb{N}, \forall n \geq n_0, \overline{x} + t_n u_n \in S\},$$

where $(t_n) \downarrow 0$ means $(t_n) \subset (0, \infty)$ and $(t_n) \to 0$.

Remark 2.2. The set $T_B(S, \overline{x})$ is a closed cone (not necessarily convex). It is also clear that $T_B(S, \overline{x}) \neq \emptyset$ if and only if $\overline{x} \in \operatorname{cl} S$.

Based on this concept, one defines an associated derivative for set-valued maps.

Definition 2.3 (Bouligand Derivative)**.** Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$. The Bouligand derivative of F at $(\overline{x}, \overline{y})$ is the set valued map $D_B F(\overline{x}, \overline{y})$ from X into Y defined by

$$\operatorname{Gr} D_B F(\overline{x}, \overline{y}) = T_B(\operatorname{Gr} F, (\overline{x}, \overline{y})).$$

We say that F is open at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if the image through F of every neighborhood of $\overline{x}$ is a neighborhood of $\overline{y}$. A stronger openness property is the openness at linear rate. We call the mapping $F : X \rightrightarrows Y$ **open at linear rate** $L > 0$ (or $L-$**open) around** $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x}), V \in \mathcal{V}(\overline{y})$ and a positive number $\varepsilon > 0$ such that, for every $(x, y) \in \operatorname{Gr} F \cap (U \times V)$ and every $\rho \in (0, \varepsilon)$,

$$B_Y(y, \rho L) \subset F(B_X(x, \rho)).$$

Apart from this "around-point" property, the following corresponding "at-point" openness is also widely used: F is said to be **open at linear rate** $L > 0$ or ($L-$**open) at** $(\overline{x}, \overline{y})$ if there exists a positive number $\varepsilon > 0$ such that, for every $\rho \in (0, \varepsilon)$,

$$B_Y(\overline{y}, \rho L) \subset F(B_X(\overline{x}, \rho)).$$

The following closely related property can also be mentioned here: The set-valued mapping F is said to have the **Aubin property** around $(\overline{x}, \overline{y})$ with constant $M > 0$ if there exist two neighborhoods $U \in \mathcal{V}(\overline{x})$, $V \in \mathcal{V}(\overline{y})$ such that, for every $x, u \in U$,

$$F(x) \cap V \subset F(u) + M \|x - u\| D_Y.$$

It is well known that the latter property is equivalent to the openness at linear rate for F^{-1} around $(\overline{y}, \overline{x})$.

On the dual spaces, we work with the constructions developed by Mordukhovich and his collaborators (see [14]). Some of these concepts, which will be used in the sequel, are briefly listed here.

Definition 2.4. Let X be a normed vector space, S be a non-empty subset of X and let $x \in S$, $\varepsilon \geq 0$. The set of $\varepsilon-$normals to S at x is

$$\widehat{N}_\varepsilon(S,x) := \left\{ x^* \in X^* \mid \limsup_{u \xrightarrow{S} x} \frac{x^*(u-x)}{\|u-x\|} \leq \varepsilon \right\}, \tag{2.1}$$

where $u \xrightarrow{S} x$ means that $u \to x$ and $u \in S$.

If $\varepsilon = 0$, the elements in the right-hand side of (2.1) are called Fréchet normals and their collection, denoted by $\widehat{N}(S,x)$, is the Fréchet normal cone to S at x.

Let $\overline{x} \in S$. The basic (or limiting, or Mordukhovich) normal cone to S at $\overline{x}$ is

$$N(S,\overline{x}) := \{x^* \in X^* \mid \exists \varepsilon_n \downarrow 0, x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}_{\varepsilon_n}(S,x_n), \forall n \in \mathbb{N}\}.$$

If X is an Asplund space, and S is closed around $\overline{x}$ (i.e., there is a neighborhood V of $\overline{x}$ such that $S \cap \operatorname{cl} V$ is closed), the formula for the basic normal cone looks as follows:

$$N(S,\overline{x}) = \{x^* \in X^* \mid \exists x_n \xrightarrow{S} \overline{x}, x_n^* \xrightarrow{w^*} x^*, x_n^* \in \widehat{N}(S,x_n), \forall n \in \mathbb{N}\}. \tag{2.2}$$

Accordingly, we recall two concepts of coderivatives for set-valued maps.

Definition 2.5. Let $F : X \rightrightarrows Y$ be a set-valued map and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$. Then the Fréchet coderivative at $(\overline{x},\overline{y})$ is the set-valued map $\widehat{D}^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in \widehat{N}(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

Similarly, the normal coderivative of F at $(\overline{x},\overline{y})$ is the set-valued map $D^*F(\overline{x},\overline{y}) : Y^* \rightrightarrows X^*$ given by

$$D^*F(\overline{x},\overline{y})(y^*) := \{x^* \in X^* \mid (x^*,-y^*) \in N(\operatorname{Gr} F,(\overline{x},\overline{y}))\}.$$

As usual, one needs some generalized compactness requirements in order to obtain nontrivial Lagrange multipliers for the objective. Let X, Y be Asplund spaces and $H : X \rightrightarrows Y$ be a set-valued map closed around $(\overline{x},\overline{y}) \in \operatorname{Gr} H$. Following [14, pages 76, 266], one says that H is **partially sequentially normally compact** ((PSNC), for short) at $(\overline{x},\overline{y})$ if

$$\left[(x_n,y_n) \xrightarrow{\operatorname{Gr} H} (\overline{x},\overline{y}), x_n^* \xrightarrow{w^*} 0, y_n^* \to 0, (x_n^*,y_n^*) \in \widehat{N}(\operatorname{Gr} H,(x_n,y_n))\right] \Rightarrow x_n^* \to 0.$$

Let $S \subset Y$ be a set closed around $\overline{y} \in S$. One says that S is **sequentially normally compact** ((SNC), for short) at $\overline{x}$ if

$$\left[y_n \xrightarrow{S} \overline{x}, y_n^* \xrightarrow{w^*} 0, y_n^* \in \widehat{N}(S,y_n)\right] \Rightarrow y_n^* \to 0.$$

Observe that in the case where $S = Q$ is a proper closed convex cone, the (SNC) property at 0 can equivalently be formulated as

$$\left[(y_n^*) \subset Q^+, y_n^* \xrightarrow{w^*} 0\right] \Rightarrow y_n^* \to 0.$$

In particular, if $\operatorname{int} Q \neq \emptyset$, then Q is (SNC) at 0.

Notice also that, if the multifunction $K : X \rightrightarrows Y$ constantly equals the closed convex cone Q, the (SNC) property of Q at 0 is exactly the (PSNC) property of K^{-1} at $(0, \overline{x}) \in \operatorname{Gr} K^{-1}$.

3. Optimality Notions

The possibly less order relation $\preceq^p_Q$ is given in the following definition (compare [3, 10]).

Definition 3.1 (Possibly less order relation $\preceq^p_Q$)**.** Let the nonempty sets $A, B, Q \subset Y$ be given. Then we define

$$A \preceq^p_Q B :\Longleftrightarrow \exists a \in A, \exists b \in B : a \in b - Q.$$

We also use the following set order relation for the comparison of sets (compare [11, 12]).

Definition 3.2 (Lower Set Less Order Relation $\preceq^l_Q$)**.** Let the nonempty sets $A, B, Q \subset Y$ be given. We define

$$A \preceq^l_Q B \ :\Longleftrightarrow \ A + Q \supseteq B.$$

Based on these set order relations, we introduce the following optimality notion for set optimization problems. For the following definition, we use Defintions 3.1 and 3.2 for $Q = K$ and $Q = \operatorname{int} K$. The symbol $\bullet$ stands for p and l, respectively.

Definition 3.3 (Local Minimal Points w.r.t. $\preceq^\bullet_K$ ($\preceq^\bullet_{\operatorname{int} K}$))**.**

(i) $\overline{x} \in X$ is a local minimal point w.r.t. $\preceq^\bullet_K$ if there is a neighborhood U of $\overline{x}$ such that there does not exist $x \in U \setminus \{\overline{x}\}$ such that $F(x) \preceq^\bullet_K F(\overline{x})$.

(ii) Let $\operatorname{int} K \neq \emptyset$. $\overline{x} \in X$ is a local minimal point w.r.t. $\preceq^\bullet_{\operatorname{int} K}$ if there is a neighborhood U of $\overline{x}$ such that there does not exist $x \in U \setminus \{\overline{x}\}$ such that $F(x) \preceq^\bullet_{\operatorname{int} K} F(\overline{x})$.

The following definition describes the vector approach as solution method for a set optimization problem.

Definition 3.4 (Local (Weak) Minimizers)**.** Let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$.

(i) $(\overline{x}, \overline{y})$ is called a local minimizer for F with respect to K if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U \setminus \{\overline{x}\}$,

$$(F(x) - \overline{y}) \cap (-K) \subseteq \{0\}.$$

(ii) Let $\operatorname{int} K \neq \emptyset$. One says that $(\overline{x}, \overline{y})$ is a local weak minimizer for F with respect to K if there is a neighborhood U of $\overline{x}$ such that, for every $x \in U \setminus \{\overline{x}\}$,

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K) = \emptyset.$$

We have the following relationships between these concepts.

Proposition 3.5. *Let* $\operatorname{int} K \neq \emptyset$*. If* $\overline{x} \in X$ *is a local minimal point w.r.t.* $\preceq^p_{\operatorname{int} K}$*, then there exists some* $\overline{y} \in F(\overline{x})$ *such that* $(\overline{x}, \overline{y})$ *is a local weak minimizer for* F *with respect to* K*.*

Proof. Let $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq^p_{\operatorname{int} K}$.
That means that

$$
\begin{aligned}
& \exists\, U(\overline{x}) \text{ s.t. } \nexists\, x \in U \setminus \{\overline{x}\} : \; F(x) \preceq^p_{\operatorname{int} K} F(\overline{x}) \\
\iff & \exists\, U(\overline{x}) \text{ s.t. } \forall\, x \in U \setminus \{\overline{x}\} : \; F(x) \npreceq^p_{\operatorname{int} K} F(\overline{x}) \\
\iff & \exists\, U(\overline{x}) \text{ s.t. } \forall\, x \in U \setminus \{\overline{x}\} : \; \forall\, y \in F(x),\ \forall\, \overline{y} \in F(\overline{x}) : \; y \notin \overline{y} - \operatorname{int} K \\
\iff & \exists\, U(\overline{x}) \text{ s.t. } \; \forall\, \overline{y} \in F(\overline{x}),\ \forall\, x \in U \setminus \{\overline{x}\} : \; \forall\, y \in F(x) : \; y \notin \overline{y} - \operatorname{int} K \\
\implies & \exists\, U(\overline{x}) \text{ s.t. } \; \exists\, \overline{y} \in F(\overline{x}),\ \forall\, x \in U \setminus \{\overline{x}\} : \; \forall\, y \in F(x) : \; y \notin \overline{y} - \operatorname{int} K.
\end{aligned}
$$

This implies that there exists some $\overline{y} \in F(\overline{x})$ s.t.

$$\exists\, U(\overline{x}) \text{ s.t. } \forall\, x \in U \setminus \{\overline{x}\} : \; \forall\, y \in F(x) : \; y \notin \overline{y} - \operatorname{int} K,$$

which is the definition of a local weak minimizer $(\overline{x}, \overline{y})$ for F with respect to K. $\square$

The following example shows that the converse implication of Proposition 3.5 is generally not fulfilled.

Example 3.6. Consider the set-valued map $F : \mathbb{R} \rightrightarrows \mathbb{R}^2$, defined as

$$F(x) = \begin{cases} D_{\mathbb{R}^2}\left((0,0), 1\right), & \text{for } x = 0, \\ D_{\mathbb{R}^2}\left((0,-1), 1/2\right), & \text{else,} \end{cases}$$

where $D_{\mathbb{R}^2}(x, \varepsilon)$ denotes the closed ball of radius ε around x. Furthermore, let $K = \mathbb{R}^2_+$. Then the pair $(\overline{x}, \overline{y}) = (0, (-1, 0))$ is a local weak minimizer for F with respect to K, but $\overline{x} = 0$ is not a local minimal point w.r.t. $\preceq^p_{\operatorname{int} K}$ (see Figure 1).

The following result can be shown like Proposition 3.5.

Proposition 3.7. *If* $\overline{x} \in X$ *is a local minimal point w.r.t.* $\preceq^p_K$*, then there exists some* $\overline{y} \in F(\overline{x})$ *such that* $(\overline{x}, \overline{y})$ *is a local minimizer for* F *with respect to* K*.*

For the next result we need the definition of strongly minimal points.

Definition 3.8 (Strongly Minimal Point)**.** Let $x \in X$ be given. We say that $y \in F(x)$ is a strongly minimal point of $F(x)$ if $y + K \supseteq F(x)$.

Proposition 3.9. *Let* $\overline{x} \in X$ *be given, and suppose that* $F(\overline{x})$ *has the strongly minimal point* $\overline{y}$*. Moreover, assume that* $\operatorname{int} K \neq \emptyset$*. If* $\overline{x}$ *is a local minimal point w.r.t.* $\preceq^l_{\operatorname{int} K}$*, then* $(\overline{x}, \overline{y})$ *is a local weak minimizer for* F *with respect to* K*.*

Proof. Let $\overline{x}$ be a local minimal point w.r.t. $\preceq^l_{\operatorname{int} K}$. Suppose that $(\overline{x}, \overline{y})$ is not a local weak minimizer for F with respect to K. Then for all neighborhoods U of $\overline{x}$ there is some $x \in U \setminus \{\overline{x}\}$ such that

$$(F(x) - \overline{y}) \cap (-\operatorname{int} K) \neq \emptyset.$$

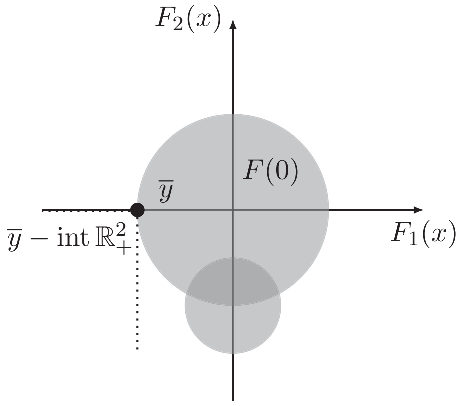

FIGURE 1. $(\overline{x}, \overline{y}) = (0, (-1, 0))$ is a local weak minimizer for F with respect to $\mathbb{R}^2_+$, but $\overline{x} = 0$ is not a local minimal point w.r.t. $\preceq^p_{\operatorname{int} \mathbb{R}^2_+}$.

This implies that $\overline{y} \in F(x) + \operatorname{int} K$. Because $\overline{y}$ is a strongly minimal point for $F(\overline{x})$, we obtain

$$F(\overline{x}) \subseteq \overline{y} + K \subseteq F(x) + K + \operatorname{int} K \subseteq F(x) + \operatorname{int} K,$$

contradicting the fact that $\overline{x}$ is a local minimal point w.r.t. $\preceq^l_{\operatorname{int} K}$. $\square$

For local minimal points w.r.t. $\preceq^l_K$, we obtain the following result.

Proposition 3.10. *Let $\overline{x} \in X$ be given, and suppose that $F(\overline{x})$ has the strongly minimal point $\overline{y}$. If $\overline{x}$ is a local minimal point w.r.t. $\preceq^l_K$, then $(\overline{x}, \overline{y})$ is a local minimizer for F with respect to K.*

4. Necessary Optimality Conditions

In this section, we will present necessary optimality conditions for set optimization problems based on the set approach by using corresponding results for set optimization problems based on the vector approach (see [4, 5, 6]).

4.1. **Approaches in primal spaces.**

For the primal approach, we derive the necessary optimality conditions in terms of the Bouligand derivative (see Definition 2.3).

4.1.1. *Solid Case.*

The following theorem describes a necessary optimality condition for local minimal point w.r.t. $\preceq^p_{\operatorname{int} K}$.

Theorem 4.1. *Let $\operatorname{int} K \neq \emptyset$ and let $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq^p_{\operatorname{int} K}$. Then there exists some $\overline{y} \in F(\overline{x})$ s.t.*

$$D_B F(\overline{x}, \overline{y})(X) \cap (-\operatorname{int} K) = \emptyset.$$

Proof. From Proposition 3.5, we know that there exists some $\overline{y} \in F(\overline{x})$ s.t. $(\overline{x}, \overline{y})$ is a local local weak minimizer for F with respect to K. By [4, Corollary 3.2, relation (29)], we immediately obtain the desired result. $\square$

Remark 4.2. If K is not a fixed ordering cone, but a set-valued map $K : X \rightrightarrows Y$ such that for each $x \in X$, $K(x)$ is a convex cone, Eichfelder and Pilecka [9] (see also [8]) obtain a corresponding result for *global* minimal points of a set optimization problem w.r.t. the possibly less order relation in terms of variable domination structures.

A necessary optimality condition based on the Bouligand derivative for local minimal points w.r.t. $\preceq^l_{\operatorname{int} K}$ is given in Theorem 4.3.

Theorem 4.3. *Let* $\operatorname{int} K \neq \emptyset$ *and let* $\overline{x} \in X$ *be a local minimal point w.r.t.* $\preceq^l_{\operatorname{int} K}$. *Moreover, assume that* $F(\overline{x})$ *has the strongly minimal point* $\overline{y}$. *Then it holds*

$$D_B F(\overline{x}, \overline{y})(X) \cap (-\operatorname{int} K) = \emptyset.$$

Proof. The proof follows immediately by taking into account Proposition 3.9 and [4, Corollary 3.2, relation (29)]. $\square$

4.1.2. *General Case.* This section is devoted to studying necessary optimality conditions of set optimization problems based on the set set approach in terms of the Bouligand derivative (see Definition 2.3), where the ordering cone K is not required to possess nonempty interior. For brevity, we define $\widetilde{F}(x) := F(x) + K$ for every $x \in X$. The following result can be found in Durea, Strugariu [6] (see also Durea, Strugariu, Tammer [7]).

Lemma 4.4. [6, Lemma 3.2] *Let* X, Y *be Banach spaces and* $F : X \rightrightarrows Y$ *be a multifunction. If* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *is a local minimizer for* F *with respect to* K *and* K *is not a linear subspace of* Y *(i.e.,* $K \setminus -K \neq \emptyset$*), then the multifunction* $\widetilde{F}$ *is not open at* $(\overline{x}, \overline{y})$. *In particular,* F *is not open at* $(\overline{x}, \overline{y})$.

The following openness result is due to Durea, Strugariu [6].

Theorem 4.5. [6, Theorem 2.6] *Let* X, Y *be Banach spaces and* $F : X \rightrightarrows Y$ *be a multifunction. Let* W *be an open subset of* $X \times Y$ *such that* $\operatorname{Gr} F \cap \operatorname{cl} W$ *is closed, and let* $\lambda > 0$ *such that for every* $(x, y) \in \operatorname{Gr} F \cap W$ *one has*

$$B_Y(0, \lambda) \subset \operatorname{cl}\left(D_B F(x, y)(B_X(0, 1))\right). \tag{4.1}$$

Then for every $\epsilon > 0$ *and for every* $(x, y) \in \operatorname{Gr} F$ *such that* $B_X(x, \epsilon) \times B_Y(y, \lambda\epsilon) \subset W$,

$$B_Y(y, \lambda\epsilon) \subset F(B_X(x, \epsilon)). \tag{4.2}$$

Now we are able to derive the result below.

Theorem 4.6. *Let* X, Y *be Banach spaces,* $F : X \rightrightarrows Y$ *is a multifunction, and let* $\overline{x} \in X$ *be a local minimal point w.r.t.* $\preceq^p_K$. *Let* $K \setminus -K \neq \emptyset$. *Suppose that for all* $\widetilde{y} \in F(\overline{x})$, $\operatorname{Gr}(\widetilde{F})$ *is closed around* $(\overline{x}, \widetilde{y})$. *Then there* *exists some* $\overline{y} \in F(\overline{x})$ *such that for every* $\varepsilon > 0$, *there there are* $(x_\varepsilon, y_\varepsilon) \in \mathrm{Gr}(\widetilde{F}) \cap (B_{X\times Y}(\overline{x}, \overline{y}), \varepsilon)$, $z_\varepsilon \in B_Y(0, \varepsilon) \setminus \{0\}$ *such that*

$$z_\varepsilon \notin \mathrm{cl}\left(D_B\widetilde{F}(x_\varepsilon, y_\varepsilon)(B_X(0,1))\right).$$

Proof. Let $\overline{x}$ be a local minimal point w.r.t. $\preceq^p_K$. From Proposition 3.7, we know that there exists some $\overline{y} \in F(\overline{x})$ s.t. $(\overline{x}, \overline{y})$ is a local minimizer of F with respect to K. Observe that, in view of Lemma 4.4, the multifunction $\widetilde{F}$ is not open at $(\overline{x}, \overline{y})$. Hence, the conclusion of Theorem 4.5 is not satisfied by $\widetilde{F}$. Because $\widetilde{F}$ is locally closed at $(\overline{x}, \overline{y})$, it follows that the other assumption of Theorem 4.5 is not satisfied, namely, for every $\varepsilon > 0$ there exist $(x_\varepsilon, y_\varepsilon) \in \mathrm{Gr}\,\widetilde{F} \cap B_{X\times Y}((\overline{x}, \overline{y}), \epsilon)$ such that

$$B_Y(0, \epsilon) \not\subset \mathrm{cl}(D_B\widetilde{F}(x_\varepsilon, y_\varepsilon)(B_X(0,1))).$$

Since $0 \in D_B(\widetilde{F}(x_\varepsilon, y_\varepsilon))(0) \subset \mathrm{cl}(D_B\widetilde{F}(x_\varepsilon, y_\varepsilon)(B_X(0,1)))$, the conclusion easily follows. □

Similarly, we derive the following result.

Theorem 4.7. *Let* X, Y *be Banach spaces,* $F : X \rightrightarrows Y$ *is a multifunction, let* $\overline{x} \in X$ *be a local minimal point w.r.t.* $\preceq^p_l$ *and suppose that* $F(\overline{x})$ *has the strongly minimal point* $\overline{y}$. *Let* $K \setminus -K \neq \emptyset$. *Suppose that* $\mathrm{Gr}\,\widetilde{F}$ *is closed around* $(\overline{x}, \overline{y})$. *Then for every* $\varepsilon > 0$, *there there are* $(x_\varepsilon, y_\varepsilon) \in \mathrm{Gr}\,\widetilde{F} \cap (B_{X\times Y}(\overline{x}, \overline{y}), \varepsilon)$, $z_\varepsilon \in B_Y(0, \varepsilon) \setminus \{0\}$ *such that*

$$z_\varepsilon \notin \mathrm{cl}\left(D_B\widetilde{F}(x_\varepsilon, y_\varepsilon)(B_X(0,1))\right).$$

4.2. **Approach in dual spaces.** For the special case that X and Y are Asplund spaces, it is possible to derive necessary optimality conditions for set optimization problems based on a set approach using the normal coderivative by Mordukhovich (see Definition 2.5).

We use the following result by Durea, Strugariu [5, Theorem 3.11 (i)] (see also Bao, Mordukhovich [2, Theorem 5.1] and Durea, Strugariu, Tammer [7, Corollary 4.13]).

Corollary 4.8. *Let* X, Y *be Asplund spaces,* $F : X \rightrightarrows Y$ *be a closed-graph multifunction,* K *be a proper closed convex cone and* $(\overline{x}, \overline{y}) \in \mathrm{Gr}\,F$ *be a local minimizer for* F *with respect to* K. *Suppose that* K *is (SNC) at* 0. *Then there exists* $y^* \in K^+ \setminus \{0\}$ *such that*

$$0 \in D^*F(\overline{x}, \overline{y})(y^*).$$

Remark 4.9. The condition "K is (SNC) at 0" can be replaced by "F^{-1} is (PSNC) at $(\overline{y}, \overline{x})$"in Corollary 4.8.

Now we are able to present the following necessary optimality condition for local minimal points w.r.t. $\preceq^p_K$.

Theorem 4.10. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a closed-graph multifunction, K be a proper closed convex cone and $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq_K^p$. Suppose that K is (SNC) at 0. Then there exists some $\overline{y} \in F(\overline{x})$ and some $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^* F(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Proposition 3.7 and Corollary 4.8. □

Remark 4.11. Notice that the condition "K is (SNC) at 0" in Theorem 4.10 can be replaced by "for all $\widetilde{y} \in F(\overline{x})$, F^{-1} is (PSNC) at $(\widetilde{y}, \overline{x})$".

The following theorem is concerned with necessary optimality condition for local minimal points w.r.t. $\preceq_K^l$.

Theorem 4.12. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a closed-graph multifunction, K be a proper closed convex cone and $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq_K^l$. Moreover, assume that $\overline{y} \in F(\overline{x})$ is a strongly minimal point of $F(\overline{x})$. Suppose that K is (SNC) at 0. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^* F(\overline{x}, \overline{y})(y^*).$$

Proof. The assertion can be obtained from Proposition 3.10 and Corollary 4.8. □

Remark 4.13. Observe that the condition "K is (SNC) at 0" in Theorem 4.12 can be replaced by "F^{-1} is (PSNC) at $(\overline{y}, \overline{x})$".

For the following results, we make use of the next corollary, which is found in [5, Theorem 3.11 (ii)] (see also [7, Corollary 4.15]).

Corollary 4.14. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a multifunction, K be a proper closed convex cone such that $\operatorname{int} K \neq \emptyset$ and $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ be a local minimizer for F with respect to K. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^* (\operatorname{cl} F)(\overline{x}, \overline{y})(y^*).$$

Theorem 4.15. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a multifunction, K be a proper closed convex cone such that $\operatorname{int} K \neq \emptyset$ and $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq_K^p$. Then there exists some $\overline{y} \in F(\overline{x})$ and $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^* (\operatorname{cl} F)(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Proposition 3.7 and Corollary 4.14. □

Theorem 4.16. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a multifunction, K be a proper closed convex cone such that $\operatorname{int} K \neq \emptyset$ and $\overline{x} \in X$ be a local minimal point w.r.t. $\preceq_K^l$. Moreover, assume that $\overline{y} \in F(\overline{x})$ is a strongly minimal point of $F(\overline{x})$. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^* (\operatorname{cl} F)(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Proposition 3.10 and Corollary 4.14. □

5. Application to Uncertain Optimization

Dealing with uncertainty in vector optimization is very important in many applications. On the one hand side, most real world optimization problems are contaminated with uncertain data, especially traffic optimization, scheduling problems, portfolio optimization, network flow and network design problems. On the other hand side, many real world optimization problems require the minimization of multiple conflicting objectives (see [16]), e.g. the maximization of the expected return versus the minimization of risk in portfolio optimization, the minimization of production time versus the minimization of the cost of manufacturing equipment, or the maximization of tumor control versus the minimization of normal tissue complication in radiotherapy treatment design.

First, we formulate a scalar optimization problem with uncertainties. Throughout this section, let $\mathcal{U} \subseteq \mathbb{R}^m$ be the uncertainty set reflecting the potential scenarios that may occur. One could think of $\xi \in \mathcal{U}$ being real numbers or real vectors. Furthermore, let X be a Banach space, $f : X \times \mathcal{U} \to \mathbb{R}$. Then an **uncertain scalar optimization problem** (uncertain OP) $P(\mathcal{U})$ is defined as a family of optimization problems

$$(P(\xi), \xi \in \mathcal{U}), \tag{5.1}$$

where for a given $\xi \in \mathcal{U}$ the optimization problem $(P(\xi))$ is given by

$$(P(\xi)) \qquad \begin{aligned} &f(x, \xi) \to \min \\ &\text{s.t. } x \in X. \end{aligned}$$

At the time the uncertain OP (5.1) has to be solved, it is not known which value $\xi \in \mathcal{U}$ is going to be realized.

In deterministic vector optimization one studies the problem

$$\text{(VOP)} \qquad \operatorname{Min}(f(X), K)$$

with a vector-valued objective function $f : X \to Y$, Y a Banach space and a proper closed convex and pointed cone $K \subset Y$.

In the following we assume that the objective function f may depend on scenarios ξ which are unknown or uncertain. As in uncertain single objective optimization, for an uncertainty set $\mathcal{U} \subseteq \mathbb{R}^m$, an **uncertain vector-valued optimization problem** $P(\mathcal{U})$ is given as the family

$$(P(\xi), \xi \in \mathcal{U})$$

of vector-valued optimization problems

$$(P(\xi)) \qquad \operatorname{Min}(f(X, \xi), K)$$

with the objective function $f : X \times \mathcal{U} \to Y$, and the notation (for $\xi \in \mathcal{U}$)

$$f(X, \xi) := \{f(x, \xi) \mid x \in X\}.$$

We call $\xi \in \mathcal{U}$ a scenario and $(P(\xi))$ an instance of $P(\mathcal{U})$.

Given an uncertain vector-valued optimization problem $P(\mathcal{U})$, the same question as in single objective optimization arises, namely, how to evaluate feasible solutions $x \in X$. For each $x \in X$ the set of objective values of it is given by

$$F(x) := \{f(x,\xi) \mid \xi \in \mathcal{U}\} \subseteq Y. \tag{5.2}$$

Dealing with an uncertain vector-valued optimization problem $P(\mathcal{U})$ leads to the following set-valued optimization problem with an objective map $F : X \rightrightarrows Y$ given in (5.2) and an order relation $\preceq$:

$$(SP - \preceq) \qquad \preceq -\text{minimize} \quad F(x), \quad \text{subject to} \quad x \in X.$$

In the following we use the l-type set-relation $\preceq^l_Q$ with $Q = K \subset Y$ and $Q = \operatorname{int} K \subset Y$, respectively:

$$A \preceq^l_Q B :\Longleftrightarrow A + Q \supseteq B,$$

where $A, B \subset Y$ are arbitrarily chosen sets. If we are dealing with $Q = \operatorname{int} K$ we suppose $\operatorname{int} K \neq \emptyset$. Consider again the set-valued map F given by (5.2). Using this notation we derive the concept of $\preceq^l_Q$-robustness.

Definition 5.1. Given an uncertain multi-objective optimization problem $P(\mathcal{U})$, a solution $\overline{x} \in X$ is called $\preceq^l_Q$-robust (with $Q = K$ and $Q = \operatorname{int} K$) for $P(\mathcal{U})$, if there is a neighborhood U of $\overline{x}$ such that there does not exist any $x \in U \setminus \{\overline{x}\}$ such that

$$F(x) \preceq^l_Q F(\overline{x}).$$

A necessary optimality condition for $\preceq^l_{\operatorname{int} K}$-robust solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$ in the sense of Definition 5.1 is given in Corollary 5.2.

Corollary 5.2. *Let* $\operatorname{int} K \neq \emptyset$ *and let* $\overline{x} \in X$ *be* $\preceq^l_{\operatorname{int} K}$*-robust. Moreover, assume that* $F : X \rightrightarrows Y$ *is given by* (5.2) *and* $\overline{y} \in F(\overline{x})$ *is a strongly minimal point of* $F(\overline{x})$*. Then it holds*

$$D_B F(\overline{x}, \overline{y})(X) \cap (-\operatorname{int} K) = \emptyset.$$

Proof. The result immediately follows taking into account Theorem 4.3. $\square$

Remark 5.3. With reference to Remark 2.2, we note that the Bouligand derivative $D_B F(\overline{x}, \overline{y})$ exists if and only if $(\overline{x}, \overline{y}) \in \operatorname{cl} \operatorname{Gr} F$, since then, we have $T_B(\operatorname{Gr} F, (\overline{x}, \overline{y})) \neq \emptyset$.

By means of Corollary 5.2, we are now able to test whether a point $\overline{x}$ is $\preceq^l_{\operatorname{int} K}$-robust, assuming that the set $F(\overline{x})$ has the strongly minimal point $\overline{y}$. The Bouligand derivative is defined for $u \in X$ as

$$\begin{aligned} D_B F(\overline{x}, \overline{y})(u) := \big\{ & v \in Y | \exists (t_n) \downarrow 0, \exists (v_n) \in Y, (v_n) \to v, \\ & \forall n \in \mathbb{N} : \ \overline{y} + t_n v_n \in F(\overline{x} + t_n u_n) \big\} \\ = \big\{ & v \in Y | \exists (t_n) \downarrow 0, \exists (v_n) \in Y, (v_n) \to v, \\ & \forall n \in \mathbb{N} : \ \overline{y} + t_n v_n \in \{ f(\overline{x} + t_n u_n, \xi) | \xi \in \mathcal{U} \} \big\}. \end{aligned}$$

Hence, if for some (t_n), $(t_n) \downarrow 0$, $(u_n) \to u$, $u \in X$ and $\xi \in \mathcal{U}$,

$$v := \lim_{n\to\infty} v_n := \lim_{n\to\infty} \frac{f(\overline{x} + t_n u_n, \xi) - \overline{y}}{t_n} \in -\operatorname{int} K,$$

then $\overline{x}$ is not $\preceq^l_{\operatorname{int} K}$-robust and can be excluded from further study.

Necessary optimality conditions for $\preceq^l_K$-robust solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$ in the sense of Definition 5.1 are given in Corollaries 5.4 and 5.6.

Corollary 5.4. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ given by* (5.2) *be a closed-graph multifunction, K be a proper closed convex cone and $\overline{x} \in X$ be $\preceq^l_K$-robust. Moreover, assume that $\overline{y} \in F(\overline{x})$ is a strongly minimal point of $F(\overline{x})$. Suppose that K is (SNC) at* 0*. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\overline{x}, \overline{y})(y^*).$$

Proof. The assertion can be obtained from Theorem 4.12. □

Remark 5.5. Observe that the condition "K is (SNC) at 0" in Corollary 5.4 can be replaced by "F^{-1} is (PSNC) at $(\overline{y}, \overline{x})$".

Corollary 5.6. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a multifunction given by* (5.2)*, K be a proper closed convex cone such that* $\operatorname{int} K \neq \emptyset$ *and $\overline{x} \in X$ be $\preceq^l_K$-robust. Moreover, assume that $\overline{y} \in F(\overline{x})$ is a strongly minimal point of $F(\overline{x})$. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*(\operatorname{cl} F)(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Theorem 4.16. □

Now we use the p-type set-relation $\preceq^p_Q$ with $Q = K$ and $Q = \operatorname{int} K$, respectively:

$$A \preceq^l_Q B :\Longleftrightarrow \exists a \in A, \exists b \in B : a \in b - Q,$$

where $A, B \subset Y$ are arbitrarily chosen sets. If we are dealing with $Q = \operatorname{int} K$ we suppose $\operatorname{int} K \neq \emptyset$. We now derive the concept of $\preceq^p_Q$-robustness using the set-valued map F given by (5.2).

Definition 5.7. Given an uncertain multi-objective optimization problem $P(\mathcal{U})$, a solution $\overline{x} \in X$ is called $\preceq^p_Q$-robust (with $Q = K$ and $Q = \operatorname{int} K$) for $P(\mathcal{U})$, if there is a neighborhood U of $\overline{x}$ such that there does not exist any $x \in U \setminus \{\overline{x}\}$ such that

$$F(x) \preceq^p_Q F(\overline{x}).$$

Necessary optimality conditions for $\preceq^p_{\operatorname{int} K}$-robust solutions of the uncertain vector-valued optimization problem $P(\mathcal{U})$ in terms of Definition 5.7 are presented in Corollary 5.8.

Corollary 5.8. *Let* $\operatorname{int} K \neq \emptyset$ *and let $\overline{x} \in X$ be $\preceq^p_{\operatorname{int} K}$-robust. Then there exists some $\overline{y} \in F(\overline{x})$ s.t.*

$$D_B F(\overline{x}, \overline{y})(X) \cap (-\operatorname{int} K) = \emptyset.$$

Proof. The result follows from Theorem 4.1. □

Based on Corollary 5.8, we are now able to test whether a point $\overline{x}$ is $\preceq^p_{\operatorname{int} K}$-robust. If for some (t_n), $(t_n) \downarrow 0$, $(u_n) \to u$, $u \in X$ and $\xi \in \mathcal{U}$,

$$v := \lim_{n\to\infty} v_n := \lim_{n\to\infty} \frac{f(\overline{x} + t_n u_n, \xi) - \overline{y}}{t_n} \in -\operatorname{int} K,$$

then $\overline{x}$ is not $\preceq^p_{\operatorname{int} K}$-robust and can be excluded from further investigation.

Necessary optimality conditions for $\preceq^p_K$-robust solutions in terms of the normal coderivative by Mordukhovich (see Definition 2.5) are presented in Corollaries 5.9 and 5.11.

Corollary 5.9. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a closed-graph multifunction given by* (5.2)*, K be a proper closed convex cone and $\overline{x} \in X$ be $\preceq^p_K$-robust. Suppose that K is (SNC) at* 0*. Then there exists $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*F(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Theorem 4.10. □

Remark 5.10. Observe that the condition "K is (SNC) at 0" in Corollary 5.9 can be replaced by "F^{-1} is (PSNC) at $(\overline{y}, \overline{x})$".

Corollary 5.11. *Let X, Y be Asplund spaces, $F : X \rightrightarrows Y$ be a multifunction given by* (5.2)*, K be a proper closed convex cone such that $\operatorname{int} K \neq \emptyset$ and $\overline{x} \in X$ be $\preceq^p_K$-robust. Then there exists some $\overline{y} \in F(\overline{x})$ and $y^* \in K^+ \setminus \{0\}$ such that*

$$0 \in D^*(\operatorname{cl} F)(\overline{x}, \overline{y})(y^*).$$

Proof. The result follows from Theorem 4.15. □

6. Conclusions

In our paper, we derived necessary optimality conditions for optimal solutions of set-valued optimization problems based on a set approach in terms of Bouligand and Mordukhovich generalized differentiation objects. Especially, we have shown our results for optimal solutions of set-valued problems with respect to the possibly and lower set less order relation. Moreover, it was possible to apply our results to vector optimization problems that are contaminated by uncertain data. The necessary optimality conditions could be useful for deriving numerical procedures for solving set-valued optimization problems or to generate robust solutions of uncertain vector optimization problems.

In a forthcoming paper we will show corresponding results for set-valued optimization problems with variable domination structure.

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