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A class of implicit peer methods for stiff systems

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Abstract

We present a class of s-stage implicit two step peer methods for the solution of stiff differential equations using in addition the function values from the previous step. This allows to increase the order to $p = s$ and to ensure zero-stability simply. Corresponding s-stage methods for $s \leq 6$ of order $p = s$ with optimal zero stability are presented and their stability is discussed. Numerical tests and comparison with `ode15s` show the high potential of this class of implicit peer methods. Under special conditions, we prove that an optimally zero-stable subclass of these methods is superconvergent of order $p = s+1$ for variable step sizes.

Keywords. Implicit peer methods, Stiff ODE systems, Zero stability, Superconvergence.

1 Introduction

Implicit peer methods for the solution of stiff initial value problems

$$y'(t) = f(t, y(t)), \quad t_0 \leq t \leq t_e, \quad y(t_0) = y_0 \tag{1}$$

where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ were introduced in [9] in the form

$$Y_{m,i} = \sum_{j=1}^{s} b_{ij} Y_{m-1,j} + h_m \sum_{j=1}^{i} g_{ij} f(t_{m,j}, Y_{m,j}). \tag{2}$$

These methods were studied in a series of papers and and adapted to special properties like parallelism or application with Krylov techniques for high dimensional problems e.g. [2, 3, 4, 8, 10, 11].

Important properties of (2) are the lack of order reduction for very stiff problems and $M(\infty) = 0$, $M(z)$ the stability matrix, implying $L(\alpha)$-stability for $A(\alpha)$-stable methods.

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On the other hand the order is restricted to $p = s - 1$ and the construction of zero-stable methods is not trivial. By a special strategy in [3] optimally zero-stable methods of order $p = s - 1$ were constructed.
In this paper, we consider s-stage implicit peer methods for stiff differential equations using in addition the function values from the previous step. This implies the loss of $M(\infty) = 0$ but allows to increase the order to $p = s$ and makes the construction of optimally zero-stable methods rather simple. We will therefore consider the following class of peer methods:

$$Y_{m,i} = \sum_{j=1}^{s} b_{ij} Y_{m-1,j} + h_m \sum_{j=1}^{s} a_{ij} f(t_{m-1,j}, Y_{m-1,j}) + h_m \sum_{j=1}^{i} g_{ij} f(t_{m,j}, Y_{m,j}). \tag{3}$$

Here $t_{m,i} = t_m + c_i h_m$ and the stage solutions $Y_{m,i}$ are approximations to the exact solution $y(t_m + c_i h_m)$, $i = 1, 2, \ldots, s$. We always assume that the nodes c_i are pairwise distinct. The coefficients a_{ij}, b_{ij} and g_{ij} with $g_{ii} = \gamma > 0$ can be collected in matrices $A = (a_{ij}), B = (b_{ij}), G = (g_{ij}) \in \mathbb{R}^{s \times s}$. Then a compact representation of the method (for simplicity for scalar equations) is as follows:

$$Y_m = B_m Y_{m-1} + h_m A_m F(t_{m-1}, Y_{m-1}) + h_m G_m F(t_m, Y_m), \tag{4}$$

where

$$Y_m = \begin{pmatrix} Y_{m1} \\ Y_{m1} \\ \vdots \\ Y_{ms} \end{pmatrix} \in \mathbb{R}^{sn} \qquad F(t_m, Y_m) = \begin{pmatrix} f(t_{m,1}, Y_{m,1}) \\ f(t_{m,2}, Y_{m,2}) \\ \vdots \\ f(t_{m,s}, Y_{m,s}) \end{pmatrix} \in \mathbb{R}^{sn}$$

and G_m is a lower triangular matrix. If G_m is a diagonal matrix, then (4) is an implicit parallel peer method [9].
Note that the coefficients will depend on the step size ratio $\sigma_m = h_m / h_{m-1}$ in general. Furthermore for the computation of Y_1 we need additional starting values Y_0.
In Section 2, we derive order conditions and consider stability properties. In the third section we prove that a special subclass of methods is superconvergent of order $p = s + 1$ and some special methods for $s = 3, 4, 5, 6$ will be presented in the fourth section. In Section 5, we will compare our methods with peer methods of the form (2) and with the MATLAB code `ode15s` [12] on test problems from literature.

2 Order conditions and stability

Order conditions can be derived by substituting the exact solution into the method and making a Taylor expansion of the residual Δ_{mi}. We obtain

$$\Delta_{mi} = \big(1 - \sum_{j=1}^{s} b_{ij}\big) y(t_m) + \sum_{k=1}^{p} \left(c_i^k - \sum_{j=1}^{s} b_{ij} \frac{(c_i - 1)^k}{\sigma_m^k} - k \sum_{j=1}^{s} a_{ij} \frac{(c_i - 1)^{k-1}}{\sigma_m^{k-1}} \right.$$
$$\left. - k \sum_{j=1}^{i} g_{ij} c_j^{k-1} \right) \frac{h_m^k}{k!} y^{(k)}(t_m) + \mathcal{O}(h_m^{p+1}). \tag{5}$$

Definition 2.1. The implicit peer method (3) is of order of consistency p if $\Delta_{mi} = \mathcal{O}(h_m^{p+1})$ for $i = 1, \dots, s$.

Note that the stage order of the methods is equal to the order of consistency.
From (5) follows

Theorem 2.2. *If the coefficients of the method* (3) *satisfy the conditions* $AB(l) = 0$, *where*

$$AB_i(l) = c_i^l - \sum_{j=1}^{s} b_{ij} \frac{(c_j - 1)^l}{\sigma_m^l} - l \sum_{j=1}^{s} a_{ij} \frac{(c_j - 1)^{l-1}}{\sigma_m^{l-1}} - l \sum_{j=1}^{i} g_{ij} c_j^{l-1}, \tag{6}$$

for all $l = 0, \dots, p$ *and* $i = 1, \dots, s$, *then method* (3) *is consistent of order* p.

In the following we will consider constant coefficient matrices B and G. With the notations

$$V_0 = \begin{pmatrix} 1 & c_1 & \cdots & c_1^{s-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & c_s & \cdots & c_s^{s-1} \end{pmatrix}, \quad V_1 = \begin{pmatrix} 1 & \cdots & (c_1 - 1)^{s-1} \\ \vdots & \ddots & \vdots \\ 1 & \cdots & (c_s - 1)^{s-1} \end{pmatrix}$$

and $D = \text{diag}(1, \dots, s)$, $S_m = \text{diag}(1, \sigma_m, \dots, \sigma_m^{s-1})$, $C = \text{diag}(c_i)$ we can characterize methods of order $p = s$. From (5) follows immediately

Corollary 2.3. *The method* (3) *is consistent of order* $p = s$ *if* $B\mathbf{1} = \mathbf{1}$ *and*

$$A_m = (CV_0 - GV_0D)D^{-1}S_mV_1^{-1} - \frac{1}{\sigma_m}B(C - I)V_1D^{-1}V_1^{-1}. \tag{7}$$

This means that for arbitrary coefficient matrices G and B satisfying the preconsistency condition $B\mathbf{1} = \mathbf{1}$ the computation of A_m by (7) guarantees order of consistency $p = s$.
For stability investigations we apply (3) to the test equation

$$y' = \lambda y, \qquad \Re\lambda \le 0 \tag{8}$$

with constant stepsize h. This gives

$$Y_m = M(z)Y_{m-1}, \quad M(z) = (I - zG)^{-1}(B + zA), \tag{9}$$

where $z = \lambda h$. So in contrast to (2), $M(\infty) = -G^{-1}A$ will be no longer zero. In the construction of methods we will try to have a small spectral radius at infinity.
Furthermore $M(0) = B$. We therefore can choose the constant matrix B to have optimal zero-stability, i.e. B has one eigenvalue equal to 1 and $s - 1$ eigenvalues equal to 0.

3 A special class of methods

In the following we prove that under certain assumptions, we can construct superconvergent methods, i.e. methods with order of convergence $p = s + 1$.

Theorem 3.1. *Let B and G be constant coefficient matrices satisfying the preconsistency condition $B\mathbf{1} = \mathbf{1}$. Let the method be optimally zero-stable and $e_s^\top B = e_s^\top$. Assume that the nodes c_i are pairwise distinct with $c_s = 1$ and A_m be computed by (7). If the starting values are of order $s + 1$ and*

$$\sum_{i=1}^{s-1} g_{si} c_i^{l-1} = \frac{1 - l\gamma}{l}, \quad l = 2, \ldots, s+1, \tag{10}$$

then the method (3) is superconvergent of order $p = s + 1$.

Proof. By the assumptions order of convergence $p = s$ follows in standard way, e.g. [8]. Furthermore by the assumptions on B holds

$$B^{s-1} = \mathbf{1} e_s^\top. \tag{11}$$

For simplicity of notations consider a scalar problem. For the global error we obtain analogously to the proof of superconvergence for explicit peer methods in [13] a recursion of the form

$$\varepsilon_m = Y(t_m) - Y_m = B\varepsilon_{m-1} + h_{m-1} H_{m-1} \varepsilon_{m-1} + \widetilde{\Delta}_m,$$

where $\widetilde{\Delta}_m = \Delta_m + \mathcal{O}(h_m^{s+2})$ and where the matrices H_{m-1} are uniformly bounded. By repeated substitution follows

$$\varepsilon_m = B^m \varepsilon_0 + \sum_{j=0}^{m-1} h_{m-j-1} B^j H_{m-j-1} \varepsilon_{m-j-1} + \sum_{j=0}^{m-1} B^j \widetilde{\Delta}_{m-j}. \tag{12}$$

The critical term is the last sum. The leading error term in Δ_{m-j} is defined by $AB(s+1)$. We have with $c_s = 1$ and (11) for $j \geq s - 1$

$$B^j AB(s+1) = \mathbf{1} e_s^\top AB(s+1) = \mathbf{1} e_s^\top \Bigg(c^{s+1} - \frac{1}{\sigma_m^{s+1}} B(c - \mathbf{1})^{s+1} \\ - \frac{s+1}{\sigma_m^s} A_m (c - \mathbf{1})^s - (s+1) G c^s \Bigg),$$

where $c^l = (c_1^l, \ldots, c_s^l)^\top$. By the structure of B and $c_s = 1$ the term with B drops and with (7) we have

$$B^j AB(s+1) = \mathbf{1} \Bigg(1 - e_s^\top \frac{s+1}{\sigma_m^s} \big((CV_0 D^{-1} - GV_0) S_m V_1^{-1} (c - \mathbf{1})^s \\ - \frac{1}{\sigma_m} B(C - I) V_1 D^{-1} V_1^{-1} (c - \mathbf{1})^s \big) - e_s^\top (s+1) G c^s \Bigg).$$

We want this expression to be zero for all σ. Because of

$$e_s^\top B(C-I) = 0$$

we get the condition

$$\sigma_m^s - (s+1)\sigma_m^s e_s^\top G c^s - (s+1) e_s^\top (CV_0 D^{-1} - GV_0) S_m V_1^{-1} (c-\mathbf{1})^s = 0.$$

With

$$e_1^\top V_1^{-1} (c-\mathbf{1})^s = e_s^\top (c-\mathbf{1})^s = 0$$

the term without σ_m vanishes. With (10) the coefficients at σ_m^l, $l = 1, \ldots, s$, are zero. We thus have for the last sum in (12)

$$\begin{aligned}\sum_{j=0}^{m-1} B^j \widetilde{\Delta}_{m-j} &= \sum_{j=0}^{s-2} B^j \widetilde{\Delta}_{m-j} + \sum_{j=s-1}^{m-1} B^j \widetilde{\Delta}_{m-j} \\ &= \mathcal{O}(h_{max}^{s+1}) + \sum_{j=s-1}^{m-1} \mathcal{O}(h_{m-j}^{s+2}) = \mathcal{O}(h_{max}^{s+1}).\end{aligned}$$

This completes the proof. □

Remark 3.2. If we require superconvergence for constant step sizes only, the conditions simplify. In this case we need not to assume that the left eigenvector to the eigenvalue one $v^\top = e_s^\top (I - B + \mathbf{1} e_s^\top)^{-1}$ is equal to $e_s^\top$, the condition $e_s^\top B = e_s^\top$ can be dropped. We then have $B^{s-1} = \mathbf{1} v^\top$ and instead of (10) we have only one condition

$$e_s^\top (I - B + \mathbf{1} e_s^\top)^{-1} AB(s+1) = 0, \tag{13}$$

where $AB(s+1)$ is considered for $\sigma_m = 1$. □

4 Construction of special methods

As already mentioned, for $A \neq 0$ the methods cannot have $M(\infty) = 0$. We therefore searched for methods with the following properties:

- Constant coefficient matrices G and B with $g_{ii} = \gamma$ satisfying $B\mathbf{1} = \mathbf{1}$ and which are optimally zero stable.
- Order of consistency $p = s$, i.e A_m is computed by (7).
- Superconvergence. We want order of convergence $p = s+1$ for variable step sizes or at least for constant step sizes. In the latter case the s conditions in (10) reduce to only one additional condition (13).
- A(α)-stability with large α.

- Small spectral radius $\rho(M(\infty))$.

In our numerical search we used the Matlab function *fmincon* with different objective functions. The following methods of order of consistency $p = s$ obtained by this search have been implemented and tested:

- 3a: $s = 3$, $\alpha = 83.9°$, $\rho(M(\infty)) = 2.1\,\mathrm{e}-1$
- 3b: $s = 3$, $\alpha = 87.6°$, $\rho(M(\infty)) = 1.4\,\mathrm{e}-1$
- 4a: $s = 4$, $\alpha = 76°$, $\rho(M(\infty)) = 3.9\,\mathrm{e}-1$
- 4b: $s = 4$, $\alpha = 85.3°$, $\rho(M(\infty)) = 7.2\,\mathrm{e}-3$
- 5: $s = 5$, $\alpha = 87.8°$, $\rho(M(\infty)) = 7.2\,\mathrm{e}-2$
- 6: $s = 6$, $\alpha = 88.3°$, $\rho(M(\infty)) = 4.9\,\mathrm{e}-1$

3a, 3b and 4a are superconvergent for variable step sizes, the others for constant step sizes only. We could not find superconvergent methods for variable step sizes for $s > 4$ with a reasonable large angle α of A(α)-stability.
For illustration we give the coefficients of 3 methods obtained in our search:

Method 3a:

$c_1 = 0.787119720456$	$c_2 = 0.626391213668$	$c_3 = 1$
$b_{11} = 0.516409350778$	$b_{12} = -0.48111516902$	$b_{13} = 0.9647058182431$
$b_{21} = 0.554292682381$	$b_{22} = -0.51640935077$	$b_{23} = 0.9621166683968$
$b_{31} = 0$	$b_{32} = 0$	$b_{33} = 1$
$g_{11} = 0.3187585854346$	$g_{12} = 0$	$g_{13} = 0$
$g_{21} = -0.038960454993$	$g_{22} = 0.318758585434$	$g_{23} = 0$
$g_{31} = -0.782161614481$	$g_{32} = 1.272202145429$	$g_{33} = 0.3187585854346$

Method 4b:

$c_1 = -0.195703077742$	$c_2 = -0.932768294639$	$c_3 = 0.280841751698$
$c_4 = 1$		
$b_{11} = 0$	$b_{12} = 0.055929542592$	$b_{13} = 0.26282166859$
$b_{14} = 0.681248788808$	$b_{21} = 0$	$b_{22} = 0$
$b_{23} = 0.531924458484$	$b_{24} = 0.468075541515$	$b_{31} = 0$
$b_{32} = 0$	$b_{33} = 0$	$b_{34} = 1$
$b_{41} = 0$	$b_{42} = 0$	$b_{43} = 0$
$b_{44} = 1$		
$g_{11} = 0.223787335842$	$g_{12} = 0$	$g_{13} = 0$
$g_{14} = 0$	$g_{21} = -0.926605683501$	$g_{22} = 0.223787335842$
$g_{23} = 0$	$g_{24} = 0$	$g_{31} = 0.375738508128$
$g_{32} = -0.121586967080$	$g_{33} = 0.223787335842$	$g_{34} = 0$
$g_{41} = 0.713026908373$	$g_{42} = -0.268812014817$	$g_{43} = 1.281930686193$
$g_{44} = 0.223787335842$		

Method 5:

$c_1 = -0.858495978259$	$c_2 = -0.485360455592$	$c_3 = 0.151533527021$
$c_4 = 0.411715083482$	$c_5 = 1$	
$b_{11} = -0.346303747960$	$b_{12} = 0.970307183469$	$b_{13} = 0.378298971565$
$b_{14} = 0.009681817299$	$b_{15} = -0.011984224373$	$b_{21} = -0.346303747960$
$b_{22} = 0.970307183469$	$b_{23} = 0.378298971565$	$b_{24} = 0.009681817299$
$b_{25} = -0.011984224373$	$b_{31} = -0.017864899147$	$b_{32} = 0.618888712428$
$b_{33} = 0.378298971565$	$b_{34} = 0.0577521826504$	$b_{35} = -0.037074967497$
$b_{41} = 0.034798774772$	$b_{42} = 0.5633121229892$	$b_{43} = 0.3782989715653$
$b_{44} = 0.009681817299$	$b_{45} = 0.0139083133733$	$b_{51} = -0.010181446862$
$b_{52} = 0.634184882371$	$b_{53} = 0.3782989715653$	$b_{54} = 0.968181729985$
$b_{55} = -0.011984224373$		
$g_{11} = 0.349137125773$	$g_{12} = 0$	$g_{13} = 0$
$g_{14} = 0$	$g_{15} = 0$	$g_{21} = 0.274954541397$
$g_{22} = 0.349137125773$	$g_{23} = 0$	$g_{24} = 0$
$g_{25} = 0$	$g_{31} = 0.164782537766$	$g_{32} = 0.682999175460$
$g_{33} = 0.349137125773$	$g_{34} = 0$	$g_{35} = 0$
$g_{41} = 0.053894296239$	$g_{42} = 0.676545952525$	$g_{43} = 0.208133669772$
$g_{44} = 0.349137125773$	$g_{45} = 0$	$g_{51} = -0.00103475750$
$g_{52} = -0.267347063005$	$g_{53} = 0.469075336314$	$g_{54} = 0.698325786726$
$g_{55} = 0.349137125773$		

5 Numerical tests

The above methods with 3–6 stages have been implemented with step size control and tested on the following standard test problems:

OREGO: Stiff system of three nonlinear ordinary differential euqations [5].

$$\begin{aligned} y_1' &= 77.27\big(y_2 + y_1(1 - 8.375 \cdot 10^{-6} y_1 - y_2)\big) \\ y_2' &= \frac{1}{77.27}\big(y_3 - (1 + y_1)y_2\big) \\ y_3' &= 0.161(y_1 - y_3) \\ y_1(0) &= 1, \quad y_2(0) = 2, \quad y_3(0) = 3, \quad t_e = 360. \end{aligned}$$

ROBER: The Robertson problem, see [5], is a stiff system of three nonlinear ordinary differential equations and describes the kinetics of an autocatalytic reaction.

$$\begin{aligned} y_1' &= -0.04y_1 + 10^4 y_2 y_3 & y_1(0) &= 1 \\ y_2' &= 0.04y_1 - 10^4 y_2 y_3 - 3 \cdot 10^7 y_2^2 & y_2(0) &= 0 \\ y_3' &= 3 \cdot 10^7 y_2^2 & y_3(0) &= 0 \end{aligned}$$

with $t_e = 1.e8$.

HIRES: Stiff system of eight nonlinear ordinary differential equations [5]. The name HIRES refers to high irradiance response and the problem originally is from plant physiology.

$$
\begin{aligned}
y_1' &= -1.71 \cdot y_1 + 0.43 \cdot y_2 + 8.32 \cdot y_3 + 0.0007 \\
y_2' &= 1.71 \cdot y_1 - 8.75 \cdot y_2 \\
y_3' &= -10.03 \cdot y_3 + 0.43 \cdot y_4 + 0.035 \cdot y_5 \\
y_4' &= 8.32 \cdot y_2 + 1.71 \cdot y_3 - 1.12 \cdot y_4 \\
y_5' &= -1.745 \cdot y_5 + 0.43 \cdot y_6 + 0.43 \cdot y_7 \\
y_6' &= -280 \cdot y_6 y_8 + 0.69 \cdot y_4 + 1.71 \cdot y_5 - 0.43 \cdot y_6 + 0.69 \cdot y_7 \\
y_7' &= 280 \cdot y_6 y_8 - 1.81 \cdot y_7 \\
y_8' &= -y_7' \\
y_1(0) &= 1, \quad y_2(0) = y_3(0) = \ldots = y_7(0) = 0, \quad y_8(0) = 0.0057,
\end{aligned}
$$

with $t_e = 321.8122$.

VDPOL: The van der Pol oscillator [5] provides a two dimensional system of ordinary differential equations and originally is from electronics which describes the behaviour of nonlinear vacuum tube circuits.

$$
\begin{aligned}
y_1' &= y_2 \\
y_2' &= \big((1 - y_1^2) y_2 - y_1\big)/\varepsilon \qquad \varepsilon = 10^{-6} \\
y_1(0) &= 2, \quad y_2(0) = 0, \quad t_e = 11.
\end{aligned}
$$

CHEMAKZO: Chemikal Akzo nobel problem is a stiff system of six nonlinear ordinary differential equations [6].

$$
\begin{aligned}
y_1' &= -37.4 \cdot y_1^4 \cdot y_2^{\frac{1}{2}} - 0.09 \cdot y_1 \cdot y_4^2 \\
&\quad + 0.58 \cdot y_3 \cdot y_4 - (0.58 \cdot y_1 \cdot y_5)/34.4 \\
y_2' &= (18.7 \cdot y_1^4 \cdot y_2^{\frac{1}{2}})/(-2) - 0.09 \cdot y_1 \cdot y_4^2 \\
&\quad + 3.3(0.9/737 - y_2) - (0.42 \cdot y_6^2 \cdot y_2^{\frac{1}{2}})/2 \\
y_3' &= 18.7 \cdot y_1^4 \cdot y_2^{\frac{1}{2}} - 0.58 \cdot y_3 \cdot y_4 + (0.58 \cdot y_1 \cdot y_5)/34.4 \\
y_4' &= -0.58 \cdot y_3 \cdot y_4 + (0.58 \cdot y_1 \cdot y_5)/34.4 - 0.18 \cdot y_1 \cdot y_4^2 \\
y_5' &= 0.58 \cdot y_3 \cdot y_4 - (0.58 \cdot y_1 \cdot y_5)/34.4 + 0.42 \cdot y_6^2 \cdot y_2^{\frac{1}{2}} \\
y_6' &= -0.42 \cdot y_6^2 \cdot y_2^{\frac{1}{2}}
\end{aligned}
$$

with $t_e = 180$.

We compare the results of the new methods with the MATLAB code `ode15s` [12] and with two peer methods `peer3` and `peer4` of the form (2) with $M(\infty) = 0$ from [1]. `Peer3` has 3 stages and is of order 2, `Peer4` has $s = 4$ and $p = 3$. Both methods are superconvergent for constant step sizes of order $p = s$.

For the peer methods, we computed numerically in each step the Jacobian. To exclude the influence of poor starting values, these were computed with `ode15s` and $atol = rtol = 5 \cdot 10^{-14}$. For the integration, we used the tolerances $atol = rtol = 10^{-2} - 10^{-8}$. Because `ode15s` sometimes did not meet the prescribed tolerances, we used for better comparison here the tolerances $atol = rtol = 10^{-2} - 10^{-12}$.

The nonlinear systems in the stage equations (3) were solved by a simplified Newton method. The iteration was stopped if

$$\max_{\mathrm{i}=1,\ldots,n} \left| \frac{\Delta \mathrm{y_{k,i}}}{atol + rtol \cdot |\mathrm{y_{m,i}}|} \right| \leq 0.01 \cdot atol$$

where $\Delta \mathrm{y_k}$ is the solution of the linear system in the k-th step of the Newton process or if maximum 10 iteration steps were performed.

The following figures give the error, $\mathrm{ERR} = \max_{\mathrm{i}=1,\ldots,n} \frac{|\mathrm{y_i} - \mathrm{y_{ref,i}}|}{1 + |\mathrm{y}_{\mathrm{ref},i}|}$, at the endpoint versus computing time where y is the numerical solution and $\mathrm{y_{ref}}$ is a reference solution computed with high accuracy.

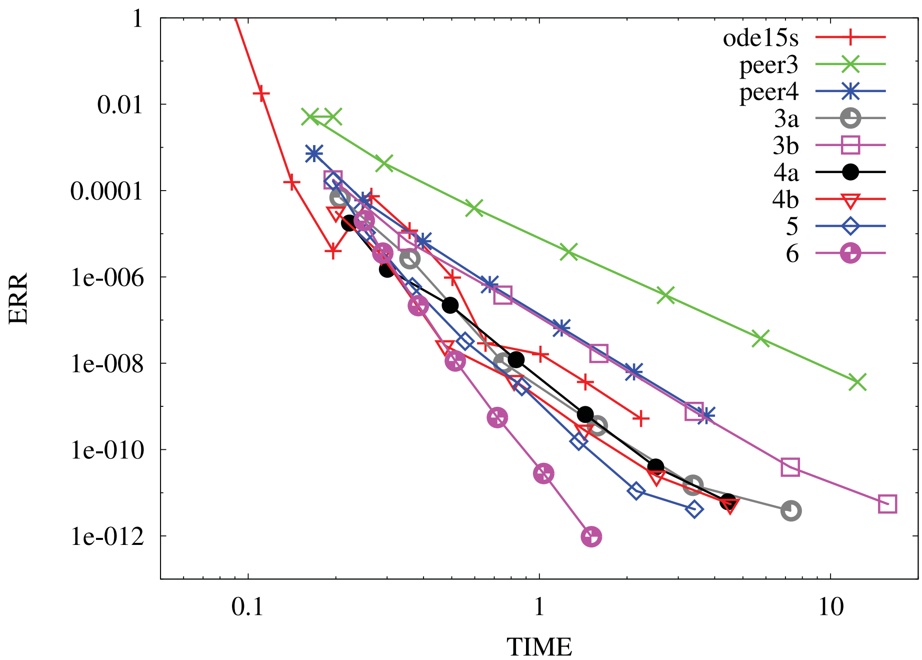

Figure 1: Results for OREGO.

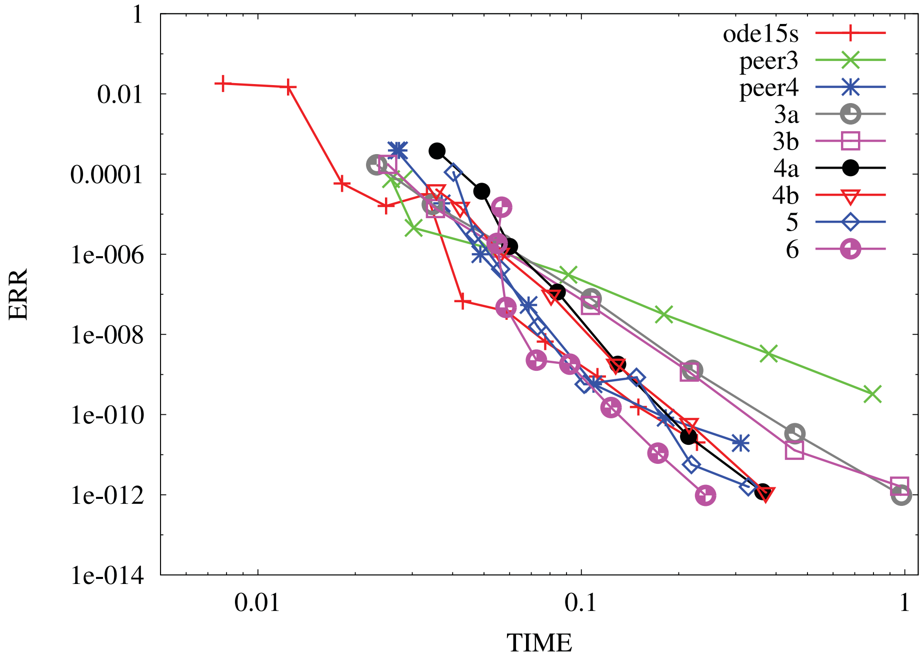

Figure 2: Results for HIRES.

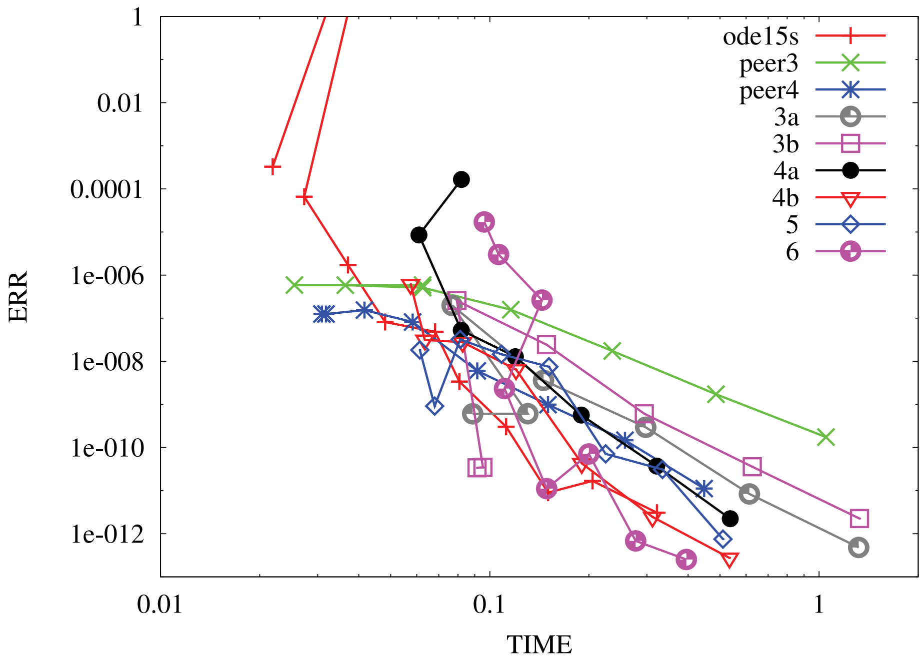

Figure 3: Results for ROBER.

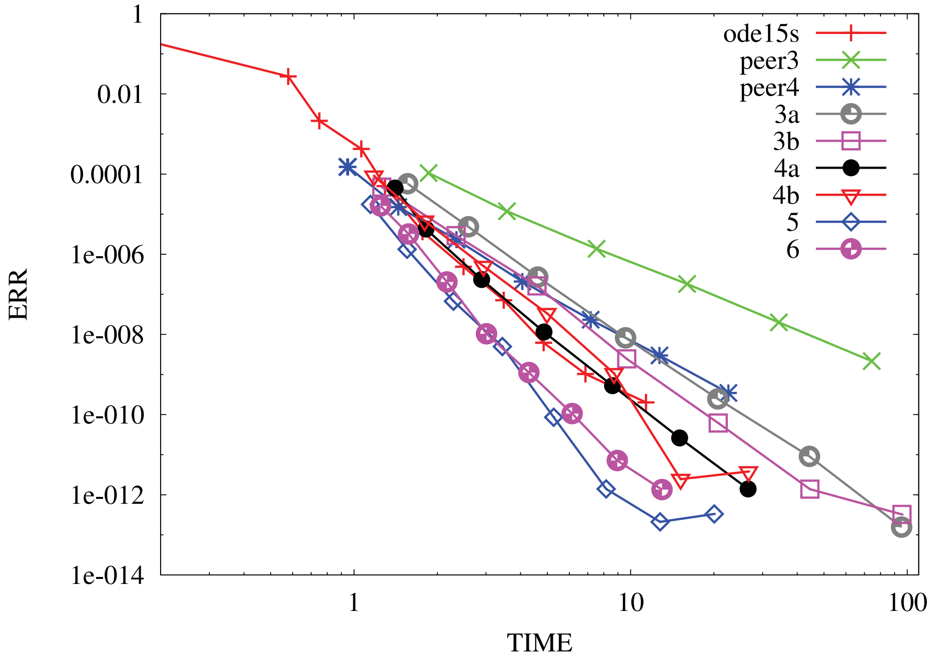

Figure 4: Results for VDPOL.

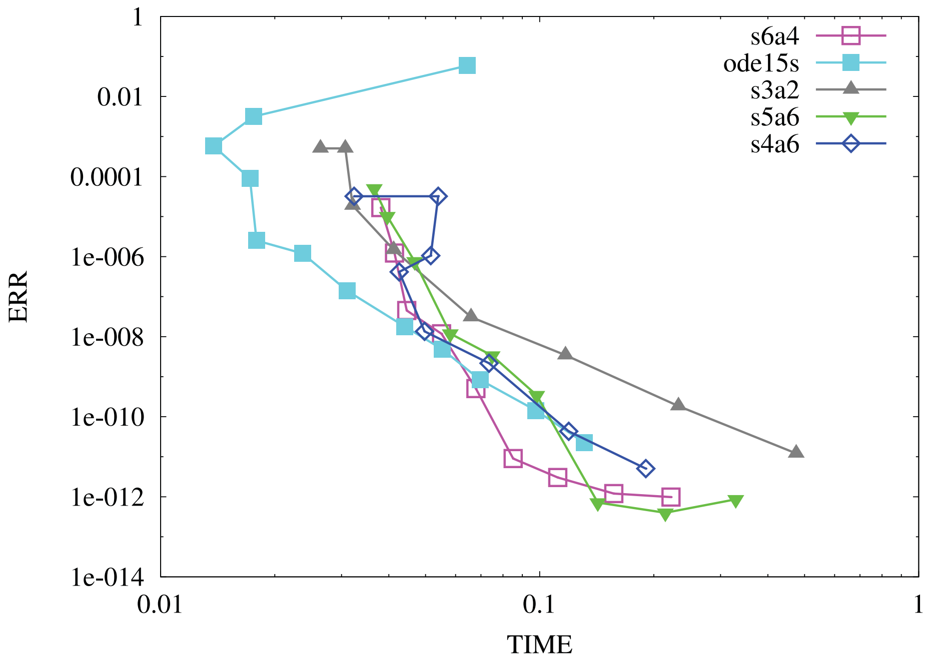

Figure 5: Results for CHEMAKZO.

6 Conclusion

The results show that the new peer methods are reliable and accurate, they outperform the old methods `Peer3` and `Peer4`. The superconvergence is clearly visible. They can compete with `ode15s`, which often misses the required accuracy especially for crude tolerances. A potential advantage of the peer methods over `ode15s` is their relatively large angles of A(α)-stability which makes them attractive for problems with large imaginary part of the eigenvalues of the Jacobian. Compared with one-step methods due to the high stage order there is no order reduction for very stiff problem. For problems with short integration interval the starting procedure takes a non-negligible fraction of time, relaxed tolerances for the computation of the starting values may decrease the effort here. A further possibility to save computing time may be to hold the Jacobian constant for several steps.

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