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Shaghaf Alzorba
Christian Günther
Christiane Tammer
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany
Email: shaghaf.alzorba@mathematik.uni-halle.de
christian.guenther@mathematik.uni-halle.de
christiane.tammer@mathematik.uni-halle.de

Nicolae Popovici
Babes-Bolyai University
Faculty of Mathematics and Computer Sciences
400084 Cluj-Napoca, Romania
Email: popovici@math.ubbcluj.ro

A new algorithm for solving planar multiobjective location problems involving the Manhattan norm

Shaghaf Alzorba[a], Christian Günther[a], Nicolae Popovici[b], Christiane Tammer[a]

[a]Martin Luther University Halle-Wittenberg, Faculty of Natural Sciences II, Institute for Mathematics, 06099 Halle (Saale), Germany; [b]Babeş-Bolyai University, Faculty of Mathematics and Computer Science, 400084 Cluj-Napoca, Romania

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Abstract

This paper deals with multiobjective location problems involving the Manhattan norm. We derive characterizations of the sets of (weakly / properly) efficient points of these location problems, present a characterization of the set of efficient solutions and prove certain topological properties of the set of efficient solutions. In order to develop an effective numerical algorithm for computing the whole set of Pareto efficient elements of the multiobjective location problem, we prove certain reduction results in order to reduce the number of objectives of the multiobjective location problem. Furthermore, we prove the correctness of our proposed algorithm and present computational results at the end of our paper.

1. Introduction

Location problems appear in many variants and with different constraints depending on the application in practice, for instance in urban development, regional planning, engineering, economics or radiotherapy treatment. Such problems and corresponding algorithms are well-studied in the literature, see the books by Love, Morris and Wesolowsky [9], Hamacher [7], Göpfert, Riahi, Tammer, Zălinescu [5] and an overview in the book chapter by Nickel, Puerto and Rodriguez-Chia [11].

In many problems of locational analysis the decision maker is looking for new facilities such that the distances between the new facilities and existing facilities are minimal in a certain sense. In our paper we will focus on planar multiobjective location problems where the new facilities are to be determined in $\mathbb{R}^2$, similarly to Wendell, Hurter, Lowe [16], Chalmet, Francis, Kolen [3], Gerth, Pöhler [4], Pelegrin, Fernandez [13], Ward [15], Lowe, Thisse, Ward and Wendell [10], Nouioua [12], Alzorba, Günther, Popovici [1], Günther [6]. An important question in modeling real-life location problems is how to choose the distances and the composition of the distances corresponding to the application. One possibility for the composition of the distances is the weighted sum (see e.g. Jahn [8]). However, for the decision maker it is often difficult to choose the weights. If he/she has chosen the weights and computes a solution of this scalar location problem it could be possible that the solution is not practicable. So it is better for the decision maker to study the original multiobjective location problem instead of the scalarized problem. In this way he/she gets an overview on the whole solution set, even on special solutions of the scalar problems, such that it is possible to better understand the problem.

The aim of our paper is to study the multiobjective location problem to find a new location $x \in \mathbb{R}^2$ such that the distances $d(x, a^i)$ between p given facilities $a^i \in \mathbb{R}^2$ $(i = 1, ..., p)$ and x are to be minimized in the sense of multiobjective optimization:

$$f(x) := \begin{pmatrix} d(x, a^1) \\ \cdots \\ d(x, a^p) \end{pmatrix} \longrightarrow \operatorname*{v-min}_{x \in \mathbb{R}^2}, \qquad (\mathcal{P})$$

where $x,\ a^i \in \mathbb{R}^2$, $(i = 1, \dots, p)$, $d(\cdot, a^i) : \mathbb{R}^2 \to \mathbb{R}_+$ is a distance function.

The decision maker has the possibility to describe the distance functions in the formulation of the location problem $(\mathcal{P})$ by a norm $|| \cdot || : \mathbb{R}^2 \to \mathbb{R}^1_+$. The distances (norms) between the new facility $x = (x_1, x_2) \in \mathbb{R}^2$ and the given facilities $a^i \in \mathbb{R}^2, i = 1, \dots, p$, can be chosen in different ways. With the choice of the norm the distance is determined, for instance as air-way in the case of the Euclidean norm. In many applications in locational analysis the road system is related to the Manhattan norm or the maximum norm. We will use the Manhattan norm in the formulation of the location problem $(\mathcal{P})$.

For multiobjective location problems it is typically that one has a small number of variables (for instance two variables in the planar case) and a large number of objectives given by the existing facilities $a^i \in \mathbb{R}^2$, $(i = 1, \dots, p)$. So it is very convenient to derive algorithms computing the set of efficient elements in the original space in contrast to the image approach in Benson's algorithm (see [2]).

In our paper we derive characterizations of the sets of (weakly / properly) efficient solutions of these location problems, present a characterization of the set of efficient solutions and prove certain topological properties, especially compactness, of the set of efficient solutions of multiobjective location problems with the Manhattan norm. We show that for these problems the set of efficient elements can be represented by a finite union of possible degenerated rectangles in the plane. Since the number of objectives is very large in multiobjective location problems it is very important to reduce the number of objectives in order to develop effective numerical algorithms. We derive a new algorithm for computing the set of Pareto efficient solutions of multiobjective location problems involving the Manhattan norm. This new algorithm includes the Jahn-Graef-Younes method (see [8, Algorithm 12.20]) for reducing the number of existing facilities in the location problem.

Our paper is organized as follows: In Section 2 we formulate the multiobjective location problem, introduce solution concepts and recall the structure of the solution set of a corresponding scalarized problem. Important properties of the sets of (weakly/properly) efficient solutions of multiobjective location problems involving the Manhattan norm are shown in Section 3. For the effectiveness of an algorithm for solving a multiobjective location problem it is important to reduce the number of objectives using the Jahn-Graef-Younes method. This method is included in the new algorithm (*Rectangular Decomposition Algorithm*) for solving the multiobjective location problem that we present in Section 5. In Section 6 we prove that the algorithm computes the whole set of efficient solutions of the multiobjective location problem. In Section 7 we present computational results obtained by implementing our algorithm in MATLAB. Section 8 contains some concluding remarks.

2. Preliminaries

In this section, we introduce some notions and basic results related to multiobjective optimization and location theory. Throughout this article, we denote the two-dimensional Euclidean space by $\mathbb{R}^2$.

First, we introduce the basic multiobjective location problem that we study in this paper. We consider p points in the plane,

$$a^1 := (a_1^1, a_2^1), \ \cdots \ , a^p := (a_1^p, a_2^p) \ \in \ \mathbb{R}^2,$$

representing some a priori given facilities. Furthermore, in the following we denote by $A := \{a^1, \dots, a^p\}$ the set of all these given facilities, and define $I_p := \{1, \cdots, p\}$ as the set of all indices of the given points from the set

A. Throughout we assume that $\operatorname{card} A = p \geq 2$, i.e., $a^1, \ldots, a^p$ are pairwise distinct.

The classical single-facility multiobjective location problem consists in minimizing the distances between a new facility $x \in \mathbb{R}^2$ and all given facilities a^i $(i \in I_p)$, where simultaneous minimization is understood in the sense of multiobjective optimization. In our paper the distances will be induced by the Manhattan norm, defined by

$$\|x\|_1 := |x_1| + |x_2| \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

We concentrate on multiobjective single-facility location problems where the distances are induced by the Manhattan norm. Therefore, we introduce the basic multiobjective location problem associated to A through

$$f_A(x) := \begin{pmatrix} \|x - a^1\|_1 \\ \cdots \\ \|x - a^p\|_1 \end{pmatrix} \to \operatorname*{v-min}_{x \in \mathbb{R}^2}. \qquad (\mathcal{P}_A)$$

Note that the objective function f_A in $(\mathcal{P}_A)$ is $\mathbb{R}^p_+$-convex, i.e., for all $x^1, x^2 \in \mathbb{R}^2$ and for all $\lambda \in [0, 1]$ it holds

$$f_A(\lambda x^1 + (1 - \lambda)x^2) \in \lambda f_A(x^1) + (1 - \lambda) f_A(x^2) - \mathbb{R}^p_+,$$

where $\mathbb{R}^p_+$ denotes the usual ordering cone in the p dimensional Euclidean space $\mathbb{R}^p$. The minimization in $(\mathcal{P}_A)$ is to understand in the sense of Pareto efficiency introduced in the following definitions:

Definition 1. *Let $F \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, be a nonempty set, and $K \subseteq \mathbb{R}^n$ be a proper pointed closed convex cone. We define the set of minimal elements in F with respect to K through*

$$\mathrm{MIN}(F, K) \;\; := \;\; \{y^0 \in F \mid F \cap (y^0 - K \backslash \{0\}) = \emptyset\}.$$

Moreover, if additionally $\operatorname{int} K \neq \emptyset$, *we define the set of weakly minimal elements in F with respect to K through*

$$\mathrm{WMIN}(F, K) \;\; := \;\; \{y^0 \in F \mid F \cap (y^0 - \operatorname{int} K) = \emptyset\}.$$

In the next definition, we want to introduce the concept of Pareto efficiency for the problem $(\mathcal{P}_A)$ with $K = \mathbb{R}^p_+$.

Definition 2. *The set of Pareto efficient solutions of problem* $(\mathcal{P}_A)$ *with respect to* $\mathbb{R}^p_+$ *is defined by*

$$\mathrm{Eff}(\mathcal{P}_A) \quad := \quad \{x^0 \in \mathbb{R}^2 \mid f_A(x^0) \in \mathrm{MIN}(f_A[\mathbb{R}^2], \mathbb{R}^p_+)\}$$

while that of weakly efficient solutions is given by

$$\mathrm{WEff}(\mathcal{P}_A) \quad := \quad \{x^0 \in \mathbb{R}^2 \mid f_A(x^0) \in \mathrm{WMIN}(f_A[\mathbb{R}^2], \mathbb{R}^p_+)\}.$$

In other words, for any point $x^0 \in \mathbb{R}^2$, *we have:*

$$x^0 \in \mathrm{Eff}(\mathcal{P}_A) \iff \nexists\, x \in \mathbb{R}^2 \ \text{ s.t.} \begin{cases} \forall\, i \in I_p & : \quad \|x - a^i\|_1 \leq \|x^0 - a^i\|_1, \\ \exists\, j \in I_p & : \quad \|x - a^j\|_1 < \|x^0 - a^j\|_1; \end{cases}$$

$$x^0 \in \mathrm{WEff}(\mathcal{P}_A) \iff \nexists\, x \in \mathbb{R}^2 \ \text{ s.t. } \forall\, i \in I_p \ : \ \|x - a^i\|_1 < \|x^0 - a^i\|_1.$$

Furthermore, we introduce the set of proper efficient solutions of the convex problem $(\mathcal{P}_A)$ *through*

$$\mathrm{PEff}(\mathcal{P}_A) \quad := \quad \bigcup_{\lambda \in \operatorname{int} \mathbb{R}^p_+} \operatorname*{argmin}_{x \in \mathbb{R}^2} \langle \lambda, f_A(x) \rangle,$$

where $\langle \cdot, \cdot \rangle$ *stands for the inner product in* $\mathbb{R}^p$.

It is well-known that we have the inclusions

$$\mathrm{PEff}(\mathcal{P}_A) \subseteq \mathrm{Eff}(\mathcal{P}_A) \subseteq \mathrm{WEff}(\mathcal{P}_A). \tag{1}$$

It is possible to use the well-known concept of the weighted sum scalarization for solving the convex problem $(\mathcal{P}_A)$. For this reason, we introduce a scalar problem

$$\langle \lambda, f_A(x) \rangle := \sum_{i=1}^{p} \lambda_i \cdot \|x - a^i\|_1 \to \min_{x \in \mathbb{R}^2}, \tag{$\mathcal{P}_A^\lambda$}$$

where $\lambda := (\lambda_1, \ldots, \lambda_p) \in \mathbb{R}^p_+ \setminus \{0\}$ holds. The solution set of the problem $(\mathcal{P}_A^\lambda)$ is denoted by

$$X^*_\lambda \quad := \quad \operatorname*{argmin}_{x \in \mathbb{R}^2} \langle \lambda, f_A(x) \rangle.$$

Note that we have $x^0 \in X^*_\lambda$ for some $\lambda \in \mathbb{R}^p_+ \setminus \{0\}$ if and only if x^0 is weakly efficient solution for the problem $(\mathcal{P}_A)$. We refer to the book by Jahn [8] for a detailed introduction to multiobjective optimization.

In preparation for our next lemma, we look at the coordinate components of the given points $a^1, \dots, a^p \subseteq \mathbb{R}^2$ and define $A_j := \{a_j^1, \dots, a_j^p\}$ for all $j = 1, 2$. Obviously, there exist uniquely determined numbers $b_1, \dots, b_{p_1} \in A_1$ with $b_1 < \dots < b_{p_1}$ and $c_1, \dots, c_{p_2} \in A_2$ with $c_1 < \dots < c_{p_2}$ for some $p_1, p_2 \in \{\tilde{p} \in \mathbb{N} \mid \tilde{p} \leq p\}$ such that $A_1 = \{b_1, \dots, b_{p_1}\}$ and $A_2 = \{c_1, \dots, c_{p_2}\}$ hold.

The following lemma characterizes the solution set X_λ^* of the problem $(\mathcal{P}_A^\lambda)$ for a weight vector $\lambda \in \operatorname{int} \mathbb{R}_+^p$:

Lemma 3. *For every weight vector $\lambda \in \operatorname{int} \mathbb{R}_+^p$, the solution set X_λ^* of the problem $(\mathcal{P}_A^\lambda)$ is an axis-parallel rectangle in $\mathbb{R}^2$, which is possibly degenerated into a line segment or a singleton. More precisely, X_λ^* has one of the following four forms:*

$$X_\lambda^* = \begin{cases} (b_i, c_j) & \text{for some } (i,j) \in \{1, \dots, p_1\} \times \{1, \dots, p_2\}, \\ \{b_i\} \times [c_j, c_{j+1}] & \text{for some } (i,j) \in \{1, \dots, p_1\} \times \{1, \dots, p_2 - 1\}, \\ [b_i, b_{i+1}] \times \{c_j\} & \text{for some } (i,j) \in \{1, \dots, p_1 - 1\} \times \{1, \dots, p_2\}, \\ [b_i, b_{i+1}] \times [c_j, c_{j+1}] & \text{for some } (i,j) \in \{1, \dots, p_1 - 1\} \times \{1, \dots, p_2 - 1\}. \end{cases}$$

For a proof of Lemma 3 see for instance [7, Section 2.1]. In particular, Lemma 3 is important for the proof of Lemma 8 and Theorem 18.

3. The properties of sets of (weakly/properly) efficient points

In this section, we focus on the properties of the set of weakly efficient points as well as the set of proper efficient points of the problem $(\mathcal{P}_A)$.

Let the maximum norm given by

$$\|x\|_\infty := \max\{|x_1|, |x_2|\} \text{ for all } x = (x_1, x_2) \in \mathbb{R}^2.$$

At first, we want to characterize the set of weakly efficient points of $(\mathcal{P}_A)$. In the next definition we introduce a special hull operator:

Definition 4. *The maximum rectangular hull $\mathcal{N}(D)$ of any nonempty bounded subset D of $\mathbb{R}^2$ is defined as the intersection of all closed balls*

$$B(x, r) := \{y \in \mathbb{R}^2 \mid \|y - x\|_\infty \leq r\}$$

with respect to the maximum norm centered in $x \in \mathbb{R}^2$ with radius $r > 0$ containing D, i.e.,

$$\mathcal{N}(D) := \bigcap \left\{ B(x, r) \;\middle|\; x \in \mathbb{R}^2,\, r > 0,\, D \subseteq B(x, r) \right\}.$$

Lemma 5. *The equation*

$$\mathcal{N}(D) = \bigcup_{x,y \in D} \mathcal{N}(\{x, y\})$$

holds for all nonempty compact sets $D \subseteq \mathbb{R}^2$*.*

A proof of Lemma 5 with Manhattan rectangular hull (Definition 4 with closed balls with respect to the Manhattan norm) can be found in [1, Lemma 4.2]. Analogously, the equation holds for the maximum rectangular hull. Note that Lemma 5 is important for the proof of Lemma 9 and Theorem 11.

Theorem 6. *The set of weakly efficient solutions is given by*

$$\mathrm{WEff}(\mathcal{P}_A) \quad = \quad \mathcal{N}(A).$$

The proof of Theorem 6 is analogously to the proof in [1, Theorem 4.3] for the multiobjective location problem with maximum norm. In particular, the Pareto reducibility (see [14]) of the problem $(\mathcal{P}_A)$ and Lemma 5 are essential for the proof of Theorem 6.

Several characterizations of efficient solutions of multiobjective location problems with Manhattan or maximum norm can be found for instance in the papers by Wendell, Hurter Jr. and Lowe [16], Chalmet, Francis and Kolen [3], Lowe *et al.* [10], or Gerth and Pöhler [4] and references therein. We will use here the characterization proposed in [4], which is based on the dual of the Manhattan norm, namely the maximum norm. Gerth and Pöhler [4] succeeded to characterize the set $\mathrm{Eff}(\mathcal{P}_A)$ by means of the maximum rectangular hull of the existing facilities and certain sets related to the structure of the subdifferential of the Manhattan norm. More precisely, for each $i \in I_p$ one defines four open sets, namely

$$\begin{array}{rcl} s_1(a^i) & := & \{x \in \mathbb{R}^2 \mid a_1^i > x_1 \ , \ a_2^i > x_2\}, \\ s_2(a^i) & := & \{x \in \mathbb{R}^2 \mid a_1^i < x_1 \ , \ a_2^i < x_2\}, \\ s_3(a^i) & := & \{x \in \mathbb{R}^2 \mid a_1^i < x_1 \ , \ a_2^i > x_2\}, \\ s_4(a^i) & := & \{x \in \mathbb{R}^2 \mid a_1^i > x_1 \ , \ a_2^i < x_2\} \end{array} \qquad (2)$$

and then, for every $r \in \{1, 2, 3, 4\}$, one constructs the set

$$S_r \quad := \quad \{x \in \mathcal{N}(A) \mid \exists\, i \in I_p \,:\, x \in s_r(a^i)\} = \mathcal{N}(A) \cap \left(\bigcup_{i \in I_p} s_r(a^i) \right). \qquad (3)$$

The following preliminary result was established in [4].

Lemma 7. *The set of efficient solutions of the location problem* $(\mathcal{P}_A)$ *admits the following representation:*

$$\begin{aligned} \operatorname{Eff}(\mathcal{P}_A) &= \big[(\operatorname{cl} S_1 \cap \operatorname{cl} S_2) \cup \big((\mathcal{N}(A) \setminus S_1) \cap (\mathcal{N}(A) \setminus S_2)\big)\big] \\ &\quad \cap \big[(\operatorname{cl} S_3 \cap \operatorname{cl} S_4) \cup \big((\mathcal{N}(A) \setminus S_3) \cap (\mathcal{N}(A) \setminus S_4)\big)\big] . \end{aligned}$$

Note that Lemma 7 is important for deriving the algorithm in Section 5.

Now we prove some useful properties of the solution set $\operatorname{Eff}(\mathcal{P}_A)$ for the multiobjective location problem $(\mathcal{P}_A)$:

Lemma 8. *Consider the location problem* $(\mathcal{P}_A)$. *Then* $\operatorname{Eff}(\mathcal{P}_A)$ *has the following properties:*

1°. $A \subseteq \operatorname{Eff}(\mathcal{P}_A)$.

2°. $\operatorname{PEff}(\mathcal{P}_A) = \operatorname{Eff}(\mathcal{P}_A)$.

2°. $\operatorname{Eff}(\mathcal{P}_A)$ *can be represented as a finite union of (possibly degenerated) rectangles in the plane.*

3°. $\operatorname{Eff}(\mathcal{P}_A)$ *is closed and bounded, i.e., compact.*

Proof. 1°. Because of $\{a^i\} = \operatorname{argmin}_{x\in\mathbb{R}^2} ||x - a^i||_1$ for all $i \in I_p$, it is easy to deduce by Definition 2 that $A \subseteq \operatorname{Eff}(\mathcal{P}_A)$.

2°. The equality between the set of all proper efficient solutions and the set of all efficient solutions is known for the problem $(\mathcal{P}_A)$ (see [16, Section 2]).

3°. Taking into account Lemma 3, we have

$$\begin{aligned} \operatorname{Eff}(\mathcal{P}_A) &= \bigcup_{\lambda \in \operatorname{int} \mathbb{R}^p_+} \operatorname*{argmin}_{x\in\mathbb{R}^2} \langle \lambda, f_A(x) \rangle \\ &\subseteq \left(\bigcup_{\substack{i=1,\dots,p_1 \\ j=1,\dots,p_2}} \{(b_i, c_j)\} \right) \cup \left(\bigcup_{\substack{i=1,\dots,p_1 \\ j=1,\dots,p_2-1}} \{b_i\} \times [c_j, c_{j+1}] \right) \\ &\quad \cup \left(\bigcup_{\substack{i=1,\dots,p_1-1 \\ j=1,\dots,p_2}} [b_i, b_{i+1}] \times \{c_j\} \right) \cup \left(\bigcup_{\substack{i=1,\dots,p_1-1 \\ j=1,\dots,p_2-1}} [b_i, b_{i+1}] \times [c_j, c_{j+1}] \right), \end{aligned}$$

and therefore it is possible to represent the set $\operatorname{Eff}(\mathcal{P}_A)$ through not more than $p_1 \cdot p_2 + p_1 \cdot (p_2 - 1) + (p_1 - 1) \cdot p_2 + (p_1 - 1) \cdot (p_2 - 1) < \infty$ rectangles. The rectangles are convex, bounded and closed sets.

4°. Because of 3° of this theorem, the set $\mathrm{Eff}(\mathcal{P}_A)$ as a finite union of closed and bounded rectangles is also closed and bounded, i.e., $\mathrm{Eff}(\mathcal{P}_A)$ is compact. □

Note that Lemma 8 is important for the proof that Algorithm 2 computes the set $\mathrm{Eff}(\mathcal{P}_A)$ (see Theorem 18).

4. Reducing the number of objectives

The main goal in this paper is to present a new algorithm which computes and also decomposes the set efficient solutions $\mathrm{Eff}(\mathcal{P}_A)$ of $(\mathcal{P}_A)$ into a finite union of rectangles. For a high complexity of the multiobjective location problem $(\mathcal{P}_A)$, i.e., when the number of existing facilities p is big, it is interesting to analyze if all existing facilities from the set A are required for the determination of the set $\mathrm{Eff}(\mathcal{P}_A)$.

In preparation of our algorithm in Section 5, we want to present reduction results, which can be used for reducing the complexity of the multiobjective location problem $(\mathcal{P}_A)$ in many cases. Note that Theorem 11 is important in order to show that Algorithm 2 generates the whole set of efficient elements of $(\mathcal{P}_A)$ (see Theorem 18).

Lemma 9. *Suppose that $a^i \in A$, $i \in I_p$, satisfies the relation*

$$a^i \in \bigcap_{t\in\{1,2,3,4\}} \mathrm{cl}\, s_t(a^{j_t})$$

with $a^{j_t} \in A \setminus \{a^i\}$ for all $t \in \{1,2,3,4\}$, where $s_t(\cdot)$ is given by (2). Then the following assertions are true:

1°. $\mathrm{WEff}(\mathcal{P}_A) = \mathrm{WEff}(\mathcal{P}_{A\setminus\{a^i\}})$.
2°. $\mathrm{Eff}(\mathcal{P}_A) = \mathrm{Eff}(\mathcal{P}_{A\setminus\{a^i\}})$.

Proof. 1°. We are going to show that $\mathcal{N}(A) = \mathcal{N}(A \setminus \{a^i\})$ holds. By Theorem 6 we conclude

$$\mathrm{WEff}(\mathcal{P}_A) = \mathcal{N}(A) = \mathcal{N}(A \setminus \{a^i\}) = \mathrm{WEff}(\mathcal{P}_{A\setminus\{a^i\}}).$$

Since $A \setminus \{a^i\} \subseteq A$, the inclusion $\mathcal{N}(A) \supseteq \mathcal{N}(A \setminus \{a^i\})$ follows immediately from Lemma 5.

Now we prove the reverse inclusion "$\subseteq$". Let $x \in \mathcal{N}(A)$ chosen arbitrarily. Because of Lemma 5, there exist two points $a^{t_1}, a^{t_2} \in A$ such that $x \in \mathcal{N}(\{a^{t_1}, a^{t_2}\})$ holds. Now let us consider three cases:

Case 1: If $a^{t_1} \neq a^i \neq a^{t_2}$, then Lemma 5 implies the inclusion $x \in \mathcal{N}(\{a^{t_1}, a^{t_2}\}) \subseteq \mathcal{N}(A \setminus \{a^i\})$.
Case 2: Suppose $a^{t_1} = a^i = a^{t_2}$. The condition $a^i \in \operatorname{cl} s_1(a^{j_1}) \cap \operatorname{cl} s_2(a^{j_2})$ and Lemma 5 imply $a^i \in \mathcal{N}(\{a^{j_1}, a^{j_2}\}) \subseteq \mathcal{N}(A \setminus \{a^i\})$. Thus, we have $\mathcal{N}(\{a^{t_1}, a^{t_2}\}) = \{a^i\} \subseteq \mathcal{N}(A \setminus \{a^i\})$.
Case 3: W.l.o.g we suppose $a^{t_1} \neq a^i = a^{t_2}$. Thus, we have $x \in \mathcal{N}(\{a^{t_1}, a^i\})$. Now there exists an index $r \in \{1,2,3,4\}$ such that $a^i \in \operatorname{cl} s_r(a^{t_1})$ holds (w.l.o.g. we suppose $r = 1$). Furthermore, because of the assumption of this theorem, there exists a point $a^{j_2} \in A \setminus \{a^i\}$ such that $a^i \in \operatorname{cl} s_2(a^{j_2})$. We observe that $a^i \in \operatorname{cl} s_1(a^{t_1}) \cap \operatorname{cl} s_2(a^{j_2})$, and therefore the inclusion $\mathcal{N}(\{a^{t_1}, a^i\}) \subseteq \mathcal{N}(\{a^{t_1}, a^{j_2}\})$ holds. By Lemma 5 and because of $a^{t_1}, a^{j_2} \in A \setminus \{a^i\}$ it follows that $\mathcal{N}(\{a^{t_1}, a^{j_2}\}) \subseteq \mathcal{N}(A \setminus \{a^i\})$. Consequently, we have $x \in \mathcal{N}(A \setminus \{a^i\})$.
In all cases we have $x \in \mathcal{N}(A \setminus \{a^i\})$, i.e., $\mathcal{N}(A) \subseteq \mathcal{N}(A \setminus \{a^i\})$.

2°. Define

$$S_r := \bigcup_{k \in I_p} (s_r(a^k) \cap \mathcal{N}(A)) \text{ and } \tilde{S}_r := \bigcup_{k \in I_p \setminus \{i\}} (s_r(a^k) \cap \mathcal{N}(A \setminus \{a^i\}))$$

for all $r = 1, \dots, 4$. In the following we prove that $S_r = \tilde{S}_r$ holds for all $r = 1, \dots, 4$. Since $\mathcal{N}(A) = \mathcal{N}(A \setminus \{a^i\})$, we conclude by Lemma 7 the equality $\operatorname{Eff}(\mathcal{P}_A) = \operatorname{Eff}(\mathcal{P}_{A \setminus \{a^i\}})$.
The inclusion $\tilde{S}_r \subseteq S_r$ is obvious. Now we prove the reverse inclusion $S_r \subseteq \tilde{S}_r$. Let an index $r \in \{1,2,3,4\}$ arbitrarily chosen. Suppose $x \in S_r$. Because of the definition of the set S_r, it exists an index $j \in I_p$ such that $x \in s_r(a^j)$. Only the case $j = i$ is of significance. Because of the assumption $a^i \in \operatorname{cl} s_r(a^{j_r})$ for some $j_r \neq i$ and the construction of the sets $s_r(\cdot)$, we have $x \in s_r(a^i) \subseteq \operatorname{cl} s_r(a^{j_r})$. Obviously, the open set $s_r(a^i)$ is included in the biggest open set $s_r(a^{j_r})$ of the set $\operatorname{cl} s_r(a^{j_r})$. Therefore, we have $x \in s_r(a^i) \subseteq s_r(a^{j_r})$, and it follows $S_r \subseteq \tilde{S}_r$. □

In what follows, we introduce four cones, namely

$$\begin{aligned} K_1 &:= \mathbb{R}_+ \times \mathbb{R}_+, & K_2 &:= \mathbb{R}_- \times \mathbb{R}_-, \\ K_3 &:= \mathbb{R}_- \times \mathbb{R}_+, & K_4 &:= \mathbb{R}_+ \times \mathbb{R}_-, \end{aligned} \tag{4}$$

where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$ and $\mathbb{R}_- := -\mathbb{R}_+$. Note that we have $K_2 = -K_1$

and $K_4 = -K_3$. Therefore, we introduce a permutation function

$$\psi : \{1,2,3,4\} \to \{1,2,3,4\}, \quad t \mapsto \psi(t) := \begin{cases} 2 & \text{for t = 1,} \\ 1 & \text{for t = 2,} \\ 4 & \text{for t = 3,} \\ 3 & \text{for t = 4.} \end{cases} \tag{5}$$

Furthermore, it is easy to see that the following equation

$$\operatorname{cl} s_j(a^i) = a^i - K_j$$

holds for all $j = 1, \dots, 4$ and for all $i \in I_p$.

In the following we use the minimality concept introduced in Definition 1. For this reason, the set $\mathrm{MIN}(A, K_t)$ represents the minimal elements in $A = \{a^1, \dots, a^p\} \subseteq \mathbb{R}^2$ with respect to K_t, $t \in \{1,2,3,4\}$.

Lemma 10. *Let $i \in I_p$. Then the following assertions are equivalent:*

1°.
$$a^i \in \bigcap_{t \in \{1,2,3,4\}} \operatorname{cl} s_t(a^{j_t})$$
with $a^{j_t} \in A \setminus \{a^i\}$ for all $t \in \{1,2,3,4\}$.

2°. *For all $t \in \{1,2,3,4\}$ exists some $a^{j_t} \in A \setminus \{a^i\}$ such that $a^{j_t} \in \operatorname{cl}\, s_t(a^i)$.*

3°.
$$a^i \in A \setminus \left(\bigcup_{t=1}^{4} \mathrm{MIN}(A, K_t) \right).$$

Proof. "(1°) $\Longrightarrow$ (2°)": Suppose that

$$a^i \in \bigcap_{t \in \{1,2,3,4\}} \operatorname{cl} s_t(a^{j_t})$$

with $a^{j_t} \in A \setminus \{a^i\}$ for all $t \in \{1,2,3,4\}$. Thus, we have $a^i \in \operatorname{cl} s_t(a^{j_t}) = a^{j_t} - K_t$ for all $t \in \{1,2,3,4\}$. Consequently, we conclude

$$a^{j_t} \in a^i + K_t = a^i - (-K_t) = a^i - K_{\psi(t)} = \operatorname{cl} s_{\psi(t)}(a^i)$$

for all $t \in \{1,2,3,4\}$.

"(2°) $\Longrightarrow$ (3°)": Note that because of the condition $\operatorname{card} A = p$ the points from the set A are pairwise distinct. Assume that there exists some $t \in \{1, 2, 3, 4\}$ such that $a^i \in \mathrm{MIN}(A, K_t)$ holds, i.e., there is no $\overline{a} \in A \setminus \{a^i\}$ with $\overline{a} \in a^i - K_t = \mathrm{cl}\, s_t(a^i)$. We arrive at a contradiction to the assumption 2°.

"(3°) $\Longrightarrow$ (1°)": Let

$$a^i \in A \setminus \left(\bigcup_{t=1}^{4} \mathrm{MIN}(A, K_t) \right).$$

For all $t = 1, \ldots, 4$ exists some $a^{j_t} \in A \setminus \{a^i\}$ such that $a^{j_t} \in a^i - K_t = \mathrm{cl}\, s_t(a^i)$. Thus, we have

$$a^i \in a^{j_t} - (-K_t) = a^{j_t} - K_{\psi(t)} = \mathrm{cl}\, s_{\psi(t)}(a^{j_t})$$

for all $t \in \{1, 2, 3, 4\}$, and we conclude the assertion in 1°. □

Example 1. Figure 1 illustrates the assertions 1° (left part of the image) and 3° (right part of the image) of Lemma 10 for an example of a location problem with eight given points $a^1, \ldots, a^8 \in \mathbb{R}^2$. By Lemma 9 we conclude for this example the equality $\mathrm{Eff}(\mathcal{P}_A) = \mathrm{Eff}(\mathcal{P}_{A \setminus \{a^8\}})$.

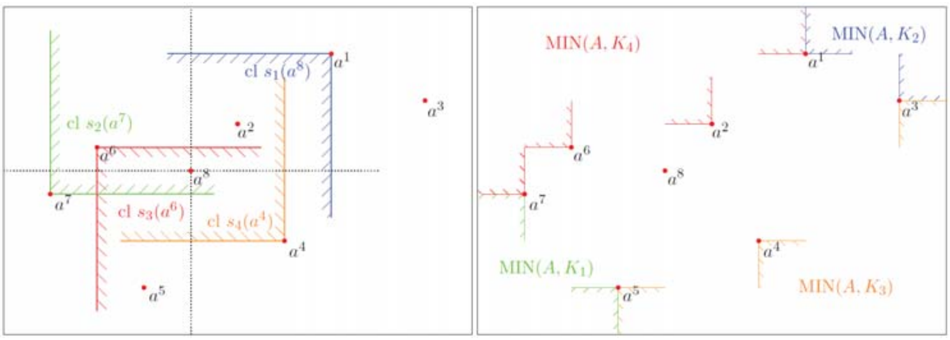

Figure 1: Visualization of assertions 1° and 3° of Lemma 10 for an example.

The following theorem gives a necessary and sufficient condition for the property that the set of efficient solutions of $(\mathcal{P}_A)$ coincides with the set of efficient solutions of the reduced location problem $(\mathcal{P}_{A \setminus \{a^i\}})$.

Theorem 11. *Let $i \in I_p$. Then the following equivalence is true:*

$$a^i \in A \setminus \left(\bigcup_{t=1}^{4} \mathrm{MIN}(A, K_t) \right) \iff \mathrm{Eff}(\mathcal{P}_A) = \mathrm{Eff}(\mathcal{P}_{A \setminus \{a^i\}}).$$

Proof. The implication "$\Longrightarrow$" follows immediately from Lemma 9 and Lemma 10. Now we prove the implication "$\Longleftarrow$". Define

$$S_r := \bigcup_{k \in I_p} (s_r(a^k) \cap \mathcal{N}(A)) \text{ and } \tilde{S}_r := \bigcup_{k \in I_p \setminus \{i\}} (s_r(a^k) \cap \mathcal{N}(A \setminus \{a^i\}))$$

for all $r = 1, \ldots, 4$. Let $\mathrm{Eff}(\mathcal{P}_A) = \mathrm{Eff}(\mathcal{P}_{A \setminus \{a^i\}})$ and assume it exists an index $r \in \{1, 2, 3, 4\}$ such that $a^i \in \mathrm{MIN}(A, K_r)$ holds. W.l.o.g we suppose $r = 1$. Thus, we have $a^i \in \mathrm{MIN}(A, K_1)$, i.e.,

$$\nexists\, \overline{a} \in A \setminus \{a^i\} :\ \overline{a} \in a^i - K_1 = \mathrm{cl}\, s_1(a^i). \tag{6}$$

We define

$$\delta_1 := \inf\{a_1^k \mid a_1^k > a_1^i, k \in I_p\} \text{ and } \delta_2 := \inf\{a_2^k \mid a_2^k > a_2^i, k \in I_p\},$$

where, by convention, $\inf \emptyset := +\infty$. If $\delta_1, \delta_2 \in \mathbb{R}$ hold, then we introduce a special point $\tilde{x} := (\tilde{x}_1, \tilde{x}_2) \in \mathbb{R}^2$ with

$$\tilde{x} = \left(\frac{1}{2}(\delta_1 + a_1^i), \frac{1}{2}(\delta_2 + a_2^i) \right). \tag{7}$$

Note that $\tilde{x} \in s_2(a^i)$ holds. By (6) and (7) it is easy to verify that there is no $\overline{a} \in A \setminus \{a^i\}$ with $\overline{a} \in \tilde{x} - K_1 = \mathrm{cl}\, s_1(\tilde{x})$, and there is no $\overline{a} \in A \setminus \{a^i\}$ with $\tilde{x} \in \overline{a} - K_2 = \mathrm{cl}\, s_2(\overline{a})$, respectively. Consequently, the relation $\tilde{x} \notin \mathrm{cl}\, \tilde{S}_2$ holds.

In the following we consider two cases:

Case 1: We have $s_2(a^i) \cap A \setminus \{a^i\} \neq \emptyset$, i.e., there exists an index $j \in I_p \setminus \{i\}$ such that $a^j \in s_2(a^i)$ holds. Now we consider two sub cases:

(a) Assume that it exist $b^1 \in \mathrm{cl}\, s_3(a^i) \cap A \setminus \{a^i\}$ and $b^2 \in \mathrm{cl}\, s_4(a^i) \cap A \setminus \{a^i\}$. Note that relation (6) imply $b^1, b^2 \notin \mathrm{cl}\, s_1(a^i)$. It is easy to verify that δ_1, $\delta_2 \in \mathbb{R}$ hold. Furthermore, by the definition of the point $\tilde{x}$ it follows that $\tilde{x}_1 < b_1^1$, $\tilde{x}_2 > b_2^1$ and $\tilde{x}_2 < b_2^2$, $\tilde{x}_1 > b_1^2$ as well as $\tilde{x} \in s_1(a^j)$. Thus, we conclude $\tilde{x} \in s_3(b^1) \cap s_4(b^2) \cap s_1(a^j) \cap s_2(a^i)$. Moreover, by Lemma 5 it follows that $\tilde{x} \in \mathcal{N}(\{b^1, b^2\}) \subseteq \mathcal{N}(A)$. Consequently, we have $\tilde{x} \in S_1 \cap S_2 \cap S_3 \cap S_4$, and by Lemma 7 we conclude $\tilde{x} \in \mathrm{Eff}(\mathcal{P}_A)$. We know that we have $\tilde{x} \notin \mathrm{cl}\, \tilde{S}_2$. By Lemma 5 we have $\tilde{x} \in \mathcal{N}(\{b^1, b^2\}) \subseteq \mathcal{N}(A \setminus \{a^i\})$ and since $\tilde{x} \in s_1(a^j)$, $a^j \neq a^i$, it follows that $\tilde{x} \in \tilde{S}_1$. Consequently, by Lemma 7 we conclude $\tilde{x} \notin \mathrm{Eff}(\mathcal{P}_{A \setminus \{a^i\}})$.

(b) Assume that we have $\operatorname{cl} s_3(a^i) \cap A \setminus \{a^i\} = \emptyset$ or $\operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\} = \emptyset$. W.l.o.g. we suppose $\operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\} = \emptyset$. Therefore, by relation (6) we conclude $(\operatorname{cl} s_1(a^i) \cup \operatorname{cl} s_4(a^i)) \cap A \setminus \{a^i\} = \emptyset$. Furthermore, it follows that $A \setminus \{a^i\} \subseteq \{x := (x_1, x_2) \in \mathbb{R}^2 \mid x_1 > a_1^i\}$ holds, and therefore we have $a_1^i < a_1^k$ for all $k \in I_p \setminus \{i\}$. In particular, we have $a_1^i < a_1^j$. Hence, by the definition of the maximum rectangular hull it follows that $a^i \notin \mathcal{N}(A \setminus \{a^i\})$. Consequently, by Lemma 7 we conclude $a^i \notin \operatorname{Eff}(\mathcal{P}_{A\setminus\{a^i\}})$. In contrast, by assertion 1° in Lemma 8 we know that $a^i \in \operatorname{Eff}(\mathcal{P}_A)$ holds.

Case 2: We have $s_2(a^i) \cap A \setminus \{a^i\} = \emptyset$. Now we consider two sub cases:

(a) Assume it exist $b^1 \in \operatorname{cl} s_3(a^i) \cap A \setminus \{a^i\}$ and $b^2 \in \operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\}$. Note that $b^1, b^2 \notin \operatorname{cl} s_1(a^i)$ and $\delta_1,\ \delta_2 \in \mathbb{R}$ hold. Analogously to case 1, (a), we have $\tilde{x} \in \mathcal{N}(\{b^1, b^2\}) \subseteq \mathcal{N}(A)$. Thus, we know that $\tilde{x} \in s_2(a^i) \cap \mathcal{N}(A)$, i.e., $\tilde{x} \in S_2$. Since $s_2(a^i) \cap A \setminus \{a^i\} = \emptyset$, we have $\tilde{x} \notin S_1$. By Lemma 7 we conclude $\tilde{x} \notin \operatorname{Eff}(\mathcal{P}_A)$. We know that $\tilde{x} \notin \operatorname{cl} \tilde{S}_2$ is true, and because of $\tilde{x} \notin S_1$ we have $\tilde{x} \notin \tilde{S}_1$. Furthermore, analogously to case 1, (a), we have $\tilde{x} \in s_3(b^1) \cap s_4(b^2)$ and $\tilde{x} \in \mathcal{N}(\{b^1, b^2\}) \subseteq \mathcal{N}(A \setminus \{a^i\})$. Hence, the relation $\tilde{x} \in \tilde{S}_3 \cap \tilde{S}_4$ is true. Consequently, we conclude $\tilde{x} \in (\mathcal{N}(A \setminus \{a^i\}) \setminus \tilde{S}_1) \cap (\mathcal{N}(A \setminus \{a^i\}) \setminus \tilde{S}_2) \cap \tilde{S}_3 \cap \tilde{S}_4$, i.e., by Lemma 7 we know $\tilde{x} \in \operatorname{Eff}(\mathcal{P}_{A\setminus\{a^i\}})$.

(b) Assume that we have $\operatorname{cl} s_3(a^i) \cap A \setminus \{a^i\} = \emptyset$ or $\operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\} = \emptyset$. Note that $\operatorname{cl} s_3(a^i) \cap A \setminus \{a^i\} = \operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\} = \emptyset$ is not possible with regard to the assumption in case 2, because of the assumption $\operatorname{card} A \geq 2$. W.l.o.g. we suppose $\operatorname{cl} s_4(a^i) \cap A \setminus \{a^i\} = \emptyset$, and therefore it exists a point $b^1 \in \operatorname{cl} s_3(a^i) \cap A \setminus \{a^i\}$. Now the proof is analogously to the proof in case 1, (b). Note that $a_1^i < b_1^1$ and $b^1 \in A \setminus \{a^i\}$.

Finally, we conclude a contradiction to $\operatorname{Eff}(\mathcal{P}_A) = \operatorname{Eff}(\mathcal{P}_{A\setminus\{a^i\}})$. □

Remark 1. If we consider the location problem $(\mathcal{P}_{\tilde{A}})$ associated to the set of existing facilities $\tilde{A} := \bigcup_{t=1}^{4} \operatorname{MIN}(A, K_t)$ a further reduction of $(\mathcal{P}_{\tilde{A}})$ is not possible. Indeed the set of efficient solutions of a further reduction doesn't coincide with the set of efficient solutions of the original problem $(\mathcal{P}_A)$ taking into account the assertion of Theorem 11.

Remark 2. Let the conditions in Theorem 11 be fulfilled. Then the equivalence

$$a^i \in A \setminus \left(\bigcup_{t=1}^{4} \operatorname{MIN}(A, K_t) \right) \iff \operatorname{WEff}(\mathcal{P}_A) = \operatorname{WEff}(\mathcal{P}_{A\setminus\{a^i\}})$$

is not true. By Lemma 9 and Lemma 10 we know that the implication "$\Longrightarrow$" is true. We give a counter example for the implication "$\Longleftarrow$": Let $A := \{a^1, a^2, a^3\}$ be the set of given facilities, where $a^1 := (0,0)$, $a^2 := (0,1)$, $a^3 := (1,1) \in \mathbb{R}^2$. Then it holds $\mathrm{WEff}(\mathcal{P}_A) = [0,1] \times [0,1] = \mathrm{WEff}(\mathcal{P}_{A\setminus\{a^2\}})$, but $a^2 \in \mathrm{MIN}(A, K_4)$.

In order to derive an efficient algorithm for solving $(\mathcal{P}_A)$ it is important to compute minimal elements of the set $A = \{a^1, \dots, a^p\}$ with respect to each cone $K \in \{K_1, K_2, K_3, K_4\}$. To this aim, we can use various methods existing in the literature for solving discrete vector optimization problems. For instance the Jahn-Graef-Younes method (see [8, Algorithm 12.20]) can be adapted for computing $\mathrm{MIN}(A, K)$:

Algorithm 1 (Jahn-Graef-Younes method).

Input: $A = \{a^1, \dots, a^p\}$ and $K \in \{K_1, K_2, K_3, K_4\}$.

1°. Let $i := 1$, $j := 2$, $u^i := a^i$ and $U := \{u^i\}$.
2°. If $a^j \notin U + K$, then $i := i + 1$, $u^i := a^j$ and $U := U \cup \{u^i\}$.
3°. If $j < p$, then $j := j + 1$ and go to step 2°, else let $T := \{u^i\}$.
4°. If $i = 1$, then go to Output, else let $i := i - 1$.
5°. If $u^i \notin T + K$, then $T := T \cup \{u_i\}$.
6°. If $i > 1$, then $i := i - 1$ and go to step 5°, else go to Output.

Output: $\mathrm{MIN}(A, K) := T$.

5. Formulation of the Rectangular Decomposition Algorithm

Now we are able to specify a new algorithm (*Rectangular Decomposition Algorithm*) to compute the set of Pareto efficient elements of the problem $(\mathcal{P}_A)$. Note, in Section 5 we formulate the algorithm and then in Section 6 we prove the correctness of our proposed algorithm.

Algorithm 2 (Rectangular Decomposition Algorithm).

Input: $(\mathcal{P}_A)$ with objective function $f_A : \mathbb{R}^2 \to \mathbb{R}^p$ and existing facilities $A = \{a^1, \dots, a^p\}$, where $a^i = (a^i_1, a^i_2) \in \mathbb{R}^2$ for all $i \in I_p$.

Step 1. Compute for all $r = 1, \dots, 4$ the set

$$T_r := \mathrm{MIN}(A, K_r)$$

using algorithms from the literature of discrete vector optimization (e.g. using Algorithm 1).

Step 2. Define the new set of existing facilities through

$$\tilde{A} := \bigcup_{r=1}^{4} T_r.$$

Let $k := \operatorname{card} \tilde{A}$ and $I_k := \{1, \ldots, k\}$. Note that $k \leq p$ holds. Then there exist $\tilde{a}^i := (\tilde{a}_1^i, \tilde{a}_2^i) \in A$, $i \in I_k$, such that

$$\tilde{A} = \{\tilde{a}^1, \ldots, \tilde{a}^k\} \subseteq \mathbb{R}^2.$$

Step 3. For all $i = 1, 2$:
Sort the components $\tilde{a}_i^1, \ldots, \tilde{a}_i^k$ of the given points from the set $\tilde{A}$ and delete duplicated values. The new values are denoted by $\overline{a}_i^1, \ldots, \overline{a}_i^{q_i}$ with $\overline{a}_i^1 < \ldots < \overline{a}_i^{q_i}$, where $q_i \in \{m \in \mathbb{N} \mid m \leq k\}$.

For simplicity of the notation, define $x_i := \overline{a}_1^i$ for all $i = 1, \ldots, q_1$ and $y_j := \overline{a}_2^j$ for all $j = 1, \ldots, q_2$.

Step 4. Define $e^1 := (x_1, y_1)$, $e^2 := (x_{q_1}, y_{q_2})$, $e^3 := (x_{q_1}, y_1)$, $e^4 := (x_1, y_{q_2})$.
Introduce the sets $\mathcal{C}_i := \emptyset$, $i = 1, \ldots, q_1$. Compute the sets

$$\begin{aligned}
\tilde{T}_1 &:= \{b \in T_2 \,|\, e^1 \in b - \operatorname{int} K_1\}, \\
\tilde{T}_2 &:= \{b \in T_1 \,|\, e^2 \in b - \operatorname{int} K_2\}, \\
\tilde{T}_3 &:= \{b \in T_4 \,|\, e^3 \in b - \operatorname{int} K_3\}, \\
\tilde{T}_4 &:= \{b \in T_3 \,|\, e^4 \in b - \operatorname{int} K_4\}.
\end{aligned}$$

For all $i = 1, \ldots, q_1$:
For all $j = 1, \ldots, q_2$:

(a) Define $bool_1 := 0$ and $bool_2 := 0$.
If there exists $\beta \in \tilde{T}_1$ such that $(x_i, y_j) \in \beta - K_1$ holds, then define $bool_1 := 1$.
If $bool_1 = 1$, then check:
If there exists $\beta \in \tilde{T}_2$ such that $(x_i, y_j) \in \beta - K_2$ holds, then define $bool_2 := 1$.

If $bool_2 = 1$, then go to (c), else go to (b).

(b) If $bool_1 = 0$, then $bool_1 := 1$, else check:
If there exists $\beta \in \tilde{T}_1$ such that $(x_i, y_j) \in \beta - \operatorname{int} K_1$ holds, then

define $bool_1 := 0$.

Define $bool_2 := 0$. If $bool_1 = 1$, then $bool_2 := 1$ and check:
If there exists $\beta \in \tilde{T}_2$ such that $(x_i, y_j) \in \beta - \operatorname{int} K_2$ holds, then define $bool_2 := 0$.

If $bool_2 = 1$, then go to (c), else choose the next point.

(c) Define $bool_1 := 0$ and $bool_2 := 0$.
If there exists $\beta \in \tilde{T}_3$ such that $(x_i, y_j) \in \beta - K_3$ holds, then define $bool_1 := 1$.
If $bool_1 = 1$, then check:
If there exists $\beta \in \tilde{T}_4$ such that $(x_i, y_j) \in \beta - K_4$ holds, then define $bool_2 := 1$.

If $bool_2 = 1$, then go to (e), else go to (d).

(d) If $bool_1 = 0$, then $bool_1 := 1$, else check:
If there exists $\beta \in \tilde{T}_3$ such that $(x_i, y_j) \in \beta - \operatorname{int} K_3$ holds, then define $bool_1 := 0$.

Define $bool_2 := 0$. If $bool_1 = 1$, then $bool_2 := 1$ and check:
If there exists $\beta \in \tilde{T}_4$ such that $(x_i, y_j) \in \beta - \operatorname{int} K_4$ holds, then define $bool_2 := 0$.

If $bool_2 = 1$, then go to (e), else choose the next point.

(e) Define $\mathcal{C}_i := \mathcal{C}_i \cup \{y_j\}$ and choose the next point.

Step 5. Define

$$\mathcal{C}_i^{\min} := \min \mathcal{C}_i \text{ and } \mathcal{C}_i^{\max} := \max \mathcal{C}_i$$

for all $i \in \{1, \ldots, q_1\}$. Now, consider two cases:

Case 1: Let $q_1 = 1$. Then define

$$\mathcal{R}_1^* := \operatorname{conv}\{(x_1, \mathcal{C}_1^{\min}), (x_1, \mathcal{C}_1^{\max})\} \text{ and } \mathcal{R}_2^* := \emptyset.$$

Case 2: Let $q_1 \geq 2$. Define

$$\overline{\mathcal{C}_i} := \max\left\{\mathcal{C}_i^{\min}, \mathcal{C}_{i+1}^{\min}\right\} \text{ and } \underline{\mathcal{C}_i} := \min\left\{\mathcal{C}_i^{\max}, \mathcal{C}_{i+1}^{\max}\right\}$$

for all $i \in \{1, \dots, q_1 - 1\}$. Moreover, define

$$\mathcal{R}_2^* := \bigcup_{i=1}^{q_1-1} \text{conv}\{(x_i, \underline{\mathcal{C}_i}),\ (x_i, \overline{\mathcal{C}_i}),\ (x_{i+1}, \overline{\mathcal{C}_i}),\ (x_{i+1}, \underline{\mathcal{C}_i})\} \ \text{ and } \ \mathcal{R}_1^* := \emptyset.$$

Now check:

If $\mathcal{C}_1^{\min} < \underline{\mathcal{C}_1}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_1, \underline{\mathcal{C}_1}), (x_1, \mathcal{C}_1^{\min})\}$.
If $\mathcal{C}_1^{\max} > \overline{\mathcal{C}_1}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_1, \overline{\mathcal{C}_1}), (x_1, \mathcal{C}_1^{\max})\}$.
If $\mathcal{C}_{q_1}^{\min} < \underline{\mathcal{C}_{q_1-1}}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_{q_1}, \underline{\mathcal{C}_{q_1-1}}), (x_{q_1}, \mathcal{C}_{q_1}^{\min})\}$.
If $\mathcal{C}_{q_1}^{\max} > \overline{\mathcal{C}_{q_1-1}}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_{q_1}, \overline{\mathcal{C}_{q_1-1}}), (x_{q_1}, \mathcal{C}_{q_1}^{\max})\}$.

Suppose that $q_1 \geq 3$ holds. In addition, check for all $i \in \{2, \dots, q_1-1\}$:

If $\mathcal{C}_i^{\min} < \mathcal{C}^* := \min\{\underline{\mathcal{C}_{i-1}}, \underline{\mathcal{C}_i}\}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_i, \mathcal{C}^*), (x_i, \mathcal{C}_i^{\min})\}$.
If $\mathcal{C}_i^{\max} > \mathcal{C}^* := \max\{\overline{\mathcal{C}_{i-1}}, \overline{\mathcal{C}_i}\}$, then $\mathcal{R}_1^* := \tilde{\mathcal{R}}_1^* \cup \text{conv}\{(x_i, \mathcal{C}^*), (x_i, \mathcal{C}_i^{\max})\}$.
If $\underline{\mathcal{C}_{i-1}} > \overline{\mathcal{C}_i}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_i, \overline{\mathcal{C}_i}), (x_i, \underline{\mathcal{C}_{i-1}})\}$.
If $\overline{\mathcal{C}_{i-1}} < \underline{\mathcal{C}_i}$, then $\mathcal{R}_1^* := \mathcal{R}_1^* \cup \text{conv}\{(x_i, \overline{\mathcal{C}_{i-1}}), (x_i, \underline{\mathcal{C}_i})\}$.

Output: The whole set of efficient solutions $\text{Eff}(\mathcal{P}_A) := \mathcal{R}_1^* \cup \mathcal{R}_2^*$ of the location problem $(\mathcal{P}_A)$ as a union of (possibly degenerated) rectangles.

Remark 3. In Step 4 of Algorithm 2 the construction set $\mathcal{C}$ defined by

$$\mathcal{C} := \bigcup_{i=1}^{q_1} \{(x_i, y) \mid y \in \mathcal{C}_i\} \tag{8}$$

is generated. In Lemma 14 we will show that the construction set is determined by all efficient grid points.

Remark 4. In Step 4 of Algorithm 2 we use Lemma 7 for computing the construction set $\mathcal{C}$ (compare Theorem 14). The advantage of the application of Lemma 7 is the very simple geometrical construction of the set $\text{Eff}(\mathcal{P}_{\tilde{A}})$. However, it would be possible to improve the complexity of the algorithm in certain cases by using well-known algorithms proposed by Chalmet, Francis and Kolen [3] or Wendell, Hurter and Lowe [16] for determining the construction set $\mathcal{C}$ in Step 4 of Algorithm 2.

Remark 5. Note that the set $\text{Eff}(\mathcal{P}_A)$ is in general not convex, but with the help of Algorithm 2 it is possible to decompose the set $\text{Eff}(\mathcal{P}_A)$ into a finite union of convex rectangles.

Example 2. Figure 2 illustrates the sets $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ for an example of a multiobjective location problem $(\mathcal{P}_A)$ with regard to eight given points $a^1, \ldots, a^8 \in \mathbb{R}^2$. Note that we have $\text{Eff}(\mathcal{P}_A) = \text{Eff}(\mathcal{P}_{A\setminus\{a^8\}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*$ in this example.

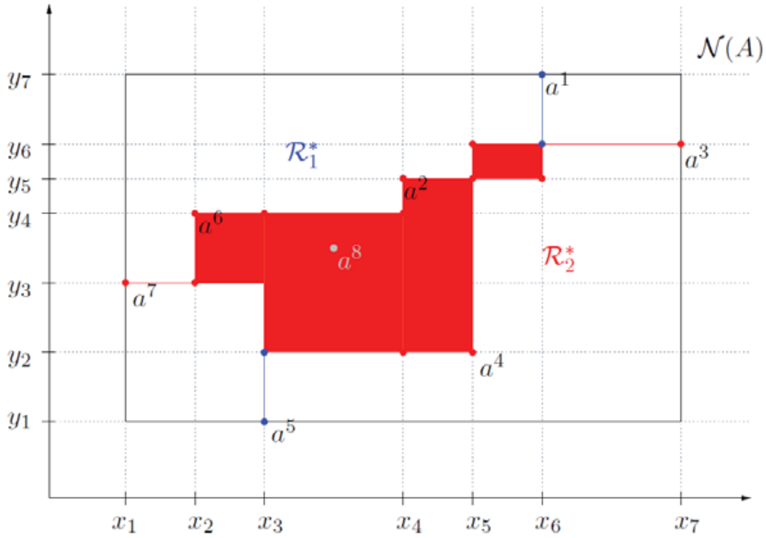

Figure 2: An example of the output of Algorithm 2.

6. Analysis of the Rectangular Decomposition Algorithm

In this section, we prove that Algorithm 2 computes the whole set of efficient solutions of the location problem $(\mathcal{P}_A)$ associated to the set of existing facilities $A = \{a^1, \ldots, a^p\}$. Throughout this section, we consider in addition the reduced multiobjective location problem $(\mathcal{P}_{\tilde{A}})$ associated to the set of existing facilities $\tilde{A} = \{\tilde{a}^1, \ldots, \tilde{a}^k\}$ generated by Algorithm 2, where $\tilde{a}^i \in A$ for all $i \in I_k = \{1, \ldots, k\}$.

Moreover, let $\tilde{A}_1 := \{x_1, \ldots, x_{q_1}\}$ and $\tilde{A}_2 := \{y_1, \ldots, y_{q_2}\}$, where $x_1, \ldots, x_{q_1}$ and $y_1, \ldots, y_{q_2}$ are generated by Algorithm 2.

By Lemma 7 we know that the set of efficient solutions for the reduced location problem $(\mathcal{P}_{\tilde{A}})$ admits the following representation:

$$\begin{aligned}\mathrm{Eff}(\mathcal{P}_{\tilde{A}}) &= \Big[(\mathrm{cl}\,\tilde{S}_1 \cap \mathrm{cl}\,\tilde{S}_2) \cup \big((\mathcal{N}(\tilde{A}) \setminus \tilde{S}_1) \cap (\mathcal{N}(\tilde{A}) \setminus \tilde{S}_2)\big)\Big] \\ &\quad \cap \Big[(\mathrm{cl}\,\tilde{S}_3 \cap \mathrm{cl}\,\tilde{S}_4) \cup \big((\mathcal{N}(\tilde{A}) \setminus \tilde{S}_3) \cap (\mathcal{N}(\tilde{A}) \setminus \tilde{S}_4)\big)\Big],\end{aligned}$$

where

$$\tilde{S}_r := \bigcup_{i \in I_k} (s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}))$$

for all $r = 1, \ldots, 4$. Note that the set $s_r(\cdot)$ is given by (2) for all $r = 1, \ldots, 4$.

In preparation of the description of the construction set $\mathcal{C}$ in Theorem 14, we need two additional lemmata:

Lemma 12. *For all $r = 1, \ldots, 4$ we have*

$$\tilde{S}_r = \bigcup_{i \in \tilde{I}^r} (s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})) \quad \textit{and} \quad \mathrm{cl}\,\tilde{S}_r = \bigcup_{i \in \tilde{I}^r} (\mathrm{cl}\,s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})),$$

where

$\tilde{I}^r_{min} := \{i \in I_k \mid \tilde{a}^i \in \mathrm{MIN}(\tilde{A}, K_{\psi(r)})\}$ *and* $\tilde{I}^r := \{i \in \tilde{I}^r_{min} \mid s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}) \neq \emptyset\}$.

Note that ψ is given by (5).

Proof. Let $r \in \{1, 2, 3, 4\}$ arbitrarily chosen. At first, notice that for all $i \in I_k$ there exists an index $j \in \tilde{I}^r_{min}$ such that $\tilde{a}^j \in \tilde{a}^i - K_{\psi(r)}$ holds. Note that we have $\tilde{a}^j \in \mathrm{MIN}(\tilde{A}, K_{\psi(r)})$. Thus, $\tilde{a}^i \in \tilde{a}^j + K_{\psi(r)} = \tilde{a}^j - K_r = \mathrm{cl}\,s_r(\tilde{a}^j)$ is true. Because of the structure of the sets $s_r(\cdot)$, it follows that $s_r(\tilde{a}^i) \subseteq s_r(\tilde{a}^j)$. Hence, it is easy to see that we have

$$\tilde{S}_r = \bigcup_{i \in I_k} (s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})) = \bigcup_{i \in \tilde{I}^r_{min}} (s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})) = \bigcup_{i \in \tilde{I}^r} (s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})). \quad (9)$$

Now we prove $\mathrm{cl}\,\tilde{S}_r = \bigcup_{i \in \tilde{I}^r} (\mathrm{cl}\,s_r(\tilde{a}^i) \cap \mathcal{N}(A))$. At first, we prove the inclusion " $\subseteq$ ". We know $s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}) \subseteq \mathrm{cl}\,s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})$ for all $i \in \tilde{I}^r$. Furthermore, the set $\bigcup_{i \in \tilde{I}^r} (\mathrm{cl}\,s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}))$ is obviously closed. Thus, the inclusion

$$\mathrm{cl}\,\tilde{S}_r \subseteq \bigcup_{i \in \tilde{I}^r} (\mathrm{cl}\,s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}))$$

is true.

Now let us prove the reverse inclusion " $\supseteq$ ". Let $z^* \in \bigcup_{i \in \tilde{I}^r} (\operatorname{cl} s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}))$ arbitrarily chosen. Then there exists some $j \in \tilde{I}^r$ such that $z^* \in \operatorname{cl} s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$ holds. Now we consider two cases:

Case 1: If $z^* \in s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$ holds, then by (9) the relation $z^* \in \tilde{S}_r \subseteq \operatorname{cl} \tilde{S}_r$ follows directly.

Case 2: Let $z^* \notin s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$. Thus, we have $z^* \in \operatorname{bd} s_r(\tilde{a}^j)$. Because of the assumption $s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A}) \neq \emptyset$, there exists an element $z \in s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$. Notice that $z^* \neq z$ holds. Because of the structure of the set $s_r(\tilde{a}^j)$, we have $\tilde{z}^\alpha := \alpha \cdot z + (1 - \alpha) \cdot z^* \in s_r(\tilde{a}^j)$ for all $\alpha \in (0, 1]$. Since $z^*, z \in \mathcal{N}(\tilde{A})$ and $\mathcal{N}(\tilde{A})$ is a convex set, it follows that $\tilde{z}^\alpha \in \mathcal{N}(\tilde{A})$ for all $\alpha \in (0, 1]$. Hence, for all $\alpha \in (0, 1]$ it holds $\tilde{z}^\alpha \in s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$. Obviously, we have $\tilde{z}^\alpha \to z^*$ for $\alpha \to 0$. Consequently, z^* is a limit point of the set $s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})$, and it follows that $z^* \in \operatorname{cl}(s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A}))$. Finally, note that

$$z^* \in \operatorname{cl}(s_r(\tilde{a}^j) \cap \mathcal{N}(\tilde{A})) \subseteq \bigcup_{i \in I_k} \operatorname{cl}(s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})) = \operatorname{cl} \tilde{S}_r.$$

□

Lemma 13. *We consider the elements e^1, e^2, e^3, e^4 like defined in Step 4 of Algorithm 2. Then for all $r = 1, \ldots, 4$ holds:*

$$s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}) \neq \emptyset \iff e^r \in s_r(\tilde{a}^i) = \tilde{a}^i - \operatorname{int} K_r.$$

Proof. Note that we have $\mathcal{N}(\tilde{A}) = \operatorname{conv}\{e^1, e^2, e^3, e^4\}$. The implication "$\Longleftarrow$" is obvious. We prove the implication "$\Longrightarrow$". Let $s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}) \neq \emptyset$, i.e., there exists an element $z \in s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A})$. Because of the structure of the maximum rectangular hull $\mathcal{N}(\tilde{A})$ and the definition of e^r, we have $z \in e^r + K_r$. Finally, we conclude $e^r \in z - K_r = \operatorname{cl} s_r(z) \subseteq s_r(\tilde{a}^i)$. □

Theorem 14. *The construction set $\mathcal{C}$ introduced in* (8) *has the following representation*

$$\mathcal{C} = \{(x, y) \in \tilde{A}_1 \times \tilde{A}_2 \mid (x, y) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})\}.$$

Proof. Taking into account Lemma 7, an element $(x, y) \in \tilde{A}_1 \times \tilde{A}_2$ is efficient if and only if

$$(x, y) \in (\operatorname{cl} \tilde{S}_1 \cap \operatorname{cl} \tilde{S}_2) \cup \big((\mathcal{N}(\tilde{A}) \setminus \tilde{S}_1) \cap (\mathcal{N}(\tilde{A}) \setminus \tilde{S}_2)\big)$$

and

$$(x, y) \in (\operatorname{cl} \tilde{S}_3 \cap \operatorname{cl} \tilde{S}_4) \cup \big((\mathcal{N}(\tilde{A}) \setminus \tilde{S}_3) \cap (\mathcal{N}(\tilde{A}) \setminus \tilde{S}_4)\big).$$

Since $\tilde{A}_1 \times \tilde{A}_2 \subseteq \mathcal{N}(\tilde{A})$, it is not necessary to check $(x,y) \in \mathcal{N}(\tilde{A})$. So by Lemma 12 we conclude that a point $(x,y) \in \tilde{A}_1 \times \tilde{A}_2$ is efficient if and only if (x,y) satisfies

$$(x,y) \in \bigcup_{i\in\tilde{I}^1} \operatorname{cl} s_1(\tilde{a}^i) \cap \bigcup_{i\in\tilde{I}^2} \operatorname{cl} s_2(\tilde{a}^i) \quad \text{or} \quad (x,y) \notin \bigcup_{i\in\tilde{I}^1} s_1(\tilde{a}^i) \cup \bigcup_{i\in\tilde{I}^2} s_2(\tilde{a}^i) \tag{10}$$

and

$$(x,y) \in \bigcup_{i\in\tilde{I}^3} \operatorname{cl} s_3(\tilde{a}^i) \cap \bigcup_{i\in\tilde{I}^4} \operatorname{cl} s_4(\tilde{a}^i) \quad \text{or} \quad (x,y) \notin \bigcup_{i\in\tilde{I}^3} s_3(\tilde{a}^i) \cup \bigcup_{i\in\tilde{I}^4} s_4(\tilde{a}^i), \tag{11}$$

where $\tilde{I}^r_{min}$ and $\tilde{I}^r$ are defined in Lemma 12 for all $r = 1,\dots,4$. By Lemma 13 it is possible to replace $s_r(\tilde{a}^i) \cap \mathcal{N}(\tilde{A}) \neq \emptyset$ in the definition of the set $\tilde{I}^r$ through $e^r \in b - \operatorname{int} K_r$ for all $r = 1,\dots,4$. Thus, the sets $\tilde{I}^r$, $r = 1,\dots,4$, are more easily computable. Furthermore, it is easy to verify that we have $T_r := \operatorname{MIN}(A, K_r) = \operatorname{MIN}(\tilde{A}, K_r)$ for all $r = 1,\dots,4$. Consequently, the sets T_1, T_2, T_3, T_4 from Step 1 in Algorithm 2 can be used in the definition of $\tilde{I}^r_{min}$ for all $r = 1,\dots,4$.

The formulations in Step 4 of Algorithm 2 are based on the relations (10) and (11). Note that we have the equality $\tilde{T}_r = \{\tilde{a}^i \in \tilde{A} \,|\, i \in \tilde{I}^r\}$ for all $r = 1,\dots,4$. Moreover, note that $(x,y) \notin \bigcup_{i\in\tilde{I}^r} \operatorname{cl} s_r(\tilde{a}^i)$ implies $(x,y) \notin \bigcup_{i\in\tilde{I}^r} s_r(\tilde{a}^i)$ for all $r = 1,\dots,4$. □

In order to show in Theorem 18 that Algorithm 2 generates the set $\operatorname{Eff}(\mathcal{P}_A)$, we need in the following three additional lemmata:

Lemma 15. *Let $z^1 := (x^1, y^1)$, $z^2 := (x^2, y^2) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ be two solutions with $x^1 = x^2$ or $y^1 = y^2$. Then, for each $\alpha \in [0,1]$ it holds $\alpha \cdot z^1 + (1-\alpha) \cdot z^2 \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$.*

Proof. This assertion directly follows from Corollary 1 in [16]. □

Lemma 16. *Suppose that $r^1, r^2, r^3, r^4 \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ are the vertices of a possibly degenerated, axially parallel rectangle $R := \operatorname{conv}\{r^1, r^2, r^3, r^4\}$. Then we have the inclusion $R \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$.*

Proof. The proof of this lemma is essentially based on the ideas in [16, Corollary 2]. If R is a degenerated rectangle, then the assertion follows

directly from Lemma 15. Now let $R := \operatorname{conv}\{r^1, r^2, r^3, r^4\}$ be a not degenerated rectangle with r^1 (bottom left vertex), r^2 (top left vertex), r^3 (top right vertex) and r^4 (bottom right vertex) as vertices of R. We choose an arbitrarily $z^* \in R \setminus \{r^1, r^2, r^3, r^4\}$ and consider two cases:
Case 1: Let $z^* \in \operatorname{bd} R$. Then z^* can be represented by a convex combination of two adjacent vertices of R. By Lemma 15 we conclude $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$.
Case 2: Let $z^* \in \operatorname{int} R$. Then it exist $z^1 := (x^1, y^1) \in \operatorname{conv}\{r^1, r^2\}$, $z^2 := (x^2, y^2) \in \operatorname{conv}\{r^3, r^4\}$ with $y^1 = y^2$ and a $\beta \in (0, 1)$ such that $z^* = \beta \cdot z^1 + (1 - \beta) \cdot z^2$ holds. Because of the assumption, we have $r^1, r^2, r^3, r^4 \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$, and therefore Lemma 15 implies $z^1, z^2 \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Consequently, by Lemma 15 we conclude $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. □

Lemma 17. *Let (x^1, y^1), $(x^2, y^2) \in \mathcal{C}$ with $x^1 < x^2$, $y^1 \neq y^2$. We assume, that there exists an index $i \in \{1, \dots, q_1 - 1\}$ such that $x^1 = x_i$ and $x^2 = x_{i+1}$ with $x_i, x_{i+1} \in \tilde{A}_1$ hold. Then there exists some $(x^3, y^3) \in \mathcal{C}$ such that*

$$x^3 = x_i \quad \text{and} \quad \begin{cases} y^1 < y^3 \leq y^2 & \text{for } y^1 < y^2, \\ y^1 > y^3 \geq y^2 & \text{for } y^1 > y^2 \end{cases}$$

or we have $(x_{i+1}, y^1) \in \mathcal{C}$.

Proof. The point $(x_i, y^1) \in \mathcal{C}$ is adjacent to the point $(x_{i+1}, y^1) \in \mathcal{C}$ (see Wendell et. al. [16]). By Corollary 3 in [16] and because of the structure of the set $\mathcal{C}$ we conclude the desired statement. □

Now we prove that Algorithm 2 computes the whole set of efficient elements $\operatorname{Eff}(\mathcal{P}_A)$ of the problem $(\mathcal{P}_A)$:

Theorem 18. *Let $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ be generated by Algorithm 2. Then we have:*

$$\operatorname{Eff}(\mathcal{P}_A) = \operatorname{Eff}(\mathcal{P}_{\tilde{A}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*.$$

Proof. By Theorem 11 we have $\operatorname{Eff}(\mathcal{P}_A) = \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Now we prove $\operatorname{Eff}(\mathcal{P}_{\tilde{A}}) = \mathcal{R}_1^* \cup \mathcal{R}_2^*$. At first, by Theorem 14 we know that the construction $\mathcal{C}$ can be represented by

$$\mathcal{C} = \bigcup_{i=1}^{q_1} \{(x_i, y) \,|\, y \in \mathcal{C}_i\} = \{(x, y) \in \tilde{A}_1 \times \tilde{A}_2 \mid (x, y) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})\}.$$

Because of Lemma 8 the inclusion $\tilde{A} \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ is true. Hence, it is easy to see that $\tilde{A}_1 = \{x_1, \dots, x_{q_1}\} = \{x \in \tilde{A}_1 \mid (x, y) \in \mathcal{C},\, y \in \tilde{A}_2\}$ holds, and therefore $\mathcal{C}_i \neq \emptyset$ for all $i = 1, \ldots, q_1$. Furthermore, for simplicity of the notation, we define $\mathcal{C}_i^{\min} := \min \mathcal{C}_i$ and $\mathcal{C}_i^{\max} := \max \mathcal{C}_i$ for all $i \in \{1, \ldots, q_1\}$. Note that it holds

$$\mathcal{R}_1^* \subseteq \bigcup_{i=1}^{q_1} \operatorname{conv}\{(x_i, \mathcal{C}_i^{\min}), (x_i, \mathcal{C}_i^{\max})\} =: \tilde{\mathcal{R}}_1^*.$$

Moreover, it is easy to see, that we have $\tilde{\mathcal{R}}_1^* \setminus \mathcal{R}_1^* \subseteq \mathcal{R}_2^*$. Therefore, it holds $\mathcal{R}_1^* \cup \mathcal{R}_2^* = \tilde{\mathcal{R}}_1^* \cup \mathcal{R}_2^*$. For this reason, in the following we prove $\operatorname{Eff}(\mathcal{P}_{\tilde{A}}) = \tilde{\mathcal{R}}_1^* \cup \mathcal{R}_2^*$.

At first we prove the inclusion "$\supseteq$". We consider two cases:
Case 1: Let $z^* \in \tilde{\mathcal{R}}_1^*$ be arbitrarily chosen. It exists some $i \in \{1, \ldots, q_1\}$ such that $z^* \in \operatorname{conv}\left\{(x_i, \mathcal{C}_i^{\min}), (x_i, \mathcal{C}_i^{\max})\right\}$. It is easy to see that we have $(x_i, \mathcal{C}_i^{\min}), (x_i, \mathcal{C}_i^{\max}) \in \mathcal{C} \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Taking into account Lemma 15 we observe that $\operatorname{conv}\left\{(x_i, \mathcal{C}_i^{\min}), (x_i, \mathcal{C}_i^{\max})\right\} \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ holds. Consequently, $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ is true.
Case 2: Let $z^* \in \mathcal{R}_2^*$ be arbitrarily chosen. Thus, we have $q_1 > 1$. We define $\underline{\mathcal{C}_i} := \max\left\{\mathcal{C}_i^{\min}, \mathcal{C}_{i+1}^{\min}\right\}$ and $\overline{\mathcal{C}_i} := \min\left\{\mathcal{C}_i^{\max}, \mathcal{C}_{i+1}^{\max}\right\}$ for all $i \in \{1, \ldots, q_1 - 1\}$. Because of the structure of the set $\mathcal{R}_2^*$, there exists some $i \in \{1, \ldots, q_1 - 1\}$ such that $z^* \in \operatorname{conv}\left\{(x_i, \underline{\mathcal{C}_i}), (x_i, \overline{\mathcal{C}_i}), (x_{i+1}, \overline{\mathcal{C}_i}), (x_{i+1}, \mathcal{C}^u)\right\}$ holds. If we can prove that $(x_i, \underline{\mathcal{C}_i}), (x_i, \overline{\mathcal{C}_i}), (x_{i+1}, \overline{\mathcal{C}_i}), (x_{i+1}, \underline{\mathcal{C}_i}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ hold, then by Lemma 16 the relation $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ follows. W.l.o.g. we suppose $\overline{\mathcal{C}_i} := \min\left\{\mathcal{C}_i^{\max}, \mathcal{C}_{i+1}^{\max}\right\} = \mathcal{C}_i^{\max}$. Obviously, we have $(x_i, \overline{\mathcal{C}_i}) = (x_i, \mathcal{C}_i^{\max}) \in \mathcal{C} \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. If $\min\left\{\mathcal{C}_i^{\max}, \mathcal{C}_{i+1}^{\max}\right\} = \mathcal{C}_i^{\max} = \mathcal{C}_{i+1}^{\max}$ also holds, then we have analogously $(x_{i+1}, \overline{\mathcal{C}_i}) = (x_{i+1}, \mathcal{C}_{i+1}^{\max}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Now suppose $\mathcal{C}_{i+1}^{\max} > \mathcal{C}_i^{\max}$. Note that we have $(x_i, \mathcal{C}_i^{\max}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ and $(x_{i+1}, \mathcal{C}_{i+1}^{\max}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. It does not exist any $y \in \tilde{A}_2$ with $y > \mathcal{C}_i^{\max}$ and $(x_i, y) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Thus, by Lemma 17 and since the point $(x_{i+1}, \overline{\mathcal{C}_i})$ is adjacent to the point $(x_i, \overline{\mathcal{C}_i})$ (see Wendell et. al. [16]), we conclude $(x_{i+1}, \overline{\mathcal{C}_i}) = (x_{i+1}, \mathcal{C}_i^{\max}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Analogously, we can prove $(x_i, \underline{\mathcal{C}_i}), (x_{i+1}, \underline{\mathcal{C}_i}) \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$. Consequently, by Lemma 16 we conclude $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$.

Now we prove the reverse inclusion "$\subseteq$". By view of Lemma 8 we infer $\operatorname{Eff}(\mathcal{P}_{\tilde{A}}) = \bigcup_{\lambda \in \operatorname{int} \mathbb{R}_+^k} X_\lambda^*$ with $X_\lambda^* := \operatorname{argmin}_{z \in \mathbb{R}^2} \langle \lambda, f_{\tilde{A}}(z) \rangle$ for all $\lambda \in \operatorname{int} \mathbb{R}_+^k$, where $f_{\tilde{A}}$ is the objective function of $(\mathcal{P}_{\tilde{A}})$. Let $z^* \in \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$ arbitrarily chosen. Thus, there exists some $\lambda \in \operatorname{int} \mathbb{R}_+^k$ such that $z^* \in X_\lambda^*$ holds. In the following we are going to show that $X_\lambda^* \subseteq \tilde{\mathcal{R}}_1^* \cup \mathcal{R}_2^*$. We distinguish four cases (compare Lemma 3):
Case 1: If $\operatorname{card} X_\lambda^* = 1$, by Lemma 3 the relation $X_\lambda^* \subseteq \tilde{A}_1 \times \tilde{A}_2$ follows. Because of $X_\lambda^* \subseteq \operatorname{Eff}(\mathcal{P}_{\tilde{A}})$, we have $X_\lambda^* \subseteq \mathcal{C}$. Obviously, the inclusion $\mathcal{C} \subseteq \tilde{\mathcal{R}}_1^*$ holds.

Case 2: If X_λ^* is a vertical line segment, then there exist $z^1 := (x^1, y^1)$, $z^2 := (x^2, y^2) \in \mathcal{C}$ with $x^1 = x^2 = x_i$ for some $i \in \{1, \ldots, q_1\}$ and it holds $X_\lambda^* = \text{conv}\{z^1, z^2\}$. Consequently, we have $X_\lambda^* = \text{conv}\{z^1, z^2\} \subseteq \text{conv}\{(x_i, \mathcal{C}_i^{\min}), (x_i, \mathcal{C}_i^{\max})\} \subseteq \tilde{\mathcal{R}}_1^*$.
Case 3: If X_λ^* is a horizontal line segment, then there exist $z^1 := (x^1, y^1)$, $z^2 := (x^2, y^2) \in \mathcal{C}$ with $y^1 = y^2$, $x^1 = x_i$, $x^2 = x_{i+1}$ for some $i \in \{1, \ldots, q_1 - 1\}$ and it holds $X_\lambda^* = \text{conv}\{z^1, z^2\}$. Note, in this case we have $q_1 > 1$. Obviously, because of the relation $\underline{\mathcal{C}_i} \leq y^1 = y^2 \leq \overline{\mathcal{C}_i}$ we conclude $\text{conv}\{z^1, z^2\} \subseteq \text{conv}\{(x_i, \underline{\mathcal{C}_i}), (x_i, \overline{\mathcal{C}_i}), (x_{i+1}, \overline{\mathcal{C}_i}), (x_{i+1}, \underline{\mathcal{C}_i})\} \subseteq \mathcal{R}_2^*$.
Case 4: If X_λ^* is a non degenerated rectangle (analougsly here $q_1 > 1$), then there exist $z^1 := (x^1, y^1)$, $z^2 := (x^2, y^2)$, $z^3 := (x^3, y^3)$, $z^4 := (x^4, y^4) \in \mathcal{C}$ with $x^1 = x^2 = x_i$ and $x^3 = x^4 = x_{i+1}$ for some $i \in \{1, \ldots, q_1 - 1\}$ as well as $y^2 = y^3 > y^4 = y^1$, and we have $X_\lambda^* = \text{conv}\{z^1, z^2, z^3, z^4\}$. Obviously, because of the relation $\underline{\mathcal{C}_i} \leq y^1 = y^4 < y^2 = y^3 \leq \overline{\mathcal{C}_i}$, we conclude $\text{conv}\{z^1, z^2, z^3, z^4\} \subseteq \text{conv}\{(x_i, \underline{\mathcal{C}_i}), (x_i, \overline{\mathcal{C}_i}), (x_{i+1}, \overline{\mathcal{C}_i}), (x_{i+1}, \underline{\mathcal{C}_i})\} \subseteq \mathcal{R}_2^*$. □

7. Computational results

The authors implemented Algorithm 2 and tested various examples for computing the set of efficient solutions of the location problem $(\mathcal{P}_A)$. In this section, we present some numerical results of the algorithm formulated in Section 5. Now let us consider $p \in \{16, 64, 256, 1024, 4096, 16384\}$ existing points, where the coordinates of the given points are chosen randomly using the function "*rand*" in MATLAB. On the one hand we apply an adapted version of Algorithm 2 with updated Step 2 (variant I), that means we do not reduce the set of existing facilities during the procedure. More precisely, we define $\tilde{A} := A$ in Step 2 of Algorithm 2. On the other hand we apply the original formulated procedure of Algorithm 2 (variant II), i.e., we put $\tilde{A} := \bigcup_{t=1}^4 \text{MIN}(A, K_t)$ in Step 2 of Algorithm 2.

Figure 3 shows the ouptut of both variants of implemented algorithms for an example problem $(\mathcal{P}_A)$ with 16 existing facilities. Moreover, the left part of Fig. 3 visualizes the decomposition of the solution set computed by variant I and the right part of Fig. 3 shows the decomposition computed by variant II, respectively.

Table 1 contains the running times (in seconds) of the above-mentioned variants of implemented algorithms. Note that the parameter p is the number of existing facilities of the original problem associated to the set A and k is the number of existing facilities of the reduced problem associated to the set $\tilde{A} = \bigcup_{t=1}^4 \text{MIN}(A, K_t)$. For each p we produced five test examples and averaged for both variants I and II the running times. Moreover, we

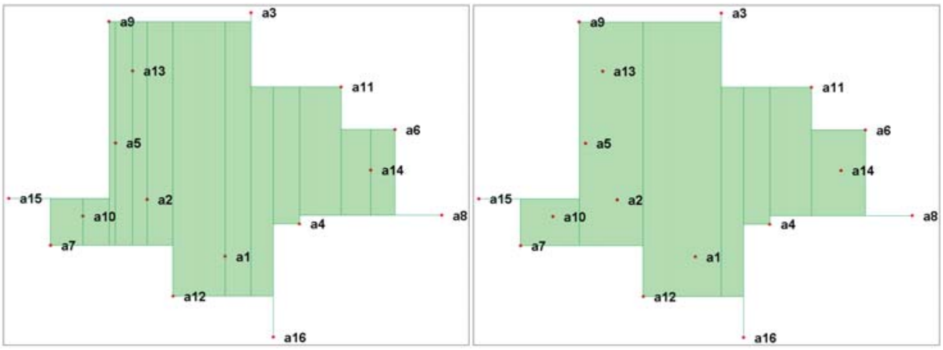

Figure 3: Visualization of the decomposition of the set of efficient solutions for both variants.

p	16	64	256	1024	4096	16384
k	8	14	22	26	33	37
I	0,01693	0,29203	5,55384	99,95128	-	-
II	0,00548	0,01848	0,04868	0,09810	0,22608	0,50471

Table 1: Running times in seconds.

averaged the parameter k. It should be mentioned that the parameter k is in most cases significantly smaller than the parameter p in our test examples. The reduction of the location problem using the Jahn-Graef-Younes method in Step 2 of Algorithm 2 causes a significant improvement concerning the running times of our algorithm in comparison with the above-mentioned adapted algorithm.

We note that the new algorithms presented in this paper are implemented in a MATLAB-based software tool (Facility Location Optimizer), see `http://www.project-flo.de`.

8. Concluding remarks

In this paper, we have derived a new algorithm (*Rectangular Decomposition Algorithm*) that computes the whole set of Pareto efficient elements of multiobjective location problems involving the Manhattan norm. Our algorithm is working in the original space which is very convenient for multiobjective location problems from the theoretical as well as practical point of view.

In comparison with other algorithms for solving multiobjective location problems the *Rectangular Decomposition Algorithm* has the following advantages:

- The algorithm generates the whole set of efficient elements in difference to algorithms based on a scalarization that generates only single points of the set of efficient elements.
- The algorithm works more effective than the algorithm generating the whole set of efficient elements (described in Lemma 7) because the number of objectives is reduced (see Table 1) .
- Because of reducing the number of objectives the algorithm allows a simple geometrical description of the solution set in comparison with other algorithms.
- A visualization of the set of efficient elements is available.
- The solution set generated by the algorithm can be used as input for decomposition algorithms for solving multiobjective location problems with additional criteria (see Alzorba, Günther, Popovici [1]).

In forthcoming papers we will extend our methods to the case of multiobjective location problems with polyhedral gauges.

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