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Treatment of set order relations by means of a nonlinear scalarization functional: A full characterization

Elisabeth Köbis* and Markus A. Köbis†

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Abstract

In this paper we show how a nonlinear scalarization functional can be used in order to characterize set order relations and which thus plays a key role in set optimization. Specifically, we use the set less order relation introduced independently by Young [23] and Nishnianidze [20], the certainly less order relation [13], the possibly less order relation [1, 13] and the minmax less order relation [13]. Our approaches do not rely on any convexity assumptions on the considered sets. Furthermore, we develop a derivative-free descent method for set optimization problems without convexity assumptions to verify the usefulness of our results.

Keywords:
set optimization set order relations nonlinear scalarizing functional descent method

1 Introduction

Set optimization has become an important research area in various fields. For an introduction to set optimization and its applications, we refer to [14]. For some applications, it is necessary to assume a feasible element to be associated to a whole set of values in the objective space instead of just one vector. For example, certain concepts of robustness for dealing with uncertainties in vector optimization can be described using approaches from set-valued optimization (see [9]). The concept of interval arithmetics for computations with strict error bounds [18] is also a special case of dealing with set-valued mappings.
Set optimization deals with the process of obtaining minimal sets, where the map to be minimized is set-valued (see [10]). In order to obtain minimal solutions

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of a set optimization problem, one uses set order relations. We are interested in characterizing certain set relations by using a very broad and manageable functional. Such characterizations of set order relations via scalarization are important for deriving numerical methods for solving set-valued optimization problems.

In Subsection 2.1, we recall the nonlinear scalarizing functional and give some important properties that this functional satisfies under very general assumptions. These properties will be used in Section 3 in order to describe set order relations by means of this particular functional. In Subsection 2.2 we very shortly review how set order relations have been discussed in the context of scalarization functionals in the literature. Section 3 deals with the characterization of certain set order relations, such as the upper, lower and set less order relation, the certainly, possibly and minmax less order relation by means of the nonlinear functional introduced in Section 2. Hereby we use the characterizing property of the nonlinear scalarization functional as a nonconvex separation functional [2] since the aforementioned set relations by definition rely on set inclusions.

Finally, in Subsection 4.1 we describe minimal solutions of set optimization problems by means of our findings. In Subsection 4.2 we propose a descent method in order to obtain an improvement of feasible points of set-valued optimization problems.

2 Nonlinear scalarization functional

2.1 Preliminaries

Let Y be a linear topological space, $k \in Y \setminus \{0\}$ and $\mathcal{F}$ and $\mathcal{C}$ be proper subsets of Y. Assume that $\mathcal{C}$ is closed and

$$\mathcal{C} + [0, +\infty) \cdot k \subset \mathcal{C}\,. \tag{1}$$

We introduce the functional $z^{\mathcal{C},k} \colon Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \bar{\mathbb{R}}$

$$z^{\mathcal{C},k}(y) := \inf\{t \in \mathbb{R} | y \in tk - \mathcal{C}\}\,. \tag{2}$$

Now we formulate the problem

$$z^{\mathcal{C},k}(y) \to \inf_{y \in \mathcal{F}}\,. \tag{$P_{k,\mathcal{C},\mathcal{F}}$}$$

Figure 1 visualizes the functional $z^{\mathcal{C},k}$, where $\mathcal{C} = \mathbb{R}^2_+$ has been taken as the natural ordering cone in $\mathbb{R}^2$ and $k \in \operatorname{int} \mathcal{C}$. We can see that the set $-\mathcal{C}$ is moved along the line $\mathbb{R} \cdot k$ up until y belongs to $tk - \mathcal{C}$. The functional $z^{\mathcal{C},k}$ assigns the smallest value t such that the property $y \in tk - \mathcal{C}$ is fulfilled. For the given set $\mathcal{C}$ and by a variation of the vector $k \in Y \setminus \{0\}$ with the property (1) all minimal elements of a vector optimization problem without any convexity assumptions can be found.

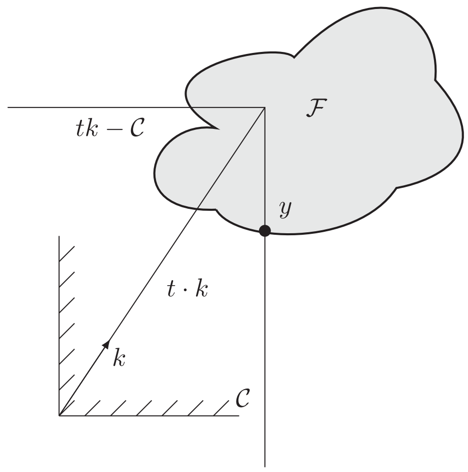

Figure 1: Illustration of the functional $z^{\mathcal{C},k}(y) := \inf\{t \in \mathbb{R} | y \in tk - \mathcal{C}\}$.

The scalarizing functional $z^{\mathcal{C},k}$ was used in [2] to prove nonconvex separation theorems. Applications of $z^{\mathcal{C},k}$ include coherent risk measures in financial mathematics (see, for instance, [8]). Properties of $z^{\mathcal{C},k}$ were studied in [2], [22] and [3]. First let us recall the D-monotonicity of a functional.

Definition 2.1. Let Y be a linear topological space, $D \subset Y$, $D \neq \emptyset$. A functional $z \colon Y \to \bar{\mathbb{R}}$ is D-**monotone**, if for

$$y_1,\ y_2 \in Y:\ y_1 \in y_2 - D \Rightarrow z(y_1) \leq z(y_2)\,.$$

Moreover, z is said to be **strictly** D-**monotone**, if for

$$y_1, y_2 \in Y:\ y_1 \in y_2 - D \setminus \{0\} \Rightarrow z(y_1) < z(y_2)\,.$$

Below we provide some properties of the functional $z^{\mathcal{C},k}$ introduced in (2).

Theorem 2.2 ([2], [3])**.** *Let Y be a linear topological space, $\mathcal{C} \subset Y$ a closed proper set and $\emptyset \neq D \subset Y$. Furthermore, let $k \in Y \setminus \{0\}$ be such that (1) is satisfied. Then the following properties hold for $z = z^{\mathcal{C},k}$:*

(a) *z is lower semi-continuous.*

(b) *(i) z is convex $\Longleftrightarrow \mathcal{C}$ is convex,*

(ii) $[\forall\ y \in Y,\ \forall\ \ r > 0:\ z(ry) = rz(y)] \Longleftrightarrow \mathcal{C}$ is a cone.

(c) *z is proper $\Longleftrightarrow \mathcal{C}$ does not contain lines parallel to k, i.e., $\forall\ y \in Y\ \exists\ r \in \mathbb{R}:\ y + rk \notin \mathcal{C}$.*

(d) z *is* D*-monotone* $\iff \mathcal{C} + D \subset \mathcal{C}$.

(e) z *is subadditive* $\iff \mathcal{C} + \mathcal{C} \subset \mathcal{C}$.

(f) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) \leq r \iff y \in rk - \mathcal{C}$.

(g) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y + rk) = z(y) + r$.

(h) z *is finite-valued* $\iff \mathcal{C}$ *does not contain lines parallel to* k *and* $\mathbb{R}k - \mathcal{C} = Y$.

Let furthermore $\mathcal{C} + (0, +\infty) \cdot k \subset \operatorname{int} \mathcal{C}$*. Then*

(i) z *is continuous.*

(j) $\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) = r \iff y \in rk - \operatorname{bd} \mathcal{C}$,
$\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) < r \iff y \in rk - \operatorname{int} \mathcal{C}$.

(k) If z *is proper, then* z *is* D*-monotone* $\iff \mathcal{C} + D \subset \mathcal{C} \iff \operatorname{bd} \mathcal{C} + D \subset \mathcal{C}$.

(l) If z *is finite-valued, then* z *is strictly* D*-monotone* $\iff \mathcal{C} + (D \setminus \{0\}) \subset \operatorname{int} \mathcal{C}$ $\iff \operatorname{bd} \mathcal{C} + (D \setminus \{0\}) \subset \operatorname{int} \mathcal{C}$.

(m) Suppose z *is proper. Then* z *is subadditive* $\iff \mathcal{C} + \mathcal{C} \subset \mathcal{C} \iff \operatorname{bd} \mathcal{C} + \operatorname{bd} \mathcal{C} \subset \mathcal{C}$.

For the proof, see [3, Theorem 2.3.1].
From now on we will restrict our attention to the case of $\mathcal{C} = C$ in the definition of the nonlinear scalarizing functional $z = z^{C,k}$ being a closed convex *cone*.

Remark 2.3. *Note that if* $\mathcal{C} = C$ *is a closed convex cone, condition* (1) *is equivalent to* $k \in C \setminus \{0\}$*. Moreover,* $C + (0, +\infty) \cdot k \subset \operatorname{int} C$ *holds if and only if* $k \in \operatorname{int} C$.

For an easier reference we will also introduce the following notation.

Definition 2.4. Let Y be a linear topological space. We denote by

$$\mathcal{P}(Y) := \{A \subseteq Y \mid A \text{ is nonempty}\}$$

the **power set** of Y.

Throughout this paper, let Y be a linear topological space.
Below we summarize the properties of the functional (2), where $C \subset Y$ is given as a proper closed convex cone.

Corollary 2.5 ([3, Corollary 2.3.5.])**.** *Let* $C \subset Y$ *be a proper closed convex cone and* $k \in \operatorname{int} C$*. Then* $z = z^{C,k}$*, defined by* (2)*, is a finite-valued continuous sublinear and strictly* $(\operatorname{int} C)$*-monotone functional such that*

$$\begin{aligned} &\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) \leq r \iff y \in rk - C, \\ &\forall\, y \in Y,\ \forall\, r \in \mathbb{R}:\ z(y) < r \iff y \in rk - \operatorname{int} C. \end{aligned} \tag{3}$$

The set relations to be defined in Section 3 rely on set inclusions where the ordering cone C is attached pointwise to the considered sets $A, B \in \mathcal{P}(Y)$. The following corollary relates $A + C$ and $B - C$ respectively by means of the functional $z^{C,k}$.

Corollary 2.6. *Let C be a proper closed convex cone and $k \in C \setminus \{0\}$. For two sets A, $B \in \mathcal{P}(Y)$ it holds*

$$\sup_{b \in B} z^{C,k}(b) = \sup_{y \in B-C} z^{C,k}(y)$$
$$\inf_{a \in A} z^{C,k}(a) = \inf_{y \in A+C} z^{C,k}(y)\,.$$

Proof. We will only give the proof of the first assertion since the second one can be proven in exactly the same way.
Let $k \in C \setminus \{0\}$, $c \in C$ and $b \in B$ be given. One can use the C-monotonicity of the functional $z^{C,k}$ (see Theorem 2.2(d) with $D = C = \mathcal{C}$) to show

$$z^{C,k}(b) \geq z^{C,k}(b - c)$$

implying

$$\sup_{b \in B} z^{C,k}(b) \geq \sup_{y \in B-C} z^{C,k}(y)$$

directly. The converse, i.e., $\sup_{b \in B} z^{C,k}(b) \leq \sup_{y \in B-C} z^{C,k}(y)$, follows directly from the definition of the supremum and $0 \in C$, or in particular $B \subseteq B - C$. $\square$

Remark 2.7 (Attainment Property)**.** *In several assertions of the following sections we suppose a certain attainment property, especially in Theorem 3.5 we assume that $\inf_{b \in B} z^{C,k_0}(a - b)$ is attained for all $a \in A$ and in Theorem 3.16 we assume that $\inf_{a \in A} z^{C,k_0}(a - b)$ is attained for all $b \in B$. Similar attainment properties are supposed in Corollary 3.24, Theorem 3.34 as well as in Attainment Properties 4.2, 4.4, 4.5 and 4.6. Sufficient conditions for these attainment properties, i.e., assertions concerning the existence of solutions of corresponding optimization problems (extremal principles) are given in the literature. The well-known Theorem of Weierstrass says that a lower semi-continuous function on a nonempty compact set has a minimum. An extension of the Theorem of Weierstrass is given by Zeidler [24, Proposition 9.13]: A proper lower semi-continuous and quasi-convex function on a nonempty closed bounded convex subset of a reflexive Banach space has a minimum.*
Taking into account that the functional z^{C,k_0} is lower semi-continuous and convex if $C \subset Y$ is a proper closed convex cone and $k_0 \in C \setminus \{0\}$ (compare Theorem 2.2), we get that the attainment property for $\inf_{a \in A} z^{C,k_0}(a - b)$ (with $b \in B$ fixed) is fulfilled if A is a nonempty closed bounded convex subset of a reflexive Banach space.

2.2 Notes on the literature

Hernández and Rodríguez-Marín [7] introduce an extension of the functional $z^{C,k}$ (see equation (2)) in order to characterize the set order relation $B \subseteq A +$

C (compare also [4, 5, 6] and the references given therein). They consider a function

$$Z^{C,k}(A,B) := \sup_{b\in B} \inf\{t \in \mathbb{R} | b \in tk + A + C\}$$
$$\left(= \sup_{b\in B} z^{-C-A,k}(b) \text{ with the notations from Section 2.1}\right)$$

and they show that

$$B \subseteq A + C \iff \text{for some } k \in -\operatorname{int} C : \; Z^{C,k}(A,B) \leq 0$$

if $A + C$ is closed, compare [7, Thm. 3.10].
Moreover, Gutiérrez et al. [4, 5, 6] proved important properties of the functional $Z^{C,k}$ in order to derive necessary and sufficient optimality conditions for set optimization problems. The advantage of the approach considered in [4, 5, 6] seems to be the inclusion of the whole sets A, B in the function $Z^{C,k}$. We will show in the next sections that it is possible to use the given functional $z^{C,k}$ (see (2)) in its traditional form to characterize the relation $B \subseteq A + C$. The benefit using our approach, as we will see in the following section, is the easy structure of the functional $z^{C,k}$ and its useful properties which allow to derive optimality conditions based on certain differentiability concepts as well as a manageable numerical treatment.
Furthermore, Jahn [11] showed that

$$A \subseteq B - C \iff \forall\, l \in C^{+} \setminus \{0\} : \; \sup_{a\in A} l(a) \leq \sup_{b\in B} l(b)$$

and

$$A \subseteq B + C \iff \forall\, l \in C^{+} \setminus \{0\} : \; \inf_{b\in B} l(b) \leq \inf_{a\in A} l(a)$$

if the sets $B - C$ and $B + C$ are closed and convex, where C^{+} is the dual cone of C.
The approach presented in this paper enables us to present a full characterization of other set order relations known from the literature, for instance the upper set less order relation, the set less order relation, the certainly less order relation, the possibly less order relation and the minmax less order relation. Our approaches do not rely on any convexity assumptions.

3 Characterizations of set order relations by means of a nonlinear scalarization functional

3.1 Upper, lower and set less order relation

A well known set order relation is the upper set less order relation introduced by Kuroiwa [16, 17].

Definition 3.1 (Upper set less order relation, [16, 17]). Let $C \subset Y$ be a proper closed pointed convex cone. The **upper set less order relation** $\preceq_C^u$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq_C^u B :\Longleftrightarrow A \subseteq B - C,$$

which is equivalent to

$$\forall\, a \in A\; \exists\, b \in B : a \leq_C b.$$

The following theorem shows a first connection between the upper set less order relation and the nonlinear scalarizing functional $z^{C,k}$.

Theorem 3.2. *Let $C \subset Y$ be a proper closed convex cone and $A, B \in \mathcal{P}(Y)$. It holds*

$$A \subseteq B - C \implies \forall\, k \in C \setminus \{0\} : \sup_{a \in A} z^{C,k}(a) \leq \sup_{b \in B} z^{C,k}(b).$$

Proof. Choose a vector $k \in C \setminus \{0\}$ arbitrary, but fixed and let $A \subseteq B - C$. The definition of the functional (2) directly leads to $\sup_{a\in A} z^{C,k}(a) \leq \sup_{y \in B-C} z^{C,k}(y)$ which together with Corollary 2.6 already gives the assertion:

$$\sup_{a\in A} z^{C,k}(a) \leq \sup_{y\in B-C} z^{C,k}(y) = \sup_{b\in B} z^{C,k}(b)\,.$$

□

The inverse implication of the assertion in Theorem 3.2 is generally not fulfilled, as the following examples illustrate.

Example 3.3. Let $Y := \mathbb{R}^2$, $A := \{(-1/4, -1/4)^T\} =: \{a\}$, $\overline{a} := (3/4, 3/4)^T$, $\overline{A} := \{\overline{a}\}$, $B := \{(s, 1-s)^T \mid s \in [0,1]\}$, $k := (k_1, k_2)^T$, k_1, $k_2 > 0$, and consider the natural ordering cone $C = \mathbb{R}^2_+$, see Figure 2. It holds (cf. Example 3.26)

$$\sup_{b\in B} z^{C,k}(b) = \sup_{s\in[0,1]} \max\left\{\frac{s}{k_1}, \frac{1-s}{k_2}\right\} = \max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} > 0$$

and

$$\sup_{\widetilde{a}\in A} z^{C,k}(\widetilde{a}) = z^{C,k}(a) = -\frac{1}{4}\min\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} < 0 < \sup_{b\in B} z^{C,k}(b)$$

corresponding to $A \subset B - C$ as well as

$$\sup_{\widetilde{a}\in\overline{A}} z^{C,k}(\widetilde{a}) = z^{C,k}(\overline{a}) = \frac{3}{4}\max\left\{\frac{1}{k_1}, \frac{1}{k_2}\right\} < \sup_{b\in B} z^{C,k}(b)$$

but clearly

$$\overline{A} \not\subseteq B - C\,.$$

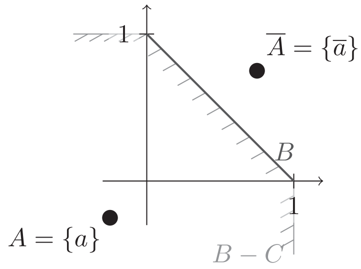

Figure 2: Illustration of Example 3.3.

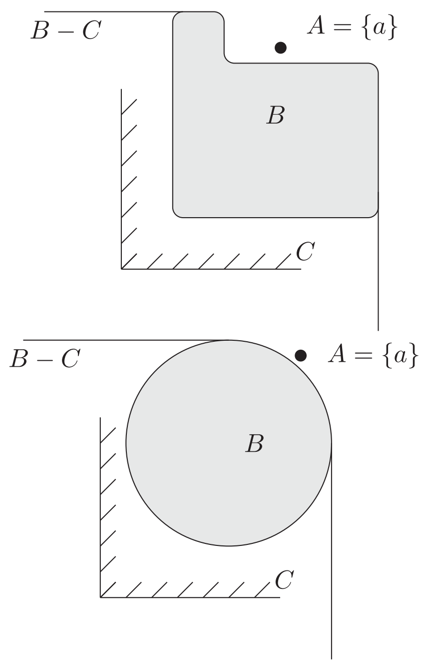

Figure 3: Illustration of the set $B - C$ and the single-valued set $A = \{a\}$.

Example 3.4. Consider again the special case $C = \mathbb{R}^2_+$, a singleton $A = \{a\}$ and the set B in Figure 3. Apparently, we have $z^{C,k}(a) \leq \sup_{b\in B} z^{C,k}(b)$ for every $k \in \operatorname{int} C$, but obviously $A \not\subseteq B - C$. The same may even hold if the set $B - C$ is convex, as the right illustration shows.

We can, however, express the inclusion $A \subseteq B - C$ for two arbitrary sets $A, B \in \mathcal{P}(Y)$ by using the nonlinear scalarization functional $z^{C,k}$, as the following theorem shows.

Theorem 3.5. *Let $C \subset Y$ be a proper closed convex cone. For two sets $A, B \in \mathcal{P}(Y)$ it holds*

$$A \subseteq B - C \quad \Longrightarrow \quad \forall k \in C \setminus \{0\} : \sup_{a\in A} \inf_{b\in B} z^{C,k}(a - b) \leq 0\,. \tag{4}$$

Assume on the other hand, that there exists a $k_0 \in C \setminus \{0\}$ such that for all $a \in A$ the infimum $\inf_{b\in B} z^{C,k_0}(a-b)$ is attained, then the converse is also true, i.e.,

$$\sup_{a\in A} \inf_{b\in B} z^{C,k_0}(a - b) \leq 0 \quad \Longrightarrow \quad A \subseteq B - C\,. \tag{5}$$

Proof. Let $A \subseteq B - C$. This corresponds to

$$\forall\, a \in A \;\; \exists\, b \in B: \; a \in b - C \quad \Longrightarrow \quad \forall\, a \in A \;\; \exists\, b \in B: \; a - b \in -C\,.$$

Because of Theorem 2.2 (f) with $r = 0$ and $y = a - b$, we have for $k \in C \setminus \{0\}$

$$\begin{aligned} &\forall\, a \in A \;\; \exists\, b \in B: \; a - b \in -C \\ \Longleftrightarrow\; &\forall\, a \in A \;\; \exists\, b \in B: \; z^{C,k}(a - b) \leq 0, \end{aligned}$$

and this implies

$$\sup_{a\in A} \inf_{b\in B} z^{C,k}(a - b) \leq 0\,.$$

Now let $k_0 \in C\setminus\{0\}$ be given such that for all $a \in A$ the infimum $\inf_{b\in B} z^{C,k_0}(a-b)$ is attained. Let

$$\sup_{a\in A} \inf_{b\in B} z^{C,k_0}(a - b) \leq 0, \tag{6}$$

but assume that $A \not\subseteq B - C$. Thus, there exists some $\overline{a} \in A$ with $\overline{a} \notin B - C$. So for all $b \in B$ it holds $\overline{a} - b \notin -C$ and with Theorem 2.2 (f) with $r = 0$ and $y = \overline{a} - b$ we obtain

$$\exists\, \overline{a} \in A \; \forall b \in B : z^{C,k_0}(\overline{a} - b) > 0 \quad \Longrightarrow \quad \exists\, \overline{a} \in A \; \inf_{b\in B} z^{C,k_0}(\overline{a} - b) > 0\,.$$

Because the last infimum is attained by assumption one concludes that

$$\sup_{\overline{a}\in A} \inf_{b\in B} z^{C,k_0}(\overline{a} - b) > 0\,,$$

a contradiction to (6). □

Remark 3.6. *Note that the existence of a* $k_0 \in C\setminus\{0\}$ *such that* $\inf_{b\in B} z^{C,k_0}(a-b)$ *is attained for all* $a \in A$ *happens if for all* $k \in C \setminus \{0\}$*,* $B - C$ *is closed, that is,* B *is* $(-C)$*-closed, and if* $\inf_{b\in B} z^{C,k}(a - b) \in \mathbb{R}$*. This condition is in the spirit of [7].*

Remark 3.7. *Note that the set inclusion* $A \subseteq B - C$ *by Theorem 3.5 also implies*

$$\sup_{k\in C\setminus\{0\}} \sup_{a\in A} \inf_{b\in B} z^{C,k}(a - b) \leq 0.$$

Remark 3.8. *If there exists an element* $k_0 \in C\setminus\{0\}$ *such that* $\inf_{b\in B} z^{C,k_0}(a-b)$ *is attained for all* $a \in A$*, then it follows from Theorem 3.5 that*

$$\begin{aligned} A \subseteq B - C &\iff \sup_{a\in A} \inf_{b\in B} z^{C,k_0}(a - b) \leq 0 \\ &\iff \sup_{k\in C\setminus\{0\}} \sup_{a\in A} \inf_{b\in B} z^{C,k}(a - b) \leq 0. \end{aligned} \tag{7}$$

Example 3.9. Consider again the setting of Example 3.3 with fixed $k_1, k_2 > 0$.

(a) For $\overline{A} = \{\overline{a}\} = (3/4, 3/4)^T$ we get

$$\sup_{\widetilde{a}\in\overline{A}} \inf_{b\in B} z^{C,k}(\widetilde{a} - b) = \min_{s\in[0,1]} \max\left\{\frac{3/4 - s}{k_1}, \frac{s - 1/4}{k_2}\right\} > 0\,,$$

see Figure 4.

(b) On the other hand, for $A = \{a\} = (-1/4, -1/4)^T$ one derives

$$\sup_{\widetilde{a}\in A} \inf_{b\in B} z^{C,k}(\widetilde{a} - b) = \min_{s\in[0,1]} \max\left\{\frac{-s - 1/4}{k_1}, \frac{s - 5/4}{k_2}\right\} < 0\,.$$

These relations clearly reveal the set inclusions $\overline{A} \not\subseteq B - C$ and $A \subseteq B - C$.

Example 3.10. Consider again Example 3.4 with $C = \mathbb{R}^2_+$. We can see in Figure 5 that for all $b \in B$, $z^{C,k}(a - b) > 0$, and thus $\inf_{b\in B} z^{C,k}(a - b) > 0$. Due to Theorem 3.5, this is equivalent to $\{a\} \not\subseteq B - C$, as it was assumed here.

Example 3.11. Consider Example 3.10 with $C = \mathbb{R}^2_+$, but now we assume that $A = \{a\} \subseteq B-C$. We can see in Figure 6 that for all $b \in B$, $z^{C,k}(a-b) \leq 0$, and thus $\inf_{b\in B} z^{C,k}(a - b) \leq 0$. Note that in this example the attainment property $(\inf_{b\in B} z^{C,k_0}(a - b) = \min_{b\in B} z^{C,k_0}(a - b))$ is fulfilled due to the compactness of B, since this implies that $B - C$ is closed. So the converse reasoning that $\sup_{a\in A} \inf_{b\in B} z^{C,k_0}(a - b) \leq 0$ implies $A \subseteq B - C$ is possible as well.

Remark 3.12. *Because* $A \subseteq B - C \iff A \preceq^u_C B$*, Theorem 3.5 gives a necessary condition and a characterization for the relation* $\preceq^u_C$*.*

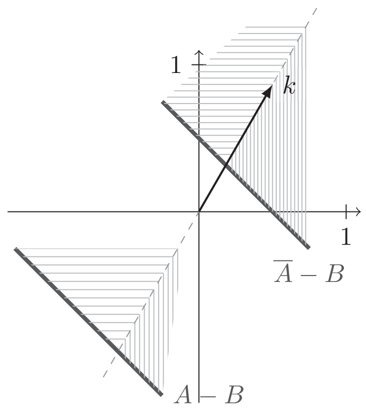

Figure 4: Illustration of the correspondence of the inclusion $A \subseteq B - C$ and $\sup_{a \in A} \inf_{b \in B} z^{C,k}(a - b) \leq 0$ in Example 3.9.

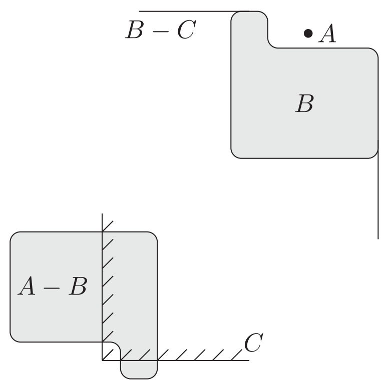

Figure 5: Illustration of the set $A - B$, where $A = \{a\}$. It is obvious that for all $k \in C \setminus \{0\}$, $\inf_{b \in B} \inf\{t \in \mathbb{R} | a - b \in tk - C\} > 0$.

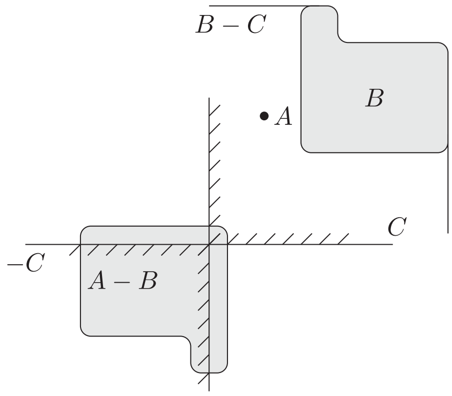

Figure 6: Illustration of the set $A-B$, where $A=\{a\}$. It can be seen that for all $k \in \operatorname{int} C$, $\inf_{b\in B} \inf\{t \in \mathbb{R} | a-b \in tk - C\} \le 0$.

In the following we will show how the characterization of the set relation $A \subseteq B - C$ by means of the functional $z^{C,k}$ can be simplified if the set $B-C$ is convex. We consider a half space $\widetilde{C}_w \subset Y$ defined as

$$\widetilde{C}_w := \{y \in Y \mid w(y) \ge 0\} \tag{8}$$

with $w \in C^+ \setminus \{0\}$, where C^+ is the dual cone of a proper pointed convex cone C in Y. Now we have the following theorem.

Theorem 3.13. *Let Y be a locally convex space, let $A, B \in \mathcal{P}(Y)$ be closed sets, and C be a proper pointed closed convex cone in Y. Suppose that for all $w \in C^+ \setminus \{0\}$, there exists some $k_w \in \operatorname{int} \widetilde{C}_w$ (defined in (8)) such that*

$$\sup_{a\in A} z^{\widetilde{C}_w,k_w}(a) \le \sup_{b\in B} z^{\widetilde{C}_w,k_w}(b). \tag{9}$$

Suppose furthermore that $B-C$ is closed and convex. Then we have $A \subseteq B-C$.

Proof. It holds for all $w \in C^+ \setminus \{0\}$ and for some $k_w \in \operatorname{int} \widetilde{C}_w$:

$$\begin{aligned}
\sup_{a\in A} z^{\widetilde{C}_w,k_w}(a) &= \sup_{a\in A} \inf\{t \in \mathbb{R} | a \in tk_w - \widetilde{C}_w\} \\
&= \sup_{a\in A} \inf\{t \in \mathbb{R} \mid w(a) \le w(tk_w)\} \\
&= \sup_{a\in A} \inf\left\{t \in \mathbb{R} \mid \frac{w(a)}{w(k_w)} \le t\right\} \\
&= \sup_{a\in A} \frac{w(a)}{w(k_w)} \\
&= \frac{1}{w(k_w)} \sup_{a\in A} w(a), \text{ because } k_w \in \operatorname{int} \widetilde{C}_w \text{ and hence } w(k_w) > 0.
\end{aligned}$$

Thus, the assertion in (9) is equivalent to

$$\forall\, w \in C^+ \setminus \{0\}\ \exists\, k_w \in \operatorname{int} \widetilde{C}_w : \ \frac{1}{w(k_w)} \sup_{a\in A} w(a) \leq \frac{1}{w(k_w)} \sup_{b\in B} w(b)$$

and this is equivalent to

$$\forall\, w \in C^+ \setminus \{0\} : \ \sup_{a\in A} w(a) \leq \sup_{b\in B} w(b). \tag{10}$$

Now we can follow the same proof as in the article by Jahn [11]. We suppose that $A \not\subseteq B - C$, thus there exists $a \in A$ such that $a \notin B - C$. Now we can separate the convex set $B - C$ and the element a by using a linear functional $w \in Y^* \setminus \{0\}$ such that

$$w(a) > \alpha \geq w(y) \ \ \forall\, y \in B - C. \tag{11}$$

This implies

$$\sup_{a\in A} w(a) > \alpha \geq \sup_{y\in B-C} w(y). \tag{12}$$

To show that $w \in C^+ \setminus \{0\}$, we assume that $w \notin C^+ \setminus \{0\}$. Thus, there exists some $c \in C$ such that $w(c) < 0$. Then the inequality (11) implies that for arbitrary $b \in B$ and $\lambda > 0$,

$$\alpha \geq w(b - \lambda c) = w(b) - \lambda w(c) \stackrel{\lambda\to\infty}{\to} \infty,$$

a contradiction. Thus, we obtain from the inequality (12)

$$\sup_{a\in A} w(a) > \alpha \geq \sup_{y\in B-C} w(y) = \sup_{b\in B} w(b) - \sup_{c\in C} w(c) = \sup_{b\in B} w(b),$$

but this is a contradiction to the assumption (10). □

Furthermore, we study the lower set less order relation (see Kuroiwa [16, 17]).

Definition 3.14 (Lower set less order relation, [16, 17])**.** Let $C \subset Y$ be a proper closed pointed convex cone. The **lower set less order relation** $\preceq^l_C$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq^l_C B :\Longleftrightarrow B \subseteq A + C,$$

which is equivalent to

$$\forall\, b \in B\ \exists\, a \in A : a \leq_C b.$$

Below we show how the set relation $B \subseteq A + C$ corresponds to the functional $z^{C,k}$. The proof is similar to the one given for Theorem 3.2 (replace A, B and C by $-A$, $-B$ and $-C$ respectively and replace suprema by infima) and is therefore skipped.

Theorem 3.15. *Let $C \subseteq Y$ be a proper closed convex cone. For two given sets $A, B \in \mathcal{P}(Y)$ it holds*

$$B \subseteq A + C \implies \forall\, k \in C \setminus \{0\} : \inf_{a \in A} z^{C,k}(a) \leq \inf_{b \in B} z^{C,k}(b)\,.$$

Theorem 3.16 slightly extends Theorem 3.10 in [7].

Theorem 3.16. *For $A, B \in \mathcal{P}(Y)$, the proper closed convex cone C and the nonlinear scalarizing functional $z^{C,k}$ it holds:*

$$B \subseteq A + C \quad \implies \quad \forall\, k \in C \setminus \{0\} : \sup_{b \in B} \inf_{a \in A} z^{C,k}(a - b) \leq 0$$

Furthermore, if there exists one *element $k_0 \in C \setminus \{0\}$ such that $\inf_{a \in A} z^{C,k_0}(a - b)$ is attained for all $b \in B$, one can deduce the implication*

$$\sup_{b \in B} \inf_{a \in A} z^{C,k_0}(a - b) \leq 0 \quad \implies \quad B \subseteq A + C\,.$$

The proof is omitted because it is similar to the one of Theorem 3.5.

Remark 3.17. *Notice that a sufficient condition for the attainment property $\inf_{a \in A} z^{C,k_0}(a - b) = \min_{a \in A} z^{C,k_0}(a - b)$ in Theorem 3.16 is the closedness of $A + C$.*

Remark 3.18. *Note that the set inclusion $B \subseteq A + C$ in Theorem 3.16 also implies $\sup_{k \in C \setminus \{0\}} \sup_{b \in B} \inf_{a \in A} z^{C,k}(a - b) \leq 0$. Furthermore, if there exists an element $k_0 \in C \setminus \{0\}$ such that $\inf_{a \in A} z^{C,k_0}(a - b)$ is attained for all $b \in B$, then the following equivalences result from Theorem 3.16*

$$\begin{aligned} B \subseteq A + C &\iff \sup_{b \in B} \inf_{a \in A} z^{C,k_0}(a - b) \leq 0 \\ &\iff \sup_{k \in C \setminus \{0\}} \sup_{b \in B} \inf_{a \in A} z^{C,k_0}(a - b) \leq 0. \end{aligned}$$

Remark 3.19. *In Theorem 3.16 we can replace $B \subseteq A + C$ by the set relation $A \preceq^l_C B$.*

Furthermore, we obtain the following result for the lower set less order relation. The proof is omitted as it follows the same lines as the proof of Theorem 3.13.

Theorem 3.20. *Let Y be a locally convex space, let $A, B \in \mathcal{P}(Y)$ closed sets, and let C be a proper pointed closed convex cone in Y. Suppose that for all $w \in C^+ \setminus \{0\}$, there exists some $k_w \in \widetilde{C}_w \setminus \{0\}$ (defined in (8)) such that*

$$\inf_{a \in A} z^{\widetilde{C}_w, k_w}(a) \leq \inf_{b \in B} z^{\widetilde{C}_w, k_w}(b).$$

Suppose furthermore that $A + C$ is closed and convex. Then we have $B \subseteq A + C$.

Example 3.21. Consider again the sets in Example 3.3 and $k_1, k_2 > 0$. Clearly it holds

$$A \preceq^l_C B \quad \text{and} \quad \overline{A} \not\preceq^l_C B.$$

Using Theorem 3.16 this leads to

$$\begin{aligned} \sup_{b\in B} \inf_{\widetilde{a}\in \overline{A}} z^{C,k}(\widetilde{a} - b) &= \max_{s\in[0,1]} \max\left\{\frac{3/4 - s}{k_1}, \frac{s - 1/4}{k_2}\right\} > 0 \\ \sup_{b\in B} \inf_{\widetilde{a}\in A} z^{C,k}(\widetilde{a} - b) &= \max_{s\in[0,1]} \max\left\{\frac{-s - 1/4}{k_1}, \frac{s - 5/4}{k_2}\right\} < 0\,, \end{aligned} \tag{13}$$

see Example 3.9 and Figure 4. Note that the inverse reasoning is also possible, such that $A \preceq^l_C B$ and $\overline{A} \not\preceq^l_C B$ can be deduced from (13).

Example 3.22. Here we again set $C = \mathbb{R}^2_+$. Consider the sets $\{a\} = A \subset B \in \mathcal{P}(Y)$ in Figure 7, and we deduce $A \subseteq B - C$ and $A \subseteq B + C$. Then, due to the symmetry of B and A being a singleton, it holds $A - B = B - A$ and we conclude with

$$\forall\, k \in \operatorname{int} C: \ \inf_{b\in B} z^{C,k}(a - b) \leq 0 \text{ and } \inf_{b\in B} z^{C,k}(b - a) \leq 0,$$

such that $A \preceq^u_C B$ and $B \preceq^l_C A$, in correspondence with Theorems 3.5 and 3.16.

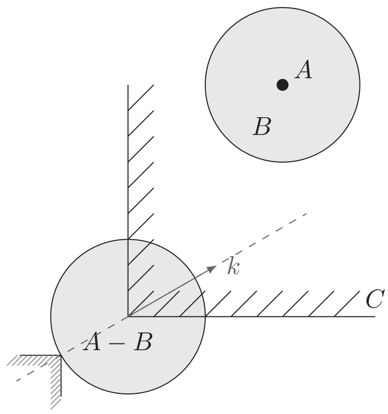

Figure 7: Illustration of the set $A - B = B - A$, where $A = \{a\} \subseteq B$.

Definition 3.23 (Set less order relation, [23, 20])**.** Let $C \subset Y$ be a proper closed pointed convex cone. The **set less order relation** $\preceq^s_C$ is defined for two sets $A, B \in \mathcal{P}(Y)$ by

$$A \preceq^s_C B :\Longleftrightarrow A \subseteq B - C \text{ and } A + C \supseteq B,$$

being equivalent to

- $(\forall\, a \in A\ \exists\, b \in B : a \leq_C b)$ and $(\forall\, b \in B\ \exists\, a \in A : a \leq_C b)$ or
- $A \preceq^u_C B$ and $A \preceq^l_C B$.

Corollary 3.24. *Let $A, B \in \mathcal{P}(Y)$. Assume that for a given proper closed pointed convex cone $C \subset Y$ there exists a $k_0 \in C \setminus \{0\}$ such that for all $a \in A$ the infima $\inf_{b \in B} z^{C,k_0}(a-b)$ are attained and a $k_1 \in C \setminus \{0\}$ such that for all $b \in B$ the infima $\inf_{a \in A} z^{C,k_1}(a-b)$ are attained. Then we have*

$$\left[\sup_{a \in A} \inf_{b \in B} z^{C,k_0}(a-b) \leq 0 \text{ and } \sup_{b \in B} \inf_{a \in A} z^{C,k_1}(a-b) \leq 0\right] \iff A \preceq^s_C B$$

as well as the equivalences

$$\begin{aligned} A \preceq^s_C B \iff & \Big[\forall\, k \in C \setminus \{0\} : \sup_{a \in A} \inf_{b \in B} z^{C,k}(a-b) \leq 0 \\ & \quad \text{and } \forall\, k \in C \setminus \{0\} : \sup_{b \in B} \inf_{a \in A} z^{C,k}(a-b) \leq 0\Big] \\ \iff & \Big[\sup_{k \in C \setminus \{0\}} \sup_{a \in A} \inf_{b \in B} z^{C,k}(a-b) \leq 0 \\ & \quad \text{and } \sup_{k \in C \setminus \{0\}} \sup_{b \in B} \inf_{a \in A} z^{C,k}(a-b) \leq 0\Big] \\ \iff & \left[\max\left(\sup_{k \in C \setminus \{0\}} \sup_{a \in A} \inf_{b \in B} z^{C,k}(a-b), \sup_{k \in C \setminus \{0\}} \sup_{b \in B} \inf_{a \in A} z^{C,k}(a-b)\right) \leq 0\right]. \end{aligned}$$

Proof. The assertions follow from Theorems 3.5 and 3.16. □

Example 3.25 (Weighted Sum Scalarization)**.** Let $Y := \mathbb{R}^m$, a vector $w := (w_1, \ldots, w_m)^T \in \mathbb{R}^m$ with $w_i \geq 0$, $i = 1, \ldots, m$, $C := \{y \in \mathbb{R}^m \mid w^T y \geq 0\}$ (note that C is a convex cone, but C is not pointed) and $k := (k_1, \ldots, k_m)^T \in \operatorname{int} C$ be given. Then we have for $a \in A \in \mathcal{P}(\mathbb{R}^m)$ and $b \in B \in \mathcal{P}(\mathbb{R}^m)$:

$$\begin{aligned} z^{C,k}(a-b) &= \inf\{t \in \mathbb{R} \mid (a-b) \in tk - C\} \\ &= \inf\{t \in \mathbb{R} \mid w^T(a-b) \leq w^T(tk)\} \\ &= \inf\{t \in \mathbb{R} \mid w^T(a-b) \leq t \cdot (w^T k)\} \\ &\overset{k \in \operatorname{int} C}{=} \inf\{t \in \mathbb{R} \mid \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i(a_i - b_i) \leq t\} \\ &= \frac{1}{w^T k} \cdot \sum_{i=1}^m w_i(a_i - b_i)\,. \end{aligned}$$

This leads to

$$\begin{aligned}\sup_{a\in A}\inf_{b\in B} z^{C,k}(a-b) &= \sup_{a\in A}\inf_{b\in B}\frac{1}{w^Tk}\cdot\sum_{i=1}^{m} w_i(a_i-b_i)\\ &= \sup_{a\in A}\frac{1}{w^Tk}\cdot\sum_{i=1}^{m} w_ia_i - \sup_{b\in B}\frac{1}{w^Tk}\cdot\sum_{i=1}^{m} w_ib_i\\ &= \frac{1}{w^Tk}\cdot\left(\sup_{a\in A}\sum_{i=1}^{m} w_ia_i - \sup_{b\in B}\sum_{i=1}^{m} w_ib_i\right).\end{aligned}$$

Hence, with the above definitions of C and k and different weights $w_i > 0$, $i = 1,\dots,m$, we obtain due to Theorems 3.5 and 3.16

$$\begin{aligned} A\subseteq B-C \quad &\iff \quad \forall\, k\in \operatorname{int} C:\ \frac{1}{w^Tk}\sup_{a\in A}\sum_{i=1}^{m} w_ia_i \le \frac{1}{w^Tk}\sup_{b\in B}\sum_{i=1}^{m} w_ib_i\\ &\iff \quad \sup_{a\in A}\sum_{i=1}^{m} w_ia_i \le \sup_{b\in B}\sum_{i=1}^{m} w_ib_i \ \text{ and, in analogy,}\\ B\subseteq A+C \quad &\iff \quad \forall\, k\in \operatorname{int} C:\ \frac{1}{w^Tk}\inf_{a\in A}\sum_{i=1}^{m} w_ia_i \le \frac{1}{w^Tk}\inf_{b\in B}\sum_{i=1}^{m} w_ib_i\\ &\iff \quad \inf_{a\in A}\sum_{i=1}^{m} w_ia_i \le \inf_{b\in B}\sum_{i=1}^{m} w_ib_i.\end{aligned}$$

Note that we only considered $k \in \operatorname{int} C$ for this example in order to exclude division by zero for this rather algorithmic example. Moreover, the attainment of the infima and suprema, respectively, is implicitly required. These two assumptions will also be needed in the next example.

Example 3.26 (Natural Ordering)**.** Let again $Y := \mathbb{R}^m$, $C := \mathbb{R}^m_+$ and $k := (k_1,\dots,k_m)^T \in \operatorname{int} C$. Then we have

$$z^{C,k}(a-b) = \sup_{i=1,\dots,m}\frac{(a-b)_i}{k_i}.$$

Hence, with the above definitions of C and k and different weights $w_i > 0$, $i = 1,\dots,m$, the assertions in Theorems 3.5 and 3.16 lead to

$$\begin{aligned} A\subseteq B-C \quad &\iff \quad \forall\, k\in \operatorname{int}\ \mathbb{R}^m:\ \sup_{a\in A}\inf_{b\in B}\sup_{i=1,\dots,m}\frac{(a-b)_i}{k_i}\le 0,\\ B\subseteq A+C \quad &\iff \quad \forall\, k\in \operatorname{int}\ \mathbb{R}^m:\ \sup_{b\in B}\inf_{a\in A}\sup_{i=1,\dots,m}\frac{(a-b)_i}{k_i}\le 0.\end{aligned}$$

Example 3.27 (Polyhedral Cones)**.** More generally, if $Y = \mathbb{R}^m$ and the cone C is given by $C := \{y \in \mathbb{R}^m \mid (Wy)_i \ge 0 \text{ for all } i = 1,\dots,l\}$ for a given matrix

$W \in \mathbb{R}^{l,m}$, $w_{ij} \geq 0$ for all $i = 1, \dots, l$, $j = 1, \dots, m$, the value of the nonlinear scalarizing functional $z^{C,k}(y)$ with $y = a - b$ can be obtained by

$$z^{C,k}(y) = \sup_{i=1,\dots,l} \frac{(Wy)_i}{(Wk)_i}.$$

Note that $k \in \operatorname{int} C$ implies $(Wk)_i \neq 0$ for all $i = 1, \dots, l$, such that this value is well defined and also that Examples 3.25 and 3.26 are special cases with $l = 1$ and $W = I_m$ (identity matrix), respectively.

3.2 Certainly less order relation

Below we recall the certainly less order relation introduced by Jahn and Ha [13].

Definition 3.28 (Certainly less order relation, [13])**.** Let $C \subset Y$ be a proper closed pointed convex cone. For arbitrary sets $A, B \in \mathcal{P}(Y)$ the **certainly less order relation** $\preceq_C^{cert}$ on $\mathcal{P}(Y)$ is defined by

$$A \preceq_C^{cert} B :\Longleftrightarrow (A = B) \text{ or } (A \neq B \text{ and } \forall\, a \in A,\ \forall\, b \in B : a \leq_C b)\,.$$

We observe the following relationship between the relation $a \leq_C b$ for all $a \in A$ and for all $b \in B$ and the nonlinear scalarizing functional $z^{C,k}$. For the proof we refer again to Theorem 3.5. Note that for this Theorem no attainment property needs to be fulfilled due to the "double supremum" in the premise.

Theorem 3.29. *Let $C \subset Y$ be a proper closed convex cone, and let $A, B \in \mathcal{P}(Y)$. Then*

$$\forall\, a \in A,\ \forall\, b \in B:\ a \in b - C \implies \forall\, k \in C \setminus \{0\} : \sup_{(a,b) \in A \times B} z^{C,k}(a - b) \leq 0.$$

On the other hand: For $k \in C \setminus \{0\}$

$$\sup_{(a,b) \in A \times B} z^{C,k}(a - b) \leq 0 \implies \forall\, a \in A,\ \forall\, b \in B:\ a \in b - C\,.$$

Remark 3.30. *Note that we are also able to deduce the implication*

$$\forall\, a \in A,\ \forall\, b \in B:\ a \in b - C \implies \sup_{k \in C \setminus \{0\}} \sup_{(a,b) \in A \times B} z^{C,k}(a - b) \leq 0.$$

due to Theorem 3.29.

Now the following relationship between the certainly less order relation and the nonlinear scalarizing functional $z^{C,k}$ is to mention.

Corollary 3.31. *Let $A, B \in \mathcal{P}(Y)$ and $C \subset Y$ be a proper closed pointed convex cone. Then*

$$\begin{aligned} A \preceq_C^{cert} B \quad &\Longleftrightarrow \quad A = B \text{ or } \forall\, k \in C \setminus \{0\} : \text{ or } \sup_{(a,b) \in A \times B} z^{C,k}(a - b) \leq 0 \\ &\Longleftrightarrow \quad A = B \text{ or } \sup_{k \in C \setminus \{0\}} \sup_{(a,b) \in A \times B} z^{C,k}(a - b) \leq 0. \end{aligned}$$

Example 3.32. Note that adding the possibility that A equals B is necessary in order to get the classification $A \preceq_C^{cert} B$. If, for example, $A = B \subset \mathbb{R}^2$ is the unit ball in $\mathbb{R}^2$ and the natural ordering is considered by $C = \mathbb{R}^2_+$, we have

$$\forall\, k \in \operatorname{int} C \;\; \exists\, a, b \in A : \; z^{C,k}(a - b) > 0,$$

but clearly $A \preceq_C^{cert} B$, see Figure 8. Also note that

$$\begin{aligned}
\sup_{a,a' \in A} z^{C,k}(a - a') \le 0 \quad &\iff \quad \forall a, a' \in A : \; z^{C,k}(a - a') \le 0 \\
&\iff \quad \forall a, a' \in A : \; a - a' \in -C \\
&\iff \quad A - A \subseteq -C \\
&\iff \quad A - A \subseteq (-C) \cap C = \{0\} \\
&\iff \quad A \text{ is a singleton.}
\end{aligned}$$

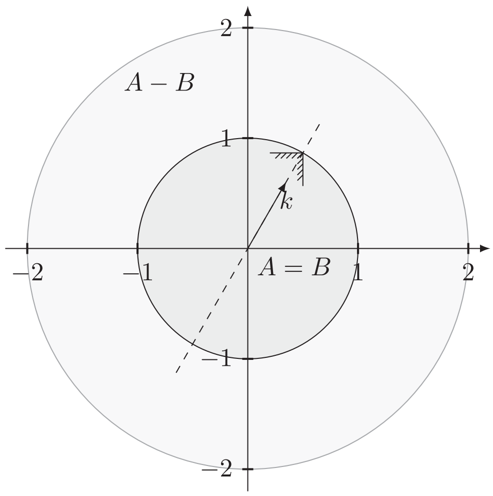

Figure 8: $\sup_{(a,b) \in A \times B} z^{C,k}(a - b) > 0$ for $A = B$ in Example 3.32.

3.3 Possibly less order relation

The possibly less order relation $\preceq_C^p$ is given in the following definition.

Definition 3.33 (Possibly less order relation, [1, 13]). Let $C \subset Y$ be a proper closed pointed convex cone. For arbitrary sets $A, B \in \mathcal{P}(Y)$ the **possibly less order relation** $\preceq_C^p$ on $\mathcal{P}(Y)$ is given by

$$A \preceq_C^p B :\iff (\exists\, a \in A, \; \exists\, b \in B : a \le_C b)\,.$$

Theorem 3.34. *Let $A, B \in \mathcal{P}(Y)$ and let $C \subset Y$ be a proper closed convex cone. Then we have*

$$\exists\, a \in A,\, \exists\, b \in B : \, a \in b - C \Longrightarrow \,\forall\, k \in C \setminus \{0\} : \inf_{(a,b) \in A \times B} z^{C,k}(a-b) \leq 0.$$

If there exists a $k_0 \in C \setminus \{0\}$ such that $\inf\limits_{(a,b) \in A \times B} z^{C,k_0}(a-b)$ is attained, the contrary holds as well:

$$\sup_{k \in C \setminus \{0\}} \inf_{(a,b) \in A \times B} z^{C,k}(a-b) \leq 0 \quad \Longrightarrow \quad \exists\, a \in A,\, \exists\, b \in B : \, a \in b - C\,.$$

We will skip the proof again for its similarity to the proof of Theorem 3.5.

Remark 3.35. *Since $A \preceq^p_C B$ is equivalent to $(\exists\, a \in A,\, \exists\, b \in B : \, a \in b - C)$, Theorem 3.34 provides a characterization of the possibly less order relation $\preceq^p_C$.*

3.4 Minmax less order relation

Throughout this subsection let $C \subset Y$ be a proper pointed closed convex cone. We work with the concepts of extremal elements. Therefore we introduce the sets

$$\begin{aligned} \mathrm{Min}(A, C) &:= \{a \in A \mid A \cap (a - (C \setminus \{0\})) = \emptyset\}, \\ \mathrm{Max}(A, C) &:= \{a \in A \mid A \cap (a + (C \setminus \{0\})) = \emptyset\} \end{aligned}$$

of minimal and maximal elements of a set $A \in \mathcal{P}(Y)$.
Furthermore, we define the set

$$\mathcal{F}_{\min,\max} := \{A \subset \mathcal{P}(Y) \mid \mathrm{Min}(A, C) \neq \emptyset \text{ and } \mathrm{Max}(A, C) \neq \emptyset\}.$$

Note that for instance in a topological real linear space Y for every compact set in $\mathcal{P}(Y)$, minimal and maximal elements exist if we assume C to be proper, closed and convex.
The following set order relation was introduced in Jahn, Ha [13] and has proven to be very useful in real-world applications (see Neukel [19]).

Definition 3.36 (Minmax less order relation, [13])**.** Let $C \subset Y$ be a proper closed pointed convex cone. Let $A, B \in \mathcal{F}_{\min,\max}$. Then the **minmax less order relation** $\preceq^m_C$ on $\mathcal{P}(Y)$ is defined by

$$\begin{aligned} A \preceq^m_C B \quad :\Longleftrightarrow \quad & \mathrm{Min}(A, C) \preceq^s_C \mathrm{Min}(B, C) \\ & \text{and } \mathrm{Max}(A, C) \preceq^s_C \mathrm{Max}(B, C). \end{aligned}$$

In analogy to Theorems 3.5 and 3.16, we obtain the following result by replacing the sets $A \in \mathcal{F}_{\min,\max}$ and $B \in \mathcal{F}_{\min,\max}$ by their respective sets of minimal and maximal elements $\mathrm{Min}(A, C)$, $\mathrm{Max}(A, C)$, $\mathrm{Min}(B, C)$ and $\mathrm{Max}(B, C)$.

Corollary 3.37. *Let $A, B \in \mathcal{F}_{\min,\max}$ and let $C \subset Y$ be a proper closed pointed convex cone. Then we have*

$$\begin{aligned} A \preceq^m_C B \quad \Longrightarrow \quad \forall\, k \in C \setminus \{0\} : \sup_{a \in \mathrm{Min}(A,C)} \inf_{b \in \mathrm{Min}(B,C)} z^{C,k}(a-b) \leq 0 \\ \textit{and} \quad \sup_{b \in \mathrm{Min}(B,C)} \inf_{a \in \mathrm{Min}(A,C)} z^{C,k}(a-b) \leq 0 \\ \textit{and} \quad \sup_{a \in \mathrm{Max}(A,C)} \inf_{b \in \mathrm{Max}(B,C)} z^{C,k}(a-b) \leq 0 \\ \textit{and} \quad \sup_{b \in \mathrm{Max}(B,C)} \inf_{a \in \mathrm{Max}(A,C)} z^{C,k}(a-b) \leq 0\,. \end{aligned}$$

If there exist $k_0, k_1, k_2, k_3 \in C \setminus \{0\}$ such that for all

$$a_0 \in \mathrm{Min}(A,C),\ b_1 \in \mathrm{Min}(B,C),\ a_2 \in \mathrm{Max}(A,C),\ b_3 \in \mathrm{Max}(A,C)$$

the infima

$$\inf_{b \in \mathrm{Min}(B,C)} z^{C,k_0}(a_0 - b), \qquad \inf_{a \in \mathrm{Min}(A,C)} z^{C,k_1}(a - b_1),$$
$$\inf_{b \in \mathrm{Max}(B,C)} z^{C,k_2}(a_2 - b), \qquad \inf_{a \in \mathrm{Max}(A,C)} z^{C,k_3}(a - b_3)$$

are attained, the converse holds as well.

We collect a summary of the characterization of the different order relations in Table 1.

4 Relations to set optimization

In the preceding sections we have shown that the nonlinear scalarizing functional $z^{C,k}$ can be used in order to represent set order relations. Since set order relations are used in set-valued optimization to determine minimal solution sets, it is our goal in this section to use the functional $z^{C,k}$ in order to characterize minimal elements of a family of sets. Note that relations between the different set order relations can also be verified using the nonlinear scalarization functional (2).

4.1 Characterization of minimal elements of set optimization problems by means of the nonlinear scalarizing functional $z^{C,k}$

In this subsection we show how minimal elements of a family of sets can be characterized by using the nonlinear scalarizing functional $z^{C,k}$. We start by giving the definition of minimal elements.

Definition 4.1 (Minimal Elements)**.** Let $\mathcal{A}$ be a family of nonempty subsets of Y and let a preorder $\preceq$ on $\mathcal{P}(Y)$ be given. $\overline{A} \in \mathcal{A}$ is called a **minimal element** of $\mathcal{A}$ w.r.t. $\preceq$ if

$$A \preceq \overline{A},\ A \in \mathcal{A} \quad \Longrightarrow \quad \overline{A} \preceq A\,.$$

Set order relation	Definition	Characterization
$A \preceq_C^u B$	$A \subseteq B - C$	$\sup\limits_{a\in A} \inf\limits_{b\in B} z^{C,k}(a-b) \le 0$ (•)
$A \preceq_C^l B$	$B \subseteq A + C$	$\sup\limits_{b\in B} \inf\limits_{a\in A} z^{C,k}(a-b) \le 0$ (••)
$A \preceq_C^s B$	$A \subseteq B - C$ and $B \subseteq A + C$	(•) and (•)
$A \preceq_C^c B$	$A = B$ or ($A \neq B$ and $\forall\, a \in A,\ \forall\, b \in B : a \le_C b$)	$\sup\limits_{(a,b)\in A\times B} z^{C,k}(a-b) \le 0$
$A \preceq_C^p B$	$\exists\, a \in A,\ \exists\, b \in B :$ $a \le_C b$	$\inf\limits_{(a,b)\in A\times B} z^{C,k}(a-b) \le 0$
$A \preceq_C^m B$	$\mathrm{Min}(A,C) \preceq_C^s \mathrm{Min}(B,C)$ and $\mathrm{Max}(A,C) \preceq_C^s \mathrm{Max}(B,C)$	$\sup\limits_{a\in \mathrm{Min}(A,C)} \inf\limits_{b\in \mathrm{Min}(B,C)} z^{C,k}(a-b) \le 0$ and $\sup\limits_{b\in \mathrm{Min}(B,C)} \inf\limits_{a\in \mathrm{Min}(A,C)} z^{C,k}(a-b) \le 0$ and $\sup\limits_{a\in \mathrm{Max}(A,C)} \inf\limits_{b\in \mathrm{Max}(B,C)} z^{C,k}(a-b) \le 0$ and $\sup\limits_{b\in \mathrm{Max}(B,C)} \inf\limits_{a\in \mathrm{Max}(A,C)} z^{C,k}(a-b) \le 0$

Table 1: Summary of the introduced characterizations for different set order relations.

For the next corollary we assume the following attainment property to be fulfilled.

Attainment Property 4.2. For A, $\overline{A} \in \mathcal{A}$, assume that for all $A \in \mathcal{A}$ there exists an element

(i) $k_0 = k_0(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a}\in\overline{A}} z^{C,k_0}(a - \overline{a})$ are attained for every $a \in A$;

(ii) $k_1 = k_1(A) \in C \setminus \{0\}$ such that the infima $\inf_{a\in A} z^{C,k_1}(\overline{a} - a)$ are attained for all $\overline{a} \in \overline{A}$.

Corollary 4.3. *Let $\mathcal{A}$ be a family of nonempty subsets of Y and $\overline{A} \in \mathcal{A}$. Suppose furthermore that the Attainment Property 4.2 is fulfilled. Then $\overline{A}$ is a minimal element of $\mathcal{A}$ w.r.t. $\preceq_C^u$ if and only if there does not exist any $A \in \mathcal{A}$ such that*

$$\sup_{a\in A} \inf_{\overline{a}\in\overline{A}} z^{C,k_0}(a - \overline{a}) \le 0 \text{ and}$$
$$\sup_{\overline{a}\in\overline{A}} \inf_{a\in A} z^{C,k_1}(\overline{a} - a) > 0.$$

Proof.

$$
\begin{aligned}
& \overline{A} \text{ is a minimal element of } \mathcal{A} \text{ w.r.t. } \preceq_C^u \\
\iff \quad & A \in \mathcal{A} : A \preceq_C^u \overline{A} \implies \overline{A} \preceq_C^u A \\
\iff \quad & \Bigg(A \in \mathcal{A} : \sup_{a \in A} \inf_{\overline{a} \in \overline{A}} z^{C,k_0}(a - \overline{a}) \leq 0 \\
& \quad \implies \sup_{\overline{a} \in \overline{A}} \inf_{a \in A} z^{C,k_1}(\overline{a} - a) \leq 0 \Bigg) \\
\iff \quad & \nexists A \in \mathcal{A} : \Bigg(\sup_{a \in A} \inf_{\overline{a} \in \overline{A}} z^{C,k_0}(a - \overline{a}) \leq 0 \\
& \quad \text{and } \sup_{\overline{a} \in \overline{A}} \inf_{a \in A} z^{C,k_1}(\overline{a} - a) > 0 \Bigg) .
\end{aligned}
$$

□

In the next corollary we collect the according result for the other set relations introduced in Section 3. For this we first need to assume the following attainment properties.

Attainment Property 4.4. For A, $\overline{A} \in \mathcal{A}$, assume that for all $A \in \mathcal{A}$ there exists an element

(i) $k_2 = k_2(A) \in C \setminus \{0\}$ such that the infima $\inf_{a \in A} z^{C,k_2}(a - \overline{a})$ are attained for every $\overline{a} \in \overline{A}$;

(ii) $k_3 = k_3(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a} \in \overline{A}} z^{C,k_3}(\overline{a} - a)$ are attained for all $a \in A$.

Attainment Property 4.5. For A, $\overline{A} \in \mathcal{A}$, assume that for all $A \in \mathcal{A}$ there exists an element

(i) $k_4 = k_4(A) \in C \setminus \{0\}$ such that the infima $\inf_{(a,\overline{a}) \in A \times \overline{A}} z^{C,k_4}(a - \overline{a})$ are attained;

(ii) $k_5 = k_5(A) \in C \setminus \{0\}$ such that the infima $\inf_{(a,\overline{a}) \in A \times \overline{A}} z^{C,k_5}(\overline{a} - a)$ are attained.

Attainment Property 4.6. For A, $\overline{A} \in \mathcal{A}$, assume that for all $A \in \mathcal{A}$ there exists an element

(i) $k_6 = k_6(A) \in C \setminus \{0\}$ such that the infima $\inf_{a \in \mathrm{Min}(A,C)} z^{C,k_6}(\overline{a} - a)$ are attained for every $\overline{a} \in \overline{A}$;

(ii) $k_7 = k_7(A) \in C \setminus \{0\}$ such that the infima $\inf_{a \in \mathrm{Max}(A,C)} z^{C,k_6}(\overline{a} - a)$ are attained for every $\overline{a} \in \overline{A}$;

(iii) $k_8 = k_8(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a} \in \mathrm{Min}(\overline{A},C)} z^{C,k_8}(\overline{a} - a)$ are attained for every $a \in A$;

(iv) $k_9 = k_9(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a}\in \mathrm{Max}(\overline{A},C)} z^{C,k_9}(\overline{a}-a)$ are attained for every $a \in A$;

(v) $k_{10} = k_{10}(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a}\in \mathrm{Min}(\overline{A},C)} z^{C,k_{10}}(a-\overline{a})$ are attained for every $a \in A$;

(vi) $k_{11} = k_{11}(A) \in C \setminus \{0\}$ such that the infima $\inf_{\overline{a}\in \mathrm{Max}(\overline{A},C)} z^{C,k_{11}}(a-\overline{a})$ are attained for every $a \in A$;

(vii) $k_{12} = k_{12}(A) \in C \setminus \{0\}$ such that the infima $\inf_{a\in \mathrm{Min}(A,C)} z^{C,k_{12}}(a-\overline{a})$ are attained for every $\overline{a} \in \overline{A}$;

(viii) $k_{13} = k_{13}(A) \in C \setminus \{0\}$ such that the infima $\inf_{a\in \mathrm{Max}(A,C)} z^{C,k_{13}}(a-\overline{a})$ are attained for every $\overline{a} \in \overline{A}$;

Corollary 4.7. *Let $\mathcal{A}$ be a family of nonempty subsets of Y. $\overline{A} \in \mathcal{A}$ is a minimal element of $\mathcal{A}$ w.r.t.*

(a) $\preceq^l_C$ *if Attainment Property 4.4 is fulfilled and if there does not exist any $A \in \mathcal{A}$ such that*

$$\sup_{\overline{a}\in\overline{A}} \inf_{a\in A} z^{C,k_2}(a-\overline{a}) \leq 0 \text{ and}$$
$$\sup_{a\in A} \inf_{\overline{a}\in\overline{A}} z^{C,k_3}(\overline{a}-a) > 0.$$

(b) $\preceq^s_C$ *if Attainment Properties 4.2 and 4.4 are fulfilled and if there does not exist any $A \in \mathcal{A}$ such that*

$$\sup_{a\in A} \inf_{\overline{a}\in\overline{A}} z^{C,k_0}(a-\overline{a}) \leq 0 \text{ and } \sup_{\overline{a}\in\overline{A}} \inf_{a\in A} z^{C,k_2}(a-\overline{a}) \leq 0 \text{ and}$$
$$\left[\sup_{\overline{a}\in\overline{A}} \inf_{a\in A} z^{C,k_1}(\overline{a}-a) > 0 \text{ or } \sup_{a\in A} \inf_{\overline{a}\in\overline{A}} z^{C,k_3}(\overline{a}-a) > 0\right].$$

(c) $\preceq^{cert}_C$ *if there does not exist any $A \in \mathcal{A}$ such that*

$$\left[A = \overline{A} \text{ or } \sup_{k\in C\setminus\{0\}} \sup_{a\in A} \sup_{\overline{a}\in\overline{A}} z^{C,k}(a-\overline{a}) \leq 0 \text{ or}\right.$$
$$\left.\sup_{k\in C\setminus\{0\}} \sup_{\overline{a}\in\overline{A}} \sup_{a\in A} z^{C,\widehat{k}}(a-\overline{a}) \leq 0\right] \text{ and}$$
$$\left[\overline{A} \neq A \text{ or } \sup_{k\in C\setminus\{0\}} \sup_{\overline{a}\in\overline{A}} \sup_{a\in A} z^{C,\widehat{k}}(\overline{a}-a) > 0 \text{ or}\right.$$
$$\left.\sup_{k\in C\setminus\{0\}} \sup_{a\in A} \sup_{\overline{a}\in\overline{A}} z^{C,k}(a-\overline{a}) > 0\right].$$

(d) $\preceq_C^p$ *if Attainment Property 4.5 is fulfilled and if there does not exist any* $A \in \mathcal{A}$ *such that*

$$\inf_{a\in A} \inf_{\overline{a}\in \overline{A}} z^{C,k_4}(a-\overline{a}) \leq 0 \textit{ and}$$
$$\inf_{\overline{a}\in \overline{A}} \inf_{a\in A} z^{C,k_5}(\overline{a}-a) > 0.$$

(e) $\preceq_C^m$ *if Attainment Property 4.6 is fulfilled and if there does not exist any* $A \in \mathcal{A}$ *such that*

$$\begin{aligned}
&\Big(\sup_{a\in \operatorname{Min}(A,C)} \inf_{\overline{a}\in \operatorname{Min}(\overline{A},C)} z^{C,k_6}(a-\overline{a}) \leq 0 \\
&\textit{and} \sup_{\overline{a}\in \operatorname{Min}(\overline{A},C)} \inf_{a\in \operatorname{Min}(A,C)} z^{C,k_7}(a-\overline{a}) \leq 0 \\
&\textit{and} \sup_{a\in \operatorname{Max}(A,C)} \inf_{\overline{a}\in \operatorname{Max}(\overline{A},C)} z^{C,k_8}(a-\overline{a}) \leq 0 \\
&\textit{and} \sup_{\overline{a}\in \operatorname{Max}(\overline{A},C)} \inf_{a\in \operatorname{Max}(A,C)} z^{C,k_9}(a-\overline{a}) \leq 0 \Big) \textit{ and} \\
&\Big(\sup_{\overline{a}\in \operatorname{Min}(\overline{A},C)} \inf_{a\in \operatorname{Min}(A,C)} z^{C,k_{10}}(\overline{a}-a) > 0 \\
&\textit{or} \sup_{a\in \operatorname{Min}(A,C)} \inf_{\overline{a}\in \operatorname{Min}(\overline{A},C)} z^{C,k_{11}}(\overline{a}-a) > 0 \\
&\textit{or} \sup_{\overline{a}\in \operatorname{Max}(\overline{A},C)} \inf_{a\in \operatorname{Max}(A,C)} z^{C,k_{12}}(\overline{a}-a) > 0 \\
&\textit{or} \sup_{a\in \operatorname{Max}(A,C)} \inf_{\overline{a}\in \operatorname{Max}(\overline{A},C)} z^{C,k_{13}}(\overline{a}-a) > 0 \,. \Big)
\end{aligned}$$

4.2 A descent method

Here we consider a set-valued optimization problem in the following setting: Let a set-valued mapping $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and an order relation $\preceq$ be given. We are looking for minimal solutions w.r.t. the order relation $\preceq$ in the sense of Definition 4.1 of the problem

$$\min_{x\in\mathbb{R}^n} F(x)\,. \tag{14}$$

The Theorems 3.5, 3.16, 3.29, 3.34 and Corollaries 3.24 and 3.37 provide us with a possibility to decide whether two sets fulfill the order relation or not in a numerical manner and even give a quantification by means of the extremal points of the functional values $z^{C,k}(a-b)$, $z^{C,k}(b-a)$, respectively. So a natural way of constructing an algorithm for the problem (14) is an iterative pattern search where in each iteration the minimal function value is determined to specify the locally best search direction. For this reason we refer to Algorithm 4.1 below as a *descent method*, cf. [12].

For the following algorithm it is very important to have an easy way to calculate the functional $z^{C,k}$. With this aim we consider a special structure of the set C in the definition of $z^{C,k}$. In order to study such a special structure we introduce a set A_γ in the following way (see Tammer, Winkler [21]):

Let γ be a norm on $\mathbb{R}^m$ which is characterized by its (closed) unit ball

$$B_\gamma := \{y \in \mathbb{R}^m \mid \ \gamma(y) \leq 1\}.$$

A norm γ is called a **block norm**, if its unit ball B_γ is polyhedral (a polytope). Let $\bar{y} \in \mathbb{R}^m$. The **reflection set** of $\bar{y}$ is defined by

$$R(\bar{y}) := \{y \in \mathbb{R}^m \mid \ |y_i| = |\bar{y}_i| \ \ \forall\, i = 1, ..., m\}.$$

A norm γ is called **absolute**, if $\gamma(y) = \gamma(\bar{y})$ for all $y \in R(\bar{y})$. A block norm γ is called **oblique**, if γ is absolute and satisfies $(y - \mathbb{R}^m_+) \cap \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma = \{y\}$ for all $y \in \mathbb{R}^m_+ \cap \operatorname{bd} B_\gamma$.
Let γ be a block norm with unit ball B_γ,

$$B_\gamma = \{y \in \mathbb{R}^m \mid \ \langle a^i, y\rangle \leq \alpha_i,\ a^i \in \mathbb{R}^m,\ \alpha_i \in \mathbb{R},\ i = 1, ..., n\}.$$

Using a^i from this formula for B_γ, we define a set $A_\gamma \subset \mathbb{R}^m$ by

$$A_\gamma := \{y \in \mathbb{R}^m \mid \ \langle a^i, y\rangle \leq \alpha_i,\ i \in I\} \tag{15}$$

with the index set

$$I := \{i \in \{1, ..., n\} \mid \ \{y \in \mathbb{R}^m :\ \langle a^i, y\rangle = \alpha_i\} \cap B_\gamma \cap \operatorname{int} \mathbb{R}^m_+ \neq \emptyset\}. \tag{16}$$

The set I is exactly the set of indices $i = 1, ..., n$ for which the hyperplanes $\langle a^i, y\rangle = \alpha_i$ are active in the positive orthant.
Let γ be an absolute block norm with unit ball B_γ and the corresponding set A_γ defined as in (15), let vectors $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be given. We define a functional $z^{A_\gamma+w,k} : \mathbb{R}^m \to \mathbb{R}$ by

$$z^{A_\gamma+w,k}(y) = \inf\{\tau \in \mathbb{R} \mid \ y \in \tau k + A_\gamma + w\}, \qquad y \in \mathbb{R}^m. \tag{17}$$

The functional $z^{A_\gamma+w,k}$ depends on the norm γ and the parameters k and w.
Let γ be an oblique block norm with unit ball B_γ and corresponding set A_γ; let $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be arbitrary. Then the functional $z^{A_\gamma+w,k}$ defined by forumla (17) is strictly $\mathbb{R}^m_+$-monotone.
For given $y \in \mathbb{R}^m$, we can calculate the value $z^{A_\gamma+w,k}(y)$ by the following formula (see Tammer, Winkler [21]):
Let γ be an absolute (oblique) block norm with unit ball B_γ and the corresponding set A_γ defined as in (15), let vectors $k \in \operatorname{int} \mathbb{R}^m_+$ and $w \in \mathbb{R}^m$ be given. We consider the functional $z^{A_\gamma+w,k} : \mathbb{R}^m \to \mathbb{R}$ defined by (17). Then $z^{A_\gamma+w,k}$ is a finite-valued, continuous, convex, $\mathbb{R}^m_+$-monotone (strictly $\mathbb{R}^m_+$-monotone) functional with

$$z^{A_\gamma+w,k}(y) = \max_{i \in I} \frac{\langle a^i, y\rangle - \langle a^i, w\rangle - \alpha_i}{\langle a^i, k\rangle}. \tag{18}$$

With the formula (18) it is very easy to compute the objective function values $z^{C,k}(a-b)$ in the following algorithm.

Algorithm 4.1. (A descent method)

Input: $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, ordering cone C with induced set order relation $\preceq_C$, starting point $x^0 \in \mathbb{R}^n$, vector $k^0 \in C \setminus \{0\}$ fulfilling the required attainment property, maximal number i_{max} of iterations, number of search directions n_s, maximal number j_{max} of iterations for the determination of the step size, initial step size h_0 and minimum step size $h_{\min}$

% initialization
$i := 0$, $h := h_0$
choose n_s points $\widetilde{x}^1, \widetilde{x}^2, \ldots, \widetilde{x}^{n_s}$ on the unit sphere around $0_{\mathbb{R}^n}$

% iteration loop
while $i \le i_{max}$ **do**
  check $F(x^i + h\widetilde{x}^j) \preceq F(x^i)$ for every $j \in \{1, \ldots, n_s\}$ by evaluating the extremal term (e. g. $\sup_{a\in A} \inf_{b\in B} z^{C,k_0}(a-b)$ for $A = F(x^i + h\widetilde{x}^j)$ and $B = F(x^i)$, when $\preceq = \preceq_C^u$). Choose the index n_0 with the smallest function value extremal$_{term}$.
  if extremal$_{term} \le 0$ **then**
    $x^{i+1} := x^i + h\widetilde{x}^{n_0}$ % new iteration point
    $j := 1$
    while $F(x^i + (j+1)h\widetilde{x}^{n_0}) \preceq F(x^i + jh\widetilde{x}^{n_0})$ and $j \le j_{max}$ **do**
      $j := j + 1$
      $x^{i+1} := x^{i+1} + h\widetilde{x}^{n_0}$ % new iteration point
    end while
  else
    $h := h/2$
    if $h \le h_{\min}$ **then**
      STOP $x := x^i$
    end if
  end if
  $i := i + 1$
end while

We emphasize that for Algorithm 4.1, we do not need any convexity assumptions on the considered sets. So in the following numerical example we turn our attention to a set-valued map with nonconvex images.

Example 4.8. Let $\Delta_t := 2\pi/40$ and $\mathcal{T} := \{j \cdot \Delta_t,\, j = 0, \ldots, 40\}$. We define the set valued mapping $F\colon \mathbb{R}^2 \rightrightarrows \mathbb{R}^2$ by

$$F(x) := \left\{ \begin{pmatrix} x_1^2 + x_2^2 \cdot \sin(2t) \\ x_2^2 + x_1^2 \cdot \cos(3t) \end{pmatrix} \;\middle|\; t \in \mathcal{T} \right\}$$

where $x = (x_1, x_2)^T$. The example is chosen such that the unique minimizer is attained at $x = 0_{\mathbb{R}^2}$.
We apply Algorithm 4.1 to the problem (14) with starting point $x^0 := (5, 6)^T$ using the natural ordering cone $C := \mathbb{R}^2_+$ and the upper set less order relation $\preceq^u_C$. Initial and minimal step lengths $h_0 := 2.1$ and $h_{\min} := 10^{-3}$ have been used.
For this discrete example the attainment property is trivially fulfilled such that any $k \in C \setminus \{0\}$ can be used in order to get the equivalences in (7). For the numerical example presented here $k^0 := (1, 1)^T$ and $n_s := 4$ search directions were chosen (compass-search).
Numerical results are depicted in Figure 9. On the diagrams to the left the iterates $x^i \in \mathbb{R}^2$ are shown with their corresponding image sets in the right diagrams. For this setup the algorithm performed 25 main iterations and the objective function F is evaluated 148 times which is the appropriate measure of computational effort for realistic problems.
For the chosen minimal step length $h_{\min}$ the algorithm terminates at $x^{25} \approx 10^{-4} \cdot (1.953, 4.395)^T$ which is clearly within a ball of radius $h_{\min}$ around the actual minimum.

5 Conclusion and outlook

In this paper we present a unifying concept for characterizing set order relations via the nonlinear scalarizing functional $z^{C,k}$. The characterization of set order relations via scalarization is an important tool for deriving numerical methods for solving set-valued optimization problems. Moreover, our results are useful for deriving optimality conditions for set-valed optimization problems.
Since set optimization is an important application of uncertain vector optimization, specifically robust vector optimization, our results can be directly applied to real-world problems. It is important to mention that the nonlinear scalarizing functional employed in this paper has already been used to describe scalar uncertain optimization problems (see [15]). Our results suggest that the functional $z^{C,k}$ can be exploited to characterize solutions of uncertain multi-objective problems as well. In addition, the scalarizing functional used in this paper corresponds to a coherent risk measure in mathematical finance, and therefore it would be interesting to discuss whether our approaches can be applied to problems in finance as well. Summing up, our results add to a wide range of applications of the nonlinear scalarizing functional $z^{C,k}$. Numerical tests presented within this manuscript show that an algorithmic realization of the findings in this paper may be carried out in a straightforward manner.

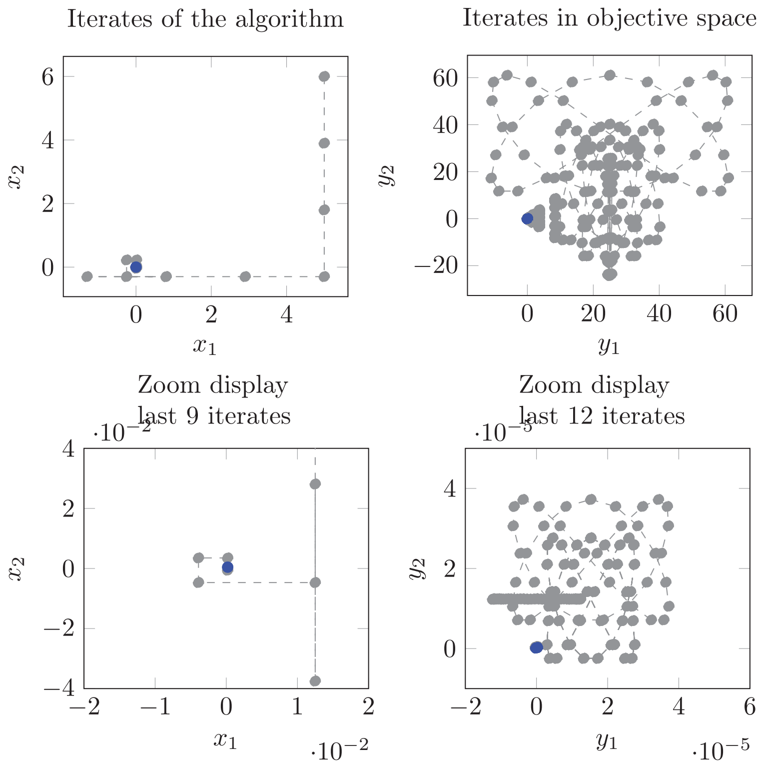

Figure 9: Numerical results for Example 4.8

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