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Strong stability preserving explicit peer methods for discontinuous Galerkin discretizations

Marcel Klinge · Rüdiger Weiner

Abstract In this paper we study explicit peer methods with the strong stability preserving (SSP) property for the numerical solution of hyperbolic conservation laws in one space dimension. A system of ordinary differential equations (ODEs) is obtained by discontinuous Galerkin (DG) spatial discretizations, which are often used in the method of lines (MOL) approach to solve hyperbolic differential equations. We present in this work the construction of explicit peer methods with stability regions that are designed for DG spatial discretizations and with SSP property. Methods of second- and third order with up to six stages are optimized. The methods constructed are tested and compared with appropriate Runge–Kutta (RK) methods. The advantage of high stage order is verified numerically.

Keywords discontinuous Galerkin · strong stability preserving · explicit peer methods

Mathematics Subject Classification (2000) 65L05 · 65L06

1 Introduction

In this paper we consider methods for the numerical solution of hyperbolic conservation laws in one dimension in space in the form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\big(f(u(x,t))\big) = 0. \tag{1}$$

Time-dependent hyperbolic differential equations and nonlinear conservation laws model many physical problems. Because of that their numerical solution has a great importance in fields of meteorology, chemical engineering, aeronautics, astrophysics financial modeling and environmental sciences. The main difficulty for the numerical solution of (1) is the appearance of shocks or discontinuities even if the initial condition is smooth. Discretization of spatial derivatives with the

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method of lines (MOL) gives rise to a system of ordinary differential equations (ODEs). In recent years discontinuous Galerkin (DG) methods have become a popular choice for the spatial discretization. This approach introduced by Reed and Hill [20] was considered in various applications, e.g. [14,15,18,19,21,25]. The ODE system generated by DG spatial discretizations is often solved in time by explicit Runge–Kutta (RK) methods. These methods developed by Cockburn and Shu [4] are known as RKDG methods.

The time steps of RKDG methods must satisfy the Courant-Friedrichs-Levy (CFL) condition. For hyperbolic differential equations there are two kinds of restrictions for the time steps, first, linear stability to ensure convergence for smooth solutions and second, some forms of nonlinear stability, e.g. total variation (TV) stability, to prevent non-physical oscillations of the numerical solution around discontinuities or shocks. In order to achieve that a so-called generalized slope limiter is applied by which RKDG methods are total variation diminishing (TVD) and total variation bounded (TVB) in the means under a suitable CFL restriction [4]. The restriction for the time step is provided by the so-called strong stability preserving (SSP) coefficient. There are many investigations in the field of SSP RK methods, an overview is given in [7].

However, Kubatko *et al.* [16] observed that for RK methods the condition on linear stability is more restrictive than the condition on nonlinear stability. This means that methods optimized with respect to the SSP property are not optimal, in general, for schemes resulting from DG spatial discretizations. Hence, in [16] SSP RK methods have been constructed that are optimal for DG spatial discretizations. So, the schemes are optimal concerning both linear and SSP stability. In this paper we develop new explicit peer methods with stability domains designed for DG spatial discretizations and nonlinear stability.

Explicit peer methods introduced by Weiner *et al.* [26] are a special class of general linear methods (GLM), see [1,10]. Explicit peer methods are considered in several papers [2,17,23,27]. These methods produce excellent results, especially in the application to nonstiff ordinary differential equations with step size control. For explicit peer methods the stage order is equal to the order of consistency. Constantinescu und Sandu [5] constructed optimal SSP GLM up to order four. In [9] Horváth *et al.* proved a sufficient condition for the SSP property of explicit peer methods and constructed SSP methods up to order 13.

The outline of the paper is as follows: In Sect. 2 we discuss the discretization of the spatial operator of a hyperbolic conservation law in one space dimension with discontinuous Galerkin method. An overview of explicit peer methods is given in Sect. 3. We mention important properties like consistency, zero-stability and convergence. A sufficient condition for strong stability preserving explicit peer methods is covered and the TVDM (TVD in the means) property for explicit peer methods is also stated. The construction of explicit peer methods optimized with respect to linear and SSP stability is detailed in Sect. 4. In Sect. 5 numerical tests are presented. We apply the peer methods to a linear transport equation and the inviscid Burgers equation and compare the results with RK methods. We show numerically the advantage of high stage order. Finally, we draw some conclusions and discuss future work in Sect. 6.

The coefficients of the optimized explicit peer methods can be found in an Appendix.

2 Discontinuous Galerkin spatial discretization

We begin by considering a time-dependent hyperbolic conservation law in one dimension in space

$$\frac{\partial}{\partial t}u(x,t)+\frac{\partial}{\partial x}\big(f(u(x,t))\big)=0,\quad x\in\Omega=(a,b)\subseteq\mathbb{R},\quad t>0 \tag{2}$$

with periodic boundary conditions $u(a,t)=u(b,t)$ and initial condition $u_0(x)=u(x,0)$. The function f is usually said to be the flux function.

First, we define for the approach of discontinuous Galerkin spatial discretization a partition of Ω

$$a=x_0<x_1<\cdots<x_N=b$$

with

$$\Omega_j=[x_{j-1},x_j]\,,\quad \Delta x_j=x_j-x_{j-1},\quad \text{for all } j=1,\dots,N.$$

The weak formulation of problem (2) is obtained by multiplying with a sufficiently smooth test function $v\colon\Omega\to\mathbb{R}$ and integrating over each Ω_j, so we have

$$\begin{aligned}0&=\int_{\Omega_j}\frac{\partial}{\partial t}u(x,t)v(x)\,\mathrm{d}x+\int_{\Omega_j}\frac{\partial}{\partial x}\big(f(u(x,t))\big)v(x)\,\mathrm{d}x\\&=\int_{\Omega_j}\frac{\partial}{\partial t}u(x,t)v(x)\,\mathrm{d}x+\big[f(u(x,t))v(x)\big]_{x_{j-1}}^{x_j}-\int_{\Omega_j}f(u(x,t))\frac{\mathrm{d}}{\mathrm{d}x}v(x)\,\mathrm{d}x,\\&j=1,\dots,N.\end{aligned}$$

The boundary flux terms are denoted by $f_j(t):=f(u(x_j,t))$, which leads to

$$\begin{aligned}0=&\int_{\Omega_j}\frac{\partial}{\partial t}u(x,t)v(x)\,\mathrm{d}x-\int_{\Omega_j}f(u(x,t))\frac{\mathrm{d}}{\mathrm{d}x}v(x)\,\mathrm{d}x+f_j(t)v(x_j)\\&-f_{j-1}(t)v(x_{j-1}),\quad j=1,\dots,N.\end{aligned} \tag{3}$$

Second, we replace the exact solution u and the test function v in (3) by the discrete functions $u_h\in V_h^k$ and $v_h\in V_h^k$, where the finite-dimensional space of functions is given by

$$V_h^k:=\left\{v\colon v|_{\Omega_j}\in\Pi_k(\Omega_j)\right\}.$$

Here $\Pi_k(\Omega_j)$ denotes the space of polynomials over Ω_j of degree at most k. By this choice the functional continuity is not ensured at the boundaries of Ω_j. We denote the left function value of $v_h\in V_h^k$ at x_j over Ω_j by $v_h(x_j^-)$ and the right function value of $v_h\in V_h^k$ at x_j over Ω_{j+1} by $v_h(x_j^+)$, see Figure 1. We replace the boundary fluxes f_j by numerical fluxes $\widehat{f}_j$. These substitutions lead to the discrete weak formulation of problem (2) for all $v_h\in V_h^k$ in the form

$$\begin{aligned}0=&\int_{\Omega_j}\frac{\partial}{\partial t}u_h(x,t)v_h(x)\,\mathrm{d}x-\int_{\Omega_j}f(u_h(x,t))\frac{\mathrm{d}}{\mathrm{d}x}v_h(x)\,\mathrm{d}x+\widehat{f}_j(t)v_h(x_j^-)\\&-\widehat{f}_{j-1}(t)v_h(x_{j-1}^+),\quad j=1,\dots,N.\end{aligned} \tag{4}$$

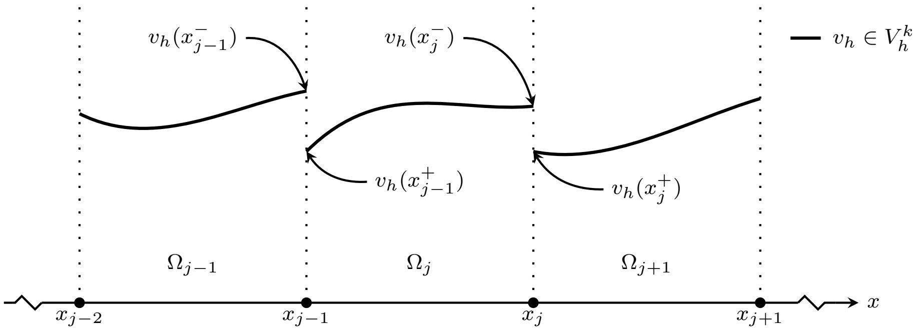


Fig. 1 Illustration of discontinuities of $v_h \in V_h^k$ at boundaries of Ω_j.

Let a set of basic functions $\Phi = (\varphi_{0,j}, \dots, \varphi_{k,j})^\top$ for the finite-dimensional space V_h^k over Ω_j be given. The discrete solution u_h over Ω_j can be expressed as

$$u_h|_{\Omega_j} = \sum_{i=0}^{k} y_{i,j}(t)\varphi_{i,j}(x).$$

With $\mathbf{y}_j = \big(y_{0,j}(t), \dots, y_{k,j}(t)\big)^\top$ and the notation

$$F_j(\varphi_{i,j}) = \int_{\Omega_j} f(u_h(x,t))\frac{\mathrm{d}}{\mathrm{d}x}\varphi_{i,j}(x)\,\mathrm{d}x + \widehat{f}_j(t)\varphi_{i,j}(x_j^-) - \widehat{f}_{j-1}(t)\varphi_{i,j}(x_{j-1}^+)$$

and

$$\mathbf{M}_j = \big(M_{il}\big)_{i,l=0}^{k} = \left(\int_{\Omega_j} \varphi_{i,j}(x)\varphi_{l,j}(x)\,\mathrm{d}x\right)_{i,l=0}^{k}, \quad \mathbf{F}_j = \big(F_j(\varphi_{0,j}), \dots, F_j(\varphi_{k,j})\big)^\top$$

the discrete weak formulation (4) can be written as

$$\mathbf{M}_j \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y}_j = \mathbf{F}_j, \quad j = 1, \dots, N.$$

The DG approach is also applied to the initial condition. By inverting the element mass matrices $\mathbf{M}_j$ we obtain with

$$\mathbf{y} = \left(\mathbf{y}_1^\top, \dots, \mathbf{y}_N^\top\right)^\top, \qquad \mathbf{L} = \left[\left(\mathbf{M}_1^{-1}\mathbf{F}_1\right)^\top, \dots, \left(\mathbf{M}_N^{-1}\mathbf{F}_N\right)^\top\right]^\top$$

a system of ordinary differential equations in the form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{y} &= \mathbf{L}(\mathbf{y}) \\ \mathbf{y}(0) &= \mathbf{y}_0 \in \mathbb{R}^{(k+1)N}, \end{aligned} \tag{5}$$

where $\mathbf{L}$ is the DG spatial operator. In the case of a linear flux function $f(u) = cu,\ c \in \mathbb{R}$, problem (5) leads to a linear initial value problem, i.e. we have $\mathbf{L}(\mathbf{y}) = \mathbf{L}\mathbf{y}$. In the linear case the integrals appearing in the DG approach can be solved exactly. Otherwise, they can be computed by using suitable numerical quadrature rules.

3 Strong stability preserving explicit peer methods

3.1 Explicit peer methods

We consider a system of ODEs in the form

$$y' = f(t,y), \quad y(t_0) = y_0 \in \mathbb{R}^n, \quad t \in [t_0, t_e]\,. \tag{6}$$

Explicit peer methods for an initial value problem (6) as introduced in [26] are given by

$$U_{m,i} = \sum_{j=1}^{s} b_{ij} U_{m-1,j} + h_m \sum_{j=1}^{s} a_{ij} f(t_{m-1,j}, U_{m-1,j}) + h_m \sum_{j=1}^{i-1} r_{ij} f(t_{m,j}, U_{m,j}), \quad i = 1,\dots,s. \tag{7}$$

Here b_{ij}, a_{ij}, c_i and r_{ij}, $i,j = 1,\dots,s$ are the parameters of the method. At each step s stage values $U_{m,i}$, $i = 1,\dots,s$ are computed. They approximate the exact solution $y(t_{m,i})$ where $t_{m,i} = t_m + c_i h_m$. The nodes c_i are assumed to be pairwise distinct, we always assume $c_s = 1$. The coefficients of the method (7) depend, in general, on the step size ratio $\sigma_m = h_m/h_{m-1}$. Defining matrices $B_m = (b_{ij})_{i,j=1}^s$, $A_m = (a_{ij})$, $R_m = (r_{ij})$ and vectors $U_m = (U_{m,i})_{i=1}^s \in \mathbb{R}^{sn}$ and $F_m = (f(t_{m,i}, U_{m,i}))_{i=1}^s$ leads to the compact form

$$U_m = (B_m \otimes I) U_{m-1} + h_m (A_m \otimes I) F_{m-1} + h_m (R_m \otimes I) F_m,$$

where R_m is strictly lower triangular. Like multistep methods peer methods need also s starting values $U_{0,i}$. We collect here some results from [26].

Definition 1 A peer method (7) is consistent of order p if

$$\Delta_{m,i} = \mathcal{O}(h_m^{p+1}), \quad i = 1,\dots,s. \qquad \square$$

By Taylor series expansion follows, see e.g. [26]

Theorem 1 *A peer method* (7) *has order of consistency p iff*

$$AB_i(l) := c_i^l - \sum_{j=1}^{s} b_{ij} \frac{(c_j - 1)^l}{\sigma_m^l} - l \sum_{j=1}^{s} a_{ij} \frac{(c_j-1)^{l-1}}{\sigma_m^{l-1}} - l \sum_{j=1}^{i-1} r_{ij} c_j^{l-1} = 0, \quad i = 1,\dots,s \tag{8}$$

is satisfied for all $l = 0,\dots,p$. $\square$

We set $\mathbb{AB}(l) := (AB_i(l))_{i=1}^s$. The condition (8) can also be written in the form

$$\exp(c\sigma_m z) - B_m \exp(z(c-\mathbb{1})) - A_m \sigma_m z \exp(z(c-\mathbb{1})) - R_m \sigma_m z \exp(c\sigma_m z) = \mathcal{O}(z^{p+1}),$$

where $\mathbb{1} = (1,\dots,1)^\top$. The exponentials are defined componentwise. The condition (8) for order $l = 0$ is referred to as preconsistency. It takes the form

$$B_m \mathbb{1} = \mathbb{1}. \tag{9}$$

Definition 2 A peer method (7) is zero stable if there is a constant $K > 0$, so that for all $m, k \geq 0$ holds

$$\|B_{m+k} \cdots B_{m+1} B_m\| \leq K. \qquad \square$$

Consistence and zero stability are necessary for convergence of peer methods. One has [27]

Theorem 2 *Let a peer method* (7) *be given. If the method is consistent of order p and zero stable and the starting values satisfy* $U_{0,i} - y(t_{0,i}) = \mathcal{O}(h_0^p)$, *then the peer method is convergent of order p.* $\square$

Let the scalar test equation [6]

$$y'(t) = \lambda y(t), \quad \lambda \in \mathbb{C}, \quad \Re\lambda \leq 0$$

be given. The application of a peer method (7) with constant step size h leads to

$$U_m = M(z) U_{m-1}, \quad z = \lambda h,$$

where $M\colon \mathbb{C} \to \mathbb{C}^{s,s}$ with $M(z) = (I - zR)^{-1}(B + zA)$ denotes the stability matrix [26]. The stability domain of a peer method (7) is defined by

$$S = \{z \in \mathbb{C}\colon \varrho(M(z)) \leq 1;\ \text{eigenvalues } \lambda_z \text{ of } M(z) \text{ with } |\lambda_z| = 1 \text{ are simple}\},$$

where $\varrho(\cdot)$ is the spectral radius.

Explicit peer methods were tested successfully with step size control. In fact, explicit peer methods of order of consistency $p = s$ and order of convergence $p = s + 1$ were constructed [27]. In the following investigations we always consider a constant time step size.

3.2 SSP property for explicit peer methods

Strong stability preserving (SSP) explicit peer methods are investigated in [9]. Optimal methods up to order 13 are constructed and tested on semidiscretized hyperbolic equations. We summarize here some results from [9].

Definition 3 Let an autonomous initial value problem (6) be given and let there exist $\Delta t_{\mathrm{FE}} > 0$ so that

$$\|y + \Delta t_{\mathrm{FE}} f(y)\| \leq \|y\|, \quad \text{for all } y \in \mathbb{R}^n \quad \text{and} \quad 0 \leq \Delta t \leq \Delta t_{\mathrm{FE}}$$

holds true, where $\|\cdot\|$ is a norm or a convex functional. An explicit peer method (7) is strong stability preserving with SSP coefficient $\mathcal{C} > 0$ if for all $0 \leq \Delta t \leq \mathcal{C}\Delta t_{\mathrm{FE}}$ the condition

$$\max_{i=1,\ldots,s} \|U_{m,i}\| \leq \max_{i=1,\ldots,s} \|U_{m-1,i}\|$$

is satisfied. $\square$

We indicate the following sufficient condition for SSP property of explicit peer methods [9]

Theorem 3 *Let an explicit peer method* (7) *be given. Assume that*

$$\mathcal{C} = \max_{r \in \mathbb{R}^+} \left\{ r\colon g(r) = (I + rR)^{-1} [R, A, B - rA] \geq 0 \right\} > 0$$

holds true, where the inequalities are meant componentwise. Then this explicit peer method is strong stability preserving with SSP coefficient $\mathcal{C}$. $\square$

3.3 TVDM property for explicit peer methods

We consider a system of ODEs in the form (6) resulting from the DG spatial discretization. Applying the forward Euler method in time and a generalized slope limiter [3] yields a scheme, which is TVDM (total variation diminishing in the means) under the condition [4]

$$\Delta t \leq \Delta t_{\mathrm{FE}} = \frac{\min_j \Delta x_j}{2(L_1 + L_2)}. \tag{10}$$

Here L_1 and L_2 denote the Lipschitz constants of the numerical flux $\widehat{f}$ with respect to the first and second argument. Considering a uniform mesh in space and a linear flux function $f(u) = cu,\ c > 0$ with the numerical upwind flux $\widehat{f_j}(t) = cu_h(x_j^-, t)$ condition (10) gives rise to

$$|c| \frac{\Delta t}{\Delta x} \leq \Delta t_{\mathrm{FE}} = \frac{1}{2}.$$

This can be applied to high order RK schemes [4]. Under the assumptions above, a RKDG method is TVDM under the condition [4]

$$|c| \frac{\Delta t}{\Delta x} \leq \nu(\mathcal{C}) = \frac{1}{2} \min_{i,j} \frac{\alpha_{ij}}{\beta_{ij}} = \frac{1}{2}\mathcal{C},$$

where $\mathcal{C}$ is referred to as SSP coefficient of the RK method. Moreover, α_{ij} and β_{ij} are the coefficients of the RK method in Shu-Osher representation [12]. Note that the ratio $\frac{\alpha_{ij}}{\beta_{ij}}$ is set to be infinite if $\beta_{ij} = 0$.

A similar result can be proved for peer methods. Considering again the assumptions above, these schemes are TVDM under the condition

$$|c| \frac{\Delta t}{\Delta x} \leq \nu(\mathcal{C}) = \frac{1}{2} \max_{r \in \mathbb{R}^+} \left\{ r \colon (I + rR)^{-1} \left[R, A, B - rA\right] \geq 0 \right\} = \frac{1}{2}\mathcal{C},$$

where $\mathcal{C}$ is the SSP coefficient of the peer method, cf. Theorem 3. A proof and more details are stated in [13].

High order RKDG schemes or DG peer methods are linearly stable under a condition in the form

$$|c| \frac{\Delta t}{\Delta x} \leq \mu(k, S), \tag{11}$$

where μ depends on the degree k of the DG spatial discretization and the absolute stability region S of the method. Condition (11) can be expressed as

$$\Delta t \lambda \in S \quad \text{for all } \lambda \in \Lambda.$$

Here Λ denotes the set of eigenvalues of the DG spatial operator $\mathbf{L}$. Then the maximum time step size for linear stability is given by

$$\mu(k, S) = \max_{\Delta t > 0} \left\{ \Delta t \colon \Delta t \lambda \in S \text{ for all } \lambda \in \Lambda \right\}.$$

SSP methods with optimal SSP coefficient are studied in many applications, e.g. summarized in [7] for RK methods and [9] for peer methods. But in the context of methods applied to ODEs resulting from the DG spatial discretization the condition

$$|c|\,\frac{\Delta t}{\Delta x} \leq \kappa = \min\left(\mu(k,S), \nu(\mathcal{C})\right) \tag{12}$$

must be satisfied to guarantee linear and SSP stability. Kubatko *et al.* [16] have constructed SSP RKDG methods up to order four that are optimal concerning linear stability. For this they optimized the stability function of the RKDG method. The free parameters of the RK scheme are used to optimize the SSP stability. For all considered stages and orders a method with $\nu \geq \mu_{\text{opt}}$ could be found, i.e. condition (12) is successfully optimized with $\kappa = \mu_{\text{opt}}$.

4 DGSSP-optimal explicit peer methods

In this section, we present an approach of constructing SSP explicit peer methods that are optimal for discontinuous Galerkin spatial discretizations.

The optimization of DG-optimal SSP RK methods is split in two main steps. First, the coefficients of the stability function are optimized with respect to linear stability depending on the degree of the DG spatial discretization. This procedure is described in detail in [11]. Second, the coefficients of the RK method are determined subject to the stability polynomial found in the previous step with the goal of maximization the SSP coefficient, see [12].

This principle cannot be applied to peer methods directly, since a stability matrix must be considered instead of a scalar stability function. We proceed therefore as follows:

Step 1: Optimization of μ: Searching for coefficients A, B, R and nodes c of a peer method so that the parameters are optimal with respect to linear stability.

Let an explicit peer method (7) of order p and a DG spatial operator $\mathbf{L}$ be given. We denote by Λ a finite set of eigenvalues of the DG spatial operator. Then the optimization problem can be written as

$$\begin{aligned} &\max_{A,B,R,c,\Delta t} \;\Delta t \\ &\text{subject to} \qquad \varrho\left(M\left(\Delta t\lambda\right)\right) \leq 1, \quad \text{for all } \lambda \in \Lambda, \\ &\qquad\qquad\qquad\qquad \mathbb{AB}(l) = 0, \quad l = 0, \ldots, p. \end{aligned} \tag{13}$$

Analogous to [11] we reformulate problem (13) in terms of the least deviation problem and apply a bisection approach. This is summarized in Algorithm 1. The output Δt_ε of Algorithm 1 satisfies

$$\lim_{\varepsilon \to 0} |c|\,\frac{\Delta t_\varepsilon}{\Delta x} = \mu.$$

Note, that Algorithm 1 is a modification of the algorithm from [16] for peer methods.

Step 2: Optimization of $\mathcal{C}$: Searching for coefficients A, B, R and nodes c of a peer method so that the parameters are optimal with respect to the SSP property and taking account of μ found in Step 1.

Algorithm 1: Optimization of μ for peer methods by bisection

Inputs: $s, p, \Lambda, \delta, \varepsilon$
Outputs: $\Delta t_\varepsilon, A, B, R, c$
$\Delta t_{\min} \leftarrow 0$
$\Delta t_{\max} \leftarrow \delta > 0$
$tol \leftarrow \varepsilon > 0$
while $\Delta t_{\max} - \Delta t_{\min} \geq tol$ **do**
  $\Delta t \leftarrow \frac{1}{2}(\Delta t_{\max} + \Delta t_{\min})$
  $PM \leftarrow \min\limits_{A,B,R,c} \left[\max\limits_{\lambda \in \Lambda} \varrho\big(M(\Delta t \lambda)\big) \right]$
  s.t. $\mathbb{AB}(l) = 0, \quad l = 0, \ldots, p$
  if $PM \leq 1$ **then**
    $\Delta t_{\min} \leftarrow \Delta t$
  else
    $\Delta t_{\max} \leftarrow \Delta t$
  end if
end while
return $\Delta t_\varepsilon \leftarrow \Delta t_{\min},\ A, B, R, c$

Explicit peer methods of order p with optimal SSP coefficient can be found by solving the following optimization problem [8, 9]:

$$\begin{aligned} &\max_{A,B,R,c,r} \quad r \\ &\text{subject to} \quad g(r) = (I + rR)^{-1}[R, A, B - rA] \geq 0, \\ &\qquad\qquad\qquad\qquad \mathbb{AB}(l) = 0, \quad l = 0, \ldots, p. \end{aligned} \tag{14}$$

To apply a bisection approach we rewrite problem (14) for a given r in the form

$$\begin{aligned} &\min_{A,B,R,c} \quad \max(\max(-g(r))) \\ &\text{subject to} \quad \mathbb{AB}(l) = 0, \quad l = 0, \ldots, p, \end{aligned} \tag{15}$$

where $\max(\max(\cdot))$ denotes the largest element of a matrix. In order to respect the linear stability we add for a given μ the constraint

$$\max_{\lambda \in \Lambda} \varrho\big(M(\lambda\mu)\big) \leq 1.$$

Our approach is given by Algorithm 2. The output r_ε satisfies then $r_\varepsilon = \mathcal{C}$ for $\varepsilon \to 0$. We denote an s-stage explicit peer method of order p with SSPEP(s, p) and, accordingly, a RK method with SSPRK(s, p). The DG spatial discretization of degree k is referred to as DG$(k + 1)$. We consider the following explicit peer methods in our search:

- SSPEP$(s, 2)$ with DG(2) spatial operator, $s = 2, 3, 4, 5, 6$ and
- SSPEP$(s, 3)$ with DG(3) spatial operator, $s = 3, 4, 5$.

The optimization problems of Algorithms 1 and 2 are solved with `fmincon` from the optimization toolbox in MATLAB. We take $\delta = s$ and $\varepsilon = 1.e{-}6$ in Algorithms 1 and 2. A finite set of eigenvalues $\Lambda \subseteq \mathbb{C}$ of the DG spatial operator with $|\Lambda| \approx 150$ is chosen. For practical computations it is useful to include constraints for the nodes c. We take [26, 27]

$$-s \leq c_i < 1, \quad i = 1, \ldots, s - 1, \quad c_s = 1.$$

Algorithm 2: Optimization of $\mathcal{C}$ for peer methods by bisection

Inputs: $s, p, \Lambda, \mu, \delta, \varepsilon$
Outputs: $r_\varepsilon, A, B, R, c$
$r_{\min} \leftarrow 0$
$r_{\max} \leftarrow \delta > 0$
$tol \leftarrow \varepsilon > 0$
while $r_{\max} - r_{\min} \geq tol$ **do**
  $r \leftarrow \frac{1}{2}(r_{\max} + r_{\min})$
  $FV \leftarrow \min\limits_{A,B,R,c} - \min(\min(g(r)))$
  s.t. $\max\limits_{\lambda \in \Lambda} \varrho\big(M(\lambda\mu)\big) \leq 1$
    $\mathbb{AB}(l) = 0, \quad l = 0, \ldots, p$
  if $FV \leq 0$ **then**
    $r_{\min} \leftarrow r$
  else
    $r_{\max} \leftarrow r$
  end if
end while
return $r_\varepsilon \leftarrow r_{\min}$, A, B, R, c

Algorithm 1 works well in practice, despite the fact that a stability matrix has to be considered instead of a scalar stability function. In all cases considered we have found the optimal time step size μ_{opt} for linear stability. Next, we proceed with Algorithm 2 and take the CFL restriction μ_{opt} from Step 1 as input. The optimization in Step 2 is successful, if the condition $\nu(\mathcal{C}) \geq \mu_{\text{opt}}$ is satisfied. In contrast to RK methods this was not achieved for the studied methods. Therefore we reduce stepwise μ and repeat with that Algorithm 2 until the condition $\nu(\mathcal{C}) \geq \mu$ is satisfied. Through this approach Algorithms 1 and 2 determine methods with optimal value $\kappa = \min(\mu, \nu)$ with $\nu \approx \mu$. The methods are neither DG-optimal nor SSP-optimal, but they are optimal with respect to condition (12). Hence we denote our new explicit peer methods DGSSP-optimal.

The results of our optimizations are presented in Tables 1 and 2. For comparison we include the CFL restrictions for the DG-optimal [16] and SSP-optimal [12] RK methods and for the SSP-optimal explicit peer methods [9]. In Tables 1 and 2 the second column shows for the SSP-optimal methods the optimal value ν_{opt} for SSP stability and the first column lists the corresponding CFL restriction μ for linear stability. The third column indicates $\kappa = \min(\mu, \nu_{\text{opt}})$, cf. condition (12). The maximal possible value μ_{opt} for linear stability is given in the fourth column. For RK methods one can find a method with $\nu \geq \mu_{\text{opt}}$, i.e. it holds $\kappa = \mu_{\text{opt}}$. For peer methods we have to decrease the value μ to find a corresponding method with $\nu \geq \mu$, i.e. one has $\kappa < \mu_{\text{opt}}$. Nevertheless, the resulting κ for the DGSSP-optimal peer methods is, except for the case $s = p = 2$ with DG(2) spatial operator, greater than for the DG-optimal RK methods. The percentage improvements in the CFL restrictions κ are denoted with $\varepsilon\kappa\%$. In all cases considered we observe an improving in the CFL restrictions compared to the existing SSP explicit peer methods. For peer methods we indicate with $\kappa/\mu_{\text{opt}}\%$ the percentage of κ relative to μ_{opt}. Note that $\kappa/\mu_{\text{opt}}\%$ increases with larger s. So, the method $s = 6,\ p = 2$ with DG(2) spatial operator is *nearly* DG-optimal.

Figures 2 and 3 illustrate the stability regions and the eigenvalues of the DG spatial operator scaled with the maximum linearly stable time step size. It shows that the stability domains of the new methods are better suited with respect to the set of eigenvalues, which allows using a larger time step size.

All peer methods are consistent of order p. Due to the SSP property it holds $B \geq 0$, cf. Theorem 3. Together with preconsistency (9) this is sufficient for zero stability, i.e. the peer methods are convergent of order p, see Theorem 2.

The parameters of the new DGSSP-optimal explicit peer methods can be found in the Appendix, Tables 5–12.

Table 1 CFL restrictions for linear stability μ and SSP stability ν of SSP-optimal and DGSSP-optimal explicit peer methods (*top*) and SSP-optimal and DG-optimal RK methods (*bottom*) of order $p = 2$ with DG(2) spatial operator.

stages	SSPEP(s,2)+DG(2) SSP-optimal			SSPEP(s,2)+DG(2) DGSSP-optimal				
s	μ	ν_{opt}	κ	μ_{opt}	ν	κ	$\varepsilon\kappa\%$	$\kappa/\mu_{\text{opt}}\%$
2	0.2950	0.3535	0.2950	0.3465	0.3159	0.3158	7.05	91.14
3	0.5504	0.9845	0.5504	0.6442	0.6242	0.6237	13.31	96.82
4	0.7030	1.5362	0.7030	0.8590	0.8784	0.8564	21.82	99.70
5	0.8153	2.0638	0.8153	1.0739	1.0789	1.0735	31.66	99.96
6	0.9050	2.5810	0.9050	1.2886	1.2890	1.2885	42.37	99.99
stages	SSPRK(s,2)+DG(2) SSP-optimal			SSPRK(s,2)+DG(2) DG-optimal				
s	μ	ν_{opt}	κ	μ_{opt}	ν	κ	$\varepsilon\kappa\%$	
2	0.3333	0.5000	0.3333	0.3333	0.5000	0.3333	0.00	
3	0.5882	1.0000	0.5882	0.5904	0.9470	0.5904	0.37	
4	0.7612	1.5000	0.7612	0.8257	1.2298	0.8257	8.47	
5	0.8966	2.0000	0.8966	1.0520	1.5392	1.0520	17.32	
6	1.0090	2.5000	1.0090	1.2740	1.8425	1.2740	26.26	

Table 2 CFL restrictions for linear stability μ and SSP stability ν of SSP-optimal and DGSSP-optimal explicit peer methods (*top*) and SSP-optimal and DG-optimal RK methods (*bottom*) of order $p = 3$ with DG(3) spatial operator.

stages	SSPEP(s,3)+DG(3) SSP-optimal			SSPEP(s,3)+DG(3) DGSSP-optimal				
s	μ	ν_{opt}	κ	μ_{opt}	ν	κ	$\varepsilon\kappa\%$	$\kappa/\mu_{\text{opt}}\%$
3	0.2062	0.5018	0.2062	0.2588	0.2463	0.2460	19.30	95.05
4	0.3064	1.0601	0.3064	0.4153	0.3963	0.3958	29.17	95.30
5	0.3606	1.4373	0.3606	0.5290	0.5233	0.5214	44.59	98.56
stages	SSPRK(s,3)+DG(3) SSP-optimal			SSPRK(s,3)+DG(3) DG-optimal				
s	μ	ν_{opt}	κ	μ_{opt}	ν	κ	$\varepsilon\kappa\%$	
3	0.2097	0.5000	0.2097	0.2097	0.5000	0.2097	0.00	
4	0.3062	1.0000	0.3062	0.3160	0.8417	0.3160	3.27	
5	0.4061	1.3253	0.4061	0.4330	1.1937	0.4330	6.62	

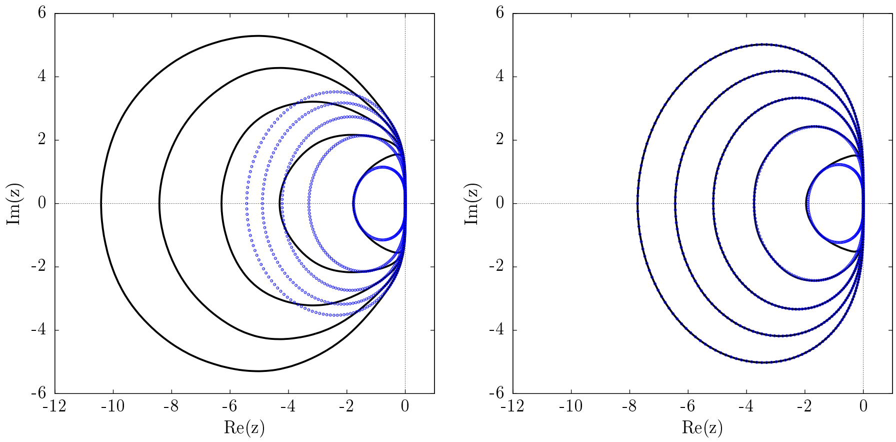


Fig. 2 Stability domains (*black*) of the SSP-optimal SSPEP methods (*left*) and DGSSP-optimal SSPEP methods (*right*) of order $p = 2$ and also the with the maximum linearly stable time step size scaled eigenvalues of the DG(2) spatial operator. (*blue, dotted*), $s = 2, \dots, 6$.

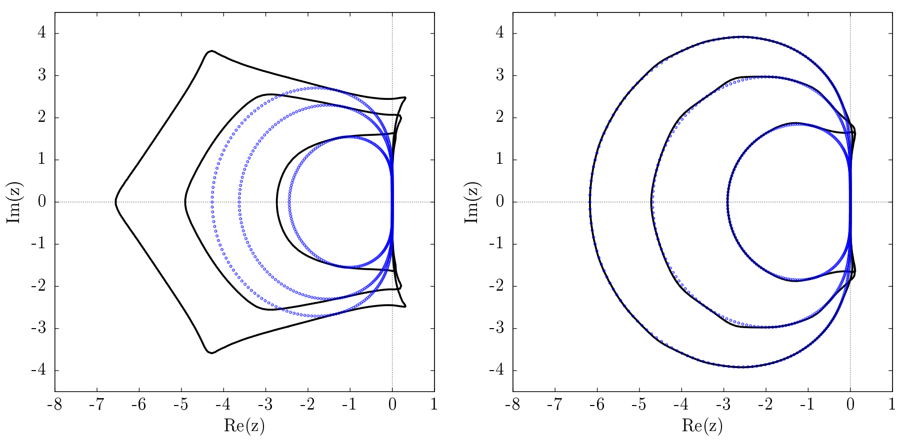


Fig. 3 Stability domains (*black*) of the SSP-optimal SSPEP methods (*left*) and DGSSP-optimal SSPEP methods (*right*) of order $p = 3$ and also the with the maximum linearly stable time step size scaled eigenvalues of the DG(3) spatial operator. (*blue, dotted*), $s = 3, 4, 5$.

5 Numerical tests

In this section, we test explicit SSP peer methods and the new DGSSP-optimal peer methods and compare them with RK methods. All numerical tests are performed in MATLAB. We always consider constant mesh grids in space and constant time steps in our experiments. Explicit peer methods require additional starting values, which are computed with `ode45` from the MATLAB ODE-suite [24] with tolerances $atol = rtol = 5.e{-}14$.

5.1 Test case 1: Linear transport equation

We consider the linear advection equation in one dimension in space [16]

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\big(cu(x,t)\big) = 0, \quad x \in [-\pi, \pi], \quad t \in (0, t_e] \tag{16}$$

with periodic boundary conditions and initial condition

$$u(x,0) = u_0(x) = \sin(2\pi x/D). \tag{17}$$

Here, $D = 2\pi$ is the length of the domain. The exact solution of the problem (16) with initial condition (17) is given by

$$u(x,t) = \sin(x - ct). \tag{18}$$

We use the advection constant $c = 1$ and integrate the problem to an end point $t_e = 315$. We consider meshes in space of $N = 50, 100, 200, 400, 800$ elements, i.e. we have $\Delta x = 2\pi/50, 2\pi/100, 2\pi/200, 2\pi/400, 2\pi/800$. The L^2-error is calculated from the solution at $t_e = 315$.

First, we test the SSP-optimal and DGSSP-optimized explicit peer method with $s = 3,\ p = 2$ and DG(2) spatial operator. The maximal stable time step for both stability properties $\kappa = 0.6237$ for the DGSSP-optimized method is used, cf. Table 1. Note that the SSP-optimal peer method does not meet this CFL restriction for linear stability. We compute both without a slope limiter and with application of the modified generalized slope limiter. We take the parameter $M = 1$ as suggested in [4]. The results are given in Table 3. Here p_{num} denotes the observed numerically order of convergence. The DGSSP-optimal peer method shows the expected order. Even though a slope limiter is applied, the SSP-optimal method displays only first order. This test case emphasizes the importance of the linear stability requirement.

Table 3 SSP and DGSSP explicit peer method with $s = 3$ and $p = 2$, paired with DG(2) spatial operator, for the linear transport equation (16), without application of a slope limiter (*top*) and with application of the modified generalized slope limiter (*bottom*).

	SSP-optimal		DGSSP-optimal	
Δx	L^2-error	p_{num}	L^2-error	p_{num}
$2\pi/50$	NaN	0.0000	8.7004e−02	0.0000
$2\pi/100$	NaN	NaN	2.1795e−02	1.9970
$2\pi/200$	NaN	NaN	5.4502e−03	1.9996
$2\pi/400$	NaN	NaN	1.3629e−03	1.9995
$2\pi/800$	NaN	NaN	3.4077e−04	1.9998
$2\pi/50$	1.6212e−01	0.0000	9.4296e−02	0.0000
$2\pi/100$	9.2417e−02	0.8109	2.2677e−02	2.0559
$2\pi/200$	4.5682e−02	1.0165	5.5967e−03	2.0185
$2\pi/400$	2.3771e−02	0.9424	1.3871e−03	2.0124
$2\pi/800$	1.1429e−02	1.0564	3.4470e−04	2.0087

Next, we apply for an order test the new DGSSP-optimal explicit peer methods and compare them with the DG-optimal RK methods from [16] and the forward

Table 4 Methods used in our numerical tests.

method	stages s	order p	μ	ν	κ	label
forward Euler	1	1	1.0000	0.5000	0.5000	`euler`
Runge–Kutta	3	2	0.5904	0.9470	0.5904	`dgrk32`
Runge–Kutta	4	3	0.3160	0.8417	0.3160	`dgrk43`
explicit peer	3	2	0.6237	0.6242	0.6237	`dgpeer32`
explicit peer	4	3	0.3958	0.3963	0.3958	`dgpeer43`

Euler method. The tested methods are listed in Table 4. We consider $\Delta x = 2\pi/N$ with $N = 20, 40, 80, 160, 320$, the maximum stable time step κ for each method and run to a final time $t_e = 2\pi$. The results are illustrated in Figure 4(a). All methods show the expected order.

The greater CFL requirements allow for peer methods using of a larger time step. Hence, the tested peer methods need less function evaluations for a similar accuracy compared to RK methods, see Figure 4(b).

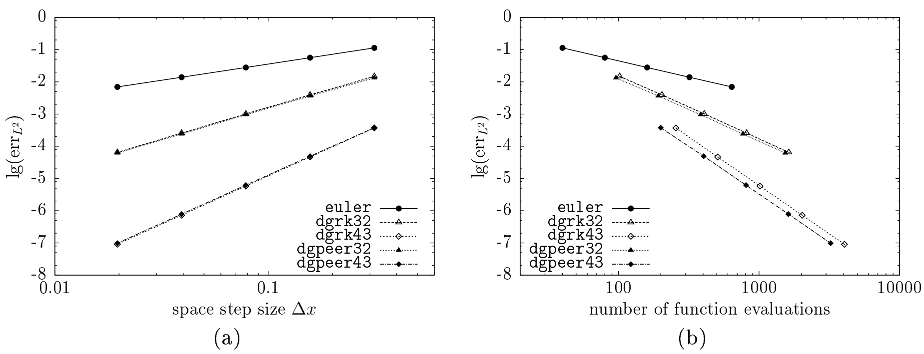


Fig. 4 Order test for the linear transport equation (16), accuracy versus space step size (*left*) and accuracy versus number of function evaluations (*right*).

5.2 Test case 2: Burgers equation

We consider a hyperbolic conservation law with nonlinear flux function, namely, Burgers equation [15]

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\left(\frac{1}{2}u(x,t)^2\right) = 0, \quad x \in [0,200], \quad t \in (0,t_e] \tag{19}$$

with periodic boundary conditions and the initial condition (17) with $D = 200$. The exact solution to the problem (19) is given by the implicit formula

$$u(x,t) = \sin\big(2\pi/D(x - tu(x,t))\big).$$

The exact solution forms a shock in $x = 100$ at time $t = 100/\pi$. We take the local Lax-Friedrichs flux for numerical flux $\widehat{f}$, see, e.g. [4].

First, we test the performance of the new DGSSP explicit peer methods. A mesh grid in space of $N = 100$ elements is taken and we use the maximum allowable time step size $\kappa = 0.6237$, cf. Table 1. We work out an advantage of applying the modified generalized slope limiter and compute for that with and without slope limiter. As proposed in [4] the parameter $M = \pi^2/10000$ is used. Figure 5(a) illustrates the results for problem (19) before a shock wave forms at time $t_e = 24$. There are no spurious oscillations in the numerical solution. The results after a shock wave builds up at time $t_e = 35$ are shown in Figure 5(b). We observe oscillations around the discontinuity in the numerical solution.

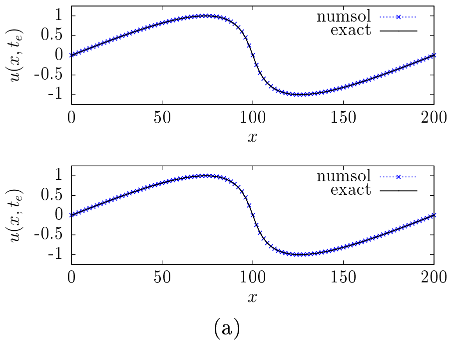


Fig. 5 Comparison of the numerical peer solution (DGSSP-optimal, $s = 3$, $p = 2$ with DG(2) spatial operator (*blue dashed lines, blue crosses*) to the exact solution (*black solid lines*) for the problem (19) with $\kappa = 0.6237$ at time $t_e = 24$ (*left*) and at time $t_e = 35$ (*right*), without application of a slope limiter (*top*) and with application of the modified generalized slope limiter (*bottom*).

For an order test we apply the methods given in Table 4. Analogous to the linear test case meshes in space of $N = 50, 100, 200, 400, 800$ elements are considered. We run in our experiment to $t_e = 24$, well before a shock wave forms. The maximum stable time step is always used. The results are presented in Figure 6. The expected orders can be seen. We can clearly observe the good properties of the new DGSSP-optimal peer methods due to the larger maximum stable time step in comparison with RK methods.

5.3 Test case 3: Test of order reduction

The stage order of explicit RK methods is only one. For explicit peer methods the stage order is equal to the order of consistency. In the next test, we show the advantage of higher stage orders. For that, we consider a hyperbolic conservation law with source term in the form [5]

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}\big(cu(x,t)\big) = b(x,t), \quad x \in [0,1]\,, \quad t \in (0,t_e]\,, \tag{20}$$

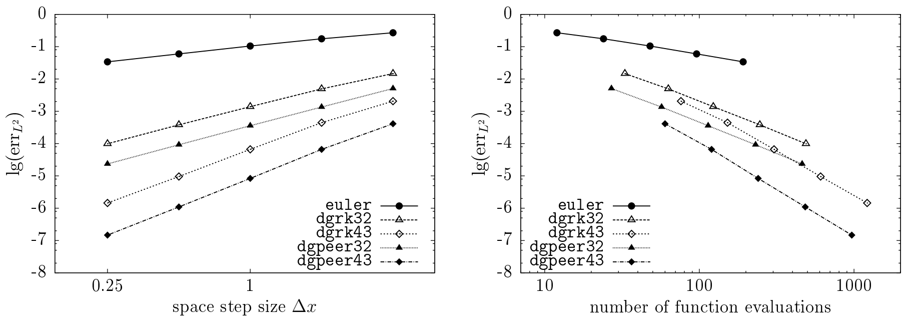


Fig. 6 Order test for the Burgers equation (19), accuracy versus space step size (*left*) and accuracy versus number of function evaluations (*right*).

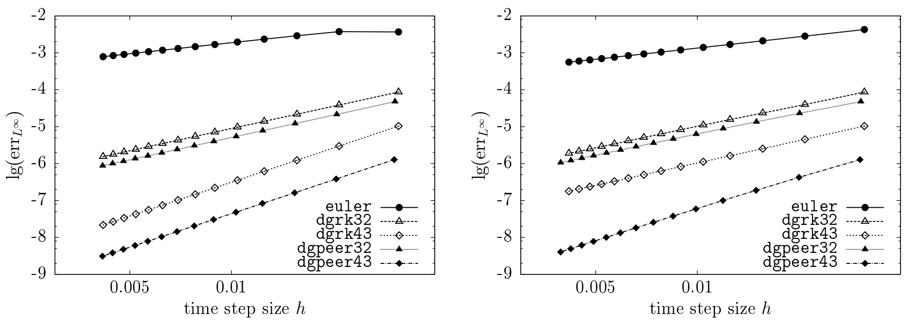
where the right-hand side of the problem (20) is given by

$$b(x,t) = \frac{t-x}{(1+t)^2}.$$

The initial condition $u_0(x) = u(x,0)$ and the right boundary condition are taken from the exact solution $u(x,t) = (1+x)/(1+t)$. Sanz-Serna *et al.* [22] show, that explicit RK methods of order $p \geq 3$ applying to problems of the form (20) suffer from order reduction. We test again the methods listed in Table 4.

First, we consider a fixed mesh in space of equal width, i.e. $N = 8$ respectively $\Delta x = 1/8$. We decrease the time step size h, or, to be more precise, we take $h = 1/nstep$, $nstep = 32, 48, 64, \dots, 240$. The L^∞-errors are calculated for the solution at $t_e = 1$. The results are presented in Figure 7(a). All methods show the classical order of convergence.

Next, we decrease the space step size Δx and the time step size h simultaneously, i.e. we choose $N = 8, 12, 16, \dots, 60$ elements in space and as stated above $nstep = 32, 48, 64, \dots, 240$ in time. Figure 7(b) shows that the RK method `dgrk43` suffers clearly from order reduction, while there is no order reduction for the peer methods.



(a) Order test. (b) Test of order reduction.

Fig. 7 Numerical tests for problem (20), order test (*left*) and test of order reduction (*right*).

6 Conclusions

In this paper, we have constructed new SSP explicit peer methods for discontinuous Galerkin spatial discretizations, which are optimal concerning both linear and SSP stability. In general, the CFL restrictions for linear stability of peer methods are greater compared to RK methods. The methods were tested successfully on linear and nonlinear test examples and have shown a good performance when compared with the SSP-optimized explicit peer methods. The advantage of high stage order was verified numerically.

New methods up to order three were constructed. The optimization of methods of higher order and the investigation of explicit DGSSP peer methods with variable step sizes will be topic of future work.

7 Appendix: Coefficients of the new DGSSP-optimal explicit peer methods

Table 5 DGSSP-optimal explicit peer method, $s = 2,\ p = 2$ with DG(2) spatial operator.

DGSSPEP(2,2)+DG(2)		
$t_{\text{opt}} = 3.1588074378967268e{-1}$	$\nu = 3.1591415405273438e{-1}$	$\mathcal{C} = 6.3182830810546875e{-1}$
$c_1 = 3.8726962960561184e{-1}$,	$c_2 = 1.0000000000000000e{+0}$,	
$b_{11} = 3.9958685046510062e{-1}$,	$b_{12} = 6.0041314953489944e{-1}$,	$b_{21} = 6.2506046658626291e{-1}$,
$b_{22} = 3.7493953341373720e{-1}$,		
$a_{11} = 3.4700795141929923e{-5}$,	$a_{12} = 6.3207392770067794e{-1}$,	$a_{21} = 6.5839010776148262e{-5}$,
$a_{22} = 3.9471578972569848e{-1}$,		
$r_{21} = 9.8821190247381530e{-1}$.		

Table 6 DGSSP-optimal explicit peer method, $s = 3,\ p = 2$ with DG(2) spatial operator.

DGSSPEP(3,2)+DG(2)		
$t_{\text{opt}} = 6.2372738968642072e{-1}$	$\nu = 6.2425704827922901e{-1}$	$\mathcal{C} = 1.2485140965584580e{+0}$
$c_1 = 2.2107967672604620e{-1}$,	$c_2 = 5.0656061487914328e{-1}$,	$c_3 = 1.0000000000000000e{+0}$,
$b_{11} = 1.7952939714598981e{-1}$,	$b_{12} = 1.0983573212036590e{-1}$,	$b_{13} = 7.1063487073364429e{-1}$,
$b_{21} = 1.6546028637722482e{-1}$,	$b_{22} = 6.8924488331683104e{-2}$,	$b_{23} = 7.6561522529109205e{-1}$,
$b_{31} = 2.8842915333335345e{-1}$,	$b_{32} = 5.8794812071492383e{-2}$,	$b_{33} = 6.5277603459515410e{-1}$,
$a_{11} = 5.0771387854624527e{-7}$,	$a_{12} = 8.7943496946377028e{-2}$,	$a_{13} = 3.2717204424969554e{-1}$,
$a_{21} = 4.3824219953913792e{-5}$,	$a_{22} = 4.5887927863455927e{-2}$,	$a_{23} = 2.0559689382563109e{-1}$,
$a_{31} = 4.4898724414544013e{-5}$,	$a_{32} = 3.9133320858599295e{-2}$,	$a_{33} = 1.7530621608580127e{-1}$,
$r_{21} = 4.1792240586620660e{-1}$,	$r_{31} = 3.5632252223207406e{-1}$,	$r_{32} = 6.8286804737201279e{-1}$.

Table 7 DGSSP-optimal explicit peer method, $s = 4,\ p = 2$ with DG(2) spatial operator.

DGSSPEP(4,2)+DG(2)		
$t_{\text{opt}} = 8.5643142648664095e{-}1$	$\nu = 8.7849845862641618e{-}1$	$\mathcal{C} = 1.7569969172528324e{+}0$
$c_1 = -1.1017990086977755e{-}2,$	$c_2 = 3.0086416786435727e{-}1,$	$c_3 = 5.8652086055378594e{-}1,$
$c_4 = 1.0000000000000000e{+}0,$		
$b_{11} = 3.1117533862087359e{-}2,$	$b_{12} = 4.6509168053559952e{-}3,$	$b_{13} = 6.3128571517354726e{-}1,$
$b_{14} = 3.3294583415900925e{-}1,$	$b_{21} = 6.8763467565832972e{-}2,$	$b_{22} = 1.1278132944089411e{-}1,$
$b_{23} = 4.5234068545279221e{-}1,$	$b_{24} = 3.6611451754048069e{-}1,$	$b_{31} = 7.4023725231017259e{-}2,$
$b_{32} = 8.1976955267757265e{-}2,$	$b_{33} = 3.6530531065842953e{-}1,$	$b_{34} = 4.7869400884279600e{-}1,$
$b_{41} = 1.3420243385704098e{-}1,$	$b_{42} = 7.5967299859526563e{-}2,$	$b_{43} = 3.3975813045900088e{-}1,$
$b_{44} = 4.5007213582443156e{-}1,$		
$a_{11} = 1.7578286279728319e{-}2,$	$a_{12} = 2.6239504152038009e{-}3,$	$a_{13} = 1.2415836181630931e{-}1,$
$a_{14} = 1.4035689478904412e{-}1,$	$a_{21} = 1.2612004135562042e{-}2,$	$a_{22} = 2.9382335254034284e{-}3,$
$a_{23} = 8.9068933371323530e{-}2,$	$a_{24} = 1.2494665236285137e{-}1,$	$a_{31} = 9.1170106617451882e{-}3,$
$a_{32} = 2.2218189554709306e{-}3,$	$a_{33} = 6.4252681383924565e{-}2,$	$a_{34} = 9.0385891751268224e{-}2,$
$a_{41} = 2.7174224172487665e{-}1,$	$a_{42} = 2.0735662657640716e{-}3,$	$a_{43} = 5.9530677838877874e{-}2,$
$a_{44} = 8.3820284681580828e{-}2,$		
$r_{21} = 4.0670235320459108e{-}1,$	$r_{31} = 2.9332670557121748e{-}1,$	$r_{32} = 4.1041522245522255e{-}1,$
$r_{41} = 2.7174224172487665e{-}1,$	$r_{42} = 3.8012579385879725e{-}1,$	$r_{43} = 5.2351717940348785e{-}1.$

Table 8 DGSSP-optimal explicit peer method, $s = 5,\ p = 2$ with DG(2) spatial operator.

DGSSPEP(5,2)+DG(2)		
$t_{\text{opt}} = 1.0735938603991406e{+}0$	$\nu = 1.0789608929793362e{+}0$	$\mathcal{C} = 2.1579217859586723e{+}0$
$c_1 = 2.7742011894826051e{-}2,$	$c_2 = 1.6740580589509829e{-}1,$	$c_3 = 3.8645514443940826e{-}1,$
$c_4 = 6.5648295647166943e{-}1,$	$c_5 = 1.0000000000000000e{+}0,$	
$b_{11} = 1.6561451717599946e{-}2,$	$b_{12} = 5.1543012462882801e{-}2,$	$b_{13} = 2.4235052989736338e{-}2,$
$b_{14} = 3.7093807640333870e{-}1,$	$b_{15} = 5.3672240642644231e{-}1,$	$b_{21} = 5.1003289257488704e{-}2,$
$b_{22} = 9.0839354925312893e{-}2,$	$b_{23} = 6.9279104843613687e{-}2,$	$b_{24} = 4.4566635507224261e{-}1,$
$b_{25} = 3.4321189590134216e{-}1,$	$b_{31} = 8.9469150942903863e{-}2,$	$b_{32} = 6.1834520299016558e{-}2,$
$b_{33} = 1.1252453345421667e{-}1,$	$b_{34} = 2.9653147102141436e{-}1,$	$b_{35} = 4.3964032428244865e{-}1,$
$b_{41} = 1.0010471672836012e{-}1,$	$b_{42} = 5.0287706382909005e{-}2,$	$b_{43} = 8.9879923092870295e{-}2,$
$b_{44} = 2.4794489219634647e{-}1,$	$b_{45} = 5.1178276159951419e{-}1,$	$b_{51} = 1.4185784754583605e{-}1,$
$b_{52} = 4.8356562311313148e{-}2,$	$b_{53} = 8.6541711605316221e{-}2,$	$b_{54} = 2.3784783003028959e{-}1,$
$b_{55} = 4.8539604850724505e{-}1,$		
$a_{11} = 6.4317723978225805e{-}4,$	$a_{12} = 2.3873232958995679e{-}2,$	$a_{13} = 1.1198546480224131e{-}2,$
$a_{14} = 7.0979582944989336e{-}2,$	$a_{15} = 1.2235673234541407e{-}1,$	$a_{21} = 1.9490070113088873e{-}2,$
$a_{22} = 4.2081142386328187e{-}2,$	$a_{23} = 3.2064655079903082e{-}2,$	$a_{24} = 4.2551326354372386e{-}2,$
$a_{25} = 8.6673201282555526e{-}2,$	$a_{31} = 2.8770766492468184e{-}2,$	$a_{32} = 2.8583535892912390e{-}2,$
$a_{33} = 1.9589831821235741e{-}2,$	$a_{34} = 3.9819710434687024e{-}2,$	$a_{35} = 1.4522379819848988e{-}1,$
$a_{41} = 2.3005603104551180e{-}2,$	$a_{42} = 2.3244404663516533e{-}2,$	$a_{43} = 1.5990526359860903e{-}2,$
$a_{44} = 3.2102608379892331e{-}2,$	$a_{45} = 1.2234823934180945e{-}1,$	$a_{51} = 2.2153614246397991e{-}2,$
$a_{52} = 2.2345370625525482e{-}2,$	$a_{53} = 1.5381011279929436e{-}2,$	$a_{54} = 3.0882207814213070e{-}2,$
$a_{55} = 1.1729006476220023e{-}1,$		
$r_{21} = 2.6536591265523851e{-}1,$	$r_{31} = 1.6381368834413768e{-}1,$	$r_{32} = 2.7002643540849952e{-}1,$
$r_{41} = 1.3356037957367423e{-}1,$	$r_{42} = 2.2057146114463183e{-}1,$	$r_{43} = 3.6517525751620039e{-}1,$
$r_{51} = 1.2835056792896077e{-}1,$	$r_{52} = 2.1190032306216625e{-}1,$	$r_{53} = 3.5173070081288632e{-}1,$
$r_{54} = 4.1295196327532063e{-}1.$		

Table 9 DGSSP-optimal explicit peer method, $s = 6,\ p = 2$ with DG(2) spatial operator.

DGSSPEP(6,2)+DG(2)		
$t_{\mathrm{opt}} = 1.2885962890624989e{+}0$	$\nu = 1.2890531122502580e{+}0$	$\mathcal{C} = 2.5781062245005160e{+}0$
$c_1 = -8.3647136532497349e{-}2$,	$c_2 = 8.7831420264993687e{-}2$,	$c_3 = 2.8213745150635866e{-}1$,
$c_4 = 4.4330817693066960e{-}1$,	$c_5 = 7.0652444866883590e{-}1$,	$c_6 = 1.0000000000000000e{+}0$,
$b_{11} = 3.8641043158428111e{-}2$,	$b_{12} = 4.4232011779101452e{-}2$,	$b_{13} = 9.9889471500482732e{-}2$,
$b_{14} = 3.0685748625566667e{-}1$,	$b_{15} = 2.1165238737722669e{-}1$,	$b_{16} = 2.9872759992909437e{-}1$,
$b_{21} = 6.0850380026999114e{-}2$,	$b_{22} = 6.8282568594205145e{-}2$,	$b_{23} = 2.1395905200081788e{-}1$,
$b_{24} = 2.1975239683638229e{-}1$,	$b_{25} = 2.1776376220817134e{-}1$,	$b_{26} = 2.1939184033342427e{-}1$,
$b_{31} = 6.2566000786014292e{-}2$,	$b_{32} = 4.6800358552182279e{-}2$,	$b_{33} = 1.6512683572752163e{-}1$,
$b_{34} = 1.6367974585222511e{-}1$,	$b_{35} = 1.6143599703058481e{-}1$,	$b_{36} = 4.0039106205147185e{-}1$,
$b_{41} = 5.4914792451267902e{-}2$,	$b_{42} = 7.0969957147218252e{-}2$,	$b_{43} = 1.8845955134871700e{-}1$,
$b_{44} = 1.7322709781768988e{-}1$,	$b_{45} = 2.0268051042812646e{-}1$,	$b_{46} = 3.0974809080698051e{-}1$,
$b_{51} = 7.0900964515960882e{-}2$,	$b_{52} = 6.2579343518510869e{-}2$,	$b_{53} = 1.6935494538856111e{-}1$,
$b_{54} = 1.5553350748289188e{-}1$,	$b_{55} = 2.1614720790293024e{-}1$,	$b_{56} = 3.2548403119114500e{-}1$,
$b_{61} = 9.5346592557543089e{-}2$,	$b_{62} = 6.0964362811776439e{-}2$,	$b_{63} = 1.6773648956268339e{-}1$,
$b_{64} = 1.5171234778306664e{-}1$,	$b_{65} = 2.0734919214222294e{-}1$,	$b_{66} = 3.1689101514270757e{-}1$,
$a_{11} = 9.9781019186123299e{-}3$,	$a_{12} = 1.7110167064243832e{-}2$,	$a_{13} = 1.2180057897467796e{-}2$,
$a_{14} = 1.1717718982741167e{-}1$,	$a_{15} = 6.6774754716575277e{-}2$,	$a_{16} = 7.9999664279063298e{-}2$,
$a_{21} = 2.0022727472542023e{-}2$,	$a_{22} = 2.3295396490237163e{-}2$,	$a_{23} = 4.1065518496485987e{-}2$,
$a_{24} = 8.3762065747880823e{-}2$,	$a_{25} = 5.9024483929204334e{-}2$,	$a_{26} = 6.0232939120099256e{-}2$,
$a_{31} = 1.3237629376995122e{-}2$,	$a_{32} = 1.6210156162414120e{-}2$,	$a_{33} = 2.6122539250869044e{-}2$,
$a_{34} = 6.2356190990293325e{-}2$,	$a_{35} = 4.2379185220326097e{-}2$,	$a_{36} = 5.5203140321455985e{-}2$,
$a_{41} = 1.3183901615091328e{-}2$,	$a_{42} = 2.5661015544193120e{-}2$,	$a_{43} = 2.4772372096300586e{-}2$,
$a_{44} = 6.5954686104383195e{-}2$,	$a_{45} = 5.0264167645655639e{-}2$,	$a_{46} = 5.2375690944478782e{-}2$,
$a_{51} = 1.1962045502518876e{-}2$,	$a_{52} = 2.2573210361251560e{-}2$,	$a_{53} = 2.2598265126438841e{-}2$,
$a_{54} = 5.9139007403067954e{-}2$,	$a_{55} = 4.4992032582584954e{-}2$,	$a_{56} = 5.0688753005090945e{-}2$,
$a_{61} = 1.1808444171549790e{-}2$,	$a_{62} = 2.1882516163669988e{-}2$,	$a_{63} = 2.2064663109906128e{-}2$,
$a_{64} = 5.7661389016855066e{-}2$,	$a_{65} = 4.3772319457984148e{-}2$,	$a_{66} = 4.9524118045136144e{-}2$,
$r_{21} = 2.6848973563223300e{-}1$,	$r_{31} = 2.0040238370394139e{-}1$,	$r_{32} = 2.0040238370394139e{-}1$,
$r_{41} = 2.1170563959111202e{-}1$,	$r_{42} = 2.0598962138841381e{-}1$,	$r_{43} = 2.0884984185923972e{-}1$,
$r_{51} = 1.8970738702010317e{-}1$,	$r_{52} = 1.8684389626107969e{-}1$,	$r_{53} = 1.9758088925136094e{-}1$,
$r_{54} = 3.2594522576213791e{-}1$,	$r_{61} = 1.8486403826946798e{-}1$,	$r_{62} = 1.8278919415720252e{-}1$,
$r_{63} = 1.9535084239777523e{-}1$,	$r_{64} = 3.1113185450651065e{-}1$,	$r_{65} = 3.4380314477186164e{-}1$.

Table 10 DGSSP-optimal explicit peer method, $s = 3,\ p = 3$ with DG(3) spatial operator.

DGSSPEP(3,3)+DG(3)		
$t_{\mathrm{opt}} = 2.4602189440780711e{-}1$	$\nu = 2.4633121489023324e{-}1$	$\mathcal{C} = 4.9266242978046648e{-}1$
$c_1 = 1.9661571175497836e{-}1$,	$c_2 = 5.0528002223731205e{-}1$,	$c_3 = 1.0000000000000000e{+}0$,
$b_{11} = 2.0890195008065693e{-}1$,	$b_{12} = 2.4702575909385212e{-}4$,	$b_{13} = 7.9085102416024922e{-}1$,
$b_{21} = 2.5112724350902732e{-}1$,	$b_{22} = 1.5099471253417268e{-}1$,	$b_{23} = 5.9787804395680011e{-}1$,
$b_{31} = 6.5276026744051063e{-}1$,	$b_{32} = 7.0419303423422813e{-}2$,	$b_{33} = 2.7682042913606658e{-}1$,
$a_{11} = 5.9869500731058144e{-}2$,	$a_{12} = 3.7500208245775736e{-}5$,	$a_{13} = 3.0465946387226572e{-}1$,
$a_{21} = 1.0819725355402451e{-}1$,	$a_{22} = 1.3039816144885929e{-}4$,	$a_{23} = 8.7883536425709521e{-}2$,
$a_{31} = 3.0773114844308813e{-}1$,	$a_{32} = 1.3344851947888053e{-}4$,	$a_{33} = 4.0812438919617153e{-}2$,
$r_{21} = 5.8552061670875277e{-}1$,	$r_{31} = 2.7104000238063714e{-}1$,	$r_{32} = 9.3953814081320286e{-}1$.

Table 11 DGSSP-optimal explicit peer method, $s = 4,\ p = 3$ with DG(3) spatial operator.

DGSSPEP(4,3)+DG(3)		
$t_{\text{opt}} = 3.9582823166165310e{-1}$	$\nu = 3.9634551296715331e{-1}$	$\mathcal{C} = 7.9269102593430663e{-1}$
$c_1 = 4.3839003669994735e{-2}$,	$c_2 = 2.6786916750388512e{-1}$,	$c_3 = 5.4953332512948561e{-1}$,
$c_4 = 1.0000000000000000e{+0}$,		
$b_{11} = 7.3546754048809115e{-2}$,	$b_{12} = 4.1817700777738304e{-5}$,	$b_{13} = 4.4229916709791539e{-1}$,
$b_{14} = 4.8411226115249789e{-1}$,	$b_{21} = 2.9274780474800307e{-2}$,	$b_{22} = 2.7520134456599044e{-1}$,
$b_{23} = 1.1912582541306244e{-1}$,	$b_{24} = 5.7639804954614682e{-1}$,	$b_{31} = 1.9034417339865942e{-1}$,
$b_{32} = 1.0525773136912379e{-1}$,	$b_{33} = 4.5708933536631198e{-2}$,	$b_{34} = 6.5868916169558567e{-1}$,
$b_{41} = 4.5099543943421483e{-1}$,	$b_{42} = 7.4772026638373540e{-2}$,	$b_{43} = 5.0180646464117427e{-2}$,
$b_{44} = 4.2405188746329436e{-1}$,		
$a_{11} = 1.5731692935638650e{-5}$,	$a_{12} = 7.0838243086917061e{-6}$,	$a_{13} = 1.7210064909648287e{-1}$,
$a_{14} = 1.4130972781309412e{-1}$,	$a_{21} = 1.1879749733938127e{-2}$,	$a_{22} = 6.2722620374107349e{-2}$,
$a_{23} = 4.6486414785873606e{-2}$,	$a_{24} = 9.0656047393412598e{-2}$,	$a_{31} = 8.1585243681312855e{-2}$,
$a_{32} = 2.4053807223148774e{-2}$,	$a_{33} = 1.7872345721743028e{-2}$,	$a_{34} = 9.3765373767690105e{-2}$,
$a_{41} = 2.3228544088457373e{-1}$,	$a_{42} = 1.7136996247339362e{-2}$,	$a_{43} = 1.2750693934711080e{-2}$,
$a_{44} = 5.9855299597249750e{-2}$,		
$r_{21} = 3.3926134244885719e{-1}$,	$r_{31} = 1.2992036651196920e{-1}$,	$r_{32} = 4.8198864450214773e{-1}$,
$r_{41} = 9.2162066058127295e{-2}$,	$r_{42} = 3.4202961192953996e{-1}$,	$r_{43} = 7.5235175512388408e{-1}$.

Table 12 DGSSP-optimal explicit peer method, $s = 5,\ p = 3$ with DG(3) spatial operator.

DGSSPEP(5,3)+DG(3)		
$t_{\text{opt}} = 5.2146838980310806e{-1}$	$\nu = 5.2331666596248216e{-1}$	$\mathcal{C} = 1.0466333319249643e{+0}$
$c_1 = -1.2896056199244499e{-1}$,	$c_2 = 1.1276392078374278e{-1}$,	$c_3 = 2.6059913091639131e{-1}$,
$c_4 = 5.6174317378048244e{-1}$,	$c_5 = 1.0000000000000000e{+0}$,	
$b_{11} = 6.0791034060216624e{-2}$,	$b_{12} = 9.5852635551357563e{-4}$,	$b_{13} = 2.0761291589709338e{-2}$,
$b_{14} = 7.6893982550132378e{-1}$,	$b_{15} = 1.4854932249323674e{-1}$,	$b_{21} = 9.2119500718257225e{-2}$,
$b_{22} = 8.3230371978585932e{-2}$,	$b_{23} = 1.1123482685038859e{-1}$,	$b_{24} = 2.3763252592429643e{-1}$,
$b_{25} = 4.7578277452847179e{-1}$,	$b_{31} = 6.8649835323895902e{-2}$,	$b_{32} = 1.8690649740809420e{-1}$,
$b_{33} = 3.9451688536070409e{-2}$,	$b_{34} = 4.0818405950290310e{-1}$,	$b_{35} = 2.9680791922903649e{-1}$,
$b_{41} = 9.6128424921121600e{-2}$,	$b_{42} = 1.9224001060936594e{-1}$,	$b_{43} = 2.0769577547696039e{-2}$,
$b_{44} = 2.1231969239092871e{-1}$,	$b_{45} = 4.7854229453088798e{-1}$,	$b_{51} = 3.0122334909900167e{-1}$,
$b_{52} = 1.4217321423769702e{-1}$,	$b_{53} = 1.5881084749237162e{-2}$,	$b_{54} = 1.5684287241466940e{-1}$,
$b_{55} = 3.8387947949939499e{-1}$,		
$a_{11} = 3.4769170227165767e{-4}$,	$a_{12} = 1.7177250110356332e{-4}$,	$a_{13} = 1.9534705415881343e{-2}$,
$a_{14} = 2.1752911879250800e{-1}$,	$a_{15} = 5.5281313260734842e{-2}$,	$a_{21} = 2.7371585237199354e{-2}$,
$a_{22} = 1.7713695470402191e{-2}$,	$a_{23} = 3.5952405868642588e{-2}$,	$a_{24} = 6.4987051933669632e{-2}$,
$a_{25} = 4.6760180866391832e{-2}$,	$a_{31} = 9.3083739401898530e{-3}$,	$a_{32} = 3.6712897752350336e{-2}$,
$a_{33} = 1.4681364872864635e{-2}$,	$a_{34} = 2.3344431655384051e{-1}$,	$a_{35} = 1.5446714488978745e{-2}$,
$a_{41} = 6.2440356373707991e{-2}$,	$a_{42} = 1.9199900681709219e{-2}$,	$a_{43} = 7.6777210563701535e{-3}$,
$a_{44} = 1.2121562382482404e{-1}$,	$a_{45} = 3.3602830148094974e{-2}$,	$a_{51} = 1.8596259860281633e{-1}$,
$a_{52} = 1.4441148030210472e{-2}$,	$a_{53} = 5.6748989961926703e{-3}$,	$a_{54} = 8.9476839801155866e{-2}$,
$a_{55} = 2.8662811204002939e{-2}$,		
$r_{21} = 2.8421447787842696e{-1}$,	$r_{31} = 9.2967942710790979e{-2}$,	$r_{32} = 3.0943072842725872e{-1}$,
$r_{41} = 4.8445608346641161e{-2}$,	$r_{42} = 1.6074044475707289e{-1}$,	$r_{43} = 4.9591576071642379e{-1}$,
$r_{51} = 3.5877294969278155e{-2}$,	$r_{52} = 1.1843396321287454e{-1}$,	$r_{53} = 3.6500090055420653e{-1}$,
$r_{54} = 7.0315997862832713e{-1}$.		

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