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T. Q. Bao

Department of Mathematics and Computer Science
Northern Michigan University
1401 Presque Isle Avenue
Marquette
MI 49855

Email: btruong@nmu.edu

Chr. Tammer
M. Hillmann

Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany

Email: christiane.tammer@mathematik.uni-halle.de
marcus.hillmann@mathematik.uni-halle.de

# Subdifferentials of Nonlinear Scalarization Functions and Applications

T.Q. Bao[1], Chr. Tammer[2], M. Hillmann[3]

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**Abstract**

In this paper, we formulate subdifferentials of nonlinear scalarization functions related to either fixed ordering cones or variable domination structures by using a calculus-based approach; i.e., using calculus rules for generalized differentiation. As an application, efficient necessary optimality conditions for multiobjective location problems involving $\ell_1$ norm are discussed.

**Key Words:** Variable domination structure, generalized differentiation, nonlinear scalarization functions, necessary conditions, location problems.

**Mathematics subject classifications (MSC 2000):** 90C29, 90C30, 90C26

## 1 Introduction

Let $X$ be an Asplund space, $\varphi : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ be an extended-real-valued function, $\Omega$ be a nonempty set in $X$, and $\overline{x} \in X$ be a local minimal solution of $\varphi$ over $\Omega$. Assume that $\Omega$ is locally closed and $\varphi$ is lower-semicontinuous (lsc) around $\overline{x}$. Assume also that either $\Omega$ is sequentially normal compact (SNC) or $\varphi$ is sequentially normal epigraphically compact (SNEC) at $\overline{x}$ and that

$$\partial_L^\infty \varphi(\overline{x}) \cap -N_L(\overline{x}; \Omega) = \{0\},$$

where $N_L(\overline{x}; \Omega)$ is the limiting normal cone to $\Omega$ at $\overline{x}$ and $\partial_L^\infty \varphi(\overline{x})$ is the singular Mordukhovich/limiting subdifferential of $\varphi$ at $\overline{x}$; all these assumptions hold if $\varphi$ is locally Lipschitz continuous around $\overline{x}$.

[1]Department of Mathematics and Computer Science, Northern Michigan University, 1401 Presque Isle Avenue, Marquette, MI 49855, email: btruong@nmu.edu

[2]Institute of Mathematics, Martin-Luther-University Halle-Wittenberg, Theodor-Lieser Str. 5, 06120 Halle, email: christiane.tammer@mathematik.uni-halle.de

[3]Institute of Mathematics, Martin-Luther-University Halle-Wittenberg, Theodor-Lieser Str. 5, 06120 Halle, email: marcus.hillmann@mathematik.uni-halle.de

Then one has

$$0 \in \partial_L \varphi(\overline{x}) + N_L(\overline{x}; \Omega),$$

where $\partial_L \varphi(\overline{x})$ is the Mordukhovich/limiting subdifferential of $\varphi$ at $\overline{x}$. The reader could find definitions of generalized differential objects in Section 2.

A natural question is: what is a necessary condition in the case of vector-valued functions? In [11] the authors introduced a nonconvex scalarization scheme which was subsequently used in [7, 8] in order to obtain multiplier rules for vector optimization problems in terms of classical (Clarke, Mordukhovich, Ioffe, etc.) subdifferentials in the case of vector optimization with *ordering cone*. The method of study used here is based on the properties of a function $\varphi_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ of the form

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} : y + tk \in A\}, \tag{1}$$

where $Y$ is a linear topological space, $A \subset Y$ is a nonempty set, $k \in Y \setminus \{0\}$ is a (nonzero) direction such that $A + k[0, +\infty) \subset A$. Important properties of the function $\varphi_{A,k}$ are shown in [11].

The aim of this paper is to determine the subdifferential of the scalarizing functional (1) for general (nonconvex) sets $A$ and to derive necessary conditions for solutions of vector optimization problems with general (fixed and variable) domination sets. Furthermore, we will apply some of the new results to derive necessary optimality conditions for certain vector-valued location problems.

— Vector optimization with a not necessarily convex domination sets was studied in [11] by using the scalarization function (1). In [15, 24] the authors dealt with a class of in general nonconvex ordering cones generated by a union of a finite number of convex and solid cones. In [2, 3], the authors used a variational approach based on extremal principles to establish necessary conditions for a broader setting with nonconvex domination sets.

— Vector optimization with variable domination structures was first studied by Yu [28]. Recently, there has been a growing interest due to its theoretical aspects and promising applications to operations research, economics, engineering design, behavioral sciences, etc.; see, e.g., [4, 6, 9, 10, 20, 25, 26].

In order to accomplish this aim, it is sufficient to formulate subdifferentials of nonlinear scalarization functions related to either fixed ordering cones or variable domination structures. In [7] the convex subdifferential of $\varphi_{k,A}$, denoted by $\partial\varphi_{k,A}$, was directly computed from definition. When $A$ is nonconvex, or $A = A(z)$ varies, it is quite difficult to proceed in the same way. In this paper we propose a new calculus-based approach to estimate $\partial\varphi_{k,A}$ in the latter general settings. Precisely, we apply known calculus rules for generalized differentiation to obtain an upper estimate of $\partial\varphi_{k,A}$ if not a closed form.

The rest of the paper is organized as follows: Section 2 provides some basic tools of generalized differentiation and a nonlinear scalarization function for separating nonconvex sets and preliminary results broadly used in this paper. Sections 3 and

4 contain singular and basic subdifferentials of versions of the nonlinear separation function $\varphi_{k,A}$ when $A$ is a fixed domination set of the image space $Y$ and $A : Y \rightrightarrows Y$ is a variable domination structure of $Y$, respectively. In the last Section 5 applications in location problems are discussed.

# 2 Preliminaries

## 2.1 Subdifferential calculus

Throughout the paper we use the standard notation of variational analysis; cf. the books in [21, 22], and assume that all the spaces under consideration are Asplund unless otherwise stated. Recall that a Banach space is Asplund if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset $U$ of $X$ is Fréchet/regular differentiable on a dense subset of $U$. The class of Asplund spaces is quite broad including every reflexive Banach space and every Banach space with a separable dual; in particular, $c_0$ and $\ell_p$, $L_p[0,1]$ for $1 < p < \infty$ are Asplund spaces. In the sequel, we present the definitions and properties of the limiting/Mordukhovich generalized differential constructions held in the Asplund space setting and enjoying a full calculus.

For an Asplund space $X$, we denote its norm by $\|\cdot\|$ and consider the dual space $X^*$ equipped with the weak* topology $w^*$, where $\langle \cdot, \cdot \rangle$ stands for the canonical pairing between $X$ and $X^*$. Given a nonempty set $\Omega$ in $X$ and $\overline{x} \in \Omega$, $\Omega$ is said to be locally closed around $\overline{x}$ if there is a neighborhood $U$ of $\overline{x}$ such that $\Omega \cap \operatorname{cl} U$ is a closed set. Assume now that $\Omega$ is locally closed around a given $\overline{x} \in \Omega$.

**Definition 2.1.** *The* **(basic, limiting, Mordukhovich) normal cone** *to* $\Omega$ *at* $\overline{x} \in \Omega$ *is defined by*

$$\begin{aligned} N_L(\overline{x};\Omega) \quad &:= \quad \operatorname*{Lim\,sup}_{x \to \overline{x}} \widehat{N}(x;\Omega) \\ &= \quad \Big\{ x^* \in X^* : \ \exists\, x_k \overset{\Omega}{\to} \overline{x},\ x_k^* \overset{w^*}{\longrightarrow} x^* \ \text{with}\ x_k^* \in \widehat{N}(x_k,\Omega) \Big\}, \end{aligned} \tag{2}$$

*where* Lim sup *stands for the sequential Painlevé-Kuratowski outer limit of Fréchet/regular normal cones to* $\Omega$ *at* $x$*, denoted by* $\widehat{N}(x;\Omega)$*, as* $x$ *tends to* $\overline{x}$ *and where* $x_k \overset{\Omega}{\to} \bar{x}$ *means* $x_k \to \bar{x}$ *and* $x_k \in \Omega$ *for all* $k \in \mathbb{N}$.

If $\overline{x} \notin \Omega$, $N_L(\overline{x};\Omega) = \emptyset$. If $\overline{x} \in \operatorname{int} \Omega$, $N_L(\overline{x};\Omega) = \{\mathbf{0}\}$. Both limiting and Fréchet cones reduce to the normal cone of convex analysis when $\Omega$ is locally convex around $\overline{x}$. The limiting normal cone is nonconvex in general.

Given a set-valued mapping $F \colon X \rightrightarrows Y$ between Asplund spaces and $\overline{y} \in F(\overline{x})$. Assume that the graph of $F$, denoted by $\operatorname{gph} F := \{(x,y) \in X \times Y : \ y \in F(x)\}$, is locally closed around $(\overline{x},\overline{y})$. We use the following construction of the (basic, normal) *coderivative* $D^*F(\overline{x},\overline{y}) \colon Y^* \rightrightarrows X^*$ of $F$ at $(\overline{x},\overline{y}) \in \operatorname{gph} F$ defined by

$$D_L^*F(\overline{x},\overline{y})(y^*) := \big\{ x^* \in X^* : \ (x^*, -y^*) \in N_L\big((\overline{x},\overline{y}); \operatorname{gph} F\big) \big\}, \tag{3}$$

which is a positively homogeneous mapping of $y^*$; we omit $\overline{y} = f(\overline{x})$ in (3) if $F = f : X \to Y$ is single-valued. If $f$ happens to be strictly differentiable at $\overline{x}$ (see, e.g., [21, Definition 1.13]; this is automatic when $f$ is $C^1$ around $\overline{x}$), then

$$D_L^* f(\overline{x})(y^*) = \{\nabla f(\overline{x})^* y^*\} \text{ for all } y^* \in Y^*.$$

Consider further an extended-real-valued function $\varphi : X \to \overline{\mathbb{R}}$ finite at $\overline{x}$ and lsc around this point, and let $\mathcal{E}_\varphi : X \rightrightarrows \mathbb{R}$ be the *epigraphical multifunction* of $\varphi$ defined by

$$\mathcal{E}_\varphi(x) := \begin{cases} [\varphi(x), +\infty) & \text{if } x \in \operatorname{dom} \varphi, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\operatorname{dom} \varphi := \{x \in \mathbb{R} : \varphi(x) < +\infty\}$. Then the *basic subdifferential* $\partial\varphi(\overline{x})$ and the *singular subdifferential* $\partial^\infty\varphi(\overline{x})$ of $\varphi$ at $\overline{x}$ could be defined as coderivatives of the epigraphical multifunction $\mathcal{E}_\varphi$ at $(\overline{x}, \varphi(\overline{x}))$, respectively,

$$\partial_L\varphi(\overline{x}) := D_L^*\mathcal{E}_\varphi\big(\overline{x}, \varphi(\overline{x})\big)(1) \text{ and } \partial_L^\infty\varphi(\overline{x}) := D_L^*\mathcal{E}_\varphi\big(\overline{x}, \varphi(\overline{x})\big)(0). \tag{4}$$

Again, we refer the reader to the book [21] for analytic representations of subdifferentials in (4) and extended calculus rules for them. Note that the singular subdifferential in (4) carries some information only for *non-Lipschitzian* functions, since $\varphi$ is locally Lipschitz continuous around $\overline{x}$ if and only if $\partial_L^\infty\varphi(\overline{x}) = \{0\}$. As mentioned e.g. in [23, Remark 5.3], it is quite easy to see that the equation $\partial_L(\alpha\varphi)(x^0) = \alpha\partial_L\varphi(x^0)$ holds for any function $\varphi$ and $\alpha \geq 0$.

We need the following calculus rules given by Mordukhovich in [21] for proving necessary optimality conditions in Section 5.2. The next lemma follows from [21, Theorem 3.36] taking into account [21, Theorem 1.26 and Corollary 1.81].

**Lemma 2.2.** *Let $X$ be an Asplund space and let $\overline{x} \in X$. Let $\varphi_i : X \to \overline{\mathbb{R}}$, $i = 1, 2$, be proper lsc functions and one of these is Lipschitzian around $\overline{x}$. Then one has*

$$\partial_L(\varphi_1 + \varphi_2)(\overline{x}) \subseteq \partial_L\varphi_1(\overline{x}) + \partial_L\varphi_2(\overline{x}).$$

The next result is shown in [21, Theorem 3.41 and Corollary 3.43].

**Lemma 2.3.** *Assume that $X$ and $Y$ are Asplund spaces and $\overline{x} \in X$. Let $f : X \to Y$ be strictly Lipschitz continuous at $\overline{x}$ and $\varphi : Y \to \overline{\mathbb{R}}$ be Lipschitz continuous around $f(\overline{x})$. Then one has*

$$\partial_L(\varphi \circ f)(\overline{x}) \subseteq \bigcup_{y^* \in \partial_L\varphi(f(\overline{x}))} \partial_L\langle y^*, f\rangle(\overline{x}).$$

For our calculations in Theorem 5.4 we will use the following helpful lemmas which are shown in [17, Proposition 4.4] and [21, Theorem 1.93], respectively.

**Lemma 2.4.** *Let $\varphi_1 : X \to \overline{\mathbb{R}}$ and $\varphi_2 : Y \to \overline{\mathbb{R}}$ be lsc near $\overline{x} \in X$ and $\overline{y} \in Y$, respectively. Consider $\varphi(x,y) := \varphi_1(x) + \varphi_2(y)$. Then one has*

$$\partial_L\varphi(\overline{x},\overline{y}) = \partial_L\varphi_1(\overline{x}) \times \partial_L\varphi_2(\overline{y}).$$

**Lemma 2.5.** *Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be lsc on its domain and $\overline{x} \in \operatorname{dom}\varphi$.*

(i) *If $\varphi$ attains a local minimum at $\overline{x}$, then $0 \in \partial_L\varphi(\overline{x})$.*

(ii) *If $\varphi$ is a convex function, then*

$$\partial_L\varphi(\overline{x}) = \partial\varphi(\overline{x}) := \{x^* \in X^* : \ \langle x^*, x - \overline{x}\rangle \leq \varphi(x) - \varphi(\overline{x}) \ \ \forall x \in X\}.$$

Although the class of Asplund spaces is quite broad, $\ell_1$ and $\ell_\infty$ are not Asplund. Another type of topological normal and subdifferential structures was developed by Ioffe, under the name of **approximate normals and subgradients**, as an extension of Mordukhovichs construction to arbitrary Banach spaces; see remarks in [21]. Assume now that $X$ is a Banach space, $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ is lsc on $X$, and $x \in \operatorname{dom}\varphi$. Ioffe [16, 17, 18] introduced the **approximate subdifferential** of $\varphi$ at $\overline{x}$:

**Definition 2.6.** *The* **approximate subdifferential** *of $\varphi$ at $\overline{x}$ is the set*

$$\partial_a\varphi(\overline{x}) := \bigcap_{L\in\mathcal{F}} \operatorname*{Lim\,sup}_{\epsilon\downarrow 0, x\to\overline{x}} \partial^-_\epsilon\varphi_{x+L}(\overline{x}),$$

*where $\mathcal{F}$ is the collection of all finite dimensional subspaces $L$ of $X$, $\varphi_{x+L}(y) = \varphi(y)$ if $y \in x + L$ and $+\infty$ otherwise, and*

$$\partial^-_\epsilon\varphi_{y+L}(y) := \{x^* \in X^* : \langle x^*, v\rangle \leq \liminf_{t\to+0} \frac{f_{y+L}(y+tv) - f_{y+L}(y)}{t} + \epsilon\|v\|, \ \forall v \in X\}.$$

*Given a nonempty set $\Omega$ in $X$. The* **approximate (Ioffe) normal cone** *to $\Omega$ at $\overline{x} \in \Omega$ is given by*

$$N_a(\overline{x};\Omega) := \cup_{\lambda>0}\lambda\partial_a d(\overline{x};\Omega),$$

*where $d(\cdot;\Omega) : X \to \mathbb{R}$ is the distance function to $\Omega$, i.e., $d(x;\Omega) := \inf_{u\in\Omega}\|x-u\|$.*

The next theorem states that in finite dimensional Banach spaces, in particular in $\mathbb{R}^n$, both the limiting and the approximate subdifferential coincide. This gives us two different approaches for calculating these subdifferentials.

**Theorem 2.7.** *Suppose that $X$ is a finite dimensional Banach space and $\varphi : X \to \overline{\mathbb{R}}$. Then, we have $\partial_L\varphi(x) = \partial_a\varphi(x)$ for all $x \in X$.*

At last, we need a solution concept for the vector optimization problems which we will consider in our paper.

**Definition 2.8.** *Let $X$ and $Y$ be arbitrary Banach spaces, $f : X \to Y$, $A \subseteq X$, $B \subseteq Y$ and $C \subset Y$ a proper closed cone with nonempty interior. We call $y_0 \in B$ a weakly minimal element of $B$ with respect to $C$ if there exists no $y \in B$ with $y_0 \in y + \operatorname{int} C$. The set of all weakly minimal elements of $B$ with respect to $C$ is denoted by* $\operatorname{WMin}(B, C)$.
*If $f(x_0) \in \operatorname{WMin}(f(A), C)$, then we call $x_0$ a weakly minimum of $f$ over $A$ with respect to $C$.*

## 2.2 Nonlinear scalarization

Let $Y$ be a real linear topological space and $C$ be a nonempty set in $Y$. The notations $\operatorname{int}(C)$, $\operatorname{cl}(C)$, and $\operatorname{bd}(C)$ stand for the topological interior, the topological closure, and the topological boundary of the set $C$, respectively. The set $C$ is said to be *solid* iff $\operatorname{int}(C) \neq \emptyset$, *proper* iff $C \neq \emptyset$ and $C \neq Y$, *pointed* iff $C \cap (-C) \subseteq \{\mathbf{0}\}$, and *a cone* iff $\lambda c \in C$ for all $c \in C$ and $\lambda \geq 0$. See [9, 12, 13, 19] for basic definitions and concepts of vector optimization, and [11, 27] for some scalarization methods in (convex and nonconvex) vector optimization with fixed domination structure/ordering cone and important properties of these methods.

Let us recall a powerful nonlinear scalarization tool from [11] by Tammer and Weidner; cf. [12] which is used in the sequel. Let $A$ be a nonempty subset of $Y$ and $k$ be a nonzero vector of $Y$. The function $\varphi_{A,k} : Y \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$\varphi_{A,k}(y) := \inf\{t \in \mathbb{R} \ : \ y \in tk - A\} \tag{5}$$

is called a *nonlinear (separating) scalarization* function (with respect to the set $A$ and the direction $k$). The following lemmas provide several important properties of $\varphi_{A,k}$.

**Lemma 2.9.** *([12, Theorem 2.3.1]) Let $Y$ be a real topological linear space, $A$ be a closed proper set in $Y$, and $k \in Y \setminus \{\mathbf{0}\}$ be a nonzero vector; namely, a direction of $Y$. Assume that the pair $(A, k)$ satisfies the following condition*

$$A + [0, +\infty)k \subseteq A. \tag{6}$$

*Then the following hold:*

(a) *The function $\varphi_{A,k}$ is lsc over its domain $\operatorname{dom} \varphi_{A,k} = \mathbb{R}k - A$. Moreover, its $t$-level set is given by*

$$\forall\, t \in \mathbb{R} : \quad \{y \in Y \ : \ \varphi_{A,k}(y) \leq t\} = tk - A, \tag{7}$$

*and the transformation of $\varphi_{A,k}$ along the direction $k$ is calculated by*

$$\forall\, y \in Y,\ \forall\, t \in \mathbb{R} : \quad \varphi_{A,k}(y + tk) = \varphi_{A,k}(y) + t. \tag{8}$$

(b) $\varphi_{A,k}$ *is convex if and only if the set* $A$ *is convex, and* $\varphi_{A,k}$ *is positively homogeneous, i.e.* $\varphi_{A,k}(ty) = t\varphi_{A,k}(y)$ *for all* $t \geq 0$ *and* $y \in Y$*, if and only if* $A$ *is a cone.*

(c) $\varphi_{A,k}$ *is proper if and only if* $A$ *does not contain lines parallel to* $k$*, i.e.*

$$\forall\, y \in Y,\ \exists\, t \in \mathbb{R}\ :\ y + tk \notin A. \tag{9}$$

(d) $\varphi_{A,k}$ *is finite-valued, i.e.* $\operatorname{dom}\varphi_{A,k} = Y$*, if and only if*

$$\mathbb{R}k - A = Y. \tag{10}$$

(e) *Given* $B \subset Y$*.* $\varphi_{A,k}$ *is* ***B-monotone****, i.e.* $\big[a \in b - B \iff \varphi_{A,k}(a) \leq \varphi_{A,k}(b)\big]$ *if and only if* $A + B \subset A$*.*

(f) $\varphi_{A,k}$ *is subadditive if and only if* $A + A \subset A$*.*

In many common situations, we need stronger properties of the function $\varphi_{A,k}$ such as continuity or even Lipschitz continuity.

**Lemma 2.10.** *([12, Theorem 2.3.1]) Let* $Y, A, B, k$*, and* $\varphi_{A,k}$ *be as in Lemma 2.9. Suppose additionally* $\operatorname{int}(A) \neq \emptyset$ *and*

$$A + (0, +\infty)k \subseteq \operatorname{int}(A). \tag{11}$$

*Then, one has:*

(g) $\varphi_{A,k}$ *is continuous and*

$$\begin{aligned} \forall\ t \in \mathbb{R}:\ \{y \in Y\ :\ \varphi_{A,k}(y) < t\} &= tk - \operatorname{int}(A), && (12)\\ \forall\ t \in \mathbb{R}:\ \{y \in Y\ :\ \varphi_{A,k}(y) = t\} &= tk - \operatorname{bd}(A). && (13) \end{aligned}$$

(h) *Assume that* $\varphi_{A,k}$ *is proper. Then,* $\varphi_{A,k}$ *is* $B$*-monotone* $\Leftrightarrow A + B \subset A \Leftrightarrow \operatorname{bd} A + B \subset A$*. Assume further that* $\varphi_{A,k}$ *is finite-valued. Then,* $\varphi_{A,k}$ *is strictly* $B$*-monotone, i.e.,* $\big[a \in b - B\ \wedge\ a \neq b \implies \varphi_{A,k}(a) < \varphi_{A,k}(b)\big] \Leftrightarrow A + (B \setminus \{\mathbf{0}\}) \subseteq \operatorname{int}(A) \Leftrightarrow \operatorname{bd}(A) + (B \setminus \{\mathbf{0}\}) \subseteq \operatorname{int}(A)$*.*

(k) *Assume that* $\varphi_{A,k}$ *is proper. Then,* $\varphi_{A,k}$ *is subadditive* $\Leftrightarrow A + A \subset A \Leftrightarrow \operatorname{bd}(A) + \operatorname{bd}(A) \subseteq A$*.*

The next lemma provides, in addition to some properties in the previous two lemmas, several important ones broadly used in vector optimization.

**Lemma 2.11.** *([12, Corollary 2.3.5 and Theorem 2.3.6] and [7, Lemma 2.1]) Let* $Y$ *be a real topological linear space,* $C \subset Y$ *be a proper, closed, convex and solid cone as an ordering cone of* $Y$*, and* $k^0 \in \operatorname{int}(C)$ *be a (positive) direction of* $Y$*. Assume that condition* (11) *holds for* $A = C$ *and* $k = k^0$*. The function* $\varphi_{C,k^0}$ *defined in* (5) *is a continuous, sublinear, and strictly-*$\operatorname{int}(C)$*-monotone function. Moreover, the following hold:*

(a) $\varphi_{C,k^0}$ *is linearly shifted along the direction* $k_0$*, i.e.,*

$$\forall y \in Y,\ \forall t \in \mathbb{R}: \quad \varphi_{C,k^0}(y + tk^0) = \varphi_{C,k^0}(y) + t. \tag{14}$$

(b) $\partial\varphi_{C,k^0}(\mathbf{0}) = \{y^* \in C^+ :\ \langle y^*, k^0\rangle = 1\}$ *with* $C^+ := \{y^* \in Y^* :\ \langle y^*, y\rangle \geq 0,\ \forall y \in C\} = -N(\mathbf{0}; C)$*, where* $N(y; C)$ *denotes the classical normal cone of convex analysis to the set* $C$ *at* $y$*.*

(c) $\partial\varphi_{C,k^0}(y) = \{y^* \in C^+ :\ \langle y^*, k^0\rangle = 1 \ \wedge\ \langle y^*, y\rangle = \varphi_{C,k^0}(y)\},\ \forall y \in Y.$

(d) *Given a nonempty set* $\Xi$ *in* $Y$*. If* $\overline{y} \in \Xi$ *is a weakly minimal point of* $\Xi$ *with respect to* $C$ *then one has*

$$\varphi_{C,k^0}(y - \overline{y}) \geq 0,\ \forall y \in \Xi.$$

**Lemma 2.12.** *([7, Theorem 2.2]) Let* $A \subseteq Y$ *be a closed convex proper set and* $k \in Y \setminus \{\mathbf{0}\}$ *such that condition* (6) *holds and for every* $y \in Y$ *there exists* $t \in \mathbb{R}$ *such that* $y + tk \in A$*. Consider the function* $\varphi_{A,k^0}(\cdot)$ *in* (5) *and let* $y \in \operatorname{dom}\varphi_{A,k^0}$*. Then one has*

$$\varphi_{A,k^0}(y) = \{y^* \in Y^* :\ y^*(k^0) = 1 \ \wedge\ y^*(d) + y^*(y) - \varphi_{A,k^0}(y) \geq 0,\ \forall\, d \in A\}. \tag{15}$$

# 3 Subdifferential of nonconvex scalarization functions

The first result provides exact subdifferential formulas for nonlinear scalarization functions $\varphi_{A,k}$ defined in (5) in the case where the set $A$ does not necessarily enjoy conical or convex properties.

**Proposition 3.1.** *Let* $Y$ *be a real topological linear space,* $A$ *be a closed proper set in* $Y$*, and* $k \in Y \setminus \{\mathbf{0}\}$ *be a nonzero vector; namely, a direction of* $Y$*. Assume that the pair* $(A, k)$ *satisfies condition* (11)*,* $A + (0, +\infty)k \subseteq \operatorname{int} A$*. Assume also that* $y \in \operatorname{dom}\varphi_{A,k} = \mathbb{R}k - A$*. Then we have:*

(i) *The subdifferential of* $\varphi_{A,k}$ *at* $y \in \operatorname{dom}\varphi_{A,k}$ *is*

$$\partial_L\varphi_{A,k}(y) = \{y^* \in Y^* :\ y^*(k) = 1 \ \wedge\ -y^* \in N_L(\varphi_{A,k}(y)k - y; \operatorname{bd} A)\}. \tag{16}$$

(ii) *The singular subdifferential of* $\varphi_{A,k}$ *at* $y \in \operatorname{dom}\varphi_{A,k}$ *satisfies* $\partial^\infty\varphi_{A,k}(y) \subseteq D^*_L\varphi_{A,k}(y)(0)$ *with*

$$D^*_L\varphi_{A,k}(y)(0) = \{y^* \in Y^* :\ y^*(k) = 0 \ \wedge\ -y^* \in N_L(\varphi_{A,k}(y)k - y; \operatorname{bd} A)\}. \tag{17}$$

*Proof.* By Lemma 2.10(g) we have

$$\varphi_{A,k}^{-1}(t) := \{y \in Y \mid \varphi_{A,k}(y) = t\} = tk - \mathrm{bd}(A), \quad \forall\ t \in \mathbb{R}.$$

Obviously, the set-valued mapping $\varphi_{A,k}^{-1} : \mathbb{R} \rightrightarrows Y$ can be decomposed into a sum of one linear function $k(t) := tk$ and a constant set-valued mapping $\mathcal{C}(\cdot; -\mathrm{bd}\, A) : \mathbb{R} \rightrightarrows Y$ with $\mathcal{C}(t; -\mathrm{bd}\, A) = -\mathrm{bd}\, A$ for all $t \in \mathbb{R}$. By employing the coderivative sum rules with equalities from [21, Theorem 1.62] to $\varphi_{A,k}^{-1}(t) = k(t) + (t; -\mathrm{bd}\, A)$, we have

$$D_L^* \varphi_{A,k}^{-1}(t, y)(y^*) = y^*(k) + D_L^* \mathcal{C}(\cdot; -\mathrm{bd}\, A)(t, y - tk)(y^*). \tag{18}$$

Since $\mathrm{gph}\, \mathcal{C}(\cdot; -\mathrm{bd}\, A) = \mathbb{R} \times (-\mathrm{bd}\, A)$, we get from the normal cone to Cartesian sets in [21, Proposition 1.2] that

$$N_L((t, y-tk); \mathbb{R}\times -\mathrm{bd}\, A) = N_L(t; \mathbb{R})\times N_L(y-tk; -\mathrm{bd}\, A) = N_L(t; \mathbb{R})\times N_L(tk-y; \mathrm{bd}\, A).$$

Thus, by the definition of the coderivative, if $D_L^* \mathcal{C}(y - tk; -\mathrm{bd}\, A)(y^*) \neq \emptyset$, then $-y^* \in N_L(tk - y; \mathrm{bd}\, A)$ and $D_L^* \mathcal{C}(y - tk; -\mathrm{bd}\, A)(y^*) = \{0\}$. By (18), for any $-y^* \in N_L(tk - y; \mathrm{bd}\, A)$, we have

$$\begin{aligned} & D_L^* \varphi_{A,k}^{-1}(t, y)(y^*) = \{y^*(k)\} \\ \Longleftrightarrow\ & (y^*(k), -y^*) \in N_L((t, y); \mathrm{gph}\, \varphi_{A,k}^{-1}) \\ \Longleftrightarrow\ & (-y^*, y^*(k)) \in N_L((y, t); \mathrm{gph}\, \varphi_{A,k}) \\ \Longleftrightarrow\ & -y^* \in D_L^* \varphi_{A,k}(y, t)(-y^*(k)). \end{aligned}$$

By the subfifferentials from coderivatives of continuous functions, [21, Theorem 1.80], we get from the relation above that

$$\begin{aligned} \partial_L \varphi_{A,k}(y) &= D_L^* \varphi_{A,k}(y, \varphi_{A,k}(y))(-1) \\ &= \{y^* \in Y^* :\ y^*(k) = 1 \ \wedge\ -y^* \in N_L(\varphi_{A,k}(y)k - y; \mathrm{bd}\, A)\} \end{aligned}$$

and

$$\begin{aligned} \partial_L^\infty \varphi_{A,k}(y) &\subseteq D_L^* \varphi_{A,k}(y, \varphi_{A,k}(y))(0) \\ &= \{y^* \in Y^* :\ y^*(k) = 0 \ \wedge\ -y^* \in N_L(\varphi_{A,k}(y)k - y; \mathrm{bd}\, A)\}. \end{aligned}$$

The proof is complete. □

When $A$ is a convex set, the closed form of the subdifferential of $\varphi_{A,k}$ obtained in (16) is more efficient than the one obtained in [7, Theorem 2.2]

$$\partial \varphi_{A,k}(\overline{y}) = \{y^* \in Y^* :\ y^*(k_0) = 1 \ \wedge\ \forall d \in A :\ y^*(d) + y^*(\overline{y}) - \varphi_{A,k}(\overline{y}) \geq 0\},$$

see (15).

**Proposition 3.2.** *Let $Y$, $A$ and $k$ satisfy condition* (11) *as in Proposition 3.1. Then, for any $y \in \operatorname{dom} \varphi_{A,k} = \mathbb{R}k - A$ and for any $t \in \mathbb{R}$ we have $\partial_L \varphi_{A,k}(y + tk) = \partial_L \varphi_{A,k}(y)$, $\partial_L^\infty \varphi_{A,k}(y + tk) = \partial_L^\infty \varphi_{A,k}(y)$, and $D_L^* \varphi_{A,k}(y + tk)(y^*) = D_L^* \varphi_{A,k}(y)(y^*)$ for all $y^* \in Y^*$.*

*Proof.* The proof is straight forward due to Lemma 2.9(a) providing that

$$\varphi_{A,k}(y + tk) = \varphi_{A,k}(y) + t, \ \forall\, y \in Y, \ \forall\, t \in \mathbb{R}.$$

□

**Corollary 3.3.** *Let $Y$, $A$ and $k$ satisfy condition* (11) *as in Proposition 3.1. Then $\varphi_{A,k}$ is locally Lipschitz continuous at a point $y \in \operatorname{dom} \varphi_{A,k}$ if and only if the point $\overline{y} := y - \varphi_{A,k}(y) \in \operatorname{bd} A$ satisfies the following condition*

$$N_L(\overline{y}; \operatorname{bd} A) \cap H(k) = \{\mathbf{0}\} \ \ \textit{with} \ \ H(k) = \{y^* \in Y^* : \ y^*(k) = 0\}. \tag{19}$$

*Proof.* By Propositions 3.1 and 3.2, we have

$$D_L^* \varphi_{A,k}(y)(0) = D_L^* \varphi_{A,k}(\overline{y})(0) = -N_L(\overline{y}; \operatorname{bd} A) \cap H(k).$$

Therefore, Theorem 4.0 in [21], which tells us that $\varphi_{A,k}$ is locally Lipschitz continuous at a point $y$ if and only if $D_L^* \varphi_{A,k}(y)(0) = \{\mathbf{0}\}$, completes the proof. □

**Example 3.4.** *Let $A = \mathbb{R}^2_+$, $e_1 = (1, 0)$, $e_2 = (0, 1)$, $k = e_1 + e_2 = (1, 1) \in \operatorname{int} A$, and $\varphi_{A,k}$ defined as in* (5)*. Obviously, $\operatorname{bd}(A) = A_1 \cup A_2$ with $A_1 = [0, +\infty) \times \{0\}$ and $A_2 = \{0\} \times [0, +\infty)$. We have*

$$N_L((a, b); \operatorname{bd} A) = \begin{cases} \mathbb{R} \cdot e_2 & \textit{if} \ \ a > 0 \ \ \textit{and} \ \ b = 0 \\ \mathbb{R} \cdot e_1 & \textit{if} \ \ a = 0 \ \ \textit{and} \ \ b > 0 \\ \mathbb{R}^2_- \cup \mathbb{R}_+ \cdot e_1 \cup \mathbb{R}_+ \cdot e_2 & \textit{if} \ \ a = 0 \ \ \textit{and} \ \ b = 0. \end{cases}$$

*By Proposition 3.2, $\partial_L \varphi_{A,k}(a, a) = \partial_L \varphi_{A,k}(0, 0)$, $\partial_L \varphi_{A,k}(a, b) = \partial_L \varphi_{A,k}(0, b - a)$ if $b > a$ and $\partial_L \varphi_{A,k}(a, b) = \partial_L \varphi_{A,k}(a - b, 0)$ if $b < a$. Then, we get from Proposition 3.1 that*

$$\partial_L \varphi_{A,k}(a, b) = \begin{cases} \{e_2\} & \textit{if} \ \ a > b \\ \{e_1\} & \textit{if} \ \ a < b \\ \{(c, d) \in \mathbb{R}^2_+ : \ c + d = 1\} & \textit{if} \ \ a = b. \end{cases}$$

*In addition, since $H(k) = \mathbb{R} \cdot (e_2 - e_1)$ has a trivial intersection with $N_L((a, b); \operatorname{bd} A)$, Corollary 3.3 ensures that the function $\varphi_{A,k}$ is locally Lipschitz continuous everywhere.*

**Example 3.5.** *Let $A = \{(a, b) \in \mathbb{R}^2 : \ |b| \leq 1 \ \wedge \ a \geq |b|\}$, $k = (1, 0) \in \operatorname{int} A$, and $\varphi_{A,k}$ defined as in* (5)*. Obviously, the pair $(A, k)$ satisfies condition* (6) *and*

*$\operatorname{bd} A = A_1 \cup A_2 \cup A_3 \cup A_4$ with $A_1 = [1, +\infty) \times \{-1\}$, $A_2 = [0,1] \cdot (e_1 - e_2)$, $A_3 = [0,1] \cdot (e_1 + e_2)$, and $A_4 = [1, +\infty) \times \{1\}$. It is not difficult to check that*

$$N_L((a,b); \operatorname{bd} A) =$$
$$\begin{cases} \mathbb{R} \cdot (0,1) \ \ \textit{if} \ \ a > 1 \ \ \textit{and} \ \ b = \pm 1; \\ \mathbb{R} \cdot (1,1) \ \ \textit{if} \ \ (a,b) = \lambda \cdot (1,-1) \ \ \textit{for some} \ \ \lambda \in (0,1); \\ \mathbb{R} \cdot (1,-1) \ \ \textit{if} \ \ (a,b) = \lambda \cdot (1,1) \ \ \textit{for some} \ \ \lambda \in (0,1); \\ \{(m,n) \in \mathbb{R}^2 : \ |n| < -m \ \vee \ |n| = |m|\} \ \ \textit{if} \ \ (a,b) = (0,0); \\ \{(m,n) \in \mathbb{R}^2 : \ n > -m \wedge m < 0\} \cup \mathbb{R} \cdot \{(0,1), (1,-1)\} \ \ \textit{if} \ \ (a,b) = (1,1); \\ \{(m,n) \in \mathbb{R}^2 : \ n < m \wedge m < 0\} \cup \mathbb{R} \cdot \{(0,1), (1,1)\} \ \ \textit{if} \ \ (a,b) = (1,-1). \end{cases}$$

*By Proposition 3.1, for all $t \in \mathbb{R}$ we have*

$$\partial_L \varphi_{A,k}(a + tk, b) = \partial_L \varphi_{A,k}(a,b) = \begin{cases} \{(1,-1)\} & \textit{if} \ \ 0 < a < 1, b = a \\ \{(1,1)\} & \textit{if} \ \ 0 < a < 1, b = -a \\ \{1\} \times (-\infty, -1] & \textit{if} \ \ (a,b) = (1,1) \\ \{1\} \times [1, +\infty) & \textit{if} \ \ (a,b) = (1,-1) \\ \{1\} \times [-1, 1] & \textit{if} \ \ (a,b) = (0,0) \\ \emptyset & \textit{otherwise,} \end{cases}$$

$$D_L^* \varphi_{A,k}(a + tk, b)(0) = D_L^* \varphi_{A,k}(a,b)(0) = \begin{cases} \{(0,0)\} & \textit{if} \ \ 0 \le a < 1 \ \ \textit{and} \ \ |b| < 1 \\ \mathbb{R} \cdot (0,1) & \textit{if} \ \ a \ge 1 \ \ \textit{and} \ \ |b| = 1 \\ \emptyset & \textit{otherwise.} \end{cases}$$

*Therefore, $\varphi_{A,k}$ is locally Lipschitz continuous at any point $(a,b)$ with $|b| < 1$ by Corollary 3.3.*

# 4 Scalarization Functions for Variable Domination Structures

In this section, we further extend the results from Section 3 to the case where the domination set is varied from one element in the image space to another; in other words, it is no longer fixed. A variable domination structure $C : Y \rightrightarrows Y$ is a set-valued mapping satisfying $\mathbf{0} \in C(y), \ \forall y \in Y$. In this setting, the nonlinear scalarization function $\varphi_{C(\cdot),k} \colon Y \mapsto \mathbb{R} \cup \{\pm\infty\}$ with respect to the variable domination structure $C(\cdot)$ and the direction $k \in Y \setminus \{\mathbf{0}\}$ is defined by

$$\varphi(y) := \varphi_{C(\cdot),k}(y) = \inf\{t \in \mathbb{R} : \ y \in tk - C(y)\} \text{ for all } \ y \in Y. \tag{20}$$

Scalarization functions related to variable domination structures are discussed in [5], [6], [9] and [25].

**Proposition 4.1.** *Let $C : Y \rightrightarrows Y$ be a set-valued mapping/variable domination structure and $k \in Y \setminus \{\mathbf{0}\}$ be a direction of $Y$. Assume that $C(y)$ is a closed and proper set in $Y$ satisfying condition (11), i.e., $C(y) + (0, +\infty)k \subseteq \operatorname{int} C(y)$ for all $y \in Y$. Then:*

(a) *If $y \in \mathbb{R}k - C(y)$; in particular, $\mathbb{R}k - C(y) = Y$ or $\operatorname{int} C(y) \neq \emptyset$, for some element $y \in Y$, then $\varphi(y)$ is finite.*

(b) *For each fixed element $y \in Y$, if $y \in tk - C(y)$ for some $t \in \mathbb{R}$, then for each $t' \geq 0$ we have $y \in (t + t')k - C(y)$.*

(c) *Assume that $\varphi(y)$ is finite for some element $y \in Y$. Then $\lambda \geq \varphi(y)$ if and only if $y \in \lambda k - C(y)$.*

*Proof.* Prove (a). Assume that $\mathbb{R}k - C(y) = Y$ holds for some fixed element $y \in Y$. By the definition of $\varphi$, we have

$$\varphi(y) = \inf\{t \in \mathbb{R} : \; y \in tk - C(y)\} = \varphi_{C(y),k}(y)$$

and thus $\varphi(y)$ is finite due to Lemma 2.9(d).

Prove (b). For each $t \in \mathbb{R}$ and $t' \geq 0$, we have

$$y \in tk - C(y) \iff y \in (t + t')k - C(y) - t'k \overset{(11)}{\subseteq} (t + t')k - C(y).$$

Prove (c). By definition of $\varphi$, the implication

$$y \in \lambda k - C(y) \implies \lambda \geq \varphi(y)$$

holds. To prove the inverse of this implication, assume that $\lambda \geq \varphi(y)$. Then we get from $\varphi(y) = \inf\{t \in \mathbb{R} : \; y \in tk - C(y)\}$ that for every $t' > 0$, we have $\lambda + t' > \varphi(y)$ and $y \in (\lambda + t')k - C(y) \iff (\lambda + t')k \in y + C(y)$. By the closedness of $C(y)$, we have $\lambda k \in y + C(y_1) = (I + C)(y)$ as $t'$ approaches zero. Therefore, the inverse implication is true as well. □

In the next proposition, the letter $k$ stands for both a vector in $Y$ and a function from $\mathbb{R}$ to $Y$. Therefore, the notation $k^{-1}$ is the inverse function of the function $k$.

**Proposition 4.2.** *Let $k \in Y$. Define a function $k : \mathbb{R} \to Y$ by $k(t) := tk$ for all $t \in \mathbb{R}$ and denote*

$$\Xi := \operatorname{epi} k^{-1}(\cdot) = \{(y, t) \in Y \times \mathbb{R} : \; \exists\, t' \leq t : \; y = t'k\}$$

*to be a convex set of $Y \times \mathbb{R}$. Then we have*

$$N((\bar{t}k, \bar{t}); \Xi) = \widehat{N}((\bar{t}k, \bar{t}); \Xi) = \{(k^*, -\gamma) \in Y^* \times \mathbb{R} : \; k^*(k) = \gamma\}. \tag{21}$$

*Therefore, the subdifferentials of the function $k^{-1}(\cdot)$ can be computed by*

$$\partial_L k^{-1}(\bar{t}k, \bar{t}) = \{k^* \in Y^* : \; k^*(k) = 1\} \text{ and } \partial^\infty k^{-1}(\bar{t}k, \bar{t}) = \{k^* \in Y^* : \; k^*(k) = 0\}. \tag{22}$$

*Proof.* By the definitions of subdifferential and singular subdifferential, (22) follows from (21). It remains to verify the formula for the normal cone to $\Xi$ at $(\bar{t}k, \bar{t})$ in (21). Obviously, $\Xi$ is a convex set. By the definition of normal cones to convex sets, $(k^*, -\gamma) \in \widehat{N}((\bar{t}k, \bar{t}); \Xi)$ iff

$$
\begin{aligned}
& \langle (k^*, -\gamma), ((tk, t) - (\bar{t}k, \bar{t})) \rangle \leq 0 \; \forall\, t \in \mathbb{R} \\
\iff \; & (t - \bar{t})\big(k^*(k) - \gamma\big) \leq 0 \; \forall\, t \in \mathbb{R} \\
\iff \; & k^*(k) - \gamma = 0 \iff k^*(k) = \gamma.
\end{aligned}
$$

The proof is complete. □

**Proposition 4.3.** *Consider the set-valued mapping $P : Y \rightrightarrows Y$ defined by $P(y) := (I + C)(y) = y + C(y)$. For any pair $(y_1, y_2) \in \operatorname{gph} P$, we have*

$$D_L^* P(y_1, y_2)(y^*) = y^* + D_L^* C(y_1, y_2 - y_1)(y^*), \; \forall\, y^* \in Y^*.$$

*Proof.* It is straight forward from [21, Theorem 1.62]. □

**Proposition 4.4.** *Let $\varphi : Y \to \mathbb{R}$ defined in (20) satisfy all the assumptions in Proposition 4.1 and $k : \mathbb{R} \to Y$ with $k(t) = tk$ for some $k \in Y$, and $\overline{y} \in \operatorname{dom} \varphi$. Denote the composition of $k$ and $\varphi$ as $\psi := k \circ \varphi : Y \to Y$. Then we have*

$$D_L^* \psi(\overline{y})(k^*) = D_L^*(k \circ \varphi)(\overline{y})(k^*) \subseteq D_L^* \varphi(\overline{y})\big(k^*(k)\big) \tag{23}$$

*provided that the pair $(\varphi, k)$ satisfies the qualification condition*

$$\big[k^*(k) = 0 \;\wedge\; k^* \in D_L^* \varphi(\overline{y})(0)\big] \implies k^* = \mathbf{0}. \tag{24}$$

*Proof.* By the chain rule for coderivatives in[21, Theorem 3.13], the coderivative formula for compositions of type $\psi \circ \varphi$ in (23) takes the form

$$
\begin{aligned}
& D_L^* \psi(\overline{y})(k^*) = D_L^*(k \circ \varphi)(\overline{y})(k^*) \\
\subseteq \quad & \bigcup \{ D_L^* \varphi(\overline{y})(\gamma) : \; \gamma \in D_L^* k(\varphi(\overline{y}))(k^*) \} \\
\overset{Prop.\ 4.2}{\subseteq} \quad & \bigcup \{ D_L^* \varphi(\overline{y})(\gamma) : \; k^*(k) = \gamma \} = D_L^* \varphi(\overline{y})\big(k^*(k)\big)
\end{aligned}
$$

since the qualification condition of the chain rule is satisfied

$$
\begin{aligned}
& k^* \in D_L^* \varphi(\overline{y}, \bar{t})(0) \cap (-D_L^* k^{-1}(\bar{t}k, \bar{t})(0)) \implies k^* = 0 \\
\iff \; & k^* \in D_L^* \varphi(\overline{y}, \bar{t})(0) \;\wedge\; -k^* \in D_L^* k^{-1}(\bar{t}k, \bar{t})(0) \\
\overset{\text{def.}}{\iff} \; & k^* \in D_L^* \varphi(\overline{y}, \bar{t})(0) \;\wedge\; (0, -k^*) \in N_L((\bar{t}k, \bar{t}); \operatorname{gph} k) \\
\overset{(23)}{\iff} \; & k^* \in D_L^* \varphi(\overline{y}, \bar{t})(0) \;\wedge\; k^*(k) = 0.
\end{aligned}
$$

and since the PSNC assumption imposed in the chain rule automatically holds in the space $\mathbb{R}$ of dimension 1. □

**Proposition 4.5.** *Let $\varphi : Y \to \mathbb{R}$ defined in (20) satisfy all the assumptions in Proposition 4.1 and $k : \mathbb{R} \to Y$ with $k(t) = tk$ for some $k \in Y$, and $\overline{y} \in \operatorname{dom} \varphi$. Then, by denoting the mapping $\Phi = (\psi, \varphi)$ with $\psi = k \circ \varphi$, we have*

$$D_L^*\Phi(\overline{y})(k^*, \gamma) \subseteq D_L^*\psi(\overline{y})(k^*) + D_L^*\varphi(\overline{y})(\gamma) \tag{25}$$

*provided that the pair $(\varphi, k)$ satisfies the qualification condition*

$$\Big[k^*(k) = 0 \ \wedge\ k^* \in D_L^*\varphi(\overline{y})(0) \cap \big( - D_L^*\varphi(\overline{y})(0)\big)\Big] \implies k^* = \mathbf{0}. \tag{26}$$

*Assume furthermore that $D_L^*\varphi(\overline{y})(0) = \{\mathbf{0}\}$. Then, we have $D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma) = D_L^*\varphi(\overline{y})(\gamma)$.*

*Proof.* Observe that the function $\Phi = (\psi, \varphi)$ can be expressed in terms of a sum of two functions $F_1, F_2 : Y \to Y \times \mathbb{R}$ with

$$F_1(y) := (\psi(y), 0) \ \text{ and } \ F_2(y) := (\mathbf{0}, \varphi(y)).$$

Obviously, $F_2$ is PSNC. By the sum rules for coderivatives in [21, Theorem 3.10], inclusion (25) holds:

$$\begin{aligned} & D_L^*(\psi, \varphi)(\overline{y})(k^*, \gamma) = D_L^*(F_1 + F_2)(\overline{y})(k^*, \gamma) \\ \subseteq\ & D_L^*F_1(k^*, \gamma) + D_L^*F_2(k^*, \gamma) = D_L^*(\psi, 0)(k^*, \gamma) + D_L^*(\mathbf{0}, \varphi)(k^*, \gamma) \\ =\ & D_L^*\psi(\overline{y})(k^*) + D_L^*\varphi(\overline{y})(\gamma) \end{aligned}$$

provided that $(F_1, F_2)$ satisfies the qualification condition of the sum rule, i.e.,

$$\begin{aligned} & y^* \in D_L^*F_1(\overline{y})(\mathbf{0}, 0) \cap \big(D_L^*F_2(\overline{y})(\mathbf{0}, 0)\big) \implies y^* = \mathbf{0} \\ \iff\ & \begin{cases} y^* \in D_L^*F_1(\overline{y})(\mathbf{0}, 0) = D_L^*\psi(\overline{y})(\mathbf{0}) \\ -y^* \in D_L^*F_2(\overline{y})(\mathbf{0}, 0) = D_L^*\varphi(\overline{y})(0) \end{cases} \implies y^* = \mathbf{0} \\ \overset{\text{Prop. 4.4}}{\impliedby}\ & y^*(k) = 0 \wedge y^* \in D_L^*\varphi(\overline{y})(0) \cap (-D_L^*\varphi(\overline{y})(0)) \implies y^* = \mathbf{0} \end{aligned}$$

which is fulfilled by the assumed qualification condition for $(\varphi, k)$ in (26).

To complete the proof, we need to consider the special case when $D_L^*\varphi(\overline{y})(\mathbf{0}) = \{\mathbf{0}\}$. In this case, we get from (25) that

$$\begin{aligned} D_L^*(\psi, \varphi)(\overline{y})(\mathbf{0}, \gamma) \quad & \subseteq \quad D_L^*\psi(\overline{y})(\mathbf{0}) + D_L^*\varphi(\overline{y})(\gamma) \\ & \overset{\text{Prop. 4.4}}{\subseteq} \quad D_L^*\varphi(\overline{y})(0) + D_L^*\varphi(\overline{y})(\gamma) = D_L^*\varphi(\overline{y})(\gamma). \end{aligned} \tag{27}$$

By applying the sum rule for coderivatives above for the sum $F_2 = \Phi + (-F_1)$ we also have

$$\begin{aligned} & D_L^*\varphi(\overline{y})(\gamma) = D_L^*F_2(k^*, \gamma) \subseteq D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma) + D_L^*(-F_1)(k^*, \gamma) \\ \subseteq\ & D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma) + D_L^*(-\psi, 0)(\mathbf{0}, \gamma) \\ =\ & D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma) + D_L^*(-\psi)(\overline{y})(k^*) \\ \overset{\text{Prop. 4.4}}{\subseteq}\ & D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma) + D_L^*\varphi(\overline{y})(\mathbf{0}) = D_L^*\Phi(\overline{y})(\mathbf{0}, \gamma). \end{aligned}$$

This together with (27) ensures that $D^*_L(\psi,\varphi)(\overline{y})(\mathbf{0},\gamma) = D^*_L\varphi(\overline{y})(\gamma)$. The proof is complete. $\square$

In the next proposition we will propose a new way to compute the subdifferential of the function $\varphi$ with respect to $C(\cdot)$ via a set-valued mapping $F$ whose graph can be expressed as the intersection of two sets.

**Proposition 4.6.** *Given a variable domination structure $C : Y \rightrightarrows Y$ and a nonzero vector $k \in Y \setminus \{\mathbf{0}\}$. Assume that $C(y)$ is a closed subset in $Y$, $\mathbb{R}k - C(y) = Y$, and $C(y) + (0,+\infty)k \subseteq \operatorname{int} C(y)$ for all $y \in Y$. Consider the function $\varphi$ defined by (20), the vector-valued function $k(\cdot) : \mathbb{R} \to Y$ defined by $k(t) := tk$, and the upper-level mapping of $C$ defined by $P(y) := (I + C)(y) = y + C(y)$. Associate the function $\varphi$ with a set-valued mapping $F : Y \rightrightarrows Y \times \mathbb{R}$ defined by*

$$F(y) := (k,1)[\varphi(y),+\infty) = \{(tk,t) \in Y \times \mathbb{R} :\ t \geq \varphi(y)\} \ \text{ for all } \ y \in Y. \tag{28}$$

*Then, the graph of $F$ can be expressed as an intersection of two sets in $Y \times Y \times \mathbb{R}$. Precisely,* $\operatorname{gph} F = \Omega_1 \cap \Omega_2$ *with*

$$\Omega_1 = Y \times \operatorname{gph}(\mathcal{E}_k{}^{-1}) \ \text{ and } \ \Omega_2 := \operatorname{gph} P \times \mathbb{R}, \tag{29}$$

*where $\mathcal{E}_k^{-1}(t)$ is the inverse of $\mathcal{E}_k(t)$ with $\mathcal{E}_k(t) = k[t,+\infty)$.*

*Proof.* By Proposition 4.1 (a), the function $\varphi$ is finite everywhere, i.e., $\operatorname{dom}\varphi = Y$. First, we show that $\operatorname{gph} F \subseteq \Omega_1 \cap \Omega_2$. For any triple $(y_1, y_2, t) \in \operatorname{gph} F$ we have $y_2 = (\varphi(y_1) + \theta)k$ for some $\theta > 0$, and $t = (\varphi(y_1) + \theta) \geq \varphi(y_1)$. Obviously, $(y_1, y_2, t) \in \Omega_1$. Since $\varphi(y_1) < +\infty$ and $C(y_1)$ is a close set, we get from Proposition 4.1 (c) that

$$y_1 \in (\varphi(y_1) + \theta)k - C(y_1)$$

and thus $y_2 = (\varphi(y_1) + \theta)k \in y_1 + C(y_1) = P(y_1)$ clearly verifying $(y_1, y_2, t) \in \Omega_2$ and thus $(y_1, y_2, t) \in \Omega_1 \cap \Omega_2$.

Next, we will show that $\Omega_1 \cap \Omega_2 \subseteq \operatorname{gph} F$. For any triple $(y_1, y_2, t) \in \Omega_1 \cap \Omega_2$, by the structure of the set $\Omega_1$ we get $y_2 = \bar{t}k$ and $t = \bar{t}$ for some $\bar{t} \in \mathbb{R}$. By the structure of the set $\Omega_2$ we have $(y_1, y_2) \in \operatorname{gph} P = \operatorname{gph}(I + C)$, i.e., $y_2 = \bar{t}k \in y_1 + C(y_1) \iff y_1 \in \bar{t}k - C(y_1)$. By the definition of $\varphi$, we get from the last inclusion that $\varphi(y_1) \leq \bar{t}$. Therefore, $(y_2, t) \in [\varphi(y_1) + (\bar{t} - \varphi(\bar{t}))] \times (k, 1)$ and thus $(y_1, y_2, t) \in \operatorname{gph} F$. The proof is complete. $\square$

We are now ready to compute the subdifferential of the function $\varphi$ with respect to a variable domination structure $C(\cdot)$.

**Theorem 4.7.** *Let $C : Y \rightrightarrows Y$ be a variable domination structure and $k \in Y \setminus \{\mathbf{0}\}$ satisfy*

$$C(y) + (0,+\infty)k \subseteq \operatorname{int} C(y) \ \textit{for all } y \in Y.$$

*Assume that $C(y)$ is a closed subset in $Y$ and $\mathbb{R}k - C(y) = Y$ for all $y \in Y$. Let $\varphi$ be defined by (20), let $F$ be defined by* (28)*, and let the sets $\Omega_i,\ i = 1, 2$ be as described in* (29)*. Given $(\overline{y}, \overline{t}k, \overline{t}) \in \operatorname{gph} F$ with $\overline{t} := \varphi(\overline{y})$, impose the following two qualification conditions:*

- *the qualification condition for the pair $(\varphi, k)$ in* (24)

$$\big[k^*(k) = 0 \ \wedge \ k^* \in D_L^*\varphi(\overline{y})(0)\big] \implies k^* = \mathbf{0},$$

- *the qualification condition for the pair $(C, k)$*

$$k^*(k) = 0 \wedge -k^* \in D_L^*C(\overline{y}, \overline{t}k - \overline{y})(k^*) \implies k^* = \mathbf{0}. \tag{30}$$

*Then we have*

$$\partial_L\varphi(\overline{y}) \subseteq \bigcup_{k^*(k)=1} \Big(D_L^*C\big(\overline{y}, \varphi(\overline{y})k - \overline{y}\big)(k^*) + k^*\Big). \tag{31}$$

*Proof.* On the one hand, the mapping $F$ described in (28) is a sum of two mappings $F(y) = (\psi, \mathcal{E}_\varphi)(y)$, where $\mathcal{E}_\varphi(y)$ is the epigraphical multifunction of $\varphi$. By Proposition 4.5, we have

$$D_L^*F(\overline{y}, \varphi(\overline{y})k, \varphi(\overline{y}))(\mathbf{0}, 1) = \partial_L\varphi(\overline{y})$$

under the assumptions made in this theorem.

On the other hand, the graph of $F$ is the intersection of two sets $\operatorname{gph} F = \Omega_1 \cap \Omega_2$, where $\Omega_i,\ i = 1, 2$ are defined in (29). Therefore, we have

$$N_L((\overline{y}, \overline{t}k, \overline{t}); \operatorname{gph} F) = N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_1 \cap \Omega_2). \tag{32}$$

Employing the intersection rule for normal cones of intersection sets in [21, Corollary 3.5] to the sets $\Omega_1$ and $\Omega_2$ ($\Omega_1$ is SNC at $(\overline{y}, \overline{t}k, \overline{t})$) we have

$$N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_1 \cap \Omega_2) \subseteq N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_1) + N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_2) \tag{33}$$

provided that the pair $\{\Omega_1, \Omega_2\}$ satisfies the normal qualification condition

$$N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_1) \cap \Big( - N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_2)\Big) = \{\mathbf{0}\}. \tag{34}$$

Let us check condition (34).
Assume that $(y^*, k^*, -\gamma) \in \Big(N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_1)\Big) \cap \Big( - N_L((\overline{y}, \overline{t}k, \overline{t}); \Omega_2)\Big)$. Then, we have

$$(y^*, k^*, -\gamma) \in N_L((\overline{y}, \overline{t}k, \overline{t}); Y \times \Xi) = \{\mathbf{0}\} \times N_L((\overline{t}k, \overline{t}); \Xi) \ \text{ and}$$
$$(-y^*, -k^*, \gamma) \in N_L((\overline{y}, \overline{t}k, \overline{t}); \operatorname{gph} P \times \mathbb{R}) = N_L((\overline{y}, \overline{t}k); \operatorname{gph} P) \times \{0\}.$$

and thus $y^* = 0$, $\gamma = 0$, $(k^*, 0) \in N_L((\bar{t}k, \bar{t}); \Xi)$ and $(\mathbf{0}, -k^*) \in N_L((\bar{y}, \bar{t}k); \mathrm{gph}\, P)$. By Proposition 4.2 and the definition of coderivative, we have

$$k^*(k) = 0 \text{ and } \mathbf{0} \in D_L^* P(\bar{y}, \bar{t}k)(k^*) = k^* + D_L^* C(\bar{y}, \bar{t}k - \bar{y})(k^*).$$

The imposed qualification condition (30) forces $k^* = \mathbf{0}$ and thus $(y^*, k^*, -\gamma) = (\mathbf{0}, \mathbf{0}, 0)$ verifying the fulfilment of the normal qualification condition (34) and thus the validity of (33).

Next, we further manipulate (33) as follows:

$$\begin{aligned}
& y^* \in D_L^* F(\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y}))(\mathbf{0}, 1) \\
\overset{Prop.\ 4.6}{\Longleftrightarrow} \quad & (y^*, \mathbf{0}, -1) \in N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \mathrm{gph}\, F\big) = N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \Omega_1 \cap \Omega_2\big) \\
\overset{(33)}{\Longrightarrow} \quad & (y^*, \mathbf{0}, -1) \in N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \Omega_1\big) + N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \Omega_2\big) \\
\Longrightarrow \quad & \exists\, k^* \in Y^*,\ (\mathbf{0}, k^*, -1) \in N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \Omega_1\big) \text{ and} \\
& \qquad (y^*, -k^*, 0) \in N_L\big((\bar{y}, \varphi(\bar{y})k, \varphi(\bar{y})); \Omega_2\big) \\
\Longrightarrow \quad & \exists\, k^* \in Y^*,\ (k^*, -1) \in N_L((\varphi(\bar{y})k, \varphi(\bar{y})); \Xi\big) \text{ and} \\
& \qquad (y^*, -k^*) \in N_L\big((\bar{y}, \varphi(\bar{y})k); \mathrm{gph}\, P\big) \\
\overset{Prop.\ 4.2,4.3}{\Longrightarrow} \quad & \exists\, k^* \in Y^*,\ k^*(\varphi(\bar{y})k) = \varphi(\bar{y}),\ y^* \in k^* + D_L^* C(\bar{y}, \varphi(\bar{y})k - \bar{y})(k^*).
\end{aligned}$$

Therefore, we have

$$\partial_L \varphi(\bar{y}) \subseteq \bigcup_{k^*(k)=1} \Big( D_L^* C\big(\bar{y}, \varphi(\bar{y})k - \bar{y}\big)(k^*) + k^* \Big).$$

The proof is complete. □

**Corollary 4.8.** *Let $Y$ be a real topological linear space, $A$ be a closed proper set in $Y$ with $\mathrm{int}\, A \neq \emptyset$, and $k \in Y \setminus \{\mathbf{0}\}$ be a nonzero vector; namely, a direction of $Y$. Assume that the pair $(A, k)$ satisfies condition (6) and $y \in \mathbb{R}k - A$. Assume also that $D_L^* \varphi(\bar{y})(0) = \{\mathbf{0}\}$. Then the subdifferential of $\varphi_{A,k}$ at $\bar{y} \in \mathrm{dom}\, \varphi_{A,k}$ is*

$$\partial_L \varphi_{A,k}(\bar{y}) \subseteq \{k^* \in Y^* :\ k^*(k) = 1 \ \wedge\ -k^* \in N_L(\varphi_{A,k}(\bar{y})k - \bar{y}; A)\}.$$

*Proof.* Consider the constant domination structure $C(y) := A$ for all $y \in Y$ with $\mathrm{gph}\, C = Y \times A$. Then, if $y^* \in D_L^* C(y_1, y_2)(k^*)$, then $y^* = \mathbf{0}$ and $k^* \in -N_L(y_2; A)$. Employing Theorem 4.7 for this constant domination structure gives

$$\begin{aligned}
\partial_L \varphi_{A,k}(\bar{y}) \ &\subseteq \ \bigcup_{k^*(k)=1} \{y^* \in D_L^* C(\bar{y}, \varphi_{A,k}(\bar{y})k - \bar{y})(k^*) - k^*\} \\
&= \ \bigcup_{k^*(k)=1} \{y^* \in \{\mathbf{0}\} + k^*,\ -k^* \in N_L(\varphi_{A,k}(\bar{y})k - \bar{y}; A)\} \\
&= \ \{k^* \in Y^* :\ k^*(k) = 1 \ \wedge\ -k^* \in N_L(\varphi_{A,k}(\bar{y})k - \bar{y}; A)\}.
\end{aligned}$$

The proof is complete. □

# 5 Applications in locational analysis

## 5.1 Computation of the subdifferential of the norm

For the computation of the subdifferential of the norm we are using the following well-known result (compare [13, Theorem 5.15]).

**Theorem 5.1. (Fenchel subdifferential of the norm).** *Suppose $X = \mathbb{R}^n$. Then the norm $\|\cdot\|_X : X \to \mathbb{R}_{\geq 0}$ is subdifferentiable and it holds*

$$\partial\|\cdot\|_X(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|_X \ \wedge\ \|x^*\|_* = 1\} \text{ for all } x \in X\backslash\{0\} \text{ and}$$

$$\partial\|\cdot\|_X(0) = \{x^* \in X^* : \|x^*\|_* \leq 1\}.$$

We apply this result for the $l_1$-norm.

**Corollary 5.2.** *For $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\varphi(x) := \|x\|_1 = \sum\limits_{i=1}^{n} |x_i|$ the Fenchel subdifferential and the limiting subdifferential are identical and given by*

$$\begin{aligned}\partial\|\cdot\|_1(x) &= \partial_L\|\cdot\|_1(x)\\ &= \left\{ x^* \in \mathbb{R}^n : \ x_i^* \in \begin{cases} \{1\} & \text{for } x_i > 0 \\ [-1,1] & \text{for } x_i = 0 \\ \{-1\} & \text{for } x_i < 0 \end{cases}, \ \forall i = 1,\dots,n \right\}.\end{aligned}$$

For calculating the limiting subdifferential of the negative $l_1$-norm, we will use Lemma 2.4 and the subdifferential of the negative absolute value function. This result is given in [14, Theorem 3.1] and [21, page 83].

**Lemma 5.3.** *For $\varphi : \mathbb{R} \to \mathbb{R}$ with*

$$\varphi(x) := -|x| = \begin{cases} -x & \text{for } x \geq 0 \\ x & \text{for } x < 0 \end{cases}$$

*the limiting as well as the approximate subdifferential has the following form:*

$$\partial_L(-|\cdot|)(x) = \begin{cases} \{-1\} & \text{for } x > 0 \\ \{-1, 1\} & \text{for } x = 0 \\ \{1\} & \text{for } x < 0 \end{cases}.$$

**Theorem 5.4.** *The limiting subdifferential of $\varphi : \mathbb{R}^n \to \mathbb{R}$ with $\varphi(x) = -\|x\|_1 = -\sum\limits_{i=1}^{n} |x_i|$ is given by*

$$\partial_L(-\|\cdot\|_1)(x) = \left\{ x^* \in \mathbb{R}^n : \ x_i^* \in \begin{cases} \{-1\} & \text{for } x_i > 0 \\ \{-1,1\} & \text{for } x_i = 0 \\ \{1\} & \text{for } x_i < 0 \end{cases}, \ \forall i = 1,\dots,n \right\}.$$

*Proof.* Notice that $\varphi(x) = \varphi_1(x_1) + \ldots + \varphi_n(x_n)$ with $\varphi_i(x_i) = -|x_i|$ for all $i = 1,\dots,n$. Hence, the assertion follows directly from Lemma 2.4 and Lemma 5.3. □

## 5.2 Necessary optimality conditions for a scalar location problem involving the $l_1$-norm

We consider the following, in general nonconvex scalar location problem

$$\varphi(x) := \varphi_w(x) := \sum_{i=1}^{m} w_i \|x - a^i\|_1 \to \min_{x \in \mathbb{R}^n} . \tag{SP$_1$}$$

The objective function $\varphi_w : \mathbb{R}^n \to \mathbb{R}$ is the weighted sum of distances between $x$ and the existing facilities $a^i \in \mathbb{R}^n$ for all $i \in \{1, \dots, m\}$, where $w = (w_1, \dots, w_m) \in \mathbb{R}^m$ is the corresponding vector of weights. We are looking for an element $x \in \mathbb{R}^n$ such that the objective function is to minimize, i.e., the distance between $x$ and $a^i$ with a positive weight should be small and the distance between $x$ and $a^i$ with a negative weight should be large.

By introducing the following sets of indexes $S := \{s \in \{1, \dots, m\} : \ w_s > 0\}$ and $T := \{t \in \{1, \dots, m\} : \ w_t < 0\}$ and $v_i := |w_i| > 0$ we get an equivalent reformulation of the problem (SP$_1$):

$$\varphi_v(x) = \sum_{s \in S} v_s ||x - a^s||_1 + \sum_{t \in T} v_t(-||x - a^t||_1) \to \min_{x \in \mathbb{R}^n} \tag{SP$_1$}$$

**Corollary 5.5.** *For every minimal solution $\overline{x} \in \mathbb{R}^n$ of (SP$_1$) holds*

$$\mathbf{0} \in \sum_{s \in S} \begin{cases} \{v_s\} & \text{for } \overline{x}_i > a_i^s \\ \{-v_s\} & \text{for } \overline{x}_i < a_i^s \\ [-v_s, v_s] & \text{for } \overline{x}_i = a_i^s \end{cases} + \sum_{t \in T} \begin{cases} \{-v_t\} & \text{for } \overline{x}_i > a_i^t \\ \{v_t\} & \text{for } \overline{x}_i < a_i^t \\ \{-v_t, v_t\} & \text{for } \overline{x}_i = a_i^t \end{cases}$$

*for all $i = 1, \dots, m$.*

*Proof.* Let $\overline{x}$ be a minimal solution of (SP$_1$), then by Proposition 2.5(i) holds

$$\begin{aligned} \mathbf{0}_{\mathbb{R}^n} &\in \partial_L \varphi(\overline{x}) \\ &= \partial_L \left( \sum_{s \in S} v_s \|x - a^s\|_1 + \sum_{t \in T} v_t(-\|x - a^t\|_1) \right) (\overline{x}). \end{aligned}$$

Using the sum rule (Lemma 2.2), Corollary 5.2, Theorem 5.4 and the fact that

$\partial_L \alpha\psi(x) = \alpha\partial_L\psi(x)$ for all $\alpha > 0$, we get

$$
\begin{aligned}
\mathbf{0}_{\mathbb{R}^n} &\in \partial_L \left( \sum_{s\in S} v_s \|x - a^s\|_1 + \sum_{t\in T} v_t(-\|x - a^t\|_1) \right)(\overline{x}) \\
&\subseteq \left( \sum_{s\in S} v_s \partial_L(\|x - a^s\|_1) + \sum_{t\in T} v_t \partial_L(-\|x - a^t\|_1) \right)(\overline{x}) \\
&= \left( \sum_{s\in S} v_s \left\{ x^* \in \mathbb{R}^n \mid x_i^* \in \begin{cases} \{1\} & \text{for } x_i - a_i^s > 0 \\ [-1,1] & \text{for } x_i - a_i^s = 0 \\ \{-1\} & \text{for } x_i - a_i^s < 0 \end{cases} \right\} \right. \\
&\quad \left. + \sum_{t\in T} v_t \left\{ x^* \in \mathbb{R}^n \mid x_i^* \in \begin{cases} \{-1\} & \text{for } x_i - a_i^t > 0 \\ \{-1,1\} & \text{for } x_i - a_i^t = 0 \\ \{1\} & \text{for } x_i - a_i^t < 0 \end{cases} \right\} \right)(\overline{x}) \\
&= \sum_{s\in S} \left\{ x^* \in \mathbb{R}^n \mid x_i^* \in \begin{cases} \{v_s\} & \text{for } \overline{x}_i > a_i^s \\ [-v_s, v_s] & \text{for } \overline{x}_i = a_i^s \\ \{-v_s\} & \text{for } \overline{x}_i < a_i^s \end{cases} \right\} \\
&\quad + \sum_{t\in T} \left\{ x^* \in \mathbb{R}^n \mid x_i^* \in \begin{cases} \{-v_t\} & \text{for } \overline{x}_i > a_i^t \\ \{-v_t, v_t\} & \text{for } \overline{x}_i = a_i^t \\ \{v_t\} & \text{for } \overline{x}_i < a_i^t \end{cases} \right\}.
\end{aligned}
$$

This implies

$$
\mathbf{0} \in \sum_{s\in S} \begin{cases} \{v_s\} & \text{for } \overline{x}_i > a_i^s \\ \{-v_s\} & \text{for } \overline{x}_i < a_i^s \\ [-v_s, v_s] & \text{for } \overline{x}_i = a_i^s \end{cases} + \sum_{t\in T} \begin{cases} \{-v_t\} & \text{for } \overline{x}_i > a_i^t \\ \{v_t\} & \text{for } \overline{x}_i < a_i^t \\ \{-v_t, v_t\} & \text{for } \overline{x}_i = a_i^t \end{cases}
$$

for $i = 1, \ldots, n$. □

## 5.3 Necessary optimality conditions for some multiobjective location problems

We now consider the nonconvex multiobjective location problem

$$
f(x) := \begin{pmatrix} \|x - a^1\|_{i_1} \\ \|x - a^2\|_{i_2} \\ \vdots \\ \|x - a^r\|_{i_r} \\ -\|x - a^{r+1}\|_{i_{r+1}} \\ \vdots \\ -\|x - a^m\|_{i_m} \end{pmatrix} \to \min_{x\in\mathbb{R}^n} \tag{MP}
$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a vector of mixed positive and negative norm distances between $x$ and the existing facilities $a^i \in \mathbb{R}^n$ for all $i \in \{1, \dots, m\}$. Again, we are looking for an element $x \in \mathbb{R}^n$ such that the objective function $f$ is minimal (now in the sense of vector optimization), i.e., the distance between $x$ and $a^i$ should be small for the first $r$ existing facilities and it should be large for the other $m - r$ facilities.

Let $A \subset \mathbb{R}^m$ be a proper closed set with nonempty interior and $0 \in \operatorname{bd} A$. Suppose that there exists a proper cone $C \subset \mathbb{R}^m$ with nonempty interior such that $A + \operatorname{int} C \subset \operatorname{int} A$ and $k \in \operatorname{int} C$. In order to prove necessary conditions for weakly minimal elements of $f(\mathbb{R}^n)$ with respect to $A$ we will use a scalarization by means of the function $\varphi_{A,k}$ given by (5) and the results concerning the limiting subdifferential of the scalarizing function $\varphi_{A,k}$ shown in Section 3.

**Theorem 5.6.** *Let $f(x_0) \in \operatorname{WMin}(f(\mathbb{R}^n), A)$, $\Phi_{A,k}(y) := \varphi_{A,k}(y - f(x_0))$ for all $y \in \mathbb{R}^m$ and assume that $\Phi_{A,k}$ is Lipschitzian around $f(x_0)$. Then there exists $y^* \in \mathcal{A} := \{y^* \in \mathbb{R}^n |\ y^*(k) = 1 \ \wedge \ -y^* \in N_L(\varphi_{A,k}(0)k; \operatorname{bd} A)\}$ with*

$$\mathbf{0} \in \partial_L \langle y^*, f \rangle (x_0). \tag{35}$$

*Proof.* By [11, Theorem 3.1] it is $f(x_0) \in \operatorname{WMin}(f(\mathbb{R}^n), A)$ if and only if $x_0$ solves the scalarized problem

$$\Phi_{A,k} \circ f(x) \to \min_{x \in \mathbb{R}^n}.$$

Using Lemma 2.5 (i) it follows $\mathbf{0} \in \partial_L (\Phi_{A,k} \circ f)(x_0)$. $f$ as a composition of norm functions is strictly Lipschitzian around $x_0$ and $\Phi_{A,k}$ is Lipschitzian around $f(x_0)$ by assumption, so we can applicate the chain rule (Lemma 2.3) and get

$$\mathbf{0} \in \partial_L (\Phi_{A,k} \circ f)(x_0) \subseteq \bigcup_{y^* \in \partial_L \Phi_{A,k}(f(x_0))} \partial_L \langle y^*, f \rangle (x_0).$$

According to Proposition 3.1 (i) it is $\partial_L \Phi_{A,k}(f(x_0)) = \partial_L \varphi_{A,k}(0) = \mathcal{A}$ which yields the assertion. □

**Remark 5.7.** *(i) The function $\langle y^*, f \rangle (x_0) = \sum_{s \in S} y^*_s \|x - a^s\|_{i_s} + \sum_{t \in T} y^*_t - \|x - a^t\|_{i_t}$ in Theorem 5.6 corresponds to the objective function of a scalar nonconvex location problem. For the case of $l_1$-norm as distance function for all existing facilities $\partial_L \langle y^*, f \rangle (x_0)$ can be calculated like in Corollary 5.5.*

*(ii) We need $\Phi_{A,k}$ to be Lipschitz around $f(x_0)$ in the proof of Theorem 5.6 in order to apply Lemma 2.3. This Lipschitz property is e.g. always verified if $A$ is a convex set and condition* (11) *is fulfilled.*

## Conclusions

This paper presents subdifferential formulas for the nonlinear scalarization functions $\varphi_{A,k}$ and $\varphi_{A(.),k}$. They allow us to study optimality conditions in vector optimization with variable domination structure. In this line of research, we will further extend our study to scalarization functions in set optimization with set-less orderings.

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