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ON GENERALIZED-CONVEX CONSTRAINED MULTI-OBJECTIVE OPTIMIZATION

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ABSTRACT. In this paper, we consider multi-objective optimization problems involving convex and nonconvex constraints, where the objective function is acting between a real linear topological pre-image space and a finite dimensional image space. The vector-valued objective function of the considered multi-objective optimization problem is assumed to be componentwise generalized-convex (e.g., semi-strictly quasi-convex or quasi-convex). For these problems with a not necessarily convex feasible set, we show that the set of efficient solutions can be computed completely using two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set in both problems. This means that it is possible to solve a problem with a nonconvex feasible set by solving two problems with convex feasible sets and to apply corresponding methods. Our approach relies on the fact that the original feasible set can be described using level sets of a certain scalar function (a kind of penalization function). At the end of the paper, we apply our approach to problems where the constraints are given by a system of inequalities with a finite number of constraint functions.

1. INTRODUCTION

Convexity plays a crucial role in optimization theory (see, e.g., the books of convex analysis by Hiriart-Urruty [12], Rockafellar [25] and Zălinescu [28]). In the last decades several new classes of functions are obtained by preserving several fundamental properties of convex functions, for instance, the following classes of generalized-convex functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (see, e.g., Cambini and Martein [3] and Popovici [22] for more details):

- *Quasi-convex functions:* The level sets of f are convex for each level (i.e., the set of minimal solutions of f on \mathbb{R}^n is a convex set in \mathbb{R}^n).
- *Semi-strictly quasi-convex functions:* Each local minimum point of f on \mathbb{R}^n is also a global minimum point.

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- *Explicitly quasi-convex functions*: Each local maximum point of f on \mathbb{R}^n is either a global minimum point.
- *Pseudo-convex functions* (e.g., in the sense of Mäkelä *et al.* [16]): 0 belongs to the Clarke subdifferential of f at a point $x \in \mathbb{R}^n$ if and only if x is global minimum point of f on \mathbb{R}^n .

Of course, as generalization of convexity, we have that every convex function is quasi-convex, semi-strictly quasi-convex and pseudo-convex. Generalized convexity assumptions appear in several branches of applications, e.g., production theory, utility theory or location theory (see, e.g., [3, Sec. 2.4]).

Since the area of multi-objective optimization has gained more and more interest, many authors studied generalized-convexity in a multi-objective optimization setting (see, e.g., Günther and Tammer [10], Mäkelä *et al.* [16], Malivert and Boissard [17], Popovici [19, 20, 21, 22], Puerto and Rodríguez-Chía [23]). In multi-objective optimization (see, e.g., the books by Ehrgott [5], Göpfert *et al.* [9], Jahn [13]), one considers optimization problems with several conflicting objective functions. Depending on the application in practice, these problems often involve certain constraints.

The aim of this paper is to study the relationships between general multi-objective optimization problems involving not necessarily convex constraints and multi-objective optimization problems involving convex constraints. In the literature, there exist techniques for solving different classes of constrained multi-objective optimization problems using corresponding unconstrained problems with an objective function that involves certain penalization terms in the component functions (see, e.g., Apetrii *et al.* [1] and Ye [26]), and, respectively, additional penalization functions (see, e.g., Durea *et al.* [4], Günther and Tammer [10], Klamroth and Tind [14]).

In the paper by Günther and Tammer [10], multi-objective optimization problems with convex constraints in finite dimensional spaces are considered and a certain gauge distance function is used as an additional penalization function. Now, we will extend and generalize the results in [10] to problems with nonconvex constraints and a real topological linear preimage space. In our approach, the vector-valued objective function of the considered multi-objective optimization problems is assumed to be componentwise generalized-convex (e.g., semi-strictly quasi-convex or quasi-convex). We show that the set of efficient solutions of a multi-objective optimization problem involving a nonempty closed (not necessarily convex) feasible set, can be computed completely using two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original (not necessarily convex) feasible set in both problems. This means, we can apply methods (see Robinson [24] for the scalar case) that use the special structure of convex feasible sets for solving both multi-objective problems with convex feasible sets in order to solve the original multi-objective optimization problem with a nonconvex feasible set. Our approach relies on

the fact that the original feasible set can be described using level sets of a certain scalar function (a kind of penalization function).

The paper is organized as follows:

In Section 2, we formulate the constrained multi-objective optimization problem considered in this paper, we introduce solution concepts and recall results about generalized-convex and semi-continuous functions. Moreover, we present an extended multi-objective optimization where we add one additional objective function (a kind of penalization function) to the objective functions given in the original multi-objective problem. We discuss useful properties of such penalization functions.

In Section 3, we present the main results of our paper, the relationships between the original multi-objective optimization problem involving a nonempty closed (not necessarily convex) feasible set, and two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set.

Furthermore, in Section 4 we apply our approach to problems where the constraints are given by a system of inequalities with a finite number of constraint functions.

Section 5 contains some concluding remarks.

2. PRELIMINARIES

Throughout this article, let \mathcal{V} be a real topological linear space. Sometimes we assume additionally that \mathcal{V} is a normed space equipped with the norm $\|\cdot\| : \mathcal{V} \rightarrow \mathbb{R}$. In this case the topology of \mathcal{V} should be produced by the metric induced by the norm $\|\cdot\|$. Moreover, let the q -dimensional normed Euclidean space denoted by \mathbb{R}^q , where $q \in \mathbb{N}$. For a nonempty set $\Omega \subseteq \mathcal{V}$, the expressions $\text{cl}\Omega$, $\text{bd}\Omega$, $\text{int}\Omega$ stand for the standard notions of closure, boundary, interior of Ω , respectively. The cardinality of the set Ω is denoted by $\text{card}(\Omega)$. The closed line segment $[a, b]$ between two points $a, b \in \mathcal{V}$ is defined by $[a, b] := \{(1 - \lambda) \cdot a + \lambda \cdot b \mid \lambda \in [0, 1]\}$. Moreover, we define $(a, b) := [a, b] \setminus \{a, b\}$, $[a, b) := [a, b] \setminus \{b\}$ and $(a, b] := [a, b] \setminus \{a\}$.

Let $\Omega \subseteq \mathcal{V}$. Considering a metric $d : \Omega \times \Omega \rightarrow \mathbb{R}$, we define the open ball around $x^0 \in \mathcal{V}$ with radius $\varepsilon > 0$ by

$$B_d(x^0, \varepsilon) := \{x^1 \in \mathcal{V} \mid d(x^0, x^1) < \varepsilon\}.$$

Moreover, the closed ball around $x^0 \in \mathcal{V}$ with radius $\varepsilon > 0$ is given as

$$\overline{B}_d(x^0, \varepsilon) := \{x^1 \in \mathcal{V} \mid d(x^0, x^1) \leq \varepsilon\}.$$

For simplicity, if d is induced by a norm $\|\cdot\|$ we write $B_{\|\cdot\|}(x^0, \varepsilon)$ and $\overline{B}_{\|\cdot\|}(x^0, \varepsilon)$.

The core (algebraic interior) of a nonempty set Ω is given by

$$\text{cor}\Omega := \{x^0 \in \Omega \mid \forall v \in \mathcal{V} \exists \delta > 0 : x^0 + [0, \delta] \cdot v \subseteq \Omega\}.$$

The following statements given in the next lemma are well-known (for more details, see Jahn [13] or Popovici [22]).

Lemma 2.1. *Let Ω be a nonempty set in a real topological linear space \mathcal{V} . Then the following statements hold:*

- 1°. *It holds $\text{int } \Omega \subseteq \text{cor } \Omega$.*
- 2°. *If $\text{int } \Omega \neq \emptyset$ and Ω is convex, then $\text{int } \Omega = \text{cor } \Omega$.*

In the proofs of Theorem 2.19 and Lemma 2.41, we will use the following property for interior points of a nonempty set Ω in a real normed space \mathcal{V} .

Lemma 2.2. *Let Ω be a nonempty set with $\text{int } \Omega \neq \emptyset$ in a real normed space $(\mathcal{V}, \|\cdot\|)$. Consider $x^0 \in \text{int } \Omega$, i.e., it exists $\varepsilon > 0$ such that $B_{\|\cdot\|}(x^0, \varepsilon) \subseteq \Omega$. Then for every $v \in \mathcal{V}$ with $\|v\| = 1$, and every $\delta \in (0, \varepsilon)$, we have*

$$[x^0 - \delta \cdot v, x^0 + \delta \cdot v] \subseteq B_{\|\cdot\|}(x^0, \varepsilon) \subseteq \Omega.$$

Remark 2.3. 1°. Note that the statements given in Lemma 2.1 and Lemma 2.2 are not true in general metric spaces. Consider the metric space (\mathbb{R}^2, d) , where $d: \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$ represents the discrete metric on \mathbb{R}^2 that is defined by $d(x, y) = 1$ for all $x, y \in \mathbb{R}^2$ with $x \neq y$ and $d(x, y) = 0$ for $x = y$. The feasible set is given by $\Omega := [-1, 1] \times [-1, 1]$. Now, it can easily be seen that the point $x^0 := (1, 1)$ belongs to the interior of Ω , since we have $B_d(x^0, \varepsilon) = \{x^0\} \subseteq \Omega$ for $\varepsilon \in (0, 1)$. However, for $v := \frac{x^1 - x^0}{\|x^1 - x^0\|}$ with $x^1 := (2, 2) \neq x^0$, we have $x^0 + \delta \cdot v \in \mathbb{R}^2 \setminus \Omega$ for all $\delta > 0$.

2°. It is important to note that the metric space (\mathbb{R}^2, d) with the discrete metric d is a topological space (considering the discrete metric topology associated with d) but not a real topological linear space as well as not a real normed space (d is not derived from a norm).

3°. If \mathcal{V} is a real linear space and d is a metric on \mathcal{V} that is invariant with respect to translation as well as homogeneous, then $d(\cdot, 0) =: \|\cdot\|: \mathcal{V} \rightarrow \mathbb{R}$ defines a norm on \mathcal{V} .

For notational convenience, for any $m \in \mathbb{N}$, we introduce the index set

$$I_m := \{1, \dots, m\}.$$

2.1. The original multi-objective optimization problem (\mathcal{P}_X) .

In the paper, we consider a multi-objective optimization problem involving m objective functions $f_1, \dots, f_m: \mathcal{V} \rightarrow \mathbb{R}$ and a nonempty (not necessarily convex) feasible set $X \subseteq \mathcal{V}$ in the topological linear space \mathcal{V} :

$$(\mathcal{P}_X) \quad f(x) := \begin{pmatrix} f_1(x) \\ \dots \\ f_m(x) \end{pmatrix} \rightarrow \underset{x \in X}{\text{v-min}}.$$

Let the set $f[X] := \{f(x) \in \mathbb{R}^m \mid x \in X\}$ be given as the image set of f over X .

For the vector-valued minimization in (\mathcal{P}_X) , we are using the concept of Pareto efficiency with respect to the natural ordering cone \mathbb{R}_+^m .

Definition 2.4. The set of *Pareto efficient solutions* of problem (\mathcal{P}_X) with respect to \mathbb{R}_+^m is defined by

$$\text{Eff}(X | f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (\mathbb{R}_+^m \setminus \{0\})) = \emptyset\},$$

while that of *weakly Pareto efficient solutions* is given by

$$\text{WEff}(X | f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (\text{int } \mathbb{R}_+^m)) = \emptyset\}.$$

The set of *strictly Pareto efficient solutions* is defined by

$$\text{SEff}(X | f) := \{x^0 \in \text{Eff}(X | f) \mid \text{card}(\{x \in X \mid f(x) = f(x^0)\}) = 1\}.$$

Remark 2.5. Obviously, we have

$$\text{SEff}(X | f) \subseteq \text{Eff}(X | f) \subseteq \text{WEff}(X | f).$$

Definition 2.6. Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a real-valued function, X be a nonempty set in \mathcal{V} and $s \in \mathbb{R}$. Then, the (strict) lower-level set and the level line of h to the level s are defined in the usual way by

$$L_{\sim}(X, h, s) := \{x \in X \mid h(x) \sim s\}$$

for all $\sim \in \{<, =, \leq\}$. Moreover, the upper-level set of h to the level s is defined by $L_{\geq}(X, h, s) := L_{\leq}(X, -h, -s)$, and the strict upper-level set of h to the level s is defined by $\bar{L}_{>}(X, h, s) := L_{<}(X, -h, -s)$.

Note, for any set Y with $X \subseteq Y \subseteq \mathcal{V}$, we have

$$L_{\sim}(X, h, s) = L_{\sim}(Y, h, s) \cap X.$$

Consider the function $f = (f_1, \dots, f_m)$ with $f_i : \mathcal{V} \rightarrow \mathbb{R}$ for all $i \in I_m$. Throughout, we define for a point $x^0 \in X$ the following intersections of (strict) lower-level sets or level lines

$$S_{\sim}(X, f, x^0) := \bigcap_{i \in I_m} L_{\sim}(X, f_i, f_i(x^0))$$

for all $\sim \in \{<, =, \leq\}$.

Useful characterizations of (strictly, weakly) efficient, efficient solutions using level lines and (strict) lower-level sets (see, e.g., [5, Th. 2.30]) are given in the next lemma.

Lemma 2.7. Let $x^0 \in X$. Consider the vector-valued function $f = (f_1, \dots, f_m) : \mathcal{V} \rightarrow \mathbb{R}^m$. Then the following hold:

$$\begin{aligned} x^0 \in \text{Eff}(X | f) &\iff S_{\leq}(X, f, x^0) \subseteq S_{=}(X, f, x^0); \\ x^0 \in \text{WEff}(X | f) &\iff S_{<}(X, f, x^0) = \emptyset; \\ x^0 \in \text{SEff}(X | f) &\iff S_{\leq}(X, f, x^0) = \{x^0\}. \end{aligned}$$

2.2. Semi-continuity and generalized-convexity properties.

In order to operate with certain generalized-convexity and semi-continuity notions, for every pair $(x^0, x^1) \in \mathcal{V} \times \mathcal{V}$, we define a function $l_{x^0, x^1} : [0, 1] \rightarrow \mathcal{V}$ by

$$l_{x^0, x^1}(\lambda) := (1 - \lambda) \cdot x^0 + \lambda \cdot x^1 \quad \text{for all } \lambda \in [0, 1].$$

Next, we recall some definitions and facts about generalized-convex and semi-continuous functions along line segments (see, e.g., Cambini and Martein [3], Giorgi *et al.* [8], Popovici [22]).

Definition 2.8. Let $X \subseteq \mathcal{V}$ be a convex set. A function $h : \mathcal{V} \rightarrow \mathbb{R}$ is called

- *upper (lower) semi-continuous along line segments* on X , if the composition $(h \circ l_{x^0, x^1}) : [0, 1] \rightarrow \mathbb{R}$ is upper (lower) semi-continuous on $[0, 1]$ for all $x^0, x^1 \in X$.
- *convex* on X , if for all $x^0, x^1 \in X$ and for all $\lambda \in [0, 1]$ it holds $h(l_{x^0, x^1}(\lambda)) \leq (1 - \lambda) \cdot h(x^0) + \lambda \cdot h(x^1)$.
- *quasi-convex* on X , if for all $x^0, x^1 \in X$ and for all $\lambda \in [0, 1]$ it holds $h(l_{x^0, x^1}(\lambda)) \leq \max(h(x^0), h(x^1))$.
- *semi-strictly quasi-convex* on X , if for all $x^0, x^1 \in X$, $h(x^0) \neq h(x^1)$ and for all $\lambda \in (0, 1)$ it holds $h(l_{x^0, x^1}(\lambda)) < \max(h(x^0), h(x^1))$.
- *explicitly quasi-convex* on X , if h is both quasi-convex and semi-strictly quasi-convex on X .

A vector-valued function $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is called componentwise upper semi-continuous along line segments (lower semi-continuous along line segments, convex, quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex) on X , if f_i is upper semi-continuous along line segments (lower semi-continuous along line segments, convex, quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex) on X for all $i \in I_m$.

Remark 2.9. Each convex function is explicitly quasi-convex and upper semi-continuous along line segments on a convex set X . Moreover, a semi-strictly quasi-convex function which is lower semi-continuous along line segments is explicitly quasi-convex on a convex set X . A semi-strictly quasi-convex function which is upper semi-continuous along line segments must not be quasi-convex.

Quasi-convex functions are characterized by the convexity of its lower-level sets, as stated in the next lemma.

Lemma 2.10. Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a function and X be a convex set in \mathcal{V} . Then the following statements are equivalent:

- 1°. h is quasi-convex on X .
- 2°. $L_{<}(X, h, s)$ is convex for all $s \in \mathbb{R}$.
- 3°. $L_{<}(X, h, s)$ is convex for all $s \in \mathbb{R}$.

Next, we present a useful equivalent characterization of semi-strictly quasi-convexity using level sets and level lines.

Lemma 2.11. *Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a function and X be a convex set in \mathcal{V} . Then the following statements are equivalent:*

- 1°. h is semi-strictly quasi-convex on X .
- 2°. For all $s \in \mathbb{R}$, $x^0 \in L_=(X, h, s)$, $x^1 \in L_<(X, h, s)$, it holds $l_{x^0, x^1}(\lambda) \in L_<(X, h, s)$ for all $\lambda \in (0, 1]$.

The next lemma (see [19, Prop. 2] or [22, Prop. 2.1.2]) is important for the proofs of Lemmata 2.23 and 2.51.

Lemma 2.12. *Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a semi-strictly quasi-convex function on a nonempty convex subset X of \mathcal{V} . Then, for every pair $(x^0, x^1) \in X \times X$, we have*

$$\text{card}(L_>((0, 1), (h \circ l_{x^0, x^1}), \max(h(x^0), h(x^1)))) \leq 1.$$

In the following lemma we recall useful equivalent characterizations of upper and lower semi-continuity.

Lemma 2.13. *Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be a function and X be a nonempty closed set in \mathcal{V} . Then the following statements are equivalent:*

- 1°. h is upper (lower) semi-continuous on X .
- 2°. $L_{\geq}(X, h, s)$ ($L_{\leq}(X, h, s)$) is closed for all $s \in \mathbb{R}$.

Proof. A proof for the case $X = \mathcal{V}$ can be found in [2, Prop. 2.5].

Let I_X be the indicator function concerning the set X , i.e., $I_X(x)$ is 0 for x in X and $+\infty$ otherwise. Since X is closed, we know that I_X is lower semi-continuous on \mathcal{V} (see [2, Cor. 2.7]). Then the following statements are equivalent (compare [29]):

- h is lower semi-continuous on X .
- $\tilde{h} := h + I_X$ is lower semi-continuous on \mathcal{V} .
- $L_{\leq}(\mathcal{V}, \tilde{h}, s)$ is closed for all $s \in \mathbb{R}$.
- $L_{\leq}(X, h, s)$ is closed for all $s \in \mathbb{R}$.

Note that we have $L_{\leq}(\mathcal{V}, \tilde{h}, s) = L_{\leq}(X, h, s)$ for every $s \in \mathbb{R}$ because of $\tilde{h}(x) = +\infty$ for all $x \notin X$.

Moreover, notice that h is upper semi-continuous on X if and only if $-h$ is lower semi-continuous on X . In view of $L_{<}(X, -h, s) = L_{\geq}(X, h, -s)$ for all $s \in \mathbb{R}$, the corresponding result for the case of upper semi-continuity follows immediately. \square

Remark 2.14. Note that the set X considered in Lemma 2.13 must be closed. This assumption is missing in [10, Lem. 5], and hence must be added. Only one result, namely [10, Th. 4], uses the characterization given in [10, Lem. 5]. In this result a closed set of the space \mathbb{R} is considered, consequently the result given in [10, Th. 4] still holds.

In Section 4, we are interested in considering the function defined by the maximum of a finite number of scalar functions $h_i : \mathbb{R}^n \rightarrow \mathbb{R}$, $i \in I_l$, $l \in \mathbb{N}$. Next, we present some important properties of this function:

Lemma 2.15. *Let a family of functions $h_i : \mathcal{V} \rightarrow \mathbb{R}$, $i \in I_l$, be given. Define the maximum of h_i , $i \in I_l$, by $h_{\max}(x) := \max_{i \in I_l} h_i(x)$ for all $x \in \mathcal{V}$. Suppose that X is a nonempty set in \mathcal{V} . Then, it holds that:*

- 1°. *Assume that X is closed. If h_i , $i \in I_l$, are lower semi-continuous on X , then h_{\max} is lower semi-continuous on X .*

Now, assume that X is additionally convex. Then the following statements hold:

- 2°. *If h_i , $i \in I_l$, are convex on X , then h_{\max} is convex on X .*
 3°. *If h_i , $i \in I_l$, are quasi-convex on X , then h_{\max} is quasi-convex on X .*
 4°. *If h_i , $i \in I_l$, are semi-strictly quasi-convex on X , then h_{\max} must not be semi-strictly quasi-convex on X .*

Proof. 1°. Due to

$$L_{\leq}(X, h_{\max}, s) = \{x \in X \mid h_{\max}(x) \leq s\} = \bigcap_{i \in I_l} L_{\leq}(X, h_i, s)$$

and the closedness of $L_{\leq}(X, h_i, s)$ for all $i \in I_l$ and all $s \in \mathbb{R}$ (compare Lemma 2.13), we obtain the lower semi-continuity of h_{\max} on X .

- 2°. Since convex functions are characterized by the convexity of their epigraphs, the assertion follows immediately by

$$\text{epi } h_{\max} = \{(x, r) \in X \times \mathbb{R} \mid h_{\max}(x) \leq r\} = \bigcap_{i \in I_l} \text{epi } h_i$$

and the convexity of $\text{epi } h_i = \{(x, r) \in X \times \mathbb{R} \mid h_i(x) \leq r\}$ for all $i \in I_l$.

- 3°. In view of Lemma 2.10, we get the assertion by

$$L_{\leq}(X, h_{\max}, s) = \{x \in X \mid h_{\max}(x) \leq s\} = \bigcap_{i \in I_l} L_{\leq}(X, h_i, s)$$

and the convexity of $L_{\leq}(X, h_i, s)$ for all $i \in I_l$ and all $s \in \mathbb{R}$.

- 4°. Consider two functions $h_i : \mathbb{R} \rightarrow \mathbb{R}$, $i \in I_2$, defined by $h_i(x) := 0$ for all $x \neq i$ and $h_i(i) = 1$. Then, it can easily be seen that h_1 and h_2 are semi-strictly quasi-convex on $X := \mathbb{R}$. Now, the function h_{\max} is given by

$$h_{\max}(x) = \begin{cases} 0 & \text{for all } x \in \mathbb{R} \setminus \{1, 2\}, \\ 1 & \text{for all } x \in \{1, 2\}. \end{cases}$$

Since $h_{\max}(0) = 0 < 1 = h_{\max}(1) = h_{\max}(2)$, we immediately get that h_{\max} is not semi-strictly quasi-convex on \mathbb{R} . □

2.3. Local generalized-convexity.

In this section, let \mathcal{V} be a normed space equipped with the norm $\|\cdot\|$.

Next, we introduce corresponding local versions of semi-strict quasi-convexity and of quasi-convexity for a function $h : \mathcal{V} \rightarrow \mathbb{R}$. These definitions will be used in Lemma 2.41, Corollary 2.46 and Corollary 2.49..

Definition 2.16. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $X \subseteq \mathcal{V}$ be an open set. A function $h : \mathcal{V} \rightarrow \mathbb{R}$ is called *locally semi-strictly quasi-convex* (*locally quasi-convex*) at a point $x^0 \in X$, if there exists $\varepsilon > 0$ such that h is semi-strictly quasi-convex (quasi-convex) on $B_{\|\cdot\|}(x^0, \varepsilon)$.

A function $h : \mathcal{V} \rightarrow \mathbb{R}$ is called *locally explicitly quasi-convex* at $x^0 \in X$, if it is both locally semi-strictly quasi-convex and locally quasi-convex at $x^0 \in X$.

Remark 2.17. Note that the set $B_{\|\cdot\|}(x^0, \varepsilon)$ is open and convex in a normed space $(\mathcal{V}, \|\cdot\|)$. We defined local semi-strict quasi-convexity and local quasi-convexity on open but not necessarily convex sets. Clearly, if h is locally semi-strictly quasi-convex at $x^0 \in X$ and lower semi-continuous on $B_{\|\cdot\|}(x^0, \varepsilon)$, then h is locally quasi-convex at $x^0 \in X$.

Lemma 2.18. *Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $X \subseteq \mathcal{V}$ be an open convex set. A function $h : \mathcal{V} \rightarrow \mathbb{R}$, which is semi-strictly quasi-convex (quasi-convex) on the set X , is locally semi-strictly quasi-convex (locally quasi-convex) at every point $x^0 \in X$. The reverse implications are not true in general.*

Proof. Obviously, the semi-strictly quasi-convexity (quasi-convexity) of h on X implies local semi-strict quasi-convexity (local quasi-convexity) at each point $x^0 \in X$. However, the reverse implications are not true, as shown in the next examples:

- Let us consider the function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$h(x) := \begin{cases} 0 & \text{for all } x \in \mathbb{R} \setminus \{1, 2\}, \\ 1 & \text{for all } x \in \{1, 2\}. \end{cases}$$

It can easily be seen that h is not semi-strictly quasi-convex on $X := (0, 3)$, but semi-strictly quasi-convex on $B_{\|\cdot\|}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for every $x \in X$ and with $\varepsilon \in (0, \min\{|x - 0|, |x - 3|, 1\})$.

- Consider the function $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$h(x) := \begin{cases} x + 1 & \text{for all } x < -1, \\ 0 & \text{for all } x \in [-1, 1], \\ x - 1 & \text{for all } x > 1. \end{cases}$$

Note that h is not quasi-convex on $X := (-2, 2)$, but quasi-convex on $B_{\|\cdot\|}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for every $x \in X$ and with $\varepsilon \in (0, \min\{|x + 2|, |x - 2|\})$.

□

Theorem 2.19. *Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $X \subseteq \mathcal{V}$ be an open convex set. Let $h : \mathcal{V} \rightarrow \mathbb{R}$ be upper semi-continuous along line segments on X . The function h is semi-strictly quasi-convex on the set X , if the following statements are fulfilled:*

- 1°. h is locally explicitly quasi-convex at each point $x^0 \in X$.
- 2°. Every local minimum of h is also global for each restriction on a line segment in X .

Proof. Assume that h is not semi-strictly quasi-convex on the set X . By Lemma 2.11, there exist $s \in \mathbb{R}$, $x^0 \in L_=(X, h, s)$ and $x^1 \in L_<(X, h, s)$ such that $x^\lambda := l_{x^0, x^1}(\lambda) \in L_\geq(X, h, s)$ for some $\lambda \in (0, 1)$. Since $h \circ l_{x^0, x^1} : [0, 1] \rightarrow \mathbb{R}$ is upper semi-continuous on $[0, 1]$, we can choose

$$\lambda_{\max} \in \left\{ \bar{\lambda} \in (0, 1) \mid h(l_{x^0, x^1}(\bar{\lambda})) = \max_{\lambda \in [0, 1]} h(l_{x^0, x^1}(\lambda)) \right\}$$

by a well-known Weierstrass type existence theorem. Now, put $x^2 := x^{\lambda_{\max}}$. Consider $\varepsilon > 0$ such that h is explicitly quasi-convex on $B_\varepsilon := B_{\|\cdot\|}(x^2, \varepsilon)$. Now, by Lemma 2.2, we get that $B_\delta := (x^2 - \delta \cdot v, x^2 + \delta \cdot v) \subseteq B_\varepsilon$ holds for $v := \frac{x^1 - x^0}{\|x^1 - x^0\|}$ (note that $x^1 \neq x^0$) and $\delta \in (0, \varepsilon)$. Define

$$\bar{\delta} := \min(\delta, \|x^2 - x^0\|, \|x^2 - x^1\|) \in (0, \delta].$$

Since $h(x^1) < s \leq h(x^2)$, we know by our assumptions that x^2 can not be a local minimum point of h on the line segment $[x^0, x^1]$. Hence, there exists $x^3 \in B_{\bar{\delta}} := (x^2 - \bar{\delta} \cdot v, x^2 + \bar{\delta} \cdot v) \setminus \{x^2\} \subseteq [x^0, x^1]$ with $h(x^3) < h(x^2)$. Obviously, we have $B_{\bar{\delta}} \subseteq B_\delta \subseteq B_\varepsilon$. Define a point $x^4 := x^2 + (x^2 - x^3) \in B_{\bar{\delta}}$. Note that $h(x^4) \leq h(x^2)$ holds. Now, we consider two cases:

Case 1: If $h(x^3) = h(x^4)$, then $x^2 \in (x^3, x^4) \subseteq L_<(B_\varepsilon, h, h(x^3))$ by the quasi-convexity of h on B_ε , a contradiction to $h(x^3) < h(x^2)$.

Case 2: If $h(x^3) < h(x^4) \leq h(x^2)$, then $x^2 \in (x^3, x^4) \subseteq L_<(B_\varepsilon, h, h(x^4))$ by the semi-strictly quasi-convexity of h on B_ε , a contradiction to $h(x^4) \leq h(x^2)$.

Case 3: The case $h(x^4) < h(x^3) \leq h(x^2)$ is analogous to Case 2.

We get that h is semi-strictly quasi-convex on the set X . \square

Corollary 2.20. *Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $X \subseteq \mathcal{V}$ be an open convex set, and let $h : \mathcal{V} \rightarrow \mathbb{R}$ be continuous along line segments on X . Then h is semi-strictly quasi-convex on X if and only if the following two statements hold:*

- 1°. h is locally semi-strictly quasi-convex at each point $x^0 \in X$.
- 2°. Every local minimum of h is also global for each restriction on a line segment in X .

Proof. Statement 1° together with the lower semi-continuity along line segments of h on X , imply the local explicit quasi-convexity of h at each point $x^0 \in X$. Hence, by Theorem 2.19, we know that both statements 1° and

2° imply the semi-strict quasi-convexity of h on X , taking into account the upper semi-continuity along line segments of h on X .

Now, we prove the reverse implication. The validity of statement 1° follows by Lemma 2.18. It can easily be seen that statement 2° holds under semi-strict quasi-convexity of h on X . \square

Remark 2.21. By [3, Th. 2.3.4], we know that statement 1° of Corollary 2.20 can be replaced by the quasi-convexity of h on X . Note that [3, Th. 2.3.4] (given for $\mathcal{V} = \mathbb{R}^n$) holds in general normed spaces too.

Corollary 2.22. *Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Let $X \subseteq \mathcal{V}$ be an open convex set, and let $h : \mathcal{V} \rightarrow \mathbb{R}$ be continuous along line segments on X . The function h is explicitly quasi-convex on X if and only if the following two statements hold:*

- 1°. h is locally explicitly quasi-convex at each point $x^0 \in X$.
- 2°. Every local minimum of h is also global for each restriction on a line segment in X .

2.4. Relationships between the problems (\mathcal{P}_X) and (\mathcal{P}_Y) .

Analogously to the original optimization problem (\mathcal{P}_X) we define the problem (\mathcal{P}_Y) , where Y is a convex subset of \mathcal{V} with $X \subseteq Y$, and \mathcal{V} is a topological linear space. Our aim is to specify some relationships between the problems (\mathcal{P}_X) and (\mathcal{P}_Y) .

In the next lemma we present some useful relationships between the problems (\mathcal{P}_X) and (\mathcal{P}_Y) . These relationships generalize the corresponding results derived in the paper by Günther and Tammer [10], where $\mathcal{V} = Y = \mathbb{R}^n$ and the topological interior of X instead of the algebraic interior of X is considered.

Lemma 2.23. *Let X be a nonempty subset of the convex set Y such that $X \subseteq Y \subseteq \mathcal{V}$. Then the following statements are true:*

- 1°. It hold

$$\begin{aligned} X \cap \text{Eff}(Y | f) &\subseteq \text{Eff}(X | f); \\ X \cap \text{WEff}(Y | f) &\subseteq \text{WEff}(X | f); \\ X \cap \text{SEff}(Y | f) &\subseteq \text{SEff}(X | f). \end{aligned}$$

- 2°. Assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . Then it hold

$$\begin{aligned} (\text{cor } X) \setminus \text{Eff}(Y | f) &\subseteq (\text{cor } X) \setminus \text{Eff}(X | f); \\ (\text{cor } X) \setminus \text{WEff}(Y | f) &\subseteq (\text{cor } X) \setminus \text{WEff}(X | f). \end{aligned}$$

- 3°. If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on Y , then

$$(\text{cor } X) \setminus \text{SEff}(Y | f) \subseteq (\text{cor } X) \setminus \text{SEff}(X | f).$$

Proof. 1°. Follows easily by Lemma 2.7.

2°. Let us prove the first inclusion. Consider $x^0 \in (\text{cor } X) \setminus \text{Eff}(Y | f)$. Since $x^0 \notin \text{Eff}(Y | f)$, there exists $x^1 \in L_{<}(Y, f_j, f_j(x^0)) \cap S_{\leq}(Y, f, x^0)$ for some $j \in I_m$. Let us consider the following two sets of indices

$$\begin{aligned} I^< &:= \{i \in I_m \mid x^1 \in L_{<}(Y, f_i, f_i(x^0))\}, \\ I^= &:= \{i \in I_m \mid x^1 \in L_{=}(Y, f_i, f_i(x^0))\}. \end{aligned}$$

Of course, we know that $I^< \neq \emptyset$ and $I^= \cup I^< = I_m$ hold.

Clearly, for $x^1 \in X$, we get immediately $x^0 \in (\text{cor } X) \setminus \text{Eff}(X | f)$. Now, assume $x^1 \in Y \setminus X$. Since $x^0 \in \text{cor } X$, by Lemma 2.1, we get $x^0 + [0, \delta] \cdot v \subseteq X$ with $v := x^1 - x^0 \neq 0$ and for some $\delta > 0$. Obviously, since $x^1 \notin X$, we have $\delta \in (0, 1)$. More precisely, we have $[x^*, x^0] \subseteq X \cap (x^0, x^1)$ for $x^* := x^0 + \delta \cdot v$. Hence, for $\lambda^* := \delta \in (0, 1)$, we have $x^* = l_{x^0, x^1}(\lambda^*)$ and it holds $x^\lambda := l_{x^0, x^1}(\lambda) \in X \cap (x^0, x^1)$ for all $\lambda \in (0, \lambda^*]$. Let us consider two cases:

Case 1: Consider $i \in I^<$. The semi-strictly quasi-convexity of f_i on Y implies $x^\lambda \in L_{<}(Y, f_i, f_i(x^0))$ for all $\lambda \in (0, 1]$ by Lemma 2.11. Because of $x^\lambda \in X$ for all $\lambda \in (0, \lambda^*]$, we get $x^\lambda \in L_{<}(X, f_i, f_i(x^0))$ for all $\lambda \in (0, \lambda^*]$.

Case 2: Consider $i \in I^=$. This means that $f_i(x^1) = f_i(x^0)$. By Lemma 2.12 (with Y in the role of X), we infer that

$$\text{card}(L_{>}((0, 1), (f_i \circ l_{x^0, x^1}), f_i(x^0))) \leq 1.$$

In the case that $\text{card}(L_{>}((0, 1), (f_i \circ l_{x^0, x^1}), f_i(x^0))) = 1$ holds, we get that there exists $\lambda_i \in (0, 1)$ such that $f_i(l_{x^0, x^1}(\lambda_i)) > f_i(x^0)$. Otherwise we define $\lambda_i := 2 \cdot \lambda^*$.

For $\bar{\lambda} := \min(\lambda^*, 0.5 \cdot \min\{\lambda_i \mid i \in I^=\})$, it follows that $x^{\bar{\lambda}} \in L_{\leq}(X, f_i, f_i(x^0))$ for all $i \in I^=$ as well as $x^{\bar{\lambda}} \in L_{<}(X, f_i, f_i(x^0))$ for all $i \in I^<$. So, we get the desired statement $x^0 \in (\text{cor } X) \setminus \text{Eff}(X | f)$ by Lemma 2.7.

The proof of the second inclusion is analogous to the proof of the first inclusion in 2°. Note that $I^< = I_m$ and $I^= = \emptyset$ hold.

3°. Consider $x^0 \in (\text{cor } X) \setminus \text{SEff}(Y | f)$. Since $x^0 \notin \text{SEff}(Y | f)$, it exists $x^1 \in Y \setminus \{x^0\}$ such that $x^1 \in S_{\leq}(Y, f, x^0)$ holds. Of course, under the assumption that $x^1 \in X$, we immediately get $x^0 \in (\text{cor } X) \setminus \text{SEff}(X | f)$. Now, let us assume $x^1 \in Y \setminus X$. Analogously to the proof of statement 2° in this lemma, there exists $\lambda^* \in (0, 1)$ such that $x^\lambda := l_{x^0, x^1}(\lambda) \in X \cap (x^0, x^1)$ for all $\lambda \in (0, \lambda^*]$.

Let $i \in I_m$ and consider two cases:

Case 1: Let f_i be semi-strictly quasi-convex on Y . Analogously to the proof of statement 2° of this lemma, we get that there exists $\lambda_i \in (0, \lambda^*]$ with $x^\lambda \in L_{\leq}(X, f_i, f_i(x^0))$ for all $\lambda \in (0, \lambda_i]$.

Case 2: Let f_i be quasi-convex on Y . By Lemma 2.10, we conclude $[x^0, x^1] \subseteq L_{\leq}(Y, f_i, f_i(x^0))$ for $x^0, x^1 \in L_{\leq}(Y, f_i, f_i(x^0))$. We put $\lambda_i := \lambda^*$.

Hence, for $\bar{\lambda} := \min\{\lambda_i \mid i \in I_m\}$, it follows that $x^{\bar{\lambda}} \in S_{\leq}(X, f, x^0) \setminus \{x^0\}$. By Lemma 2.7, we get the desired statement $x^0 \in (\text{cor } X) \setminus \text{SEff}(X \mid f)$. \square

Remark 2.24. Note that the proof of Lemma 2.23 uses the ideas given in the corresponding results in [10]. Moreover, notice that we can not replace the semi-strictly quasi-convexity assumption with respect to f by a quasi-convexity assumption in 2° of Lemma 2.23 (see [10, Ex. 2] for the case $Y = \mathcal{V} = \mathbb{R}^n$ and \mathcal{V} is a normed space).

Corollary 2.25. *Let X be a nonempty subset of the convex set Y such that $X \subseteq Y \subseteq \mathcal{V}$. Then the following statements are true:*

1°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y , then we have*

$$X \cap \text{Eff}(Y \mid f) \subseteq \text{Eff}(X \mid f) \subseteq [X \cap \text{Eff}(Y \mid f)] \cup \text{bd } X;$$

$$X \cap \text{WEff}(Y \mid f) \subseteq \text{WEff}(X \mid f) \subseteq [X \cap \text{WEff}(Y \mid f)] \cup \text{bd } X.$$

2°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on Y , then it holds*

$$X \cap \text{SEff}(Y \mid f) \subseteq \text{SEff}(X \mid f) \subseteq [X \cap \text{SEff}(Y \mid f)] \cup \text{bd } X.$$

Corollary 2.26. *Let X be a nonempty open subset of the convex set Y such that $X \subseteq Y \subseteq \mathcal{V}$. Then the following statements are true:*

1°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y , then we have*

$$X \cap \text{Eff}(Y \mid f) = \text{Eff}(X \mid f);$$

$$X \cap \text{WEff}(Y \mid f) = \text{WEff}(X \mid f).$$

2°. *If $f : \mathcal{V} \rightarrow \mathbb{R}^m$ be componentwise semi-strictly quasi-convex or quasi-convex on Y , then it holds*

$$X \cap \text{SEff}(Y \mid f) = \text{SEff}(X \mid f).$$

2.5. The extended multi-objective optimization problem (\mathcal{P}_Y^E) .

Throughout, we suppose that the following assumption is fulfilled:

$$(2.1) \quad \left\{ \begin{array}{l} \text{Let } \mathcal{V} \text{ be a real topological linear space;} \\ \text{let } X \subseteq \mathcal{V} \text{ be a nonempty closed set with } X \neq \mathcal{V} \text{ and } \text{bd } X \neq \emptyset; \\ \text{let } Y \text{ be a convex set with } X \subseteq Y \subseteq \mathcal{V}. \end{array} \right.$$

Remark 2.27. Note that we have

$$(2.2) \quad \text{int } X \subseteq X \subseteq \text{cl } X = \text{int } X \cup \text{bd } X.$$

By the closedness of X , we know $X = \text{cl } X$, hence by (2.2) it follows that X is not open if and only if $\text{bd } X \neq \emptyset$. Moreover, for a nonopen set X , we immediately get $\emptyset \neq X \neq \mathcal{V}$ in a topological space \mathcal{V} .

If the space \mathcal{V} considered in (2.1) is a connected topological space, then the assumption $\text{bd } X \neq \emptyset$ can be removed in (2.1). This follows by the fact that a topological space \mathcal{V} is connected if and only if the only closed and open sets are the empty set and \mathcal{V} . Hence, the closedness of X and the assumption $\emptyset \neq X \neq \mathcal{V}$ imply that X is not open or equivalently, $\text{bd } X \neq \emptyset$.

For the case $\text{bd } X = \emptyset$ (hence X is open), we refer the reader to Corollary 2.26.

By adding one additional objective function $f_{m+1} : \mathcal{V} \rightarrow \mathbb{R}$ (a kind of penalization function) to the original objective function f of the problem (\mathcal{P}_Y) , we obtain an extended multi-objective optimization problem

$$(\mathcal{P}_Y^E) \quad f^E(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \\ f_{m+1}(x) \end{pmatrix} \rightarrow \underset{x \in Y}{\text{v-min}}.$$

Moreover, in certain results we need that the feasible set X can be described using a level set (level 0) of the function f_{m+1} :

$$(2.3) \quad X = L_{\leq}(Y, f_{m+1}, 0).$$

Remark 2.28. Let X be a nonempty closed feasible set with $X \neq \mathcal{V}$. In our approach it is important to choose a suitable function f_{m+1} (with certain useful properties) that fulfils the level set condition in (2.3). For instance, if the set X is convex, then it is appropriate to choose a function that is quasi-convex on the convex set Y , or even on the whole space \mathcal{V} (e.g., a certain gauge function in the case that $\text{int } X \neq \emptyset$; see Günther and Tammer [10]).

In the following we formulate some additional assumptions concerning the lower-level sets and level lines of the objective function f_{m+1} :

A1 If $x^0 \in \text{bd } X$, then

$$L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = X.$$

A2 If $x^0 \in \text{bd } X$, then

$$L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) = \text{bd } X.$$

A3 If $x^0 \in X$, then

$$L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) = L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = X.$$

Remark 2.29. Note that we have $\text{int } X = L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$ for all $x^0 \in \text{bd } X$ under both Assumptions A1 and A2. Under Assumption A3 it follows for the strict level set $L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset$ for all $x^0 \in X$. Moreover, if we assume $\text{int } X = \emptyset$, then Assumptions A1 and A2 together

are equivalent to Assumption A3. It is easy to see that Assumption A3 implies Assumption A1.

Now, we are interested in sufficient conditions such that

$$(2.4) \quad L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \subseteq X \quad \text{for all } x^0 \in X$$

holds.

Lemma 2.30. *Let (2.1) be satisfied. Condition (2.4) holds, if one of the following statements is fulfilled:*

- 1°. (2.3) holds.
- 2°. Assumptions A1 is fulfilled.

Proof. First, we assume that 1° holds. For all $x^0 \in X = L_{\leq}(Y, f_{m+1}, 0)$, we have $f_{m+1}(x^0) \leq 0$, hence (2.4) holds.

Assume that 2° holds. Let $x^1 \in \text{bd } X$. For all $x^0 \in X = L_{\leq}(Y, f_{m+1}, f_{m+1}(x^1))$, we have $f_{m+1}(x^0) \leq f_{m+1}(x^1)$. This means that (2.4) is true. \square

In the following, we are looking for conditions such that Assumptions A1 and A2 are fulfilled for the penalization function f_{m+1} .

Lemma 2.31. *Let (2.1) be satisfied. Consider an Assumption $A \in \{A1, A2, A3\}$. Let $h : f_{m+1}[Y] \rightarrow \mathbb{R}$ be a strict monoton increasing function. Moreover, assume that $h^{-1} : (h \circ f_{m+1})[Y] \rightarrow f_{m+1}[Y]$ is strict monoton increasing too. Then the function $f_{m+1} : \mathcal{V} \rightarrow \mathbb{R}$ fulfils Assumption A if and only if the function $\tilde{f}_{m+1} := h \circ f_{m+1} : \mathcal{V} \rightarrow \mathbb{R}$ fulfils Assumption A.*

Proof. For $x^0 \in X$, it holds

$$\begin{aligned} L_{\sim}(Y, \tilde{f}_{m+1}, \tilde{f}_{m+1}(x^0)) &= \{x \in Y \mid (h \circ f_{m+1})(x) \sim (h \circ f_{m+1})(x^0)\} \\ &= \{x \in Y \mid f_{m+1}(x) \sim f_{m+1}(x^0)\} \\ &= L_{\sim}(Y, f_{m+1}, f_{m+1}(x^0)) \end{aligned}$$

for every $\sim \in \{=, \leq\}$. In view of Assumptions A1, A2 and A3, we get the desired statements immediately. \square

Corollary 2.32. *Let (2.1) be satisfied. Consider an Assumption $A \in \{A1, A2, A3\}$. The function f_{m+1} fulfils Assumption A if and only if the function $f_{m+1} + c$ with $c \in \mathbb{R}$, fulfils Assumption A.*

Lemma 2.33. *Let (2.1) be satisfied. Consider an Assumption $A \in \{A1, A2, A3\}$. Assume that f_{m+1} fulfils the Assumption A for $Y = \mathcal{V}$. Then f_{m+1} fulfils Assumption A even for each set Y with $X \subseteq Y \subseteq \mathcal{V}$.*

Proof. Since $\text{bd } X \subseteq X \subseteq Y \subseteq \mathcal{V}$ and

$$L_{\sim}(Y, f_{m+1}, f_{m+1}(x^0)) = L_{\sim}(\mathcal{V}, f_{m+1}, f_{m+1}(x^0)) \cap Y$$

for all $\sim \in \{=, \leq\}$, the assertions follow immediately. \square

In the following results of this section we assume that the assumption

$$(2.5) \quad \left\{ \begin{array}{l} \text{Let (2.1) and (2.3) be fulfilled.} \end{array} \right.$$

is satisfied.

Remark 2.34. If f_{m+1} fulfils Assumptions A1 and A2, then the function \tilde{f}_{m+1} defined by $\tilde{f}_{m+1}(x) := f_{m+1}(x) - f_{m+1}(x^0)$ for all $x \in Y$ and for some $x^0 \in \text{bd } X$ satisfies the condition (2.3). Moreover, the function \tilde{f}_{m+1} fulfils Assumptions A1 and A2 too (see Corollary 2.32).

Lemma 2.35. *Let (2.5) be satisfied and let Y be closed. Assume that f_{m+1} fulfils Assumptions A1 and A2 concerning the set X . Consider the set $\tilde{X} := Y \setminus (\text{int } X) = L_{\geq}(Y, f_{m+1}, 0) \subseteq Y$, and assume that $\text{int } \tilde{X} = Y \setminus X$ holds. Then $\tilde{f}_{m+1} := -f_{m+1}$ fulfils Assumptions A1 and A2 concerning the set \tilde{X} (where \tilde{X} is in the role of X in the formulation of the Assumptions A1 and A2).*

Proof. We show that \tilde{f}_{m+1} fulfils Assumptions A1 and A2 concerning the set \tilde{X} :

Since $L_{<}(Y, f_{m+1}, 0) = \text{int } X$ we have $L_{\geq}(Y, f_{m+1}, 0) \cup (\mathcal{V} \setminus Y) = \mathcal{V} \setminus (\text{int } X)$, hence $L_{\leq}(Y, \tilde{f}_{m+1}, 0) = (\mathcal{V} \setminus (\text{int } X)) \setminus (\mathcal{V} \setminus Y)$. Moreover, due to $(\mathcal{V} \setminus (\text{int } X)) \setminus (\mathcal{V} \setminus Y) = Y \setminus (\text{int } X)$, we get

$$(2.6) \quad L_{\leq}(Y, \tilde{f}_{m+1}, 0) = Y \setminus (\text{int } X) = \tilde{X}.$$

By the closedness of Y and the openness of $\text{int } X$, we get that \tilde{X} is a closed set in \mathcal{V} .

By $L_{\leq}(Y, f_{m+1}, 0) = X$, we immediately infer that $L_{>}(Y, f_{m+1}, 0) \cup (\mathcal{V} \setminus Y) = \mathcal{V} \setminus X$, hence $L_{<}(Y, \tilde{f}_{m+1}, 0) = (\mathcal{V} \setminus X) \setminus (\mathcal{V} \setminus Y)$. Obviously, we have $(\mathcal{V} \setminus X) \setminus (\mathcal{V} \setminus Y) = Y \setminus X$, and by our assumption $\text{int } \tilde{X} = Y \setminus X$, we obtain

$$(2.7) \quad L_{<}(Y, \tilde{f}_{m+1}, 0) = Y \setminus X = \text{int } \tilde{X}.$$

Consequently, by (2.6) and (2.7), we get

$$L_{=}(Y, \tilde{f}_{m+1}, 0) = \text{bd } \tilde{X},$$

and therefore $\tilde{f}_{m+1}(x^0) = 0$ for all $x^0 \in \text{bd } \tilde{X}$.

This means that \tilde{f}_{m+1} fulfils Assumptions A1 and A2 concerning the set \tilde{X} . \square

Lemma 2.36. *Let (2.5) be satisfied, let Y be open, and let X be convex. Assume $\text{int } X \neq \emptyset$. Then we have $\text{int } \tilde{X} = Y \setminus X$ for the set $\tilde{X} := Y \setminus (\text{int } X)$.*

Proof. First we prove

$$(2.8) \quad \text{int } (\mathcal{V} \setminus \text{int } X) = \mathcal{V} \setminus X.$$

Since $\mathcal{V} \setminus X$ is open and $\mathcal{V} \setminus X \subseteq \mathcal{V} \setminus (\text{int } X)$ the inclusion “ \supseteq ” in (2.8) follows immediately.

Now, we prove the reverse inclusion “ \subseteq ”. Assume that there is $x^0 \in \text{int } (\mathcal{V} \setminus \text{int } X)$ with $x^0 \notin \mathcal{V} \setminus X$, i.e., $x^0 \in X$. Of course, it holds $x^0 \notin \text{int } X$,

hence $x^0 \in \text{bd } X$. Consider an interior point $\tilde{x} \in \text{int } X$. By the convexity of X , we know that $(x^0, \tilde{x}] \subseteq \text{int } X$ (see, e.g., [28, Th. 1.1.2]). This means, for every $\delta \in (0, 1]$, we have $x^0 + (0, \delta] \cdot (\tilde{x} - x^0) \subseteq \text{int } X$, hence $x^0 \notin \text{cor } (\mathcal{V} \setminus \text{int } X)$. This implies $x^0 \notin \text{int } (\mathcal{V} \setminus \text{int } X)$ by Lemma 2.1, a contradiction.

Consequently, by the openness of Y , by condition (2.8), and by a rule concerning the intersection of the interiors of sets, we get

$$\begin{aligned} \text{int } \tilde{X} &= \text{int } (Y \setminus (\text{int } X)) \\ &= \text{int } (Y \cap (\mathcal{V} \setminus \text{int } X)) \\ &= \text{int } Y \cap \text{int } (\mathcal{V} \setminus \text{int } X) \\ &= \text{int } Y \cap (\mathcal{V} \setminus X) \\ &= Y \cap (\mathcal{V} \setminus X) \\ &= Y \setminus X. \end{aligned}$$

□

Corollary 2.37. *Let (2.5) be satisfied, let $Y = \mathcal{V}$, let X be convex and assume $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2 concerning the set X . Consider the set $\tilde{X} := \mathcal{V} \setminus (\text{int } X) = L_{\geq}(\mathcal{V}, f_{m+1}, 0)$. Then $\tilde{f}_{m+1} := -f_{m+1}$ fulfils Assumptions A1 and A2 concerning the set \tilde{X} (where \tilde{X} is in the role of X in the formulation of the Assumptions A1 and A2).*

Proof. Follows by Lemma 2.35 and Lemma 2.36. □

Lemma 2.38. *Let (2.5) be satisfied and let Y be open. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$ holds. Then we have*

$$(2.9) \quad \emptyset \neq L_{<}(Y, f_{m+1}, 0) \subseteq \text{int } X,$$

hence X has a nonempty interior.

Proof. In view of (2.3), we have

$$\emptyset \neq L_{<}(Y, f_{m+1}, 0) = L_{<}(\mathcal{V}, f_{m+1}, 0) \cap Y \subseteq L_{\leq}(\mathcal{V}, f_{m+1}, 0) \cap Y = X.$$

Due to the upper semi-continuity of f_{m+1} on \mathcal{V} , the set $L_{<}(\mathcal{V}, f_{m+1}, 0)$ is open (see Lemma 2.13), hence the intersection of $L_{<}(\mathcal{V}, f_{m+1}, 0)$ with the open set Y is open too. Clearly, we conclude (2.9). □

In the formulation of Lemma 2.38 one can not remove the openness assumption concerning the set Y , as to see in the next example.

Example 2.39. Let h be the maximum norm defined on \mathbb{R}^2 , i.e., $h := \|\cdot\|_{\infty} : \mathbb{R}^2 \rightarrow \mathbb{R}$. Now, define $f_{m+1} := h - 1$. Note that f_{m+1} is a convex function on \mathbb{R}^2 (hence explicitly quasi-convex as well as continuous on \mathbb{R}^2). Moreover, put $x^1 := (0, 0)$, $x^2 := (1, 0)$ and $Y := \overline{B}_{\|\cdot\|_{\infty}}(x^2, 1)$.

First, we have

$$\begin{aligned} L_{\leq}(Y, f_{m+1}, 0) &= L_{\leq}(\mathbb{R}^2, \|\cdot\|_{\infty}, 1) \cap Y \\ &= \overline{B}_{\|\cdot\|_{\infty}}(x^1, 1) \cap \overline{B}_{\|\cdot\|_{\infty}}(x^2, 1) \\ &= [0, 1] \times [-1, 1] \\ &=: X. \end{aligned}$$

Moreover, it can easily be seen that

$$\begin{aligned} L_{<}(Y, f_{m+1}, 0) &= L_{<}(\mathbb{R}^2, h, 1) \cap Y \\ &= B_{\|\cdot\|_{\infty}}(x^1, 1) \cap \overline{B}_{\|\cdot\|_{\infty}}(x^2, 1) \\ &= [(0, 1) \times (-1, 1)] \cup [\{0\} \times (-1, 1)] \\ &\supsetneq (0, 1) \times (-1, 1) \\ &= \text{int } X. \end{aligned}$$

Hence, the inclusion given in (2.9) of Lemma 2.38 does not hold. This means that the openness assumption of Y in Lemma 2.38 can not be removed.

Lemma 2.40. *Let (2.5) be satisfied, let Y be open, and let f_{m+1} be upper semi-continuous on \mathcal{V} . Assume that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$ holds. Then for every $x^0 \in \text{bd } X$ we have $f_{m+1}(x^0) = 0$. Furthermore, Assumption A1 is fulfilled.*

Proof. Let $x^0 \in \text{bd } X$. We are going to show that $f_{m+1}(x^0) = 0$ holds, hence Assumption A1 follows by the fact that $X = L_{\leq}(Y, f_{m+1}, 0)$.

Assume the contrary, i.e., $f_{m+1}(x^0) < 0$. In view of Lemma 2.38 this means that

$$x^0 \in L_{<}(Y, f_{m+1}, 0) \subseteq \text{int } X,$$

in contradiction to $x^0 \in \text{bd } X$. \square

The next lemma uses the definition of local semi-strict quasi-convexity and local quasi-convexity of the function f_{m+1} (see Definition 2.16).

Lemma 2.41. *Let (2.5) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Furthermore, assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$. For every $x^0 \in \text{int } X \cap L_{=}(Y, f_{m+1}, 0)$, we suppose that there exists $\varepsilon > 0$ such that f_{m+1} is explicitly quasi-convex (can be replaced by semi-strict quasi-convexity and lower semi-continuity along line segments) on $B_{\|\cdot\|}(x^0, \varepsilon)$. Moreover, we assume that there exists*

$$(2.10) \quad x^1 \in B_{\|\cdot\|}(x^0, \varepsilon) \cap L_{<}(Y, f_{m+1}, 0).$$

Then Assumptions A1 and A2 are fulfilled.

Proof. The validity of Assumption A1 follows by Lemma 2.40. We are going to prove that Assumption A2 holds.

For $x^0 \in \text{bd } X$, we know (see the proof of Lemma 2.40) that $f_{m+1}(x^0) = 0$ holds, hence $x^0 \in L_{=}(Y, f_{m+1}, 0)$ is fulfilled.

Now, consider $x^0 \in L_=(Y, f_{m+1}, 0) \subseteq X$. Assume that the contrary $x^0 \in \text{int } X$ holds. Since $x^0 \in \text{int } X$, there is $\bar{\varepsilon} > 0$ such that $B_{\|\cdot\|}(x^0, \bar{\varepsilon}) \subseteq X$. Obviously, it holds $B_{\tilde{\varepsilon}} := B_{\|\cdot\|}(x^0, \tilde{\varepsilon}) \subseteq X$ for $\tilde{\varepsilon} := \min(\bar{\varepsilon}, \varepsilon)$. By Lemma 2.2, we know that

$$B_\delta := (x^0 - \delta \cdot v, x^0 + \delta \cdot v) \subseteq B_{\tilde{\varepsilon}}$$

for $\delta \in (0, \tilde{\varepsilon})$ and $v := \frac{x^1 - x^0}{\|x^1 - x^0\|}$ (note that $x^1 \neq x^0$). Due to the semi-strictly quasi-convexity on $B_\varepsilon := B_{\|\cdot\|}(x^0, \varepsilon)$ and the fact that $x^0 \in L_=(Y, f_{m+1}, 0)$ and $x^1 \in L_<(Y, f_{m+1}, 0)$, we can choose $x^2 \in B_\delta \cap (x^0, x^1]$ with $x^2 \in L_<(Y, f_{m+1}, 0)$. For $x^3 := x^0 + (x^0 - x^2)$, we have $x^3 \in B_\delta \subseteq B_{\tilde{\varepsilon}} \subseteq X$ and $x^0 \in (x^2, x^3)$. Now, since we have $x^3 \in X$, we can consider two cases:

Case 1: Let $f_{m+1}(x^3) = 0$, i.e., $x^3 \in L_=(Y, f_{m+1}, 0)$. Under the semi-strictly quasi-convexity of f_{m+1} on B_ε , we get $x^0 \in (x^2, x^3) \subseteq L_<(B_\varepsilon, f_{m+1}, 0)$. Since $(x^2, x^3) \subseteq B_\delta$, it follows $x^0 \in L_<(B_\delta, f_{m+1}, 0) \subseteq L_<(Y, f_{m+1}, 0)$, a contradiction to $x^0 \in L_=(Y, f_{m+1}, 0)$.

Case 2: Assume $f_{m+1}(x^3) < 0$, i.e., $x^3 \in L_<(Y, f_{m+1}, 0)$. Since $x^2, x^3 \in L_<(B_\varepsilon, f_{m+1}, 0)$, it follows $x^0 \in (x^2, x^3) \subseteq L_<(B_\varepsilon, f_{m+1}, 0)$ by the quasi-convexity of f_{m+1} on B_ε . Hence, $x^0 \in L_<(B_\delta, f_{m+1}, 0) \subseteq L_<(Y, f_{m+1}, 0)$, again a contradiction to $x^0 \in L_=(Y, f_{m+1}, 0)$.

We get that $x^0 \in \text{bd } X$. Consequently, Assumption A2 holds. \square

Remark 2.42. It is important to note that in Lemma 2.40 and Lemma 2.41 the openness assumption concerning the set Y can not be removed, as shown in the next example.

Example 2.43. Assume that Assumption A2 holds for the problem considered in Example 2.39. Then we have $L_=(Y, f_{m+1}, f_{m+1}(x^2)) = \text{bd } X$ for the point $x^2 \in \text{bd } X$. Since $f_{m+1}(x^2) = 0$ we have $f_{m+1}(x) = 0$ for all $x \in \text{bd } X$. However, for the point $x^1 \in \text{bd } X$ we obtain $f_{m+1}(x^1) = -1$, a contradiction.

If we assume that Assumption A1 holds for the problem considered in Example 2.39, then for the point $x^2 \in \text{bd } X$ we have $x^2 \notin L_<(Y, f_{m+1}, f_{m+1}(x^1)) = X$, again a contradiction.

Consequently, for the problem given in Example 2.39 the Assumptions A1 and A2 do not hold. This means that the openness assumption concerning the set Y can not be removed in Lemma 2.40 and Lemma 2.41.

Corollary 2.44. *Let (2.5) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Furthermore, assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $\tilde{x} \in L_<(Y, f_{m+1}, 0) \neq \emptyset$. If f_{m+1} is explicitly quasi-convex (can be replaced by semi-strictly quasi-convexity and lower semi-continuity along line segments) on $B_{\|\cdot\|}(\tilde{x}, \varepsilon)$, where*

$$\varepsilon > \max\{\|x - \tilde{x}\| \mid x \in X\},$$

then Assumptions A1 and A2 hold.

Proof. The assertion follows by Lemma 2.41, since $\tilde{x} \in L_{<}(Y, f_{m+1}, 0)$ and

$$\|x^0 - \tilde{x}\| \leq \max\{\|x - \tilde{x}\| \mid x \in X\} < \varepsilon,$$

hence

$$\tilde{x} \in B_{\|\cdot\|}(x^0, \varepsilon) \cap L_{<}(Y, f_{m+1}, 0).$$

Note that $\max\{\|x - \tilde{x}\| \mid x \in X\}$ exists because of the closedness of X . \square

Lemma 2.45. *Let (2.5) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Furthermore, assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$. Consider two points $\tilde{x} \in L_{<}(Y, f_{m+1}, 0)$ and $x^0 \in \text{int } X \cap L_{=}(Y, f_{m+1}, 0)$. Then there exists $x^1 \in \text{int } X$ such that condition (2.10) holds, if one of the following statements is true:*

- 1°. *Every local minimum point of f_{m+1} on $\text{int } X$ is also global.*
- 2°. *Assume X is convex. Every local minimum of f_{m+1} is also global for each restriction on a line segment in $\text{int } X$.*

Proof. Let 1° be fulfilled. Assume that there is no $x^1 \in \text{int } X$ such that (2.10) holds. Then x^0 is a local minimum of f_{m+1} on $\text{int } X$, hence under 1° also global on $\text{int } X$. This is a contradiction, since we have $\tilde{x} \in L_{<}(Y, f_{m+1}, 0) \subseteq \text{int } X$ (see Lemma 2.40) and $f_{m+1}(\tilde{x}) < 0 = f_{m+1}(x^0)$.

Now, let 2° be fulfilled. Since $x^0 \in \text{cor } X$ by Lemma 2.1, for $v := x^0 - \tilde{x} \neq 0$ there exists $\delta > 0$ such that $x^0 + [0, \delta] \cdot v \subseteq X$. Define $x^2 := x^0 + \delta \cdot v$. By $\tilde{x} \in L_{<}(Y, f_{m+1}, 0) \subseteq \text{int } X$ and the convexity of X , we know that $x^0 \in (\tilde{x}, x^2) \subseteq \text{int } X$ (see, e.g., [28, Th. 1.1.2]). Choose $x^3 \in (x^0, x^2)$. Assume that there is no $x^1 \in \text{int } X$ such that (2.10) holds. Then $x^0 \in (\tilde{x}, x^3)$ is a local minimum of f_{m+1} on the line segment $[\tilde{x}, x^3] \subseteq \text{int } X$. By assumption 2° of this lemma, we know that x^0 is also global minimum of f_{m+1} on $[\tilde{x}, x^3]$, in contradiction to $f_{m+1}(\tilde{x}) < 0 = f_{m+1}(x^0)$. \square

Corollary 2.46. *Let (2.5) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Furthermore, assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$ (i.e., $\text{int } X \neq \emptyset$). The function f_{m+1} fulfils the Assumptions A1 and A2, if both of the following statements hold:*

- 1°. *Every local minimum point of f_{m+1} on $\text{int } X$ is also global.*
- 2°. *f_{m+1} is locally explicitly quasi-convex on $\text{int } X$.*

Proof. Directly follows by Lemmata 2.41 and 2.45. \square

Note that every local minimum point of a semi-strictly quasi-convex function on a convex set is also global (see, e.g., [22, Sec. 2.1]).

Corollary 2.47. *Let (2.5) be satisfied and let Y be open. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$ holds. If f_{m+1} is explicitly quasi-convex (can be replaced by semi-strictly quasi-convexity and lower semi-continuity along line segments) on Y , then Assumptions A1 and A2 hold.*

Proof. If \mathcal{V} is additionally normed, then we get the statement of this corollary by Corollary 2.46. Now, let us assume that \mathcal{V} is not necessarily normed.

By Lemma 2.40, we know that Assumption A1 is fulfilled. We are going to prove that Assumption A2 holds. In view of the proof of Lemma 2.41 we know $\text{bd } X \subseteq L_=(Y, f_{m+1}, 0)$. We are going to show $L_=(Y, f_{m+1}, 0) \cap \text{int } X = \emptyset$. Consider $x^0 \in L_=(Y, f_{m+1}, 0) \cap \text{int } X$ and $\tilde{x} \in L_<(Y, f_{m+1}, 0)$. We know that $x^0 \in \text{cor } X$ holds by Lemma 2.1. Hence, there exists $\delta > 0$ such that $x^1 := x^0 + \delta \cdot (x^0 - \tilde{x}) \in X$. Note that $x^0 \in (\tilde{x}, x^1)$. Now, we look at two cases:

Case 1: Let $f_{m+1}(x^1) < 0$. Then the quasi-convexity of f_{m+1} on Y implies $x^0 \in (\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, 0)$, a contradiction to $f_{m+1}(x^0) = 0$.

Case 2: Let $f_{m+1}(x^1) = 0$. By the semi-strictly quasi-convexity of f_{m+1} on Y , we get $x^0 \in (\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, 0)$, again a contradiction to $f_{m+1}(x^0) = 0$. \square

Corollary 2.48. *Let (2.5) be satisfied and let $Y = \mathcal{V}$. Assume that f_{m+1} is semi-strictly quasi-convex as well as continuous on \mathcal{V} . Suppose that $L_<(\mathcal{V}, f_{m+1}, 0) \neq \emptyset$. Then Assumptions A1 and A2 hold.*

The following two corollaries give interesting connections to generalized-concavity.

Corollary 2.49. *Let (2.5) be satisfied. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Assume that f_{m+1} is upper semi-continuous on $Y = \mathcal{V}$ and that $L_<(\mathcal{V}, f_{m+1}, 0) \neq \emptyset$. Moreover, suppose that the following statements hold:*

- 1°. *Every local minimum point of f_{m+1} on $\text{int } X$ is also global.*
- 2°. *f_{m+1} is locally explicitly quasi-convex on $\text{int } X$.*

Consider the set $\tilde{X} := \mathcal{V} \setminus (\text{int } X) = L_{\geq}(\mathcal{V}, f_{m+1}, 0)$. Then $\tilde{f}_{m+1} := -f_{m+1}$ fulfils Assumptions A1 and A2 concerning the set \tilde{X} (where \tilde{X} is in the role of X in the formulation of the Assumptions A1 and A2).

Proof. This statement follows by Corollary 2.37 and Corollary 2.46. \square

Corollary 2.50. *Let (2.5) be satisfied and let $Y = \mathcal{V}$. Assume that f_{m+1} is semi-strictly quasi-convex as well as continuous on \mathcal{V} and that $L_<(\mathcal{V}, f_{m+1}, 0) \neq \emptyset$ holds. Consider the set $\tilde{X} := \mathcal{V} \setminus (\text{int } X) = L_{\geq}(\mathcal{V}, f_{m+1}, 0)$. Then $\tilde{f}_{m+1} := -f_{m+1}$ fulfils Assumptions A1 and A2 concerning the set \tilde{X} (where \tilde{X} is in the role of X in the formulation of the Assumptions A1 and A2).*

Proof. Follows immediately by Corollary 2.37 and Corollary 2.48. \square

The next lemma shows that under Assumption A1 a semi-strictly quasi-convex or quasi-convex function f_{m+1} on Y guarantees that the set X is convex. Hence, in order to describe a nonconvex set X using the level set of the function f_{m+1} to the level 0, it is necessary that f_{m+1} is not a semi-strictly quasi-convex or a quasi-convex function on Y .

Lemma 2.51. *Let (2.5) be satisfied. Assume that f_{m+1} is semi-strictly quasi-convex on the convex set Y . Then $X = L_{\leq}(Y, f_{m+1}, 0)$ is a convex set too.*

Proof. Assume the contrary, this means that there exist $x^1, x^2 \in X$, $\lambda \in (0, 1)$ such that $x^3 := l_{x^1, x^2}(\lambda) \notin X$. We know that

$$(2.11) \quad L_{\leq}(Y, f_{m+1}, 0) = X.$$

Consider the complement of X , i.e, the set

$$(2.12) \quad X^c := \mathcal{V} \setminus X = L_{>}(Y, f_{m+1}, 0) \cup (\mathcal{V} \setminus Y).$$

The convexity of Y ensures $x^3 \in (x^1, x^2) \subseteq Y$, and therefore,

$$(2.13) \quad x^3 \in L_{>}(Y, f_{m+1}, 0).$$

Since X is closed, the set X^c is open, hence by (2.13) and Lemma 2.1 we know $x^3 \in \text{cor } X^c$ holds. Therefore, for $v := x^1 - x^3 \neq 0$, it exists $\delta > 0$ such that $x^3 + [0, \delta] \cdot v \subseteq X^c$. Moreover, we have $x^3 + [0, 1] \cdot v = [x^3, x^1] \subseteq Y$. Hence, by (2.12) it follows

$$(2.14) \quad x^3 + [0, \bar{\delta}] \cdot v \subseteq L_{>}(Y, f_{m+1}, 0) \cap (x^1, x^2)$$

for $\bar{\delta} := \min(\delta, 0.5) > 0$. By (2.11) and due to $x^1, x^2 \in X$, we have

$$(2.15) \quad \max(f_{m+1}(x^1), f_{m+1}(x^2)) \leq 0.$$

In view of (2.14) and (2.15), it follows

$$\begin{aligned} x^3 + [0, \bar{\delta}] \cdot v &\subseteq L_{>}((0, 1), (f_{m+1} \circ l_{x^1, x^2}), 0) \\ &\subseteq L_{>}((0, 1), (f_{m+1} \circ l_{x^1, x^2}), \max(f_{m+1}(x^1), f_{m+1}(x^2))). \end{aligned}$$

Note that $\text{card}(x^3 + [0, \bar{\delta}] \cdot v) > 1$ holds. However, in view of Lemma 2.12, under the semi-strictly quasi-convexity of f_{m+1} on the convex set Y , it follows

$$\text{card}(L_{>}((0, 1), (f_{m+1} \circ l_{x^1, x^2}), \max(f_{m+1}(x^1), f_{m+1}(x^2)))) \leq 1,$$

a contradiction. This completes the proof. \square

Remark 2.52. Since $X = L_{\leq}(Y, f_{m+1}, 0)$ by assumption (2.3), we know that the quasi-convexity of f_{m+1} implies convexity of X . If f_{m+1} is additionally lower semi-continuous along line segments on Y , then semi-strictly quasi-convexity of f_{m+1} implies quasi-convexity of f_{m+1} on Y . Note that the proof of Lemma 2.51 does not use lower semi-continuity along line segments of f_{m+1} on Y .

2.6. Examples for the penalization function f_{m+1} .

Concerning penalization in multi-objective optimization theory, many authors (see, e.g., Apetrii *et al.* [1], Durea *et al.* [4], Ye [26], and references therein) use a penalization function $\phi : \mathcal{V} \rightarrow \mathbb{R}$ (penalty term concerning X) which fulfils Assumption A3 concerning $Y = \mathcal{V}$ (ϕ in the role of f_{m+1}), i.e., for $x^0 \in Y = \mathcal{V}$ we have

$$x^0 \in X \iff \phi(x^0) = 0$$

and

$$x^0 \in \mathcal{V} \setminus X \iff \phi(x^0) > 0.$$

Such a penalization function ϕ will be given in Example 2.55 with $\phi = f_{m+1}$ (compare with *Clarke's Exact Penalty Principle* in optimization; see Ye [26] for more details).

In the following, we will present some examples in order to illustrate how the function f_{m+1} can be chosen.

Example 2.53. Suppose that the feasible set $X \subsetneq \mathcal{V}$ is convex and closed with $\tilde{x} \in \text{int } X \neq \emptyset$. Moreover, suppose that $Y = \mathcal{V}$. Then the function $f_{m+1}(\cdot) := \mu(\cdot - \tilde{x})$, defined using a gauge function

$$\mu(z) := \inf\{\lambda > 0 \mid z \in \lambda \cdot (-\tilde{x} + X)\} \quad \text{for all } z \in \mathcal{V},$$

fulfils Assumptions A1 and A2 (see Günther and Tammer [10]).

Example 2.54. Assume that a set $F \subsetneq \mathcal{V}$ is convex and closed with $\tilde{x} \in \text{int } F \neq \emptyset$. Now, consider the feasible set $X := \mathcal{V} \setminus \text{int } F$. Note that X is a nonconvex set. In addition suppose that $Y = \mathcal{V}$ holds. Then the function $f_{m+1}(\cdot) := -\mu(\cdot - \tilde{x})$, defined using a gauge function

$$\mu(z) := \inf\{\lambda > 0 \mid z \in \lambda \cdot (-\tilde{x} + F)\} \quad \text{for all } z \in \mathcal{V},$$

fulfils Assumptions A1 and A2 (follows by Example 2.53 and Corollary 2.37).

Example 2.55. Let X be a nonempty closed set in a normed space $(\mathcal{V}, \|\cdot\|)$, and let the function $f_{m+1} : \mathcal{V} \rightarrow \mathbb{R}$ be given by a distance function

$$f_{m+1}(x) := d_X(x) := \inf\{\|x - z\| \mid z \in X\} \quad \text{for all } x \in \mathcal{V}.$$

Several authors pointed out important properties of such distance functions (see, e.g., Mordukhovich and Nam [18], and references therein). We will recall some important properties of f_{m+1} :

- d_X is convex on \mathcal{V} if and only if the closed set X is convex in \mathcal{V} .
- d_X is Lipschitz continuous on \mathcal{V} with constant 1, hence lower and upper semi-continuous on \mathcal{V} .
- Due to the closedness of X , it holds $L_{\leq}(\mathcal{V}, d_X, 0) = L_{=}(\mathcal{V}, d_X, 0) = X$.
- Since we have $d_X(x^0) = 0$ for all $x^0 \in X$, the function $f_{m+1} = d_X$ fulfils condition (2.3) and Assumption A3 for $Y = \mathcal{V}$.

Example 2.56. Let X be a nonempty closed set with $\emptyset \neq X \neq \mathcal{V}$ in a normed space $(\mathcal{V}, \|\cdot\|)$. Based on the distance function $d_X : \mathcal{V} \rightarrow \mathbb{R}$ (see Example 2.55) one can introduce a function $\Delta_X : \mathcal{V} \rightarrow \mathbb{R}$ by

$$f_{m+1}(x) := \Delta_X(x) := d_X(x) - d_{\mathcal{V} \setminus X}(x) = \begin{cases} d_X(x) & \text{for } x \in \mathcal{V} \setminus X, \\ -d_{\mathcal{V} \setminus X}(x) & \text{for } x \in X. \end{cases}$$

The *signed distance function* (*Hiriart-Urruty function*) Δ_X was introduced by Hiriart-Urruty [11]. Next, we recall some well-known properties of Δ_X (see Hiriart-Urruty [11], Eichfelder [6], Liu *et al.* [15], Zaffaroni [27]):

- Δ_X is Lipschitz continuous on \mathcal{V} with constant 1.
- Δ_X is convex on \mathcal{V} if and only if X is convex.
- If X is a cone (i.e., $\lambda \cdot X \subseteq X$ for all $\lambda \geq 0$), then Δ_X is positively homogeneous.
- Let us define the dual unit sphere in the dual space \mathcal{V}^* of \mathcal{V} by $S^* = \{x^* \in \mathcal{V}^* \mid \|x^*\|_* = 1\}$, where $\|\cdot\|_* : \mathcal{V}^* \rightarrow \mathbb{R}$ denotes the dual norm of $\|\cdot\|$. If X is additionally convex with nonempty interior, then

$$\Delta_X(x) = \sup_{x^* \in S^*} \inf_{z \in X} x^*(x - z) \quad \text{for all } x \in \mathcal{V}.$$

- It holds $L_{<}(\mathcal{V}, \Delta_X, 0) = \text{int } X$, $L_{=}(\mathcal{V}, \Delta_X, 0) = \text{bd } X$ and $L_{\leq}(\mathcal{V}, \Delta_X, 0) = X$, this means that $f_{m+1} = \Delta_X$ fulfils condition (2.3) and Assumptions A1 and A2 for $Y = \mathcal{V}$.

Example 2.57. Let X be a nonempty closed set in a real topological linear space \mathcal{V} with $X \neq \mathcal{V}$. Consider $k \in \mathcal{V}$ such that the pair (X, k) fulfils $X + [0, +\infty) \cdot k \subseteq X$. Now, we consider the function $\phi_{X,k} : \mathcal{V} \rightarrow \mathbb{R} \cup \{\pm\infty\}$ (with respect to the set X and a direction k) by

$$\phi_{X,k}(x) := \inf\{s \in \mathbb{R} \mid x \in s \cdot k - X\} \quad \text{for all } x \in \mathcal{V}.$$

In the following, we recall some important properties of the function $\phi_{X,k}$ (see Gerth and Weidner [7] and Göpfert *et al.* [9, Th. 2.3.1]):

- $\phi_{X,k}$ is lower semi-continuous on \mathcal{V} and for every $s \in \mathbb{R}$ it holds

$$L_{\leq}(\mathcal{V}, \phi_{X,k}, s) = s \cdot k - X.$$

- $\phi_{X,k}$ is convex if and only if X is convex.
- $\phi_{X,k}(\lambda \cdot x) = \lambda \cdot \phi_{X,k}(x)$ for all $x \in \mathcal{V}$ and all $\lambda > 0$ if and only if X is a cone.
- $\phi_{X,k}$ is subadditive if and only if $X + X \subseteq X$.
- $\phi_{X,k}$ is proper if and only if X does not contain lines parallel to k , i.e.,

$$\forall x \in \mathcal{V} \exists t \in \mathbb{R} : x + t \cdot k \notin X.$$

- $\phi_{X,k}$ is finite-valued if and only if X does not contain lines parallel to k and

$$\mathbb{R} \cdot k - X = \mathcal{V}.$$

- Suppose, furthermore, that $X + (0, +\infty) \cdot k \subseteq \text{int } X$ holds. Then the function $\phi_{X,k}$ is continuous and for every $s \in \mathbb{R}$ we have

$$\begin{aligned} L_{<}(\mathcal{V}, \phi_{X,k}, s) &= s \cdot k - \text{int } X, \\ L_{=}(\mathcal{V}, \phi_{X,k}, s) &= s \cdot k - \text{bd } X. \end{aligned}$$

Let us consider the function $f_{m+1} : \mathcal{V} \rightarrow \mathbb{R}$ defined by $f_{m+1}(x) := \phi_{X,k}(-x)$ for every $x \in \mathcal{V}$, where we assume that $\phi_{X,k}$ is finite-valued. Then we have:

- Condition (2.3) holds, i.e., we have

$$L_{\leq}(\mathcal{V}, f_{m+1}, 0) = L_{\leq}(\mathcal{V}, \phi_{X,k}(-\cdot), 0) = X.$$

- Suppose, furthermore, that $X + (0, +\infty) \cdot k \subseteq \text{int } X$ holds. Then it hold

$$\begin{aligned} L_{<}(\mathcal{V}, f_{m+1}, 0) &= L_{<}(\mathcal{V}, \phi_{X,k}(-\cdot), 0) = \text{int } X, \\ L_{=}(\mathcal{V}, f_{m+1}, 0) &= L_{=}(\mathcal{V}, \phi_{X,k}(-\cdot), 0) = \text{bd } X, \end{aligned}$$

hence both Assumptions A1 and A2 are fulfilled for $Y = \mathcal{V}$.

Remark 2.58. Note, Examples 2.54, 2.55, 2.56 and 2.57 show that it is possible to consider a nonconvex set X in our approach. Let X be an arbitrarily nonempty closed set with $\emptyset \neq X \neq \mathcal{V}$. In a normed space $(\mathcal{V}, \|\cdot\|)$, we know that the Hiriart-Urruty function Δ_X fulfils Assumptions A1 and A2, and moreover, the function d_X fulfils Assumption A3. Hence, the results obtained in this paper extend and generalize the results given in the paper by Günther and Tammer [10], where X is supposed to be a convex set and a gauge function is used as penalization function (see Example 2.53).

3. RELATIONSHIPS BETWEEN THE PROBLEMS (\mathcal{P}_X) , (\mathcal{P}_Y) AND (\mathcal{P}_Y^E)

In this section, under the standard assumption given in (2.5), we study the relationships between the original multi-objective optimization problem (\mathcal{P}_X) with a nonempty closed (not necessarily convex) feasible set $X \subsetneq \mathcal{V}$ and two related multi-objective optimization problems (\mathcal{P}_Y) and (\mathcal{P}_Y^E) with a convex feasible set $Y \subseteq \mathcal{V}$ for that $X \subseteq Y$ holds.

Note, throughout this section, we do not assume the convexity of the feasible set X . If we need the convexity of X , then we will mention it in the formulation of the corresponding results.

The importance of the results in this section is given by the fact that the solution sets of certain vector optimization problems with a not necessarily convex feasible set can be computed by solving two multi-objective optimization problems with convex feasible sets. These results are very useful because they offer a way to solve nonconvex problems using algorithms for convex problems.

3.1. Set of efficient solutions of (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) .

An important result of our paper is given in the next theorem that is a generalization of [10, Th. 1]. In [10] the special case $\mathcal{V} = Y = \mathbb{R}^n$, X closed and convex with $\text{int } X \neq \emptyset$, and f_{m+1} defined as a gauge function is considered.

Theorem 3.1. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumptions A1 and A2. Then the following statements are true:*

1°. *It holds*

$$[X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)] \subseteq \text{Eff}(X | f).$$

2°. *If $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . Then, it holds that*

$$[X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)] \supseteq \text{Eff}(X | f).$$

Proof. 1°. The inclusion $X \cap \text{Eff}(Y | f) \subseteq \text{Eff}(X | f)$ follows by Lemma 2.23. Consider $x^0 \in \text{bd } X \cap \text{Eff}(Y | f^E)$. By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)) and Assumptions A1 and A2, it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &= S_{=}(Y, f, x^0) \cap \text{bd } X \\ &\subseteq S_{=}(Y, f, x^0) \cap X \\ &= S_{=}(X, f, x^0). \end{aligned}$$

Hence, we get $x^0 \in \text{Eff}(X | f)$ by Lemma 2.7 (applied for the problem (\mathcal{P}_X)).

2°. Let $x^0 \in \text{Eff}(X | f) \subseteq X$. Of course, $x^0 \in X \cap \text{Eff}(Y | f)$ is possible as a first case. Furthermore, we consider $x^0 \in X \setminus \text{Eff}(Y | f)$ as the second case. Obviously, we have $x^0 \in \text{bd } X$ for the case $\text{int } X = \emptyset$. Due to Corollary 2.25, we obtain $x^0 \in \text{bd } X$ under the assumption $\text{int } X \neq \emptyset$. By Lemma 2.7 (applied for the problem (\mathcal{P}_X)) and Assumption A1, we can conclude

$$\begin{aligned} S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(X, f, x^0) \\ &\subseteq S_{=}(X, f, x^0) \\ &= S_{=}(Y, f, x^0) \cap X. \end{aligned}$$

Now, we will prove the equation

$$(3.1) \quad S_{=}(Y, f, x^0) \cap X = S_{=}(Y, f, x^0) \cap \text{bd } X.$$

For the nontrivial case $\text{int } X \neq \emptyset$, it is sufficient to prove $S_=(Y, f, x^0) \cap \text{int } X = \emptyset$ in order to get the validity of (3.1). Indeed, if we suppose that there exists $x^1 \in \text{int } X$ with $x^1 \in S_=(Y, f, x^0)$, then we have to discuss following two cases:

Case 1: If $x^1 \in (\text{int } X) \setminus \text{Eff}(Y | f)$ holds, then we get $x^1 \in (\text{int } X) \setminus \text{Eff}(X | f)$ by Lemma 2.23. This implies $x^0 \in X \setminus \text{Eff}(X | f)$ because of $x^1 \in S_=(X, f, x^0)$. This is a contradiction to the assumption $x^0 \in \text{Eff}(X | f)$.

Case 2: Now, we assume $x^1 \in \text{Eff}(Y | f)$, such that we get $x^0 \in \text{Eff}(Y | f)$ by $x^1 \in S_=(Y, f, x^0)$ in contradiction to $x^0 \in X \setminus \text{Eff}(Y | f)$.

This means that (3.1) holds.

Furthermore, we obtain by Assumption A2 and taking into account $x^0 \in \text{bd } X$,

$$S_=(Y, f, x^0) \cap \text{bd } X = S_=(Y, f, x^0) \cap L_=(Y, f_{m+1}, f_{m+1}(x^0)).$$

From Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), we conclude $x^0 \in \text{Eff}(Y | f^E)$. □

Remark 3.2. Note that the proof of Theorem 3.1 is a slight modification of the proof of [10, Th. 1]. Moreover, notice that one can not replace the semi-strictly quasi-convexity assumption with respect to f in 2° of Theorem 3.1 by a quasi-convexity assumption (see [10, Ex. 2] and [10, Ex. 3] for the case $Y = \mathcal{V} = \mathbb{R}^n$).

Corollary 3.3. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumptions A1 and A2. Furthermore, if $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . Then, it holds that*

$$[X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)] = \text{Eff}(X | f).$$

Remark 3.4. It is important to mention that the following inclusions do not hold in general under the assumptions supposed in Corollary 3.3 (see [10, Ex. 1] and [10, Ex. 5]):

$$(3.2) \quad \text{Eff}(X | f) \subseteq X \cap \text{Eff}(Y | f^E);$$

$$(3.3) \quad \text{bd } X \cap \text{Eff}(Y | f) \subseteq \text{bd } X \cap \text{Eff}(Y | f^E);$$

$$(3.4) \quad \text{Eff}(X | f) \subseteq [\text{int } X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)].$$

Corollary 3.5. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumptions A1 and A2. Furthermore, if $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . If $X \cap \text{Eff}(Y | f) = \emptyset$, then*

$$\text{Eff}(X | f) = \text{bd } X \cap \text{Eff}(Y | f^E) \subseteq \text{bd } X.$$

Remark 3.6. Corollary 3.5 shows that the additional assumption $X \cap \text{Eff}(Y \mid f) = \emptyset$ is sufficient for the validity of (3.2) under Assumptions A1 and A2.

Now, we present further sufficient conditions for the validity of the inclusion (3.2).

Lemma 3.7. *Let (2.1) and (2.4) be satisfied. Assume that $x^0 \in \text{Eff}(X \mid f)$ holds. If we have*

$$S_=(X, f, x^0) \subseteq L_=(Y, f_{m+1}, f_{m+1}(x^0)),$$

then $x^0 \in X \cap \text{Eff}(Y \mid f^E)$ is true.

Proof. Consider $x^0 \in \text{Eff}(X \mid f)$. First, note that the following statements are equivalent:

- $S_=(X, f, x^0) \subseteq L_=(Y, f_{m+1}, f_{m+1}(x^0))$.
- $S_=(Y, f, x^0) \cap X \subseteq L_=(Y, f_{m+1}, f_{m+1}(x^0))$.
- $S_=(Y, f, x^0) \cap X \subseteq S_=(Y, f, x^0) \cap L_=(Y, f_{m+1}, f_{m+1}(x^0))$.

Hence, by Lemma 2.7 (applied for the problem (\mathcal{P}_X)) and by (2.4), we get

$$\begin{aligned} S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) &\subseteq S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(X, f, x^0) \\ &\subseteq S_=(X, f, x^0) \\ &= S_=(Y, f, x^0) \cap X \\ &\subseteq S_=(Y, f, x^0) \cap L_=(Y, f_{m+1}, f_{m+1}(x^0)). \end{aligned}$$

By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), it follows $x^0 \in X \cap \text{Eff}(Y \mid f^E)$. \square

Theorem 3.8. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumptions A1 and A2. If $\text{Eff}(X \mid f) \subseteq \text{bd } X$, then it holds that*

$$\text{Eff}(X \mid f) = \text{bd } X \cap \text{Eff}(Y \mid f^E).$$

Proof. First, we prove the inclusion “ \subseteq ”. Consider $x^0 \in \text{Eff}(X \mid f)$. Because of $x^0 \in \text{Eff}(X \mid f) \subseteq \text{bd } X$ and Assumption A2, we get

$$S_=(X, f, x^0) \subseteq \text{Eff}(X \mid f) \subseteq \text{bd } X = L_=(Y, f_{m+1}, f_{m+1}(x^0)).$$

Hence, it follows $x^0 \in \text{Eff}(Y \mid f^E)$ by Lemma 3.7. Note that (2.4) is satisfied by Lemma 2.30.

The reverse inclusion “ \supseteq ” follows from 1° of Theorem 3.1. (the inclusion “ \supseteq ” holds without assuming $\text{Eff}(X \mid f) \subseteq \text{bd } X$). \square

Theorem 3.9. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A3. Then the following statement is true:*

$$\text{Eff}(X \mid f) = X \cap \text{Eff}(Y \mid f^E).$$

Proof. Let $x^0 \in X \cap \text{Eff}(Y \mid f^E)$. By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)) and Assumption A3, it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &= S_{=}(Y, f, x^0) \cap X \\ &= S_{=}(X, f, x^0). \end{aligned}$$

Consequently, applying Lemma 2.7 for the problem (\mathcal{P}_X) , we get $x^0 \in \text{Eff}(X \mid f)$.

Now, let $x^0 \in \text{Eff}(X \mid f)$. Due to

$$\begin{aligned} S_{=}(X, f, x^0) &= S_{=}(Y, f, x^0) \cap X \\ &= S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq L_{=}(Y, f_{m+1}, f_{m+1}(x^0)), \end{aligned}$$

it follows $x^0 \in X \cap \text{Eff}(Y \mid f^E)$ by Lemma 3.7. Note that condition (2.4) is satisfied under Assumption A3 (see Remark 2.27 and Lemma 2.30). \square

Remark 3.10. Under certain additional assumptions given in Corollary 3.5, Lemma 3.7, Theorem 3.8 and Theorem 3.9, one can see that the inclusion (3.2) holds.

3.2. Set of weakly efficient solutions of (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) .

In the first part of this section, we present some relationships between the problems (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) . In the second part, we study the concept of *Pareto reducibility*, that was introduced by Popovici [19] for these problems.

3.2.1. Some relationships.

In the next theorem, we present some relationships between the problems (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) for the concept of weak efficiency. This theorem is a generalization of [10, Th. 3] where the special case $\mathcal{V} = Y = \mathbb{R}^n$, X closed and convex with $\text{int } X \neq \emptyset$, and f_{m+1} defined as a gauge function is considered.

Theorem 3.11. *Let (2.1) and (2.4) be satisfied. The following statements are true:*

1°. *It holds*

$$X \cap \text{WEff}(Y \mid f) \subseteq \text{WEff}(X \mid f) \subseteq X \cap \text{WEff}(Y \mid f^E).$$

2°. *If $S_{<}(Y, f, x^0) \subseteq L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$ for all $x^0 \in X \cap \text{WEff}(Y \mid f^E)$, then*

$$X \cap \text{WEff}(Y \mid f) = \text{WEff}(X \mid f) = X \cap \text{WEff}(Y \mid f^E).$$

Proof. 1°. In view of Corollary 2.23, it follows $X \cap \text{WEff}(Y \mid f) \subseteq \text{WEff}(X \mid f)$. Now, let us prove the second inclusion $\text{WEff}(X \mid f) \subseteq X \cap \text{WEff}(Y \mid f^E)$.

Consider $x^0 \in \text{WEff}(X \mid f) \subseteq X$. By Lemma 2.7 (applied for the problem (\mathcal{P}_X)) and under the assumption (2.4), we get

$$\begin{aligned} \emptyset &= S_{<}(X, f, x^0) \\ &= S_{<}(Y, f, x^0) \cap X \\ &\supseteq S_{<}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\supseteq S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)). \end{aligned}$$

By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), it follows $x^0 \in X \cap \text{WEff}(Y \mid f^E)$, i.e., assertion 1° holds.

2°. Consider $x^0 \in X \cap \text{WEff}(Y \mid f^E)$. By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), we know that

$$(3.5) \quad \emptyset = S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)).$$

The assumption $S_{<}(Y, f, x^0) \subseteq L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$ implies

$$(3.6) \quad S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = S_{<}(Y, f, x^0).$$

Hence, it holds $x^0 \in X \cap \text{WEff}(Y \mid f)$ by (3.5), (3.6) and Lemma 2.7 (applied for the problem (\mathcal{P}_X)). Consequently, the inclusions in 1° of this theorem are equalities such that assertion 2° holds. \square

Remark 3.12. Let (2.1) be satisfied. Assume $\text{int } X \neq \emptyset$. Under Assumptions A1 and A2, we have

$$L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = L_{<}(Y, f_{m+1}, f_{m+1}(x^1)) = \text{int } X$$

for given points $x^0, x^1 \in \text{bd } X$. Hence, the assumption of semi-strictly quasi-convexity of f_{m+1} on Y (applying Lemma 2.11) ensures

$$\forall x^0, x^1 \in \text{bd } X, \forall \tilde{x} \in \text{int } X : [\tilde{x}, x^1] \subseteq L_{<}(Y, f_{m+1}, f_{m+1}(x^0)).$$

Especially, since $\text{int } X \neq \emptyset$, this means that

$$(3.7) \quad \forall x^0, x^1 \in \text{bd } X \exists \tilde{x} \in \text{int } X, \bar{\lambda}_{m+1} \in (0, 1] \forall \lambda \in (0, \bar{\lambda}_{m+1}) : \\ l_{x^1, \tilde{x}}(\lambda) \in L_{<}(Y, f_{m+1}, f_{m+1}(x^0)).$$

Now, we are able to formulate another important result of the paper concerning the set of weakly efficient solutions of (\mathcal{P}_X) (a generalization of [10, Th. 4]). This result shows that it is possible to compute the set of weakly efficient solutions of problems with nonconvex feasible sets by solving two optimization problems with convex feasible sets.

Theorem 3.13. *Let (2.1) be satisfied. The following statements are true:*

1°. Suppose $\text{int } X \neq \emptyset$. Assume that f is componentwise upper semi-continuous along line segments on Y . Furthermore, we suppose that f_{m+1} fulfils Assumptions A1 and A2, as well as condition (3.7). Then, it holds that

$$[X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)] \subseteq \text{WEff}(X | f).$$

2°. Assume (2.4) is true. If $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . Then, it holds

$$[X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)] \supseteq \text{WEff}(X | f).$$

Proof. Consider $i \in I_m$. The following statements are equivalent (see Definition 2.8 and Lemma 2.13):

- f_i is upper semi-continuous on line segments in Y .
- $L_{\geq}([0, 1], (f_i \circ l_{a,b}), s)$ is closed for all $s \in \mathbb{R}$ and all $a, b \in Y$.
- $L_{<}([0, 1], (f_i \circ l_{a,b}), s) \cup (\mathbb{R} \setminus [0, 1])$ is open for all $s \in \mathbb{R}$ and all $a, b \in Y$.

1°. In view of Corollary 2.23, it follows $X \cap \text{WEff}(Y | f) \subseteq \text{WEff}(X | f)$. Now, let us consider $x^0 \in \text{bd } X \cap \text{WEff}(Y | f^E)$. By Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), it follows

$$(3.8) \quad \emptyset = S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)).$$

Now, we will prove that

$$(3.9) \quad S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = S_{<}(Y, f, x^0) \cap X.$$

By (3.8) and (3.9), we get $S_{<}(X, f, x^0) = \emptyset$, hence $x^0 \in \text{WEff}(X | f)$ by Lemma 2.7 (applied for the problem (\mathcal{P}_X)).

In view of Assumption A1, the inclusion “ \subseteq ” in (3.9) follows immediately.

Let us prove the reverse inclusion “ \supseteq ” in (3.9). Assume the contrary holds, i.e., it exists $x^1 \in S_{<}(Y, f, x^0) \cap X$ such that $x^1 \notin L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$. This means that $x^1 \in L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) = \text{bd } X$ by Assumption A1. By (3.7), we know that for $x^0, x^1 \in \text{bd } X$ there exists $\tilde{x} \in \text{int } X$ such that $l_{x^1, \tilde{x}}(\lambda) \in L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$ for all $\lambda \in (0, \bar{\lambda}_{m+1}]$.

Consider $i \in I_m$. Since $x^1 \in L_{<}(Y, f_i, f_i(x^0))$, we get

$$0 \in L_{<}([0, 1], (f_i \circ l_{x^1, \tilde{x}}), f_i(x^0)) \cup (\mathbb{R} \setminus [0, 1]).$$

The openness of the set $L_{<}([0, 1], (f_i \circ l_{x^1, \tilde{x}}), f_i(x^0)) \cup (\mathbb{R} \setminus [0, 1])$ implies that there exists a $\bar{\lambda}_i \in \mathbb{R}$ with $0 < \bar{\lambda}_i < 1$ such that $f_i(l_{x^1, \tilde{x}}(\lambda)) < f_i(x^0)$ for all $\lambda \in (0, \bar{\lambda}_i]$.

Now, the point $x^2 := l_{x^1, \tilde{x}}(\min\{\bar{\lambda}_i | i \in I_{m+1}\})$ fulfils

$$x^2 \in S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0))$$

in contradiction to (3.8).

Consequently, we infer that (3.9) holds.

2°. Consider $x^0 \in \text{WEff}(X | f) \subseteq X$. Of course, we can have $x^0 \in \text{WEff}(Y | f)$ and therefore $x^0 \in X \cap \text{WEff}(Y | f)$. Let us assume that $x^0 \in X \setminus \text{WEff}(Y | f)$ holds. In view of Theorem 3.11, we know that $x^0 \in \text{WEff}(X | f)$ implies $x^0 \in X \cap \text{WEff}(Y | f^E)$. Consider two cases:

Case 1: Assume that $\text{int } X \neq \emptyset$. By Corollary 2.25, we get $x^0 \in \text{bd } X$ because of the componentwise semi-strictly quasi-convexity of f on Y .

Case 2: Assume that $\text{int } X = \emptyset$. Obviously, we have $x^0 \in \text{bd } X$.

Finally we obtain the desired statement $x^0 \in \text{bd } X \cap \text{WEff}(Y | f^E)$.

□

Remark 3.14. Note that the proof of Theorem 3.13 is a slight modification of the proof of [10, Th. 4]. Furthermore, notice that one can not replace the semi-strictly quasi-convexity assumption with respect to f in 2° of Theorem 3.13 by a quasi-convexity assumption (see [10, Ex. 2] and [10, Ex. 3] for the case $Y = \mathcal{V} = \mathbb{R}^n$).

Corollary 3.15. *Let (2.1) be satisfied and let $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2, as well as condition (3.7). Furthermore, suppose that f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y . Then, it holds that*

$$\begin{aligned} \text{WEff}(X | f) &= [X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)] \\ &= [\text{int } X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)]. \end{aligned}$$

Proof. Follows directly by Lemma 2.30, Theorem 3.11 and Theorem 3.13.

□

Corollary 3.16. *Let (2.1) be satisfied and let $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2, and moreover, let condition (3.7) be satisfied. Furthermore, suppose that f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y . If $\text{int } X \cap \text{WEff}(Y | f) = \emptyset$, then*

$$\text{WEff}(X | f) = \text{bd } X \cap \text{WEff}(Y | f^E) \subseteq \text{bd } X.$$

Theorem 3.17. *Let (2.1) be satisfied and let $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2, and moreover, let condition (3.7) be satisfied. Furthermore, suppose that f is componentwise upper semi-continuous along line segments on Y . If $\text{WEff}(X | f) \subseteq \text{bd } X$, then it holds that*

$$\text{WEff}(X | f) = \text{bd } X \cap \text{WEff}(Y | f^E).$$

Proof. The inclusion “ \supseteq ” follows by 1° of Theorem 3.13.

The reverse inclusion “ \subseteq ” follows by 1° of Theorem 3.11, taking into account the assumption $\text{WEff}(X | f) \subseteq \text{bd } X$ and Lemma 2.30. □

Theorem 3.18. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A3. Then the following statement is true:*

$$\text{WEff}(X | f) \subseteq X \cap \text{WEff}(Y | f^E) = X.$$

Proof. In view of Remark 2.27 and Lemma 2.30, the inclusion $\text{WEff}(X | f) \subseteq X \cap \text{WEff}(Y | f^E)$ follows by 1° of Theorem 3.11.

Now, we prove $X \cap \text{WEff}(Y | f^E) = X$. The inclusion “ \subseteq ” is obvious. Let us prove the reverse inclusion “ \supseteq ”.

Consider $x^0 \in X$. We have $L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset$ by Assumption A3, and it follows

$$(3.10) \quad S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset.$$

By (3.10) and Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), we infer the desired statement $x^0 \in X \cap \text{WEff}(Y | f^E)$. \square

Corollary 3.19. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A3. If $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y . Then, it holds that*

$$\begin{aligned} \text{WEff}(X | f) &\subseteq [X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)] \\ &= [\text{int } X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)] \\ &= [\text{int } X \cap \text{WEff}(Y | f)] \cup \text{bd } X. \end{aligned}$$

Proof. The assertion follows by statement 1° of Theorem 3.11, by statement 2° of Theorem 3.13 and by Theorem 3.18. Note that condition (2.4) is satisfied under Assumption A3 (see Remark 2.27 and Lemma 2.30) and that Theorem 3.18 implies $\text{bd } X \cap \text{WEff}(Y | f^E) = \text{bd } X$. \square

Remark 3.20. Obviously, the inclusion $\text{bd } X \subseteq \text{WEff}(X | f)$ does not hold for convex multi-objective location problems in general (see, e.g., [10, Ex. 5]). Hence, under Assumption A3 the inclusion

$$\text{WEff}(X | f) \supseteq [X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)]$$

does not hold in general, in contrast to the assumptions given in statement 1° of Theorem 3.13.

Due to the results given Theorem 3.13, Corollary 3.15, Theorem 3.18 and Corollary 3.19, it seems to be more appropriate to work with a function f_{m+1} that fulfils Assumptions A1 and A2 instead of Assumption A3 in order to compute the set $\text{WEff}(X | f)$.

3.2.2. Pareto reducibility.

In this section, our aim is to use the concept of *Pareto reducibility* of multi-objective optimization problems in order to prove some interesting results. According to Popovici in [19], the multi-objective optimization problem (\mathcal{P}_X) is called *Pareto reducible* if the set of weakly efficient solutions of (\mathcal{P}_X) can be represented as the union of the sets of efficient solutions of its subproblems.

Consider a selection of indices $I \subseteq I_{m+1}$ with cardinality $\text{card}(I) = k \geq 1$. Then we introduce a function

$$f_I = (f_{i_1}, \dots, f_{i_k}) : \mathcal{V} \rightarrow \mathbb{R}^k,$$

where the indices i_1, \dots, i_k are implicitly defined by $I = \{i_1 < \dots < i_k\}$. Note that $f_{I_m} = f$ and $f_{I_{m+1}} = f^E$ hold.

Let us introduce the following optimization problem:

$$(\mathcal{P}_X^I) \quad f_I(x) \rightarrow \text{v-min}_{x \in X}.$$

The problem (\mathcal{P}_X^I) is a single-objective optimization problem when I is a singleton set, otherwise being a multi-objective optimization problem. Moreover, for $\emptyset \neq I \subseteq I_m$, the problem (\mathcal{P}_X^I) can be seen as a subproblem of (\mathcal{P}_X) .

In addition, for $\emptyset \neq I \subseteq I_m$, we introduce the following subproblem of the extended problem (\mathcal{P}_Y^E) :

$$(\mathcal{P}_Y^{I^E}) \quad f_I^E(x) := \begin{pmatrix} f_I(x) \\ f_{m+1}(x) \end{pmatrix} \rightarrow \text{v-min}_{x \in Y}.$$

Next, we will recall sufficient conditions for Pareto reducibility (see the papers by Popovici [19, Prop. 4] and [21, Cor. 4.5]):

Proposition 3.21 ([19, 21]). *Consider the space \mathcal{V} given in (2.1) and assume that X is a nonempty convex set in \mathcal{V} . If f is componentwise semi-strictly quasi-convex and upper semi-continuous along line segments on X , then*

$$\text{WEff}(X | f) = \bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(X | f_I).$$

If \mathcal{V} is the n -dimensional Euclidean space \mathbb{R}^n and f is additionally componentwise lower semi-continuous along line segments on X , then

$$\text{WEff}(X | f) = \bigcup_{\substack{\emptyset \neq I \subseteq I_m; \\ \text{card } I \leq n+1}} \text{Eff}(X | f_I).$$

Now, we will give an alternative proof for the statement given in Corollary 3.15 using Corollary 3.3 and the Pareto reducibility given in Proposition 3.21.

Theorem 3.22. *Let (2.1) be satisfied, let X be convex, and let $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2. Furthermore, assume that f^E is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on the convex set Y . Then, it holds that*

$$\text{WEff}(X | f) = [X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)].$$

Proof. By Corollary 3.3, we have

$$(3.11) \quad [X \cap \text{Eff}(Y | f_I)] \cup [\text{bd } X \cap \text{Eff}(Y | f_I^E)] = \text{Eff}(X | f_I)$$

for all $\emptyset \neq I \subseteq I_m$. Moreover, under the Assumptions A2 and A3, it follows

$$\begin{aligned} \text{Eff}(Y \mid f_{m+1}) &= \text{argmin}_{x \in Y} f_{m+1}(x) \\ &\subseteq L_{\leq}(Y, f_{m+1}, f_{m+1}(\tilde{x})) \\ &\subseteq L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &= \text{int } X \end{aligned}$$

for points $\tilde{x} \in \text{int } X \neq \emptyset$ and $x^0 \in \text{bd } X$. This implies

$$(3.12) \quad \text{bd } X \cap \text{Eff}(Y \mid f_{m+1}) = \emptyset.$$

Consequently, applying Proposition 3.21 for the problems (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) , we get

$$\begin{aligned} \text{WEff}(X \mid f) &= \bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(X \mid f_I) \\ &\stackrel{(3.11)}{=} X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I) \right] \cup \text{bd } X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I^E) \right] \\ &\stackrel{(3.12)}{=} X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I) \right] \cup \text{bd } X \cap \left[\bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1}; \\ \{m+1\} \subseteq I}} \text{Eff}(Y \mid f_I) \right] \\ &= X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I) \right] \\ &\cup \text{bd } X \cap \left[\bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1}; \\ \{m+1\} \subseteq I}} \text{Eff}(Y \mid f_I) \cup \bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I) \right] \\ &= X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y \mid f_I) \right] \cup \text{bd } X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_{m+1}} \text{Eff}(Y \mid f_I) \right] \\ &= [X \cap \text{WEff}(Y \mid f)] \cup [\text{bd } X \cap \text{WEff}(Y \mid f^E)]. \end{aligned}$$

□

Note that the proof of Theorem 3.22 is a slight modification of the proof of [10, Th. 6].

The next theorem presents a Pareto reducibility type result for multi-objective optimization problems.

Theorem 3.23. *Let (2.1) be satisfied and let $\text{int } X \neq \emptyset$. Suppose that f_{m+1} fulfils Assumptions A1 and A2. Moreover, assume that f^E is component-wise semi-strictly quasi-convex as well as upper semi-continuous along line*

segments on Y . Then, it holds that

$$\text{WEff}(X | f) = \text{int } X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y | f_I) \right] \cup \text{bd } X \cap \left[\bigcup_{\emptyset \neq I \subseteq I_{m+1}} \text{Eff}(Y | f_I) \right].$$

If \mathcal{V} is the n -dimensional Euclidean space \mathbb{R}^n and f^E is additionally lower semi-continuous along line segments on Y , then

$$\text{WEff}(X | f) = \text{int } X \cap \left[\bigcup_{\substack{\emptyset \neq I \subseteq I_m; \\ \text{card } I \leq n+1}} \text{Eff}(Y | f_I) \right] \cup \text{bd } X \cap \left[\bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1}; \\ \text{card } I \leq n+1}} \text{Eff}(Y | f_I) \right].$$

Proof. By Corollary 3.15, it follows that

$$\text{WEff}(X | f) = [\text{int } X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)].$$

Applying Proposition 3.21 for both problems (\mathcal{P}_Y) and (\mathcal{P}_Y^E) , we get the desired statements in this theorem. \square

Remark 3.24. Let (2.3) be fulfilled, i.e., it holds $X = L_{\leq}(Y, f_{m+1}, 0)$ (see also Remark 2.34). By Lemma 2.51, we know that the assumptions given in Theorem 3.23 imply that the set X must be convex. Hence, all ideas to prove Theorem 3.23 are also given in the proof of Theorem 3.22.

If $Y = \mathcal{V}$, then Theorem 3.22 presents a representation of the set of weakly efficient solutions of the constrained problem (\mathcal{P}_X) using the sets of weakly efficient solutions of families of unconstrained (free) optimization problems.

Theorem 3.25. *Let (2.1) be satisfied and let X be convex. Suppose that f_{m+1} fulfils Assumption A3. Moreover, assume that f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y . Then, it holds that*

$$\text{WEff}(X | f) = X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1}; \\ \text{card } I \geq 2}} \text{Eff}(Y | f_I).$$

Proof. By Theorem 3.9, we have

$$(3.13) \quad X \cap \text{Eff}(Y | f_I^E) = \text{Eff}(X | f_I) \quad \text{for all } \emptyset \neq I \subseteq I_m.$$

Consequently, we obtain

$$\begin{aligned} \text{WEff}(X | f) &= \bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(X | f_I) \\ &\stackrel{(3.13)}{=} X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \text{Eff}(Y | f_I^E) \\ &= X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1}; \\ \text{card } I \geq 2}} \text{Eff}(Y | f_I) \end{aligned}$$

by Proposition 3.21.

□

Remark 3.26. Note, under Assumption A3, we have

$$\text{Eff}(Y | f_{m+1}) = L_=(Y, f_{m+1}, f_{m+1}(x^0)) = X$$

for every $x^0 \in X$. Consequently, it follows $X \cap \text{Eff}(Y | f_{m+1}) = X$ such that

$$\text{WEff}(X | f) \subseteq X = X \cap \bigcup_{\{m+1\} \subseteq I \subseteq I_{m+1}} \text{Eff}(Y | f_I).$$

3.3. Set of strictly efficient solutions of (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) .

The next theorem presents some relationships between the problems (\mathcal{P}_X) , (\mathcal{P}_Y) and (\mathcal{P}_Y^E) for the concept of strict efficiency. This theorem is a generalization of [10, Th. 7] where the special case $\mathcal{V} = Y = \mathbb{R}^n$, X closed and convex with $\text{int } X \neq \emptyset$, and f_{m+1} defined as a gauge function is considered.

Theorem 3.27. *Let (2.1) and (2.4) be satisfied. The following statements are true:*

1°. *It holds*

$$X \cap \text{SEff}(Y | f) \subseteq \text{SEff}(X | f) \subseteq X \cap \text{SEff}(Y | f^E).$$

2°. *If $S_{\leq}(Y, f, x^0) \subseteq L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0))$ for all $x^0 \in X \cap \text{SEff}(Y | f^E)$, then*

$$X \cap \text{SEff}(Y | f) = \text{SEff}(X | f) = X \cap \text{SEff}(Y | f^E).$$

Proof. 1°. By Corollary 2.23, we get $X \cap \text{SEff}(Y | f) \subseteq \text{SEff}(X | f)$. We now show the second inclusion.

Consider $x^0 \in \text{SEff}(X | f) \subseteq X$. In view of Lemma 2.7 (applied for the problem (\mathcal{P}_X)) and the assumption (2.4), we get

$$\begin{aligned} S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) &\subseteq S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(X, f, x^0) = \{x^0\}. \end{aligned}$$

Therefore, it follows $x^0 \in X \cap \text{SEff}(Y | f^E)$ by Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), i.e., assertion 1° holds.

2°. Consider some $x^0 \in X \cap \text{SEff}(Y | f^E)$. By the assumption $S_{\leq}(Y, f, x^0) \subseteq L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0))$ and by Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)), we infer

$$\{x^0\} = S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = S_{\leq}(Y, f, x^0).$$

So, it follows $x^0 \in X \cap \text{SEff}(Y | f)$ by Lemma 2.7 (applied for the problem (\mathcal{P}_X)).

Consequently, the inclusions in 1° of this theorem are equalities such that assertion 2° holds.

□

Now, we are able to formulate another important result of the paper concerning the set of strictly efficient solutions of (\mathcal{P}_X) (a generalization of [10, Th. 8]).

Theorem 3.28. *Let (2.1) be satisfied. The following statements are true:*

1°. *Under Assumption A1, we have*

$$[X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)] \subseteq \text{SEff}(X | f).$$

2°. *Assume (2.4) is true. If $\text{int } X \neq \emptyset$, assume that $f : \mathcal{V} \rightarrow \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on Y . We have*

$$[X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)] \supseteq \text{SEff}(X | f).$$

Proof. 1°. In view of Corollary 2.23, we have $X \cap \text{SEff}(Y | f) \subseteq \text{SEff}(X | f)$. Consider $x^0 \in \text{bd } X \cap \text{SEff}(Y | f^E)$. Taking into account Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)) and Assumption A1, it holds that

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = \{x^0\}. \end{aligned}$$

From Lemma 2.7 (applied for the problem (\mathcal{P}_X)), we get $x^0 \in \text{SEff}(X | f)$.

2°. Consider $x^0 \in \text{SEff}(X | f) \subseteq X$. If $x^0 \in \text{SEff}(Y | f)$, then $x^0 \in X \cap \text{SEff}(Y | f)$. We now suppose that $x^0 \in X \setminus \text{SEff}(Y | f)$ holds. By 1° of Theorem 3.27, we immediately get $x^0 \in X \cap \text{SEff}(Y | f^E)$. Let us consider two cases:

Case 1: If $\text{int } X \neq \emptyset$, then we conclude $x^0 \in \text{bd } X$ because of Corollary 2.25.

Case 2: If $\text{int } X = \emptyset$, then clearly it follows $x^0 \in \text{bd } X$. So, we infer that $x^0 \in \text{bd } X \cap \text{SEff}(Y | f^E)$ holds. □

Remark 3.29. Note that the proof of Theorem 3.28 is a slight modification of the proof of [10, Th. 8]. In contrast to 1° in Theorem 3.1 as well as 1° in Theorem 3.13, we only need the Assumption A1 concerning the level sets of the function f_{m+1} in 1° of Theorem 3.28. In accordance to 2° in Theorem 3.13, only condition (2.4) concerning the level sets of f_{m+1} must be fulfilled in 2° of Theorem 3.28. In 2° of Theorem 3.1 both assumptions A1 and A2 must be fulfilled.

Corollary 3.30. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A1. If $\text{int } X \neq \emptyset$, assume that f is componentwise semi-strictly quasi-convex or quasi-convex on Y . Then, it holds that*

$$\begin{aligned} \text{SEff}(X | f) &= [X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)] \\ &= [\text{int } X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)]. \end{aligned}$$

Proof. Follows immediately by 1° of Theorem 3.27 and Theorem 3.28. Note that condition (2.4) holds under Assumption A1 by Lemma 2.30. □

Corollary 3.31. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A1. If $\text{int } X \neq \emptyset$, assume that f is componentwise semi-strictly quasi-convex or quasi-convex on Y . If $\text{int } X \cap \text{SEff}(Y | f) = \emptyset$, then it holds that*

$$\text{SEff}(X | f) = \text{bd } X \cap \text{SEff}(Y | f^E) \subseteq \text{bd } X.$$

Theorem 3.32. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A1. If $\text{SEff}(X | f) \subseteq \text{bd } X$, then it holds that*

$$\text{SEff}(X | f) = \text{bd } X \cap \text{SEff}(Y | f^E).$$

Proof. Due to statement 1° of Theorem 3.28, it holds $\text{bd } X \cap \text{SEff}(Y | f^E) \subseteq \text{SEff}(X | f)$ under Assumption A1. By the assumption $\text{SEff}(X | f) \subseteq \text{bd } X$ and by 1° of Theorem 3.27, we get $\text{SEff}(X | f) \subseteq \text{bd } X \cap \text{SEff}(Y | f^E)$. Note that condition (2.4) is fulfilled under the Assumption A1 (see Lemma 2.30) \square

Next, we present a corresponding result to Theorem 3.9 for the concept of strict efficiency.

Theorem 3.33. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A3. Then the following statement is true:*

$$\text{SEff}(X | f) = X \cap \text{SEff}(Y | f^E).$$

Proof. First, we show the inclusion “ \supseteq ”, therefore consider $x^0 \in X \cap \text{SEff}(Y | f^E)$. Because of Lemma 2.7 (applied for the problem (\mathcal{P}_Y^E) instead of (\mathcal{P}_X)) and Assumption A3 it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = \{x^0\}. \end{aligned}$$

By Lemma 2.7 (applied for the problem (\mathcal{P}_X)), we conclude $x^0 \in \text{SEff}(X | f)$.

In view of statement 1° in Theorem 3.27, we get immediately the reverse inclusion “ \subseteq ”. Note that condition (2.4) holds by Lemma 2.30. \square

Corollary 3.34. *Let (2.1) be satisfied. Suppose that f_{m+1} fulfils Assumption A3. If $\text{int } X \neq \emptyset$, assume that f is componentwise semi-strictly quasi-convex or quasi-convex on Y . Then, it holds that*

$$\begin{aligned} \text{SEff}(X | f) &= [X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)] \\ &= [\text{int } X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)] \\ &= X \cap \text{SEff}(Y | f^E) \\ &= [\text{int } X \cap \text{SEff}(Y | f^E)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)]. \end{aligned}$$

Hence,

$$\text{int } X \cap \text{SEff}(Y | f) = \text{int } X \cap \text{SEff}(Y | f^E).$$

Proof. The proof follows by Corollary 3.30 and Theorem 3.33. \square

4. PROBLEMS INVOLVING CONSTRAINTS GIVEN BY A SYSTEM OF INEQUALITIES

In the previous section, the feasible set $X \subseteq \mathcal{V}$ was always represented by certain level sets of one scalar (penalization) function f_{m+1} (see Assumptions A1 and A3). However, in many cases the feasible set X is given by a system of inequalities, i.e., we have

$$X := \{x \in Y \mid g_1(x) \leq 0, \dots, g_q(x) \leq 0\} = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0)$$

for some constraint functions $g_1, \dots, g_q : \mathcal{V} \rightarrow \mathbb{R}$, $q \in \mathbb{N}$, and a convex set $Y \subseteq \mathcal{V}$. For notational convenience, let us consider $g := (g_1, \dots, g_q) : \mathcal{V} \rightarrow \mathbb{R}^q$ as the vector-valued constraint function.

In some results, we need the well-known *Slater condition* that is given by

$$(4.1) \quad \bigcap_{i \in I_q} L_{<}(Y, g_i, 0) \neq \emptyset.$$

4.1. Defining f_{m+1} by the maximum of the components of the constrained function g .

In order to apply results from Section 3, for the additional function f_{m+1} considered in (\mathcal{P}_Y^E) , we put

$$f_{m+1} := \max(g_1, \dots, g_q).$$

Then condition (2.3) is satisfied, i.e., we have

$$(4.2) \quad X = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0) = L_{\leq}(Y, f_{m+1}, 0).$$

Remark 4.1. If the function $g = (g_1, \dots, g_q) : \mathcal{V} \rightarrow \mathbb{R}^q$ is componentwise convex (quasi-convex) on Y , then f_{m+1} is convex (quasi-convex) on Y too. If Y is additionally a closed set, then componentwise lower semi-continuity of g implies that f_{m+1} is lower semi-continuous on Y (see Lemma 2.15). The componentwise quasi-convexity of g on Y implies that the set X is convex. Componentwise lower semi-continuity of g on a closed set Y ensures that the set X is closed.

Note that the standard assumption given in (2.1) is satisfied. Especially this means that the feasible set X is a nonempty and closed set that fulfils (4.2), and the set Y is convex.

Corollary 4.2. *Let (2.1) be satisfied and let Y be open. The upper semi-continuity of f_{m+1} on \mathcal{V} and the Slater condition (4.1) imply that*

$$\emptyset \neq \bigcap_{i \in I_q} L_{<}(Y, g_i, 0) = L_{<}(Y, f_{m+1}, 0) \subseteq \text{int } X,$$

hence X has a nonempty interior.

Proof. The Slater condition (4.1) guarantees that $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$, hence this result directly follows by Lemma 2.38. \square

In the following, we are looking for sufficient conditions such that Assumptions A1 and A2 are fulfilled for the function f_{m+1} .

The next six corollaries are an immediate consequence of Lemmata 2.40 and 2.41, as well as, of Corollaries 2.44, 2.46, 2.47, 2.48 and 4.2, respectively.

Corollary 4.3. *Let (2.1) be satisfied, let Y be open, and let f_{m+1} be upper semi-continuous on \mathcal{V} . Assume that the Slater condition (4.1) holds. Then Assumption A1 is fulfilled.*

Corollary 4.4. *Let (2.1) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that the Slater condition (4.1) holds. For every $x^0 \in \text{int } X \cap L_=(Y, f_{m+1}, 0)$, we suppose that there exists $\varepsilon > 0$ such that f_{m+1} is semi-strictly quasi-convex and quasi-convex (can be replaced by semi-strict quasi-convexity and lower semi-continuity along line segments) on $B_{\|\cdot\|}(x^0, \varepsilon)$. Moreover, we assume that there exists $x^1 \in B_{\|\cdot\|}(x^0, \varepsilon) \cap L_<(Y, f_{m+1}, 0)$. Then Assumptions A1 and A2 are fulfilled.*

Corollary 4.5. *Let (2.1) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that the Slater condition (4.1) is fulfilled. If f_{m+1} is explicitly quasi-convex (can be replaced by semi-strictly quasi-convexity and lower semi-continuity along line segments) on $B_{\|\cdot\|}(\tilde{x}, \varepsilon)$, where*

$$\varepsilon > \max\{\|x - \tilde{x}\| \mid x \in X\} \quad \text{and} \quad \tilde{x} \in L_<(Y, f_{m+1}, 0),$$

then Assumptions A1 and A2 hold.

Corollary 4.6. *Let (2.1) be satisfied and let Y be open. Suppose that $(\mathcal{V}, \|\cdot\|)$ is a normed space. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that the Slater condition (4.1) holds. The function f_{m+1} fulfils the Assumptions A1 and A2, if both of the following statements hold:*

- 1°. *Every local minimum point of f_{m+1} on $\text{int } X$ is also global.*
- 2°. *f_{m+1} is locally explicitly quasi-convex on $\text{int } X$.*

Corollary 4.7. *Let (2.1) be satisfied and let Y be open. Assume that f_{m+1} is upper semi-continuous on \mathcal{V} and that the Slater condition (4.1) is fulfilled. Moreover, suppose that f_{m+1} is explicitly quasi-convex on Y . Then Assumptions A1 and A2 hold.*

Corollary 4.8. *Let (2.1) be satisfied and let $Y = \mathcal{V}$. Assume that f_{m+1} is semi-strictly quasi-convex as well as continuous on \mathcal{V} and that the Slater condition (4.1) holds. Then both Assumptions A1 and A2 hold.*

Next, we present relationships between the original problem (\mathcal{P}_X) with constraint set X given by a system of inequalities and the objective function

$$f = (f_1, \dots, f_m),$$

and two related problems (\mathcal{P}_Y) and (\mathcal{P}_Y^E) with a convex feasible set Y and the objective functions

$$f = (f_1, \dots, f_m),$$

and

$$f^E = (f_1, \dots, f_m, f_{m+1}) = (f_1, \dots, f_m, \max(g_1, \dots, g_q)),$$

respectively.

Theorem 4.9. *Let (2.1) be satisfied, let Y be open, and let f_{m+1} be upper semi-continuous on \mathcal{V} . Assume that the Slater condition (4.1) holds. Then we have:*

1°. *Let the Assumption A2 be fulfilled. If f is componentwise semi-strictly quasi-convex on Y , then*

$$(4.3) \quad \text{Eff}(X | f) = [X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)].$$

2°. *Assume that Assumption A2 and condition (3.7) hold. If f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y , then*

$$(4.4) \quad \text{WEff}(X | f) = [\text{int } X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)].$$

3°. *If f is componentwise semi-strictly quasi-convex or quasi-convex on Y , then*

$$(4.5) \quad \text{SEff}(X | f) = [\text{int } X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)].$$

Proof. By Corollary 4.3, we know that Assumption A1 is fulfilled. The Slater condition (4.1) implies that $\text{int } X \neq \emptyset$ by Corollary 4.2. Hence, this theorem follows directly by Corollaries 3.3, 3.15 and 3.30. Note, in view of (4.2) and Lemma 2.30, the condition (2.4) is fulfilled. \square

Corollary 4.10. *Let (2.1) be satisfied, let Y be open, and let f_{m+1} be upper semi-continuous on \mathcal{V} and explicitly quasi-convex on Y . Assume that the Slater condition (4.1) holds. Then we have:*

1°. *If f is componentwise semi-strictly quasi-convex on Y , then (4.3) holds.*

2°. *If f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y , then (4.4) holds.*

3°. *If f is componentwise semi-strictly quasi-convex or quasi-convex on Y , then (4.5) holds.*

Proof. Follows directly by Theorem 4.9. Note that Assumption A2 is fulfilled by Corollary 4.7 and condition (3.7) is satisfied for the semi-strictly quasi-convex function f_{m+1} on Y . \square

Remark 4.11. Note that the set X considered in Theorem 4.9 and Corollary 4.10 must be closed (e.g., if Y is closed and f_{m+1} is lower semi-continuous on Y).

Corollary 4.12. *Let (2.1) be satisfied, let $Y = \mathcal{V}$, and assume that f_{m+1} is semi-strictly quasi-convex and continuous on \mathcal{V} . Furthermore, suppose that the Slater condition (4.1) holds. Then we have:*

1°. *If f is componentwise semi-strictly quasi-convex on \mathcal{V} , then (4.3) holds for $Y = \mathcal{V}$.*

- 2°. If f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on \mathcal{V} , then (4.4) holds for $Y = \mathcal{V}$.
- 3°. If f is componentwise semi-strictly quasi-convex or quasi-convex on \mathcal{V} , then (4.5) holds for $Y = \mathcal{V}$.

4.2. Defining f_{m+1} by a gauge function for a convex feasible set X .

Under the assumption (2.1), we assume that the set $X = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0)$ is additionally convex. Moreover, let the Slater condition (4.1) be satisfied. Consequently, we know by Lemma 2.38 that there exists

$$\tilde{x} \in \bigcap_{i \in I_q} L_{<}(Y, g_i, 0) \subseteq \text{int } X,$$

if Y is open and g is componentwise upper semi-continuous on \mathcal{V} .

Remark 4.13. Note that the convexity of X does not imply that $L_{\leq}(Y, g_i, 0)$ is convex for every $i \in I_q$, as shown by the following Example 4.14.

Example 4.14. For $Y = \mathbb{R}^2$, we consider the constraint set X given by the two inequalities

$$\begin{aligned} g_1(x) &:= -x_2 \leq 0 \quad \text{for every } x = (x_1, x_2) \in \mathbb{R}^2, \\ g_2(x) &:= \min\{-x_1, -x_2\} \leq 0 \quad \text{for every } x = (x_1, x_2) \in \mathbb{R}^2. \end{aligned}$$

The function g_1 is quasi-convex on \mathbb{R}^2 , but g_2 is not quasi-convex on \mathbb{R}^2 . However, we have

$$\begin{aligned} X &= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid -x_2 \leq 0 \wedge (-x_2 \leq 0 \vee -x_1 \leq 0)\} \\ &= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0 \vee x \in \mathbb{R}_+^2\} \\ &= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_2 \geq 0\}, \end{aligned}$$

hence X is a convex set in \mathbb{R}^2 .

In the following, we consider a point $\tilde{x} \in \text{int } X$. Using the ideas from the paper [10], we can consider a gauge function $\mu : \mathcal{V} \rightarrow \mathbb{R}$, with an associated unit ball $B_\mu := -\tilde{x} + X$, defined in the following way

$$\mu(x) := \inf\{\lambda > 0 \mid x \in \lambda \cdot B_\mu\} \quad \text{for all } x \in \mathcal{V}.$$

In order to apply results from Section 3, for the additional function f_{m+1} considered in (\mathcal{P}_Y^E) , we now put

$$f_{m+1}(\cdot) := \mu(\cdot - \tilde{x}).$$

Lemma 4.15. *Let (2.1) be satisfied, let X be convex, and let $\tilde{x} \in \text{int } X \neq \emptyset$. The function $f_{m+1}(\cdot) = \mu(\cdot - \tilde{x})$ fulfils the Assumptions A1 and A2 for every $Y \subseteq \mathcal{V}$ with $X \subseteq Y$.*

Proof. Due to the convexity and continuity of the function $\tilde{f}_{m+1}(\cdot) := \mu(\cdot - \tilde{x}) - 1$ on $Y = \mathcal{V}$, we know by Corollary 2.48 that \tilde{f}_{m+1} fulfils Assumptions A1 and A2 concerning $X = L_{<}(\mathcal{V}, \tilde{f}_{m+1}, 0)$. Note that $\tilde{x} \in L_{<}(\mathcal{V}, \tilde{f}_{m+1}, 0)$ holds. Hence, by Corollary 2.32, we get that $f_{m+1}(\cdot) = \mu(\cdot - \tilde{x})$ fulfils Assumptions A1 and A2 too. By Lemma 2.33, we get that the Assumptions A1 and A2 are fulfilled for every set $Y \subseteq \mathcal{V}$ with $X \subseteq Y$. Note that X is a closed set. \square

Note that some properties of the function f_{m+1} are discussed in the paper by Günther and Tammer [10] for the case $Y = \mathcal{V}$.

Next, under the assumption (2.1), we present relationships between the original problem (\mathcal{P}_X) and two related problems (\mathcal{P}_Y) and (\mathcal{P}_Y^E) with convex feasible set $Y \subseteq \mathcal{V}$. The objective function of the problem (\mathcal{P}_Y^E) is now given by

$$f^E = (f_1, \dots, f_m, f_{m+1}) = (f_1, \dots, f_m, \mu(\cdot - \tilde{x})).$$

Theorem 4.16. *Let (2.1) be satisfied, let $\tilde{x} \in \text{int } X \neq \emptyset$ (e.g., if, under the Slater condition (4.1), the set Y is open and $g : \mathcal{V} \rightarrow \mathbb{R}^q$ is componentwise upper semi-continuous on \mathcal{V}), and let X be convex (e.g., if $g : \mathcal{V} \rightarrow \mathbb{R}^q$ is componentwise quasi-convex on Y). Then we have:*

1°. *If f is componentwise semi-strictly quasi-convex on Y , then*

$$(4.6) \quad \text{Eff}(X | f) = [X \cap \text{Eff}(Y | f)] \cup [\text{bd } X \cap \text{Eff}(Y | f^E)].$$

2°. *If f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y , then*

$$(4.7) \quad \text{WEff}(X | f) = [\text{int } X \cap \text{WEff}(Y | f)] \cup [\text{bd } X \cap \text{WEff}(Y | f^E)].$$

3°. *If f is componentwise semi-strictly quasi-convex or quasi-convex on Y , then*

$$(4.8) \quad \text{SEff}(X | f) = [\text{int } X \cap \text{SEff}(Y | f)] \cup [\text{bd } X \cap \text{SEff}(Y | f^E)].$$

Proof. By Lemma 4.15, we know that f_{m+1} fulfils Assumptions A1 and A2 concerning a set $Y \subseteq \mathcal{V}$ with $X \subseteq Y$. Moreover, condition (3.7) is fulfilled for the convex function f_{m+1} . Consequently, Theorem 4.16 follows directly by Corollaries 3.3, 3.15 and 3.30. \square

Note that in Theorem 4.16 the feasible set X must be closed (e.g., if Y is closed and g is componentwise lower semi-continuous on Y).

Corollary 4.17. *Let (2.1) be satisfied, let $Y = \mathcal{V}$, and let X be convex (e.g., if $g : \mathcal{V} \rightarrow \mathbb{R}^q$ is componentwise quasi-convex on \mathcal{V}). Moreover, assume that $g : \mathcal{V} \rightarrow \mathbb{R}^q$ is componentwise continuous on \mathcal{V} . Suppose that the Slater condition (4.1) holds (i.e., $\tilde{x} \in \bigcap_{i \in I_q} L_{<}(Y, g_i, 0) \subseteq \text{int } X$). Then we have:*

1°. *If f is componentwise semi-strictly quasi-convex on \mathcal{V} , then (4.6) holds for $Y = \mathcal{V}$.*

- 2°. If f is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on \mathcal{V} , then (4.7) holds for $Y = \mathcal{V}$.
- 3°. If f is componentwise semi-strictly quasi-convex or quasi-convex on \mathcal{V} , then (4.8) holds for $Y = \mathcal{V}$.

5. CONCLUDING REMARKS

In this paper, we derived a new approach for solving generalized-convex multi-objective optimization problems involving (not necessarily convex) constraints. These results extend and generalize the results given by Günther and Tammer in [10]. We showed that the set of efficient solutions (in an arbitrarily real topological linear space) of a multi-objective optimization problem involving a nonempty closed (not necessarily convex) feasible set, can be computed completely using two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set. Our approach relies on the fact that the original feasible set can be described using level sets of a certain scalar function (see Assumptions A1, A2 and A3). We applied our approach to problems where the feasible set is given by a system of inequalities with a finite number of constraint functions. For deriving our new results, we assumed that the well-known Slater constraint condition is fulfilled.

In a forthcoming paper, we apply our results to special types of nonconvex multi-objective optimization problems. It is interesting to study problems where the nonconvex feasible set is given by a union of convex sets, as well as problems involving multiple forbidden regions. Such problems can be motivated by several models in location theory. Moreover, we aim to publish in a forthcoming work new optimality conditions for (strictly, weakly) efficient solutions of multi-objective optimization problem using our results presented in this paper.

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