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A Vector-Valued Ekeland's Variational Principle in Vector Optimization with Variable Ordering Structures

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Abstract

In this paper, we are dealing with Ekeland's variational principle for vector optimization problems with variable ordering structures. Many generalizations of Ekeland's variational principle for vector optimization problems with fixed ordering structures are given in recent books and papers. Recently, certain variational principles for approximate solutions of vector optimization problems with variable ordering structures are derived in the literature. Here, using nonlinear scalarization techniques, we give some new generalizations of Ekeland's variational principle for approximate minimizers and nondominated solutions of vector optimization problems with variable ordering structures. These generalizations can be used for deriving necessary conditions for approximate solutions of vector optimization problems with variable ordering structures.

Keywords. Nonconvex vector optimization, Variable ordering structures, Approximate solutions, Ekeland's variational principle.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C46,90C48.

1 Introduction

Ekeland's variational principle is one of the most important results in nonlinear analysis. It is an assertion concerning the existence of an exact solution of a perturbed problem in a neighborhood of an approximate solution of the original problem without convexity and without compactness assumptions. It is a useful tool in order to drive necessary conditions for approximate solutions of problems in optimization, optimal control theory, game theory, nonlinear equations and dynamical systems. Generalizations of Ekeland's variational principle for vector optimization problems with fixed ordering structures have been extensively

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studied by many authors in the literature, see, e.g. [1] and references therein. Some generalizations of Ekeland's variational principle for approximately minimal solutions of vector optimization problems with variable ordering structures for both solid and nonsolid cases are given in [2, 3], see [3–14, 16] for an introduction to vector optimization problems with variable ordering structures and some recent works in this subject. In the following, we introduce concepts for approximately optimal solutions of vector optimization problems with variable ordering structures. These concepts are generalizations of the concept of ϵ-efficiency for vector optimization problems with fixed ordering structure by Loridan [17].

In the following we impose the following assumptions:

Assumption (A1). Y is a Hausdorff topological linear space, $\varepsilon \geq 0$, $k^0 \in Y \backslash \{\mathbf{0}\}$ and $C : Y \rightrightarrows Y$ is a set-valued map where $C(y)$ is a proper, pointed ($C(y) \cap -C(y) = \{\mathbf{0}\}$), solid ($\operatorname{int} C(y) \neq \emptyset$) and closed set which satisfies $C(y) + [0, +\infty[k^0 \subseteq C(y)$ for all $y \in Y$.

Additionally to (A1), we impose the following assumptions in certain cases.

(A2) X is a real complete metric space, Ω is a closed subset of X and $f : \Omega \to Y$ is a vector-valued function.

(A3) $B : Y \rightrightarrows Y$ is a cone-valued map such that $C(y) + (B(y) \backslash \{\mathbf{0}\}) \subseteq \operatorname{int} C(y)$ and $k^0 \in \operatorname{int} B(y)$ for all $y \in Y$.

(A4) For all $y \in Y$, $\quad C(y) + C(y) \subseteq C(y)$.

Under the assumptions (A1) $-$ (A2), we consider the following vector optimization problem with respect to a variable ordering structure given by $C : Y \rightrightarrows Y$:

$$\varepsilon k^0 - \operatorname{Min}(f, \Omega, C(\cdot)), \tag{VVOP}$$

where ϵk^0-minimality stands for three different kinds of concepts for approximate solutions; see [14, 15].

In [14,15], it is shown that the sets of different kinds of approximate solutions do coincide in vector optimization problems with fixed ordering structures but not in vector optimization problems with variable ordering structures.

In this paper we establish new Ekeland's variational principles for two kinds of solutions, named approximate minimizers and approximately nondominated solutions, by using a nonlinear scalarization technique in Sections 4 and 5. Our technique is based on the nonlinear scalarization technique used in [18] for vector optimization problems with fixed ordering structure. In the third section, we give some properties of the nonlinear scalarization functional defined by Chen, et al. in [4] which is a generalization of functional defined by Tammer and Weidner in [19]. In Section 4, we establish a new Ekeland's variational principle for

εk^0-minimizers and Section 5 is devoted to results related to approximately nondominated solutions of vector optimization problems with variable ordering structures.

2 Preliminaries

Let S be a nonempty subset of Hausdorff linear topological space Y. We denote the topological interior of the set S by $\operatorname{int} S$, $\operatorname{cl} S$ denotes the topological closure, $\operatorname{bd} S$ the topological boundary of S, $\operatorname{conv} S$ denotes the convex hull of a set S and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. A set C is called a cone iff $\lambda c \in C$ for all $\lambda \geq 0$ and $c \in C$. A nonempty set $C \subseteq Y$ is said to be convex if $\lambda c^1 + (1-\lambda)c^2 \in C$ for all $c^1, c^2 \in C$ and $0 \leq \lambda \leq 1$. A set $C \subseteq Y$ is said to be solid iff $\operatorname{int} C \neq \emptyset$ and a set $C \subseteq Y$ is proper iff $\emptyset \neq C \neq Y$. See [20–23] for basic definitions and solution concepts for vector optimization problems and [19,24–26] for some scalarization methods for solving vector optimization problems with respect to a fix ordering and some properties of these scalarization methods.
One of the important tools which will be used in this paper is the following nonlinear scalarization functional which is an extension of the nonlinear separating functional (see Tammer and Weidner [19]). Let Y be a Hausdorff linear topological space, $k^0 \in Y \setminus \{\mathbf{0}\}$ and $C : Y \rightrightarrows Y$ be a cone-valued map. Chen and Yang in [4] assumed that for all $y \in Y$, $C(y)$ is a closed, solid and convex cone and for each $y, z \in Y$, defined $\xi(z, y) : Y \times Y \to \overline{\mathbb{R}}$ as following:

$$\xi(z, y) := \inf\{t \in \mathbb{R} \mid z \in tk^0 - C(y)\}, \tag{1}$$

i.e., if $\xi(z, y) = t$, then $z \in tk^0 - C(y)$.

Obviously, we can see that if $C = C(y)$ for all $y \in Y$ and $z = y$, then ξ coincides with the nonlinear separating functional $\theta(z) := \inf\{t \in \mathbb{R} \mid z \in tk^0 - C\}$ discussed in [19].

In [4], authors showed that the functional (1) is well-defined and lower bounded [4, Proposition 2.1]. Furthermore, in [4, Lemma 2.3], they proved the following theorem under the assumptions that $C(y)$ is a closed, solid and convex cone for all $y \in Y$. The proof for the general case where $C(y)$ $(y \in Y)$ is not supposed be a cone can be found in [27, Theorem 4.2.7].

Theorem 2.1. *Let assumptions* (A1) *and* (A3) *be fulfilled. Then the following assertions hold for all $y, z \in Y$ and $t \in \mathbb{R}$:*

$$\xi(z, y) > t \Leftrightarrow z \notin tk^0 - C(y), \tag{2}$$

$$\xi(z, y) \geqq t \Leftrightarrow z \notin tk^0 - \operatorname{int} C(y), \tag{3}$$

$$\xi(z, y) = t \Leftrightarrow z \in tk^0 - \operatorname{bd} C(y), \tag{4}$$

$$\xi(z, y) \leqq t \Leftrightarrow z \in tk^0 - C(y), \tag{5}$$

$$\xi(z,y) < t \Leftrightarrow z \in tk^0 - \operatorname{int} C(y). \tag{6}$$

Now, we recall one of the most important results in nonlinear analysis given by Ekeland [28] in 1972.

Theorem 2.2. *[28] Let X be a complete metric space, and $g : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function, not identical to $+\infty$, and bounded from below on a closed set Ω in X. Let $\varepsilon > 0$ be given, and $\overline{x} \in \Omega$ such that $g(\overline{x}) \leq \inf_{x\in\Omega} g(x) + \varepsilon$. Then there exists an element $x_\varepsilon \in \operatorname{dom} g \cap \Omega$ such that*

(i) $g(x_\varepsilon) \leq g(\overline{x}) \leq \inf_{x\in\Omega} g(x) + \varepsilon$,

(ii) $d(x_\varepsilon, \overline{x}) \leq \sqrt{\varepsilon}$,

(iii) $g(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) > g(x_\varepsilon),\ \forall x \in \Omega \setminus \{x_\varepsilon\}$.

Remark 2.3. *[28]* **(Strong form of Ekeland's variational principle).** *Theorem 2.2 is known as the weak version of Ekeland's variational principle since we can find an element $x_\varepsilon \in \operatorname{dom} g \cap \Omega$ which satisfies, in addition to* (i)–(iii)*, the following condition*

(i$'$) $g(x_\varepsilon) + \sqrt{\varepsilon} d(\overline{x}, x_\varepsilon) \leq g(\overline{x})$.

Obviously, (i$'$) *implies* (i) *and* (ii)*.*

Extensions of Theorem 2.2 for approximately minimal solutions (see Definition 2.4) for vector optimization problems with variable ordering structures are given in the papers [2, 3]. We will recall such a type of variational principle for (VVOP) using the solution concept given in the following definition.

Definition 2.4. *Let assumptions* (A1) *and* (A2) *be fulfilled and consider (VVOP).*

1. *$x_\varepsilon \in \Omega$ is said to be an εk^0-minimal solution of* (VVOP) *with respect to the ordering map $C : Y \rightrightarrows Y$ iff*

$$(f(x_\varepsilon) - \varepsilon k^0 - (C(f(x_\varepsilon))\setminus\{\mathbf{0}\})) \cap f(\Omega) = \emptyset.$$

2. *Let $\operatorname{int} C(f(x_\varepsilon)) \neq \emptyset$. $x_\varepsilon \in \Omega$ is said to be a weakly εk^0-minimal solution of* (VVOP) *with respect to the ordering map $C : Y \rightrightarrows Y$ iff*

$$(f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x_\varepsilon))) \cap f(\Omega) = \emptyset.$$

When $\varepsilon = 0$, Definition 2.4 coincides with the usual definition of (weakly) minimal solutions; see, e.g. [6, 12]. We denote the sets of εk^0-minimal and weakly εk^0-minimal solutions by εk^0-M(Ω, f, C) and εk^0-WM(Ω, f, C), respectively. Variational principles for approximately minimal solutions of vector optimization problems with variable orderings structures are given in [2, 3, 29].

Theorem 2.5. *[2] Consider problem* (VVOP)*, let $\overline{x} \in \Omega$ be an approximately minimal solution of* (VVOP) *and set $\overline{y} := f(\overline{x})$. Assume that in addition to (A1) the following conditions hold:*

(i) *$\overline{C} := C(\overline{y})$ is a proper, closed, pointed, and solid set satisfying $\mathbb{R}k^0 - \overline{C} = Y$.*

(ii) *There exists a cone-valued mapping $B : Y \rightrightarrows Y$ such that $k^0 \in \operatorname{int} \overline{B}$ with $\overline{B} := B(\overline{y})$, $\overline{C} + \overline{B} \setminus \{\mathbf{0}\} \subset \operatorname{int} \overline{C}$, and $B(f(x)) \subset \overline{B}$ for all $x \in \Omega$ with $\|x - \overline{x}\| \leq \sqrt{\varepsilon}$.*

(iii) *f is bounded from below over Ω with respect to $\underline{y} \in Y$ and $\overline{C}$, i.e., $f(\Omega) \subseteq \underline{y} + \overline{C}$.*

(vi) *f is $(k^0, \overline{C})$-lower semicontinuous over Ω, i.e., $M(t) := \{u \in \Omega \mid f(u) \in t \cdot k^0 - \overline{C}\}$ is closed in X for all $t \in \mathbb{R}$.*

Then, there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that

1. *$x_\varepsilon \in \varepsilon k^0 - \mathrm{M}(\Omega, f, B)$, i.e., $\big(f(x_\varepsilon) - \varepsilon k^0 - B(f(x_\varepsilon)) \setminus \{\mathbf{0}\}\big) \cap f(\Omega) = \emptyset$,*
2. *$\|x_\varepsilon - \overline{x}\| \leq \sqrt{\varepsilon}$,*
3. *$x_\varepsilon \in \mathrm{M}(\Omega, f_{\varepsilon k^0}, B)$, where $f_{\varepsilon k^0}(x) := f(x) + \sqrt{\varepsilon}\|x - x_\varepsilon\| k^0$.*

In order to show the closedness of subsets of topological spaces, we will use Moore-Smith-sequences $\{x_a\}_{a \in A}$ where A is an index set which is more general than $\mathbb{N}$. For more details see Zeidler [30].

Definition 2.6.
1. *A set A is called* **directed***, if there is a $\leq$-relation defined on certain pairs (a, b) with $a, b \in A$, such that for all elements of A:*
   (i) *$a \leq a$ (reflexivity),*
   (ii) *if $a \leq b$ and $b \leq c$, then $a \leq c$ (transivity),*
   (iii) *for $a, b \in A$ there exists an element $d \in A$ such that $a \leq d$ and $b \leq d$.*
2. *Let X be a topological space and A a directed set. A* **Moore-Smith-sequence** *(M-S-sequence) $\{x_a\}_{a \in A}$ is given by a map that assigns to each $a \in A$ an element $x_a \in X$.*

Remark 2.7. *Let X be a Hausdorff topological space. Every sequence $\{x_n\}$, $n \in \mathbb{N}$, is a M-S-sequence. The convergence for M-S-sequences is defined analogously to the convergence for sequences $\{x_n\}$, $n \in \mathbb{N}$. The limit of an M-S-sequence is unique. A set $S \subset X$ is closed, if it is M-S-sequentially closed. The point x belongs to the closure $\operatorname{cl} S$, if there is a M-S-sequence $\{x_a\}_{a \in A}$ in S such that $x_a \to x$.*

In the following section, we prove some properties of our scalarizing functional and these properties will be used in the next section in order to characterize approximately optimal solutions of vector optimization problems with variable ordering structures and later for the proof of variational principles of vector optimization problems with variable ordering structures.

3 Properties of Nonlinear Scalarization Functionals

In this section, we will prove that the scalarizing functional defined in (1) is lower semi-continuous, subadditive, positively homogenous, monotone and continuous in the case that some assumptions hold. These properties are important for us and they will be used in the next sections in order to show generalizations of Ekeland's variational principle for vector optimization problems with variable ordering structures.

Lemma 3.1. *Let assumption* (A1) *be fulfilled,* $z, y \in Y$ *and* $\xi(z, y) = t_1$*. Then for any* $t_2 \geq t_1$*,*

$$z \in t_2 k^0 - C(y).$$

Proof. By $C(y) + [0, +\infty[k^0 \subseteq C(y)$, $z \in t_1 k^0 - C(y)$ and $t_2 - t_1 \geq 0$, we can write

$$z \in t_1 k^0 - C(y) = t_2 k^0 - [(t_2 - t_1)k^0 + C(y)] \subseteq t_2 k^0 - C(y)$$

and this completes the proof. □

In the following theorems, we show that our scalarizing functional defined in (1) is finite-valued and positively homogenous under some assumptions. These theorem was proven by Göpfert et. al. for the case of fixed ordering; see [21, Theorem 2.3.1].

Theorem 3.2. *Let the assumptions* (A1) *and* (A3) *be fulfilled. Then the functional* $\xi(\cdot, y)$ *is finite-valued for all fixed elements* $y \in Y$*.*

Proof. Assumption (A3) and [21, Proposition 2.3.4] implies that $C(y)$ does not contain lines parallel to k^0. Now, suppose that $\xi(z, y) = -\infty$, then by Lemma 3.1, for any $t > -\infty$, we have $z \in tk^0 - C(y)$ and $\{tk^0 - z \mid t \in \mathbb{R}\} \subset C(y)$ and this means that there exists $y \in Y$ such that $C(y)$ contains a line parallel to k^0 and this leads to a contradiction. □

Theorem 3.3. *Let* assumptions (A1) *and* (A3) *be fulfilled. For each fixed element* $y \in Y$*,* $\xi(\cdot, y)$ *defined by (1) is positively homogeneous if and only if* $C(y)$ *is a cone.*

Proof. Assume that $\lambda \geq 0$, then for any $y \in Y$, we have

$$\xi(\lambda z, y) = \inf \{t \in \mathbb{R} \mid \lambda z \in tk^0 - C(y)\}.$$

First we consider $\lambda = 0$ and prove $\xi(\mathbf{0}, y) = 0$. By $C(y) + [0, +\infty[k^0 \subseteq C(y)$ and the pointedness of $C(y)$ for all $y \in Y$, we have $t_0 = \xi(0, y) \leq 0$. If $t_0 < 0$, then by (1), we get $t_0 k^0 \in C(y) \backslash \{\mathbf{0}\}$. Also $\mathbf{0} \in \operatorname{bd} C(y)$ and $C(y) + [0, +\infty[k^0 \subset C(y)$ for all $y \in Y$, we get $-t_0 k^0 \in C(y)$ and this implies $t_0 k^0 \in C(y) \backslash \{\mathbf{0}\} \cap (-C(\overline{y}))$. But this is a contradiction to the pointedness of $C(y)$ in assumption (A1) and therefore $t_0 = 0$ and $\xi(\mathbf{0}, y) = 0$. Now, we consider the case $\lambda > 0$. Since $C(y)$ is a cone, we have $C(y) = \lambda C(y)$ and

$$\xi(\lambda z, y) = \inf\{t \in \mathbb{R} \mid \lambda z \in tk^0 - \lambda C(y)\} = \lambda \inf\{\frac{t}{\lambda} \in \mathbb{R} \mid z \in \frac{t}{\lambda} k^0 - C(y)\},$$

so by $t' = \frac{t}{\lambda}$, we can write

$$\xi(\lambda z, y) = \lambda \inf\{t' \in \mathbb{R} \mid z \in t'k^0 - C(y)\} = \lambda \xi(z, y).$$

Now, assume that $\xi(\cdot, y)$ is positively homogenous and take $c^1 \in C(y)$. Obviously, $0 \in C(y)$ and by (5) of Theorem 2.1, $\xi(-c^1, z) \leq 0$. Taking into account that $\xi(\cdot, y)$ is positively homogeneous, we obtain

$$\xi(-\lambda c^1, z) \leq \lambda \xi(-c^1, z) \leq 0.$$

Again by (5) of Theorem 2.1, $\lambda c^1 \in C(y)$ and $\lambda C(y) \subseteq C(y)$.

Now, suppose that $c^1 \in C(y)$, then by (5) of Theorem 2.1

$$\xi(-c^1, y) \leq 0 \Rightarrow \lambda \xi(-\frac{c^1}{\lambda}, y) \leq 0.$$

By $\lambda > 0$, we get $\frac{c^1}{\lambda} \in C(y)$ and $c^1 \in \lambda C(y)$ and this implies $C(y) \subseteq \lambda C(y)$. Hence, $C(y) = \lambda C(y)$ for any $\lambda > 0$, $y \in Y$ and $C(y)$ is a cone. $\square$

The subadditivity of the scalarizing functional is important for us and we need this property in the next section for the characterization of approximate minimizers and approximately nondominated solutions. Furthermore, subadditivity is an important property for deriving a variational principle for vector optimization problems with a variable ordering structures.

Theorem 3.4. *Let assumptions* (A1), (A3) *be fulfilled and* $y \in Y$ *be fixed. Then the functional* $\xi(\cdot, y)$ *defined by (1) is subadditive if and only if* (A4) *holds, i.e.,* $C(y) + C(y) \subseteq C(y)$.

Proof. Assume that $C(y) + C(y) \subseteq C(y)$ for $y \in Y$. Let $z^1, z^2 \in Y$ and $t_1, t_2 \in \mathbb{R}$ such that $\xi(z^1, y) = t_1$ and $\xi(z^2, y) = t_2$. By (5) of Theorem 2.1

$$\xi(z^1, y) = t_1 \Rightarrow z^1 \in t_1 k^0 - C(y). \tag{7}$$

$$\xi(z^2, y) = t_2 \Rightarrow z^2 \in t_2 k^0 - C(y). \tag{8}$$

By (7), (8) and the inclusion $C(y) + C(y) \subseteq C(y)$ in assumption (A4) , we get

$$z^1 + z^2 \in (t_1 + t_2)k^0 - (C(y) + C(y)) \subseteq (t_1 + t_2)k^0 - C(y).$$

Again, by (5) of Theorem 2.1, $\xi(z^1 + z^2, y) \leq t_1 + t_2 = \xi(z^1, y) + \xi(z^2, y)$.

Now, we prove $C(y) + C(y) \subseteq C(y)$ assuming that $\xi(\cdot, y)$ is subadditive. Take $c^1, c^2 \in C(y)$. By (5) of Theorem 2.1 and $c^1, c^2 \in C(y)$, we get $\xi(-c^1, y) \leq 0$ and $\xi(-c^2, y) \leq 0$. Taking into account that $\xi(\cdot, y)$ is subadditive, we obtain

$$\xi(-c^1 - c^2, y) \leq \xi(-c^1, y) + \xi(-c^2, y) \leq 0.$$

Again by (5), we get $c^1 + c^2 \in C(y)$ and this completes our proof. $\square$

In the following theorem we show that our scalarizing functional $\xi(\cdot, y)$ defined by (1) is convex.

Definition 3.5. *Suppose that x is a linear space. A functional $h : X \to \overline{\mathbb{R}}$ is convex if its epigraph is a convex set.*

Theorem 3.6. *Let assumptions* (A1) *and* (A3) *be fulfilled. For all fixed $y \in Y$, $\xi(\cdot, y)$ is convex if and only if $C(y)$ is a convex set.*

Proof. Assume that $y \in Y$ is fixed, $\lambda \in [0, 1]$ and $z^1, z^2 \in Y$ such that $\xi(z^1, y) = t_1$ and $\xi(z^2, y) = t_2$. By (5) of Theorem 2.1, $z^1 \in t_1k^0 - C(y)$ and $z^2 \in t_2k^0 - C(y)$ and since $C(y)$ is a convex set, we can write,

$$\begin{aligned}\lambda z^1 + (1-\lambda)z^2 &\in \lambda t_1 k^0 + (1-\lambda)t_2k^0 - (\lambda C(y) + (1-\lambda)C(y)) \\ &\subseteq (\lambda t_1 + (1-\lambda)t_2)k^0 - C(y).\end{aligned}$$

Therefore

$$\xi(\lambda z^1 + (1-\lambda)z^2, y) \le \lambda \xi(z^1, y) + (1-\lambda)\xi(z^2, y),$$

and this means that $\xi(\cdot, y)$ is convex.

Now, suppose that $\xi(\cdot, y)$ is convex for all $y \in Y$, $c^1, c^2 \in C(y)$ and $\lambda \in]0, 1[$. By $c^1, c^2 \in C(y)$ and (5), we get $\xi(-c^1, y) \le 0$ and $\xi(-c^2, y) \le 0$. By convexity of $\xi(\cdot, y)$, we obtain

$$\xi(-(\lambda c^1 + (1-\lambda)c^2), y) \le \lambda\xi(-c^1, y) + (1-\lambda)\xi(-c^2, y) \le 0.$$

Again by (5), we get $\lambda c^1 + (1-\lambda)c^2 \in C(y)$ and $C(y)$ is convex. □

In the following theorem we are dealing with Moore-Smith-sequences (see Definition 2.6) in order to show the closedness of certain level sets.

Definition 3.7. *Suppose that x is a linear space. A functional $h : X \to \overline{\mathbb{R}}$ is lower semicontinuous if its epigraph is closed.*

Lemma 3.8. *Let Y be a topological space and let $\theta : Y \to \mathbb{R}$. The following conditions are equivalent.*

1. *The functional θ is lower semicontinuous on Y.*
2. *For any $t \in \mathbb{R}$, the set $\{y \in Y \mid \theta_\omega(y) > t\}$ is an open set in Y.*
3. *For any $t \in \mathbb{R}$, the set $\{y \in Y \mid \theta_\omega(y) \le t\}$ is a closed set in Y.*

Proof. See [31, Theorem 7.1.1]. □

Theorem 3.9. *Let* assumptions (A1) *and* (A3) *be fulfilled and $y \in Y$ be fixed. Then the functional $\xi(\cdot, y)$ defined by (1) is continuous.*

Proof. 1^0 : Suppose that Y is a metric space. We prove that $\xi(\cdot, y)$ is upper and lower semicontinuous for all $y \in Y$. First we show that the functional $\xi(\cdot, y)$ for fixed $y \in Y$ is lower semicontinuous and for this we prove that for any $t \in \mathbb{R}$,

$$S_t := \{z \in Y \mid \xi(z, y) \leq t\}$$

is a closed set. Suppose $\{z^n\}_{n\in\mathbb{N}}$ is a sequence with $z^n \to z^0$, $z^n \in S_t$ and $\xi(z^n, y) \leq t$. By (5),

$$z^n \in tk^0 - C(y) \Rightarrow \; tk^0 - z^n \in C(y).$$

Since $C(y)$ is a closed set, the limit point of the sequence $tk^0 - z^n \to tk^0 - z^0$ belongs to $C(y)$ and $z^0 \in tk^0 - C(y)$ and again by (5), we get $\xi(z^0, y) \leq t$. This means that S_t is a closed set for any $t \in \mathbb{R}$ and $\xi(\cdot, y)$ is lower semicontinuous for any $y \in Y$. Now, we show that $\xi(\cdot, y)$ is upper semicontinuous for all $y \in Y$ and for any $t \in \mathbb{R}$,

$$\overline{S}_t := \{z^1 \in Y | \; \xi(z^1, y) \geq t\}$$

is a closed set. Suppose that $z^n \to z^0$ is a sequence and $z^n \in \overline{S}_t$. By $z^n \in \overline{S}_t$, we get $\xi(z^n, y) \geq t$ and by (5), we have

$$z^n \notin tk^0 - \operatorname{int} C(y) \Rightarrow tk^0 - z^n \notin \operatorname{int} C(y) \Rightarrow tk^0 - z^n \in (\operatorname{int} C(y))^c.$$

Since $\operatorname{int} C(y)$ is an open set, its complement $(\operatorname{int} C(y))^c$ is a closed set and includes all the limit points. Therefore $tk^0 - z^0 \in (\operatorname{int} C(y))^c$ and this means

$$tk^0 - z^0 \notin \operatorname{int} C(y) \Rightarrow z^0 \notin tk^0 - \operatorname{int} C(y).$$

Again by (5), we get $\xi(z^0, y) \geq t$ and this implies that $\overline{S}_t$ is a closed set and $\xi(\cdot, y)$ is upper semicontinuous. Since $\xi(\cdot, y)$ is also lower semicontinuous, $\xi(\cdot, y)$ is continuous.
2^0 : The same arguments as in 1^0 can be used for Moore-Smith-sequences in Hausdorff topological linear spaces. □

Theorem 3.10. *Let assumptions* (A1) *and* (A3) *be fulfilled. Then we get the following properties of* ξ*:*

1. $\xi(z + tk^0, y) = \xi(z, y) + t \quad \forall y \in Y, \forall t \in \mathbb{R}, \forall z \in Y.$
2. $\xi(\cdot, y)$ *is strictly* B*-monotone if and only if* $C(y) + B(y)\backslash\{\mathbf{0}\} \subseteq C(y)$ *for each* $y \in Y$.

Proof. See Theorem 3.8 of [3]. □

The following nonconvex separation theorem will be used in the next section for our proofs; see [21, Theorem 2.3.6] for vector optimization problems with fixed ordering structures.

Theorem 3.11. *Suppose that assumptions* (A1) *and* (A3) *be fulfilled,* $S \subseteq Y$ *a nonempty set and for each* $y \in Y$, $S \cap (-\operatorname{int} C(y)) = \emptyset$. *Then for all* $y \in Y$, $\xi(\cdot, y)$ *defined by (1) is a proper continuous functional and*

$$\xi(-z, y) < 0 \leq \xi(s, y) \qquad \forall y \in Y, \forall z \in \operatorname{int} C(y),\ \forall s \in S.$$

Proof. By Theorem 3.2 and Theorem 3.9, $\xi(\cdot, y)$ is proper and continuous. Also obviously by (6), $-\operatorname{int} C(y) = \{z \in Y \mid \xi(z, y) < 0\}$. By $S \cap (-\operatorname{int} C(y)) = \emptyset$ for all $y \in Y$, we get

$$\xi(-z, y) < 0 \leq \xi(s, y) \qquad \forall y \in Y, \forall z \in \operatorname{int} C(y),\ \forall s \in S$$

and this completes the proof. □

In the last theorem of this section, we recall some monotonicity properties of our scalarization functional and these properties will be used in the next section in order to characterize approximately optimal solutions of vector optimization problems with variable ordering structures and later for the proof of variational principle of vector optimization problems with variable ordering structures; see Theorem 2.3.1 of [21] for the case of fixed ordering case. First we recall definition of monotonicity.

Definition 3.12. *Suppose that* Y *is a Hausdorff linear topological space,* $D : Y \rightrightarrows Y$ *is a set-valued map. We say that* $\xi(\cdot, y)$ *is a monotone functionalin* z *with respect to the set-valued map* $D : Y \rightrightarrows Y$ *if for fixed* $y \in Y$ *and all* $z^1, z^2 \in Y$

$$z^1 \in z^2 + D(y) \backslash \{\mathbf{0}\} \ \text{ implies } \ \xi(z^1, y) \geqq \xi(z^2, y).$$

Also, we say $\xi(\cdot, y)$ *is strictly* D*-monotone, if for fixed* $y \in Y$ *and all* $z^1, z^2 \in Y$

$$z^1 \in z^2 + D(y) \backslash \{\mathbf{0}\} \ \text{ implies } \ \xi(z^1, y) > \xi(z^2, y).$$

Theorem 3.13. *Let assumptions* (A1) *and* (A3) *be fulfilled, the functional* $\xi : Y \times Y \to \overline{\mathbb{R}}$ *is strictly monotone with respect to the set-valued map* $B : Y \rightrightarrows Y$ *in the first variable, i.e., if for* $z^1, z^2 \in Y$, $z^1 \in z^2 + \operatorname{int} B(y)$, *then* $\xi(z^1, y) > \xi(z^2, y)$.

Proof. By Theorem 3.2, the functional ξ is finite-valued. Let $z^1, z^2 \in Y$, $z^1 \in z^2 + \operatorname{int} B(y)$ and $t_1 := \xi(z^1, y)$. By Theorem 2.1, we have

$$z^2 \in z^1 - \operatorname{int} B(y) \subseteq (t_1 k^0 - C(y)) - \operatorname{int} B(y).$$

Now, by assumption (A3) we get $C(y)) + \operatorname{int} B(y) \subset \operatorname{int} C(y)$ and $z^2 \in z^1 - \operatorname{int} C(y)$. Again by Theorem 2.1 we have $\xi(z^1, y) < t_1 = \xi(z^1, y)$ and this completes the proof. □

4 Variational Principles for εk^0-Minimizer of (VVOP)

In this section, we give an extension of Ekeland's theorem for εk^0-minimizers of vector optimization problems with variable ordering structures. It is important to emphasize that there is no difference between εk^0-minimizers, εk^0-nondominated and εk^0-minimal solutions in the case of fixed ordering structures. The reader can find many examples illustrating that this statement is in general not true in the case of variable ordering structure in [6, 7, 14]. First, we introduce the concept of approximate minimizers of vector optimization problems with variable ordering structures and in order to prove the main theorem of this section, we prove the following lemmas.
We already introduced the first solution concept concerning approximately minimal solution of vector optimization problems with variable ordering structures in Definition 2.4. Now, we introduce the second solution concept as follows:

Definition 4.1. *Let* assumptions (A1) *and* (A2) *be fulfilled and consider (VVOP).*

1. *$x_\varepsilon \in \Omega$ is said to be an εk^0-minimizer of the problem* (VVOP) *with respect to the ordering map $C : Y \rightrightarrows Y$ iff*

$$\forall x, x^1 \in \Omega : \qquad (f(x_\varepsilon) - \varepsilon k^0 - (C(f(x))\backslash\{\mathbf{0}\})) \cap \{f(x^1)\} = \emptyset.$$

2. *Let* $\operatorname{int} C(f(x)) \neq \emptyset$ *for all $x \in \Omega$. $x_\varepsilon \in \Omega$ is said to be a weak εk^0- minimizer of* (VVOP) *with respect to the ordering map $C : Y \rightrightarrows Y$ iff*

$$\forall x, x^1 \in \Omega : \qquad (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap \{f(x^1)\} = \emptyset.$$

We denote the sets of εk^0-minimizers and weak εk^0-minimizers by εk^0-MZ(Ω, f, C) and εk^0-WMZ(Ω, f, C) respectively. If $\varepsilon = 0$, then these definitions are the definitions of the exact and weak minimizers. More details and properties of these points are given in [14, 15].

Example 4.2. *Let $\varepsilon = \frac{1}{100}$ and $k^0 = (1, 0)^T$. Also suppose that*

$$\Omega = \big\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\big\}$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_1 \geq 0,\ d_2 \leq 0\}, & \textit{for } (-1, 0)^T \\ \mathbb{R}^2_+, & \textit{otherwise.} \end{cases}$$

Then $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -1 + \varepsilon\}$ is the set of εk^0-minimal elements but just the elements of the set

$$\{(y_1, y_2) \in \Omega \mid y_1 < -1 + \varepsilon\} \cup \{(-1 + \varepsilon, 0)\}$$

are εk^0-nondominated and εk^0-minimizers (see Fig. 1).

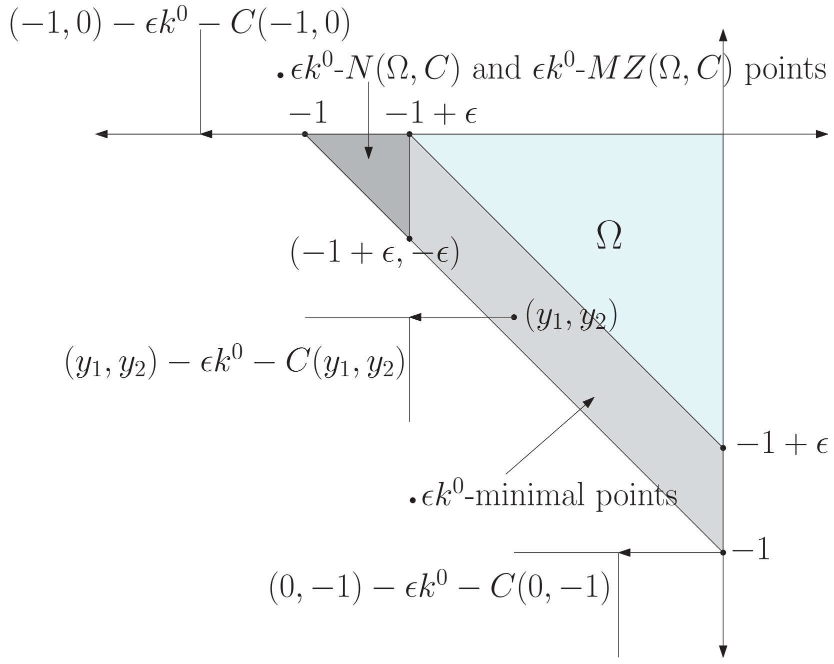

Figure 1: Example 4.2 where there exists an εk^0-minimal element of Ω which is neither εk^0-nondominated element nor εk^0-minimizer.

Lemma 4.3. *Let assumptions* (A1)–(A3) *be fulfilled and consider the problem* (VVOP)*. If $x_\varepsilon \in \varepsilon k^0$-MZ$(\Omega, f, C)$, then for each element $\omega \in f(\Omega)$, there exists a continuous functional $\xi(\cdot,\omega) : Y \to \mathbb{R}$ which is strictly B-monotone in the sense of Definition 3.12 and*

$$\forall x \in \Omega, \omega \in f(\Omega): \qquad \xi(f(x_\varepsilon),\omega) \leqq \xi(f(x)+\varepsilon k^0,\omega).$$

Moreover if (A4) *holds, then for each $\omega \in f(\Omega)$, $\xi(\cdot,\omega)$ is subadditive on Y and*

$$\forall x \in \Omega, \omega \in f(\Omega): \qquad \xi(f(x_\varepsilon),\omega) \leqq \xi(f(x),\omega) + \xi(\varepsilon k^0,\omega).$$

Proof. Suppose that $k^0 \in Y\backslash\{\mathbf{0}\}$, $\varepsilon > 0$ and $x_\varepsilon \in \varepsilon k^0$-MZ$(\Omega, f, C)$. This means for all $\omega \in f(\Omega)$, $(f(x_\varepsilon) - \varepsilon k^0 - C(\omega)\backslash\{\mathbf{0}\}) \cap f(\Omega) = \emptyset$ and therefore

$$\forall \omega \in f(\Omega): \qquad (f(x_\varepsilon) - C(\omega)\backslash\{\mathbf{0}\}) \cap (f(\Omega)+\varepsilon k^0) = \emptyset.$$

We apply Theorem 3.11 to the sets $(f(x_\varepsilon)-C(\omega)\backslash\{\mathbf{0}\})$ and $f(\Omega)+\varepsilon k^0$. Taking into account (A3) and applying Theorem 3.11 and Theorem 3.13 , we get desired functionals $\xi : Y\times Y \to \mathbb{R}$. Therefore for any fixed $\omega \in f(\Omega)$, there exist a continuous functional $\xi(\cdot,\omega) : Y \to \mathbb{R}$ such that

$$\xi(f(x_\varepsilon),\omega) \leqq \ \xi(f(\Omega)+\varepsilon k^0,\omega).$$

Now, if (A4) holds, then $\xi(\cdot,\omega)$ is subadditive for all $\omega \in f(\Omega)$ and

$$\xi(f(x_\varepsilon),\omega) \leqq \xi(f(\Omega),\omega) + \xi(\varepsilon k^0,\omega)$$

and the proof is complete. $\square$

The following lemma gives some properties of functionals in Lemma 4.3 and these properties will be used later in the proof of other lemmata and our main theorem about an extension of Ekeland's theorem for εk^0-minimizers of vector optimization problems with variable ordering structures.

Lemma 4.4. *Let assumptions* (A1)–(A3) *be fulfilled, then for each fixed* $\omega \in f(\Omega)$*, the functional* $\xi(\cdot, \omega) : Y \to \mathbb{R}$ *in* (1) *has the following properties:*

1. $\xi(k^0, \omega) = 1$.
2. $\xi(\mathbf{0}, \omega) = 0$.
3. $\xi(\varepsilon k^0, \omega) = \varepsilon$ and $\xi(-\varepsilon k^0, \omega) = -\xi(\varepsilon k^0, \omega) = -\varepsilon$.

Proof. 1. By definition of the separating functional $\xi(\cdot, \omega)$ in (1), for each $\omega \in f(\Omega)$,

$$\xi(y, \omega) = \inf\{t \mid y \in tk^0 - C(\omega)\}.$$

By pointedness of $C(\omega)$, (A3) and [21, Proposition 2.3.4], we get $\mathbf{0} \in \operatorname{bd} C(\omega)$ and $k^0 \in k^0 - \operatorname{bd} C(\omega)$ for all $\omega \in f(\Omega)$. Therefore by (4), we get $\xi(k^0, \omega) = 1$.

2. By (4) and $\mathbf{0} \in \operatorname{bd} C(\omega)$ for all $\omega \in f(\Omega)$, we get $\xi(\mathbf{0}, \omega) = 0$ for all $\omega \in f(\Omega)$.

3. By the first part of Theorem 3.10, we know that for all $y \in Y$, $t \in \mathbb{R}$, $\omega \in f(\Omega)$ the following equation holds:

$$\xi(y + tk^0, \omega) = \xi(y, \omega) + t,$$

therefore $\xi(\mathbf{0} + \varepsilon k^0, \omega) = \xi(\mathbf{0}, \omega) + \varepsilon$ and by the second part $\xi(\varepsilon k^0, \omega) = \varepsilon$. Proofs of other parts are similar.

$\square$

Lemma 4.5. *Let* X *be a complete metric space,* $\Omega \subset X$*,* $x_\varepsilon \in \Omega$*,* Y *be a Hausdorff topological linear space,* $\varepsilon \geq 0$ *and* $k^0 \in Y \backslash \{\mathbf{0}\}$*. Let* $f : X \to Y$ *be a vector-valued function with* $\operatorname{dom} f \neq \emptyset$*,* $B : Y \rightrightarrows Y$ *be a cone-valued map where* $k^0 \in B(y)$ *for all* $y \in Y$*.*
(i) Furthermore, suppose that for any $\omega \in f(\Omega)$ *and for a strictly* B*-monotone (in the sense of Definition 3.12), continuous, subadditive functional* $\xi(\cdot, \omega) : Y \to \mathbb{R}$ *the following inequality holds*

$$\forall x \in \Omega, \omega \in f(\Omega) : \qquad \xi(f(x_\varepsilon), \omega) \leqq \xi(f(x), \omega) - \xi(-\varepsilon k^0, \omega).$$

Then there is a set-valued map $C : Y \rightrightarrows Y$ *such that* $\operatorname{cl} C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$ *and* $B(\omega)\backslash\{\mathbf{0}\} \subseteq C(\omega)$ *for all* $\omega \in f(\Omega)$ *and* $x_\varepsilon \in \varepsilon k^0$-$\mathrm{WMZ}(\Omega, f, C)$*.*

Proof. For each $\omega \in f(\Omega)$, we define $C(\omega)$ and functional $\hat{\xi}(\cdot,\omega) : Y \to \mathbb{R}$ as follows,

$$C(\omega) := \{y \in Y \mid \xi(-y + f(x_\varepsilon) - \varepsilon k^0, \omega) < \xi(f(x_\varepsilon) - \varepsilon k^0, \omega)\}, \tag{9}$$

$$\hat{\xi}(y,\omega) := \xi(y + f(x_\varepsilon) - \varepsilon k^0, \omega). \tag{10}$$

By (10) and (i) and since $\xi(\cdot,\omega)$ is subadditive for all $\omega \in f(\Omega)$, we get

$$\begin{aligned}&\hat{\xi}(f(\Omega) + \varepsilon k^0 - f(x_\varepsilon), \omega) = \xi(f(\Omega), \omega) \geqq \\ &\xi(f(x_\varepsilon), \omega) + \xi(-\varepsilon k^0, \omega) \geqq \\ &\xi(f(x_\varepsilon) - \varepsilon k^0, \omega) = \hat{\xi}(\mathbf{0}, \omega).\end{aligned}$$

Now, by (9) and (10), we get

$$\hat{\xi}(-C(\omega), \omega) = \xi(-C(\omega) + f(x_\varepsilon) - \varepsilon k^0, \omega) < \xi(f(x_\varepsilon) - \varepsilon k^0, \omega) = \hat{\xi}(\mathbf{0}, \omega),$$

therefore for each $\omega \in f(\Omega)$,

$$(-\operatorname{int} C(\omega)) \cap (f(\Omega) + \varepsilon k^0 - f(x_\varepsilon)) = \emptyset \implies (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(\omega)) \cap f(\Omega) = \emptyset.$$

Now, we show that $\operatorname{cl} C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$. Choose $y \in \operatorname{cl} C(\omega)$ and $b \in y + B(\omega)\backslash\{\mathbf{0}\}$. Since $\hat{\xi}(\cdot,\omega)$ is strictly B-monotone and $y \in \operatorname{cl} C(\omega) \subseteq \{y \mid \hat{\xi}(-y,\omega) \leqq \hat{\xi}(\mathbf{0},\omega)\}$, we have

$$\hat{\xi}(-b,\omega) < \hat{\xi}(-y,\omega) \leqq \hat{\xi}(\mathbf{0},\omega).$$

Therefore $b \in C(\omega)$ and $\operatorname{cl} C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$. Now, by $\mathbf{0} \in \operatorname{cl} C(\omega)$, we get $B(\omega)\backslash\{\mathbf{0}\} \subseteq C(\omega)$. Furthermore by the inclusion $\operatorname{cl} C(\omega) + (B(\omega)\backslash\{\mathbf{0}\}) \subseteq C(\omega)$ and the assumption $k^0 \in B(\omega)$, we get $C(\omega) + \varepsilon k^0 \subseteq C(\omega)$. □

In the following, we introduce the concept of functions that are bounded from below on a set with respect variable ordering structures. Suppose the map $C : Y \rightrightarrows Y$ is the ordering map which for any $x \in \Omega$ assigns $C(f(x))$.

Definition 4.6. *Let X be a complete metric space, Y be a Hausdorff topological linear space and $C : Y \rightrightarrows Y$ be a set-valued map. We say that $f : X \to Y$ is bounded from below over Ω with respect to the set-valued map C if for any $y \in f(\Omega)$ there exists $y^0 \in Y$ such that $f(\Omega) \subseteq y^0 + C(y)$.*

Lemma 4.7. *Let assumptions* (A1) – (A3) *be fulfilled. Suppose $f : X \to Y$ is bounded from below over Ω with respect to C in the sense of Definition 4.6, then $\xi(\cdot, y) \circ f$ is bounded below for all $y \in f(\Omega)$.*

Proof. By Definition 4.6, we know that there exists $y^0 \in Y$ such that $f(\Omega) \subset y^0 + C(y)$ for all $y \in f(\Omega)$. By the first part of [21, Proposition 2.3.4], there exists $\hat{t}$ such

$$\hat{t}k^0 - y^0 \notin C(y). \tag{11}$$

Assume there exist $\overline{y} \in f(\Omega)$ and $x \in \Omega$ such that $\xi(f(x), \overline{y}) < \hat{t}$ and $\xi(\cdot, \overline{y}) \circ f$ is not bounded from below. Taking into account that f is bounded from below, there exists $c_1 \in C(\overline{y})$ such that $f(x) = y^0 + c_1$. By $\xi(f(x), \overline{y}) < \hat{t}$, Lemma 3.1 and Theorem 2.1, we have

$$f(x) \in \hat{t}k^0 - C(\overline{y}) \implies y^0 + c_1 \in \hat{t}k^0 - C(\overline{y}) \implies y^0 \in \hat{t}k^0 - (C(\overline{y}) + c_1).$$

By $C(\overline{y}) + c_1 \subseteq C(\overline{y})$, we get $y^0 \in \hat{t}k^0 - C(\overline{y})$ which is a contradiction to (11). This completes the proof and $\xi(\cdot, y) \circ f$ is bounded below for all $y \in f(\Omega)$. □

Note that in many Ekeland-type results in the literature; see, e.g. [29, 32, 33] and the references therein, the function f is assumed to be C-level-closed, known also as C-lower semicontinuous and C-semicontinuous [34, Definition 2.4], where C is a fixed ordering cone of the ordered image space. Therefore we introduce the following concepts of lower semicontinuity concerning variable ordering structures.

Definition 4.8. *Consider problem (VVOP), $\overline{x} \in \Omega \cap \operatorname{dom} f$, $\overline{y} := f(\overline{x})$ and $\overline{C} := C(\overline{y})$ is fixed. The function f is $(k^0, \overline{C})$-lower semicontinuous over Ω iff the sets*

$$M(t) := \left\{x \in \Omega | \ f(x) \in tk^0 - \overline{C}\right\}$$

are closed for all $t \in \mathbb{R}$.

Definition 4.9. *We say that $f : X \to Y$ is lower semicontinuous with respect to the ordering map $C : Y \rightrightarrows Y$, $k^0 \in Y \backslash \{\mathbf{0}\}$ and $\Omega \subseteq X$ (for short (k^0, C, Ω)-lsc), if*

$$M^X_{(\omega,t)} := \{x \in \Omega \mid f(x) \in tk^0 - \operatorname{cl} C(\omega)\}$$

is a closed set for all $\omega \in f(\Omega)$ and each $t \in \mathbb{R}$.

If $C = C(\omega_1) = C(\omega_2)$ is a fixed set, then Definition 4.9 coincides with the definition in [18]. Moreover, if $Y = \mathbb{R}$, then our definition coincide with the standard definition of lower semicontinuity. In order to prove the main theorem of this section, first we have to prove the following lemmata.

Lemma 4.10. *Let $C : Y \rightrightarrows Y$ be a set-valued map and assumptions* (A1) − (A3) *be fulfilled. For each fixed $\omega \in f(\Omega)$, consider the functional $\xi(\cdot, \omega)$ defined by (1). If the function $f : X \to Y$ in (VVOP) is (k^0, C, Ω)-lsc, then $(\xi(\cdot, \omega) \circ f)(\cdot) = \xi(f(\cdot), \omega)$ is a lower semicontinuous functional for each $\omega \in f(\Omega)$.*

Proof. Since the function $f : X \to Y$ is (k^0, C, Ω)-lsc, the set

$$M^X_{(\omega,t)} = \{x \in \Omega \mid f(x) \in tk^0 - C(\omega)\}$$

is closed for all $\omega \in f(\Omega)$ and $t \in \mathbb{R}$. Now, consider the set $M^Y_{(\omega,t)} := tk^0 - C(\omega) \subseteq Y$. By (A3) and Theorem 3.9, we know that $\xi(\cdot, \omega) : Y \to (-\infty, \infty)$ is a continuous functional for each $\omega \in f(\Omega)$ and by Theorem 2.1, we get

$$\begin{aligned}&M^Y_{(\omega,t)} = tk^0 - C(\omega) = \{y \in Y \mid y \in tk^0 - C(\omega)\} = \\ &\{y \in Y \mid \xi(y, \omega) \leqq \xi(tk^0, \omega)\} = \{y \in Y \mid \xi(y, \omega) \leqq t\} =: M^Y_{(\xi(\cdot,\omega),t)}\end{aligned}$$

for each $\omega \in f(\Omega)$ and $t \in \mathbb{R}$. This means for all $\omega \in f(\Omega)$ and $t \in \mathbb{R}$,

$$\begin{aligned}&M^X_{(\xi(\cdot,\omega),t)} = \{x \in \Omega \mid \xi(f(x),\omega) \leqq t\} = \\ &\{x \in \Omega \mid f(x) \in M^Y_{(\xi(\cdot,\omega),t)}\} = \{x \in \Omega \mid f(x) \in M^Y_{(\omega,t)}\} = M^X_{(\omega,t)}\end{aligned}$$

is a closed set and $\xi(\cdot,\omega) \circ f$ is lower semicontinuous for all $\omega \in f(\Omega)$. □

We now are ready to present an extension of Ekeland's theorem for εk^0-minimizers of vector optimization problem (VVOP) with a variable ordering structure.

Theorem 4.11. *Consider the problem* (VVOP) *and let* $\overline{x} \in \varepsilon k^0\text{-MZ}(\Omega, f, C)$. *Impose in addition to* (A1)–(A4) *the following assumptions:*

(A5) f *is* (k^0, C, Ω)*-lower semicontinuous over* Ω *in the sense of Definition 4.9.*

(A6) f *is bounded from below over* Ω *with respect to* C *in the sense of Definition 4.6.*

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ *such that*

1. $x_\varepsilon \in \varepsilon k^0\text{-WMZ}(\Omega, f, B)$,
2. $d(\overline{x}, x_\varepsilon) \leqq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \text{WMZ}(\Omega, f_{\varepsilon k^0}, B) \quad \text{with } f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) k^0.$ *(12)*

Proof. Let $\overline{x} \in \varepsilon k^0\text{-MZ}(\Omega, f, C)$. By the definition of εk^0-minimizers (Definition 4.1), we get

$$\forall \omega \in f(\Omega): \qquad (f(\overline{x}) - \varepsilon k^0 - C(\omega)\backslash\{\mathbf{0}\}) \cap f(\Omega) = \emptyset.$$

Now, suppose that $\overline{f} := f - f(\overline{x})$, then we have

$$\forall \omega \in f(\Omega): \qquad (\overline{f}(\overline{x}) - \varepsilon k^0 - C(\omega)\backslash\{\mathbf{0}\}) \cap \overline{f}(\Omega) = \emptyset.$$

By (A4), Lemma 4.3 and 4.4, for all fixed $\omega \in f(\Omega)$, the functional $\xi(\cdot,\omega) : Y \to \mathbb{R}$ in (1) is a strictly B-monotone, continuous and subadditive functional such that

$$\forall x \in \Omega: \qquad \xi(\overline{f}(\overline{x}),\omega) \leqq \xi(\overline{f}(x),\omega) + \xi(\varepsilon k^0,\omega) = \xi(\overline{f}(x),\omega) + \varepsilon.$$

This means that for all $\omega \in f(\Omega)$,

$$\xi(\overline{f}(\overline{x}),\omega) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x),\omega) + \varepsilon, \qquad \varepsilon > 0.$$

Observe that the validity of (A5)–(A6) ensures (k^0, C, Ω)-lower semicontinuity and the boundedness from below of f and $\overline{f}$. By Lemma 4.10, Lemma 4.7, Theorem 2.2 and Remark 2.3, there exists $x_\varepsilon \in \Omega$ such that for all $\omega \in f(\Omega)$,

1. $\xi(\overline{f}(x_\varepsilon),\omega) \leqq \xi(\overline{f}(\overline{x}),\omega) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x),\omega) + \varepsilon,$ (13)

2. $d(x_\varepsilon, \overline{x}) \leqq \sqrt{\varepsilon}$,

3. for all $x, \omega \in \Omega$: $\xi(\overline{f}(x_\varepsilon), \omega) \leqq \xi(\overline{f}(x), \omega) + \sqrt{\varepsilon} d(x, x_\varepsilon)$, (14)

4. $\xi(\overline{f}(x_\varepsilon), \omega) + \sqrt{\varepsilon} d(\overline{x}, x_\varepsilon) \leq \xi(\overline{f}(\overline{x}), \omega)$. (15)

By Lemma 4.4 and (13), for all $x \in \Omega, \omega \in f(\Omega)$, we get

$$\begin{gathered} \xi(\overline{f}(x_\varepsilon), \omega) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x), \omega) + \varepsilon \leqq \\ \xi(\overline{f}(x), \omega) + \xi(\varepsilon k^0, \omega) = \xi(\overline{f}(x), \omega) - \xi(-\varepsilon k^0, \omega). \end{gathered}$$

By Lemma 4.5, assumption (A3) and $\overline{f} = f - f(\overline{x})$, we get

$$(\overline{f}(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(\omega) \backslash \{\mathbf{0}\}) \cap \overline{f}(\Omega) = \emptyset.$$

This implies that $x_\varepsilon \in \varepsilon k^0$-WMZ$(\Omega, f, B)$. Now, we prove (12) and for this, suppose that there exist elements $x, \omega \in \Omega$ such that

$$\begin{aligned} & f(x) \in f(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0 - \operatorname{int} B(\omega) \\ \Longrightarrow \quad & \overline{f}(x) \in \overline{f}(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0 - \operatorname{int} B(\omega). \end{aligned}$$

Since for all fixed $\omega \in f(\Omega)$, $\xi(\cdot, \omega)$ is a strictly B-monotone continuous subadditive functional, we have

$$\xi(\overline{f}(x, \omega)) < \xi(\overline{f}(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0, \omega) \leqq \xi(\overline{f}(x_\varepsilon), \omega) + \xi(-\sqrt{\varepsilon} d(x, x_\varepsilon) k^0, \omega).$$

Now, by Lemma 4.4, we get

$$\xi(-\sqrt{\varepsilon} d(x, x_\varepsilon) k^0, \omega) = -\sqrt{\varepsilon} d(x, x_\varepsilon) \implies \xi(\overline{f}(x_\varepsilon), \omega) > \xi(\overline{f}(x), \omega) + \sqrt{\varepsilon} d(x, x_\varepsilon),$$

but this yields a contradiction because of (14). □

In the special case that $C : Y \rightrightarrows Y$ is a solid, closed, pointed and convex cone-valued map, we have the following corollary.

Corollary 4.12. *Suppose that $C : Y \rightrightarrows Y$ is a cone-valued map where $C(\omega)$ is a closed solid convex cone for all $\omega \in f(\Omega)$, $k^0 \in \bigcap_{\omega\in f(\Omega)} \operatorname{int} C(\omega)$ and $\varepsilon > 0$. Consider the problem* (VVOP) *and let $\overline{x} \in \varepsilon k^0$-MZ$(\Omega, f, C)$. Impose the following assumptions:*

(A5) f is (k^0, C, Ω)-lower semicontinuous over Ω in the sense of Definition 4.9.

(A6) f is bounded from below over Ω with respect to C in the sense of Definition 4.6.

Then there exists an element $x_\varepsilon \in \Omega$ such that

1. $x_\varepsilon \in \varepsilon k^0$-WMZ$(\Omega, f, C)$,

2. $d(\overline{x}, x_\varepsilon) \leqq \sqrt{\varepsilon}$,

3. $x_\varepsilon \in \mathrm{WMZ}(\Omega, f_{\varepsilon k^0}, C)$ *with* $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) k^0$.

Because of the relationships between approximate minimizers and weak approximate minimizers, the results in this section hold for approximate minimizers of vector optimization problems with variable ordering structures too. In the case of fixed ordering structure, we get very well known variational principle for vector optimization problems with fixed ordering structures; see [18, 35, 36].

5 Variational principles for εk^0-Nondominated Solutions

In this section, we introduce the third solution concept for vector optimization problems with variable ordering structures called approximately nondominated solution (see Definition 5.1) and we will give an extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structures. It is important to emphasize that there is no difference between εk^0-minimizers and εk^0-nondominated solutions in the case of fixed ordering structure. The reader can find many examples illustrating that this statement is in general not true in the case of variable ordering structure in [6, 7, 14]. Variational principles for nondominated solutions of vector optimization problems with variable ordering structures are already shown in [2, Theorem 4.7] in a scalar form. In difference to this papers we will show the variational principle in a vector-valued form.

Definition 5.1. *Let* assumptions (A1) − (A2) *be fulfilled and consider problem (VVOP).*

1. $x_\varepsilon \in \Omega$ *is said to be an* εk^0*-nondominated solution of the problem* (VVOP) *with respect to the ordering map* $C : Y \rightrightarrows Y$ *iff*

$$\forall x \in \Omega : \qquad (f(x_\varepsilon) - \varepsilon k^0 - (C(f(x)) \backslash \{\mathbf{0}\})) \cap \{f(x)\} = \emptyset.$$

2. *Let* $\operatorname{int} C(f(x)) \neq \emptyset$ *for all* $x \in \Omega$. $x_\varepsilon \in \Omega$ *is said to be a weakly* εk^0*-nondominated solution of* (VVOP) *with respect to the ordering map* $C : Y \rightrightarrows Y$ *iff*

$$\forall x \in \Omega : \qquad (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap \{f(x)\} = \emptyset.$$

Sets of εk^0-nondominated and weakly εk^0-nondominated solutions will be denoted by εk^0-$\mathrm{N}(\Omega, f, C)$ and εk^0-$\mathrm{WN}(\Omega, f, C)$ respectively. If $\varepsilon = 0$, then all these definitions coincide with the usual definitions of nondominated solutions [6, 16].

Example 5.2. *Let* $\varepsilon = \frac{1}{100}$ *and* $k^0 = (1, 0)^T$. *Also suppose that*

$$\Omega = \left\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\right\}$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_2 \geq 0,\ d_1 + d_2 \geq 0\}, & \text{for } (y_1, y_2) = (0, 0) \\ \mathbb{R}^2_+, & \text{otherwise.} \end{cases}$$

Then $\left\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{99}{100}\right\}$ is the set of εk^0-minimal and εk^0-nondominated points. But only points of the set
$\left\{(y_1, y_2) \in \Omega \middle| \, y_1 + y_2 < -\frac{99}{100}\right\}$ are εk^0-minimizers and points of

$$\left\{(y_1, y_2) \in \Omega \middle| \, y_1 + y_2 = -\frac{99}{100}\right\}$$

are not εk^0-minimizers. This shows that there exist elements of Ω which are both εk^0-nondominated and εk^0-minimal but not εk^0-minimizer (see Fig. 2).

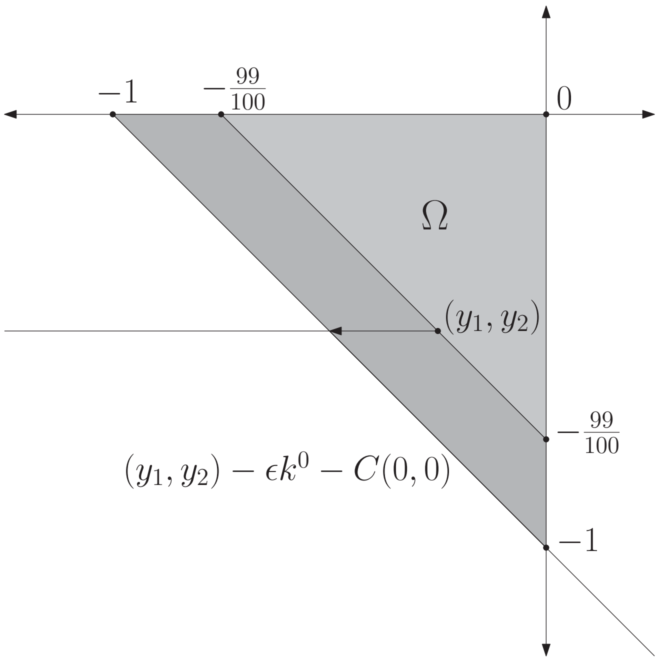

Figure 2: Example 5.2 where there exists an element which is both εk^0-nondominated and εk^0-minimal element but not εk^0-minimizer.

Lemma 5.3. *Let assumptions* (A1)–(A3) *be fulfilled and consider the problem (VVOP). If* $x_\varepsilon \in \varepsilon k^0\text{-N}(\Omega, f, C)$*, then for every element* $x \in \Omega$*, there exists a continuous functional* $\xi(\cdot, f(x)) : Y \to \mathbb{R}$ *which is strictly B-monotone in the sense of Definition 3.12 and*

$$\forall x \in \Omega : \qquad \xi(f(x_\varepsilon), f(x)) \leqq \xi(f(x) + \varepsilon k^0, f(x)).$$

Moreover, if (A4) *holds, then* $\xi(\cdot, f(x))$ *is subadditive on* Y *for each* $x \in \Omega$ *and*

$$\forall x \in \Omega : \qquad \xi(f(x_\varepsilon), f(x)) \leqq \xi(f(x), f(x)) + \xi(\varepsilon k^0, f(x)).$$

Proof. Suppose that $k^0 \in Y \backslash \{\mathbf{0}\}$, $\varepsilon > 0$ and $x_\varepsilon \in \varepsilon k^0\text{-N}(\Omega, f, C)$. This means that for all $x \in \Omega$,

$$(f(x_\varepsilon) - \varepsilon k^0 - C(f(x)) \backslash \{\mathbf{0}\}) \cap f(x) = \emptyset \;\Rightarrow\; (f(x_\varepsilon) - C(f(x)) \backslash \{\mathbf{0}\}) \cap (f(x) + \varepsilon k^0) = \emptyset.$$

We apply Theorem 3.11 to the sets $(f(x_\varepsilon) - C(f(x))\backslash\{\mathbf{0}\})$ and $f(x) + \varepsilon k^0$. Taking into account (A3) and Theorem 3.11, we get desired functionals $\xi : Y \times Y \to \mathbb{R}$. Therefore for any fixed $x \in \Omega$, there exist a continuous functional $\xi(\cdot, f(x)) : Y \to \mathbb{R}$ such that

$$\xi(f(x_\varepsilon), f(x)) \leqq \ \xi(f(x) + \varepsilon k^0, f(x)).$$

Now, if (A4) holds, then for all $x \in \Omega$, $\xi(\cdot, f(x))$ is subadditive and

$$\xi(f(x_\varepsilon), f(x)) \leqq \xi(f(x), f(x)) + \xi(\varepsilon k^0, f(x))$$

and proof is complete. □

The following lemma gives some properties of the functional in Lemma 5.3 and these properties will be used later in the proof of other lemmas and our main theorem about extension of Ekeland's theorem for εk^0-nondominated solutions of vector optimization problems with variable ordering structures.

Lemma 5.4. *Let assumptions* (A1)–(A3) *be fulfilled, then for each fixed* $x \in \Omega$*, the functional* $\xi(\cdot, f(x)) : Y \to \mathbb{R}$ *in Lemma 5.3 has the following properties:*

1. $\xi(k^0, f(x)) = 1$.
2. $\xi(\mathbf{0}, f(x)) = 0$.
3. $\xi(\varepsilon k^0, f(x)) = \varepsilon$ and $\xi(-\varepsilon k^0, f(x)) = -\xi(\varepsilon k^0, f(x)) = -\varepsilon$.

Proof. The proof is similar to that of Lemma 4.4. □

Lemma 5.5. *Let* X *be a complete metric space,* $\Omega \subset X$*,* $x_\varepsilon \in \Omega$*,* Y *be a Hausdorff topological linear space,* $\varepsilon \geq 0$*,* $k^0 \in Y\backslash\{\mathbf{0}\}$*,* $f : X \to Y$ *is a vector-valued function with* $\operatorname{dom} f \neq \emptyset$ *and* $B : Y \rightrightarrows Y$ *be a cone-valued map where* $k^0 \in B(y)$ *for all* $y \in Y$*.*
(j) Furthermore, suppose that for any $x \in \Omega$ *and a strictly* B*-monotone (in the sense of Definition 3.12), continuous, subadditive functionals* $\xi(\cdot, f(x)) : Y \to \mathbb{R}$ *the following inequality holds*

$$\forall x \in \Omega : \qquad \xi(f(x_\varepsilon), f(x)) \leqq \xi(f(x), f(x)) - \xi(-\varepsilon k^0, f(x)).$$

Then there is a set-valued map $C : Y \rightrightarrows Y$ *such that* $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$ *and* $B(f(x))\backslash\{\mathbf{0}\} \subseteq C(f(x))$ *for all* $x \in \Omega$ *and* $x_\varepsilon \in \varepsilon k^0\text{-WN}(\Omega, f, C)$*.*

Proof. For each $x \in \Omega$, we define $C(f(x))$ and functional $\hat{\xi}(\cdot, f(x)) : Y \to \mathbb{R}$ as follows,

$$C(f(x)) := \{y \in Y \mid \xi(-y + f(x_\varepsilon) - \varepsilon k^0, f(x)) < \xi(f(x_\varepsilon) - \varepsilon k^0, f(x))\}, \tag{16}$$

$$\hat{\xi}(y, f(x)) := \xi(y + f(x_\varepsilon) - \varepsilon k^0, f(x)). \tag{17}$$

By (17) and (j) and since $\xi(\cdot, f(x))$ is subadditive for all $x \in \Omega$, we get

$$\begin{aligned}
&\hat{\xi}(f(x) + \varepsilon k^0 - f(x_\varepsilon), f(x)) = \xi(f(x), f(x)) \geqq \\
&\xi(f(x_\varepsilon), f(x)) + \xi(-\varepsilon k^0, f(x)) \geqq \\
&\xi(f(x_\varepsilon) - \varepsilon k^0, f(x)) = \hat{\xi}(\mathbf{0}, f(x)).
\end{aligned}$$

Now, by (16) and (17), we have

$$\hat{\xi}(-C(f(x)), f(x)) = \xi(-C(f(x)) + f(x_\varepsilon) - \varepsilon k^0, f(x)) < \xi(f(x_\varepsilon) - \varepsilon k^0, f(x)) = \hat{\xi}(\mathbf{0}, f(x)),$$

and therefore for each $x \in \Omega$,

$$(-\operatorname{int} C(f(x))) \cap (f(x) + \varepsilon k^0 - f(x_\varepsilon)) = \emptyset \;\Rightarrow\; (f(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} C(f(x))) \cap f(x) = \emptyset.$$

Now, we show that $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$. Choose $y \in \operatorname{cl} C(f(x))$ and $b \in y + B(f(x))\backslash\{\mathbf{0}\}$. Because the functional $\hat{\xi}(\cdot, f(x))$ is strictly B-monotone and by $y \in \operatorname{cl} C(f(x)) \subseteq \{y \mid \hat{\xi}(-y, f(x)) \leqq \hat{\xi}(\mathbf{0}, f(x))\}$, we get

$$\hat{\xi}(-b, f(x)) < \hat{\xi}(-y, f(x)) \leqq \hat{\xi}(\mathbf{0}, f(x)).$$

Therefore $b \in C(f(x))$ and $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$. Now, by $\mathbf{0} \in \operatorname{cl} C(f(x))$, we get $B(f(x))\backslash\{\mathbf{0}\} \subseteq C(f(x))$. Furthermore by the inclusion $\operatorname{cl} C(f(x)) + (B(f(x))\backslash\{\mathbf{0}\}) \subseteq C(f(x))$ and the assumption $k^0 \in B(f(x))$, we get $C(f(x)) + \varepsilon k^0 \subseteq C(f(x))$. □

In the following theorem, we give a generalizations of the Ekeland's variational principle for εk^0-nondominated solutions of (VVOP) provided that $f : X \to Y$ is bounded from below and (k^0, C, Ω)-lower semicontinuous.

Theorem 5.6. *Consider the problem* (VVOP) *and let* $\overline{x} \in \varepsilon k^0\text{-N}(\Omega, f, C)$. *Impose in addition to* (A1)–(A4) *the following assumptions:*

(A5) f *is* (k^0, C, Ω)*-lower semicontinuous over* Ω *in the sense of Definition 4.9.*

(A6) f *is bounded from below over* Ω *with respect to* C *in the sense of Definition 4.6.*

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ *such that*

1. $x_\varepsilon \in \varepsilon k^0\text{-WN}(\Omega, f, B)$,
2. $d(\overline{x}, x_\varepsilon) \leqq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \mathrm{WN}(\Omega, f_{\varepsilon k^0}, B)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) k^0$. *(18)*

Proof. Let $\overline{x} \in \varepsilon k^0\text{-N}(\Omega, f, C)$, then by the definition of approximately nondominated solutions (Definition 5.1), we have $(f(\overline{x}) - \varepsilon k^0 - C(f(x))\backslash\{\mathbf{0}\}) \cap f(x) = \emptyset$ for all $x \in \Omega$. Now, suppose that $\overline{f} := f - f(\overline{x})$, then we have

$$(\overline{f}(\overline{x}) - \varepsilon k^0 - C(f(x))\backslash\{\mathbf{0}\}) \cap \overline{f}(x) = \emptyset.$$

By (A4), Lemma 5.3 and Lemma 5.4, the functional $\xi(\cdot, f(x)) : Y \to \mathbb{R}$ defined by (1) is strictly B-monotone, continuous and subadditive for all fixed $x \in \Omega$. Furthermore,

$$\forall\, x \in \Omega : \quad \xi(\overline{f}(\overline{x}), f(x)) \leqq \xi(\overline{f}(x), f(x)) + \xi(\varepsilon k^0, f(x)) = \xi(\overline{f}(x), f(x)) + \varepsilon.$$

This means that for all $x \in \Omega$,

$$\xi(\overline{f}(\overline{x}), f(x)) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x), f(x)) + \varepsilon, \qquad \varepsilon > 0.$$

Observe that the validity of (A5)–(A6) ensures the boundedness from below and (k^0, C, Ω)-lower semicontinuity of f and $\overline{f}$. By Lemma 4.10, Lemma 4.7, Theorem 2.2 and Remark 2.3, there exists $x_\varepsilon \in \Omega$ such that for all $x \in \Omega$,

1. $\xi(\overline{f}(x_\varepsilon), f(x)) \leqq \xi(\overline{f}(\overline{x}), f(x)) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x), f(x)) + \varepsilon,$ (19)
2. $d(x_\varepsilon, \overline{x}) \leqq \sqrt{\varepsilon},$
3. for all $x \in \Omega$: $\quad \xi(\overline{f}(x_\varepsilon), f(x)) \leqq \xi(\overline{f}(x), f(x)) + \sqrt{\varepsilon} d(x, x_\varepsilon).$ (20)

By Lemma 5.4 and (19), for all $x \in \Omega$, we get

$$\begin{aligned}
&\xi(\overline{f}(x_\varepsilon), f(x)) \leqq \inf_{x\in\Omega} \xi(\overline{f}(x), f(x)) + \varepsilon \leqq \\
&\xi(\overline{f}(x), f(x)) + \xi(\varepsilon k^0, f(x)) = \xi(\overline{f}(x), f(x)) - \xi(-\varepsilon k^0, f(x)).
\end{aligned}$$

Now, by Lemma 5.5, assumption (A3) and $\overline{f} = f - f(\overline{x})$,

$$\forall x \in \Omega : \qquad (\overline{f}(x_\varepsilon) - \varepsilon k^0 - \operatorname{int} B(f(x))) \cap \{\overline{f}(x)\} = \emptyset.$$

This implies that $x_\varepsilon \in \varepsilon k^0$-WN$(\Omega, f, B)$. Now, we prove (18) and for this, suppose that there exists an element $x \in \Omega$ such that

$$\begin{aligned}
& f(x) \in f(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0 - \operatorname{int} B(f(x)) \\
\Longrightarrow\quad & \overline{f}(x) \in \overline{f}(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0 - \operatorname{int} B(f(x)).
\end{aligned}$$

Since for all fixed $x \in \Omega$, $\xi(\cdot, f(x))$ is a strictly B-monotone continuous subadditive functional, we have

$$\begin{aligned}
&\xi(\overline{f}(x), f(x)) < \xi(\overline{f}(x_\varepsilon) - \sqrt{\varepsilon} d(x, x_\varepsilon) k^0, f(x)) \leqq \\
&\xi(\overline{f}(x_\varepsilon), f(x)) + \xi(-\sqrt{\varepsilon} d(x, x_\varepsilon) k^0, f(x)).
\end{aligned}$$

Now, by Lemma 5.4, we get

$$\begin{aligned}
& \xi(-\sqrt{\varepsilon} d(x, x_\varepsilon) k^0, f(x)) = -\sqrt{\varepsilon} d(x, x_\varepsilon) \\
\Longrightarrow\quad & \xi(\overline{f}(x_\varepsilon), f(x)) > \xi(\overline{f}(x), f(x)) + \sqrt{\varepsilon} d(x, x_\varepsilon),
\end{aligned}$$

but this yields a contradiction because of (20). □

Note that the third condition in the variational principle in [2, Theorem 4.7] is given in scalarized form but here this condition is in vector form. In the paper [2], the variational principle is derived using a scalarizing functional

$$\varphi : Y \to \overline{\mathbb{R}} : \quad \varphi(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 - C(y)\}$$

defined in [6] such that it is only possible to have the third condition in a scalarized form. However, in Theorem 5.6, the functional defined by (1) is used in order to get a stronger result in the third part.

In the special case that $C : Y \rightrightarrows Y$ is a solid, closed, pointed and convex cone-valued map, we have the following corollary.

Corollary 5.7. *Let $C : Y \rightrightarrows Y$ be a cone-valued map where $C(f(x))$ is a closed solid convex cone for all $x \in \Omega$, $k^0 \in \bigcap_{x\in\Omega} \operatorname{int} C(f(x))$ and $\varepsilon > 0$. Consider the problem* (VVOP) *and furthermore, let $\overline{x} \in \varepsilon k^0\text{-N}(\Omega, f, C)$. Impose the following assumptions:*

(A5) f is (k^0, C, Ω)-lower semicontinuous over Ω in the sense of Definition 4.9.

(A6) f is bounded from below over Ω with respect to C in the sense of Definition 4.6.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f \cap \Omega$ such that

1. $x_\varepsilon \in \varepsilon k^0\text{-WN}(\Omega, f, C)$,
2. $d(\overline{x}, x_\varepsilon) \leqq \sqrt{\varepsilon}$,
3. $x_\varepsilon \in \text{WN}(\Omega, f_{\varepsilon k^0}, C)$ *with* $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon} d(x, x_\varepsilon) k^0$.

Because of the relationships between approximately nondominated and weakly approximate nondominated solutions, the results in this section hold for approximately nondominated solutions of vector optimization problems with variable ordering structures too.

6 Applications for Deriving Necessary Conditions for Approximate Solutions of (VVOP)

In this section, we use the variational principles presented in the previous sections in order to prove necessary conditions for approximately minimizers and nondominated solutions of (VVOP). In the whole section we will assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)_Y$ are Banach spaces where the metric is induced by the norm and f is Gâteaux differentiable which is defined as follows.

Definition 6.1. *The Gâteaux derivative $df(x, h)$ of $f : X \to Y$ at $x \in \Omega$ in the direction $h \in X$ is defined as*

$$df(x, h) := \lim_{t\to 0} \frac{f(x + th) - f(x)}{t}.$$

If the limit exists for all direction $h \in X$, then one says f is Gâteaux differentiable at x.

The following theorem is the direct consequence of Theorem 4.11 if we suppose the Gâteaux derivative of the objective function $f : X \to Y$.

Theorem 6.2. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)_Y$ be Banach spaces and $f : X \to Y$ be a Gâteaux differentiable function. Let $\overline{x} \in \varepsilon k^0$-MZ$(X, f, C)$ be an approximately minimizer of $(VVOP)$. Impose in addition to* (A1)–(A4) *the following assumptions:*

(A5) f is (k^0, C, X)-lower semicontinuous over X in the sense of Definition 4.9.

(A6) f is bounded from below over X with respect to C in the sense of Definition 4.6.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f$ such that the following holds for all $x \in X$ and all $h \in X$ with $\|h\| = 1$,

$$df(x_\varepsilon, h) \notin -\sqrt{\varepsilon}k^0 - \operatorname{int} B(f(x)).$$

Proof. Let $\overline{x} \in \varepsilon k^0$-MZ$(X, f, C)$. By condition 3. in Theorem 4.11, there exists $x_\varepsilon \in \operatorname{dom} f$ such that $x_\varepsilon \in$ WMZ$(X, f_{\varepsilon k^0}, B)$ with $f_{\varepsilon k^0}(x) = f(x) + \sqrt{\varepsilon}\,\|x - x_\varepsilon\|\, k^0$, i.e.,

$$\forall x \in X : \quad f_{\varepsilon k^0}(X) \cap (f_{\varepsilon k^0}(x_\varepsilon) - \operatorname{int} B(f(x))) = \emptyset.$$

This means there exists no $x, x_1 \in X$ such that $f_{\varepsilon k^0}(x_1) \in f_{\varepsilon k^0}(x_\varepsilon) - \operatorname{int} B(f(x))$ which implies

$$\nexists x, x_1 \in X : \quad f(x_1) + \sqrt{\varepsilon}\,\|x_1 - x_\varepsilon\|\, k^0 \in f(x_\varepsilon) - \operatorname{int} B(f(x)).$$

By choosing $x_1 = x_\varepsilon + th$, $\|h\| = 1$ and $t > 0$, we get

$$\Rightarrow \qquad \nexists h \in X : \quad f(x_\varepsilon + th) + \sqrt{\varepsilon}\,\|x_\varepsilon + th - x_\varepsilon\|\, k^0 \in f(x_\varepsilon) - \operatorname{int} B(f(x))$$

$$\Rightarrow \qquad f(x_\varepsilon + th) - f(x_\varepsilon) \notin -\sqrt{\varepsilon}tk^0 - \operatorname{int} B(f(x))$$

$$\Rightarrow \qquad \frac{f(x_\varepsilon + th) - f(x_\varepsilon)}{t} \notin -\sqrt{\varepsilon}k^0 - \operatorname{int} B(f(x))$$

$$\Rightarrow \qquad \lim_{t \to 0^+} \frac{f(x_\varepsilon + th) - f(x_\varepsilon)}{t} \notin -\sqrt{\varepsilon}k^0 - \operatorname{int} B(f(x))$$

and this means $df(x_\varepsilon, h) \notin -\sqrt{\varepsilon}k^0 - \operatorname{int} B(f(x))$ for all $x \in X$. □

Similar results for approximate nondominated solutions of $(VVOP)$ are as following:

Theorem 6.3. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|)_Y$ be Banach spaces and $f : X \to Y$ be a Gâteaux differentiable function. Let $\overline{x} \in \varepsilon k^0$-N$(X, f, C)$ be an approximately nondominated solution of $(VVOP)$. Impose in addition to* (A1)–(A4) *the following assumptions:*

(A5) f is (k^0, C, X)-lower semicontinuous over X in the sense of Definition 4.9.

(A6) f is bounded from below over X with respect to C in the sense of Definition 4.6.

(A7) There exists an element $\widehat{x} \in X$ such that $B(f(\widehat{x})) \subseteq B(f(x))$ for all $x \in X$.

Then there exists an element $x_\varepsilon \in \operatorname{dom} f$ such that the following holds for all $h \in X$ with $\|h\| = 1$:

$$df(x_\varepsilon, h) \notin -\sqrt{\varepsilon}k^0 - \operatorname{int} B(f(\widehat{x})).$$

The proof is similar to that of Theorem 6.2 using Theorem 5.6 instead of Theorem 4.11.

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