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The Lipschitzianity of convex vector and set-valued functions

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Abstract

It is well known that every scalar convex function is locally Lipschitz on the interior of its domain in finite dimensional spaces. The aim of this paper is to extend this result for both vector functions and set-valued mappings acting between infinite dimensional spaces with an order generated by a proper convex cone C. Under the additional assumption that the ordering cone C is w-normal, we prove that a locally C-bounded C-convex vector function is Lipschitz on the interior of its domain by two different ways. Moreover, we derive necessary conditions for Pareto minimal points of vector-valued optimization problems where the objective function is C-convex and C-bounded.

1 Introduction

Nowadays, methods of variational analysis have been studied deeply and applied widely, the literature on this field as well as its applications is very abundant. The tools of variational analysis are getting more and more varied, one can list here several of its advanced tools such as normal cones, subdifferentials and coderivatives. One uses these tools for deriving necessary and sufficient conditions for solutions of optimization problems.

In this paper, we consider vector-valued as well as set-valued optimization problems where the objective function is acting between infinite dimensional spaces. The optimality conditions will be derived for different solution concepts. For applying methods of generalized differentiation with corresponding calculus it is important to show certain Lipschitz properties of the objective function. Under C-convexity assumptions of the vector-valued as well as set valued objective function we will derive Lipschitz properties. These results yield optimality conditions for solutions of vector optimization problems with a C-convex objective function and a not necessarily convex feasible set.

We express the optimality conditions in terms of generalized derivatives such as generalized subdifferentials and coderivatives. To reach these results there are two ways. The

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first one is to use directly the properties of the subdifferentials and coderivatives, not to use scalarization at all. The second possibility is to use scalarization methods in order to transform the vector-valued optimization problem into a scalar optimization problem such that one can find optimality conditions for scalar problems. To scalarize, one uses two different scalarizing functions which are the oriented distance function [13, 30] as well as nonlinear separating functionals introduced in [9], each of them has its advantages and disadvantages. In this paper, we will only use the second scalarizing function to deal completely with Lagrange multiplier rules in both cases that the interior of the ordering cone is empty or nonempty.

For scalar functions it is well known that a convex function bounded from above on a neighborhood of a point of its domain is locally Lipschitz continuous on the interior of its domain [25, 31]. This is also important for vector-valued function not only in finite dimensional spaces but also in infinite dimensional spaces, because almost all subdifferentials and coderivatives are defined for the class of (Lipschitz) continuous functions. Hence, if C-convex vector-valued functions are also Lipschitz continuous, then we can calculate their subdifferential and coderivative in some certain senses. The main aim of this paper is to prove the Lipschitz continuity of C-convex vector-valued functions and set-valued functions in infinite dimensional spaces in order to derive necessary optimality conditions for vector and set-valued optimization problems with C-convex objective functions.

The paper is organized as follows. In Section 2, we recall some well-known results about Lipschitz continuity of scalar convex functions [25, 31] as well as C-convex vector-valued functions [16]. We emphasize the result by Borwein [5], who proved the Lipschitz continuity in the case that C is a normal cone, we will present a new proof for this result, and show the Lipschitz constant more precisely in Theorem 2.14, and extend it to w-normal cones C in Theorem 2.16. Some related results are known from Papageorgiou [22] in the case of vector lattice spaces, or Reiland [24] and Thibault [28] with other concepts of Lipschitz mappings. In Section 3, we derive new results concerning Lipschitz properties for C-convex set-valued mappings. To do this, we recall the notations of C-convex, C-Lipschitz, Lipschitz-like functions already introduced by Tan and Minh in [18, 19] and by Mordukhovich in [20]. Section 4 contains applications for vector optimization problems, we derive Lagrange multiplier rules based on the full calculus of coderivatives introduced by Mordukhovich and Fenchel subdifferentials.

2 The Lipschitzianity of convex vector functions

Throughout the following let X and Y be normed vector spaces with the corresponding norms $||.||_X, ||.||_Y$, unless otherwise stated; we denote the closed unit balls in X and Y by U_X, U_Y, respectively. For a nonempty set A, we write $\operatorname{int} A$ and $\operatorname{cl} A$ for the interior and closure of A, respectively.

We denote a closed ball centered at x_0 of radius $r > 0$ contained in X by

$$B(x_0, r) := \{x \in X \mid \ ||x - x_0||_X \leq r\}.$$

For a proper convex cone $C \subset Y$ we are using the following notation

$$\forall x, y \in Y, \qquad x - y \in C \qquad \Longleftrightarrow \qquad x \geq_C y.$$

The cone C is called pointed if $C \cap -C = \{0\}$.
We denote the topological dual space of Y by Y^*. The positive dual cone C^+ of C is:

$$C^+ := \{y^* \in Y^* \mid \forall c \in C : \quad y^*(c) \geq 0\}.$$

The multiplication of a set with a scalar and the sum of sets are given by

$$\alpha A := \{\alpha a \mid a \in A\}, \qquad A + B := \{a + b \mid a \in A, b \in B\}.$$

We are using the following basic conventions for any real number α, a set A and $y^* \in C^+$,

$$\alpha \cdot \emptyset = \emptyset, \quad \emptyset + A = A + \emptyset = \emptyset, \quad y^*(\emptyset) = +\infty.$$

We adjoin a maximal element $+\infty$ to Y ($+\infty \notin Y$) such that $+\infty \geq_C y$ for all $y \in Y$, and we use the notation $Y^\bullet := Y \cup \{+\infty\}$. The infinity element satisfies

$$\alpha \cdot (+\infty) = +\infty, \quad y + (+\infty) = +\infty, \quad 0 \cdot (+\infty) = 0, \quad y^*(+\infty) = +\infty$$

for any positive real α, any y in Y and any $y^* \in C^+$.
Furthermore, we use the notation $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$. The set of positive integers is denoted by $\mathbb{N}^* := \{1, 2, \ldots\}$.
We consider a function $f : X \to Y^\bullet$, and denote the domain of f by $\operatorname{dom} f := \{x \in X \mid f(x) \in Y\}$. In the sequel, we always assume that $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$.

Definition 2.1 *Consider $f : X \to Y^\bullet$, and let $C \subset Y$ be a proper convex cone. The function f is said to be **C-convex** if for all $x, y \in \operatorname{dom} f, \lambda \in (0, 1)$, one has*

$$\lambda f(x) + (1 - \lambda) f(y) \geq_C f(\lambda x + (1 - \lambda) y).$$

In the case $Y = \mathbb{R}$ and $C = \mathbb{R}^+ := \{\alpha \in \mathbb{R} \mid \alpha \geq 0\}$, Definition 2.1 reduce to the classical definition of convexity for functionals. Obviously, the convexity of f implies that $\operatorname{dom} f$ is convex.

Definition 2.2 *Consider $f : X \to Y^\bullet$;*

*(i) f is **Lipschitz on** $U \subset X$ if $U \subset \operatorname{dom} f$, and there exists $l \geq 0$ such that*

$$||f(x) - f(x')||_Y \leq l||x - x'||_X, \qquad \forall x, x' \in U.$$

(ii) f *is said to be* ***Lipschitz around*** x *if there is a neighborhood* U_x *of* x *such that* f *is Lipschitz on* U_x *(in particular* $x \in \mathrm{int}(\mathrm{dom}\, f)$*).*

(iii) f *is said to be* ***locally Lipschitz*** *on a nonempty subset* D *of* X*, if* f *is Lipschitz around every point* $x \in D$*. Hence* D *is open and* $D \subset \mathrm{dom}\, f$*.*

It is known that in a finite dimensional normed linear space, every proper scalar convex function is Lipschitz around any interior point of its domain (see Rockafellar [26, Theorem 10.4] and Roberts, Varberg [25, Theorem A]). But a scalar convex function on an infinite dimensional normed space may be locally unbounded, so we need a mild additional condition on f, more precisely the boundedness from above of the function on a nonempty open set.

Lemma 2.3 *([25, Lemma B])*
Let $(X, \|.\|_X)$ *be a normed vector space,* $f : X \to \overline{\mathbb{R}}$ *be a proper convex function. Assume that* $U \subset \mathrm{dom}\, f$ *is an open convex set. If* f *is bounded from above in a neighborhood of just one point of* U*, then* f *is locally bounded on* U*.*

For a proof of the Lipschitzianity on an infinite dimensional normed space see Roberts and Varberg ([25], Theorem B); Zălinescu ([31, Corollary 2.2.12]).

Lemma 2.4 *([31, Corollary 2.2.12])*
Let $(X, \|.\|_X)$ *be a normed vector space, and let* $f : X \to \overline{\mathbb{R}}$ *be a proper convex function on* X*. Suppose that* $x_0 \in dom f$ *and for some* $\theta > 0$*,* $m \geq 0$*,*

$$\forall x \in B(x_0, \theta) : f(x) \leq f(x_0) + m;$$

then

$$\forall \theta' \in (0, \theta), \forall x, x' \in B(x_0, \theta') : |f(x) - f(x')| \leq \frac{m}{\theta} \cdot \frac{\theta + \theta'}{\theta - \theta'} \cdot \|x - x'\|_X.$$

Lemma 2.4 not only shows the Lipschitz property, but also estimates the Lipschitz constant.

In the case of C-convex vector functions $f : \mathbb{R}^n \to \mathbb{R}^m$, Luc, Tan and Tinh in [16] proved the Lipschitz property in finite dimensional spaces.

Lemma 2.5 *([16, Theorem 3.1]) Assume that the closure* $\mathrm{cl}\, C$ *of the proper convex cone* $C \subset \mathbb{R}^m$ *is pointed,* D *is a proper convex subset of* $\mathbb{R}^n$*, and* $f : D \to \mathbb{R}^m$ *is a* C*-convex vector function. Then* f *is locally Lipschitz on the relative interior of* D*.*

A natural question arises whether the above result is true in infinite dimensional spaces. We now show that if the cone C satisfies certain properties related to the topology and order, then all locally C-bounded C-convex vector functions will be locally Lipschitz in infinite dimensional spaces.

We will give the definition of a normal cone which shows a connection between topology and order of the space Y. At first, we say that the nonempty set A of the linear space Y is full with respect to the convex cone $C \subset Y$ if $A = [A]_C$, where

$$[A]_C := (A + C) \cap (A - C);$$

note that $[A]_C$ is full with respect to C for every nonempty subset A of Y.

Definition 2.6 *Let Y be a normed vector space, and let $C \subset Y$ be a proper convex cone. Then C is called* ***normal*** *if the origin $0 \in Y$ has a neighborhood base formed by full sets with respect to C.*

Remark 2.7 *If the neighborhood base of the origin in Definition 2.6 is taken in the weak topology of Y, then C is called* w***-normal****.*

Lemma 2.8 *([10, Proposition 2.2.9 (ii), (iii)]) Let Y be a normed vector space, and $C \subset Y$ be a proper convex cone.*

(i) C is w-normal if and only if C^+ is reproducing; i.e., $Y^* = C^+ - C^+$.

(ii) If C is normal, then C^+ is reproducing.

Lemma 2.9 *Let $(Y, \|.\|_Y)$ be a normed vector space, and $C \subset Y$ be a normal cone. Then there exists $\rho > 0$ such that $\rho U_{Y^*} \subset C_1^+ - C_1^+$, where $C_1^+ := U_{Y^*} \cap C^+$.*

This result can be deduced from Jameson's book [15]. We provide its proof for reader's convenience. In the following proof we are dealing with the polar set of $\emptyset \neq A \subset Y$ defined by

$$A^0 := \{y^* \in Y^* \mid y^*(y) \leq 1, \forall y \in A\}.$$

Proof. Because C is normal, there exists $\rho > 0$ such that $(\rho U_Y + C) \cap (\rho U_Y - C) \subset U_Y$. Since $(\rho U_Y + C)^0 = C^+ \cap \rho^{-1} U_{Y^*} = \rho^{-1} C_1^+$ is convex and w^*-compact, we get

$$\begin{aligned} U_{Y^*} = (U_Y)^0 &= [(\rho U_Y + C) \cap (\rho U_Y - C)]^0 = \mathrm{cl}^*(\mathrm{conv}[(\rho U_Y + C)^0 \cup (\rho U_Y - C)^0]) \\ &= \cup_{\lambda \in [0,1]} [\lambda (\rho U_Y + C)^0 + (1 - \lambda)(\rho U_Y - C)^0] \subset \rho^{-1}(C_1^+ - C_1^+); \end{aligned}$$

where $\mathrm{cl}^*(\mathrm{conv}\, E)$ is the **closed convex hull** of the subset E of the vector space Y with respect to weak* topology. This completes the proof. □

In the following we present C-boundedness notions of a mapping $f : X \to Y^\bullet$, where $C \subset Y$ is a proper convex cone.

Definition 2.10 *Consider $f : X \to Y^\bullet$;*

(i) f is said to be ***C-bounded from above (resp. below)*** *on a subset A of X if there exists a constant $\mu > 0$ such that*

$$f(A) \subset \mu U_Y - C, \quad (\textit{resp. } f(A) \subset \mu U_Y + C).$$

*(ii) f is said to be **C-bounded** on a subset A of X if it is C-bounded from above and C-bounded from below on A.*

The following result for vector-valued functions is similar to Lemma 2.3 in the case $Y = \mathbb{R}$.

Proposition 2.11 *Let X, Y be two normed vector spaces, $C \subset Y$ be a proper convex cone, and let $f : X \to Y^\bullet$ be C-convex. If f is C-bounded from above on a neighborhood V of $x_0 \in \operatorname{dom} f$ then for every $x \in \operatorname{int}(\operatorname{dom} f)$, f is C-bounded on a neighborhood of x.*

Proof. As $x_0 \in \operatorname{int}(\operatorname{dom} f)$, we take $\theta, \mu_0 > 0$ such that $U := x_0 + \theta U_X \subset \operatorname{dom} f$ and $f(U) \subset \mu_0 U_Y - C$. Fix some $x \in \operatorname{int}(\operatorname{dom} f)$. Then there exist $x' \in \operatorname{dom} f$ and $\lambda \in (0, 1)$ such that $x = (1 - \lambda)x' + \lambda x_0$. Then for $u \in U_X$ we have that $x + \lambda\theta u = (1-\lambda)x' + \lambda(x_0 + \theta u)$, and so $f(x+\lambda\theta u) \in (1-\lambda)f(x') + \lambda f(x_0+\theta u) - C \subset B_0 - C$, where $B_0 := (1-\lambda)f(x') + \lambda\mu_0 U_Y$. Therefore, f is C-bounded from above on a neighborhood of x.

From the proof above, there exist a constant $\mu > 0$ and a neighborhood $U = B(x, r)$ of x such that $f(U) \subset \mu U_Y - C$, so $-f(U) \subset \mu U_Y + C$. It is sufficient to prove that f is C-bounded from below on a neighborhood of x.
For every x' in U, we can take $x'' \in U$ so that $x = \frac{1}{2}x' + \frac{1}{2}x''$, and so $f(x) \in \frac{1}{2}f(x') + \frac{1}{2}f(x'') - C$. Hence $f(x') \in 2f(x) - f(x'') + C \subset 2f(x) + \mu U_Y + C$. This completes the proof. □

Remark 2.12 *If we assume that $f : X \to Y^\bullet$ is topologically bounded on a neighborhood U of x_0, i.e., there is a positive real μ such that $f(U) \subset \mu U_Y$, it is obvious that f is C-bounded on U. Conversely, if X, Y are normed spaces, and C is normal cone, any C-bounded function around x_0 is topologically bounded around this point. Indeed, from Proposition 2.11, there exist a neighborhood U of x_0 and a constant $\mu > 0$ such that*

$$f(U) \subset \mu U_Y + C \text{ and } f(U) \subset \mu U_Y - C,$$

that is

$$f(U) \subset [\mu U_Y].$$

Because C is normal, we can take μ' such that

$$\frac{1}{\mu'} f(U) \subset \frac{1}{\mu'}[\mu U_Y] \subset U_Y,$$

so f is topologically bounded around x_0.

Borwein [5] proved the following result.

Theorem 2.13 *([5, Corollary 2.4]) Let X, Y be normed spaces, C be a normal cone in Y, and $f : X \to Y^\bullet$ be C-convex. If f is bounded above on a neighbourhood of a point $x_0 \in X$, then f is Lipschitz around this point.*

Next, by Remark 2.12 we will present a result similar to Theorem 2.13, and prove it by another way. We obtain a more accurate Lipschitz constant than the one in [5].

Theorem 2.14 *Let X, Y be two normed vector spaces, $C \subset Y$ be a proper convex cone, and let $f : X \to Y^\bullet$ be C-convex. Suppose that C is a normal cone, and f is C-bounded from above on a neighborhood U of $x_0 \in \operatorname{int}(\operatorname{dom} f)$, i.e., $f(U) \subset \mu U_Y - C$ for a real number $\mu > 0$; then f is Lipschitz around x_0.*

Proof. Without loss of generality we suppose that $x_0 = 0$ and $f(0) = 0$. Let $\theta, \mu > 0$ be such that $U := \theta U_X$ and $f(U) \subset \mu U_Y - C$.
Let x be arbitrary in U, then $f(x) = \mu y - c$ with $\|y\|_Y \leq 1$ (as $y \in U_Y$) and $c \in C$. Take $y^* \in C_1^+$, that is $\|y^*\|_* \leq 1$ and $y^* \in C^+$. We obtain that

$$y^*(f(x)) = y^*(\mu y - c) = \mu y^*(y) - y^*(c) \leq \mu y^*(y) \leq \mu \|y^*\|_* \|y\|_Y \leq \mu.$$

So $y^*(f(x)) \leq \mu$ for all $y^* \in C_1^+$, $x \in U$ (μ does not depend on y^*).
Since $y^* \circ f$ is proper and convex, for $\theta' \in (0, \theta)$ and $y^* \in C_1^+$, applying Lemma 2.4, we get

$$|y^*(f(x) - f(x'))| \leq L' \|x - x'\|_X, \quad \forall x, x' \in \theta' U_X, \tag{1}$$

where $L' := \mu(\theta + \theta')/[\theta(\theta - \theta')]$. As C is normal, we take $\rho > 0$ provided by Lemma 2.9 and $y^* \in U_{Y^*}$, we find $y_1^*, y_2^* \in C_1^+$ such that $\rho y^* = y_1^* - y_2^*$. From (1), we get

$$\rho |y^*(f(x) - f(x'))| = |y_1^*(f(x) - f(x')) - y_2^*(f(x) - f(x'))| \leq 2L' \|x - x'\|_X,$$

for all $x, x' \in \theta' U_X$. We have

$$\|f(x) - f(x')\|_Y = \sup_{y^* \in U_{Y^*}} |y^*(f(x) - f(x'))| \leq 2\rho^{-1} L' \|x - x'\|_X, \quad \forall x, x' \in \theta' U_X.$$

This shows that f is Lipschitz on $\theta' U_X$ with the Lipschitz constant $L = 2\rho^{-1}\mu(\theta + \theta')/[\theta(\theta - \theta')]$. □

Proposition 2.15 *Let X, Y be two normed vector spaces, and let $f : X \to Y^\bullet$ be a mapping. If $y^* \circ f$ is Lipschitz around $x \in \operatorname{dom} f$ for every linear function $y^* \in Y^*$, then f is also Lipschitz around x.*

Proof. We suppose that f is not Lipschitz around x, i.e., f is not Lipschitz on $B(x, \frac{1}{n})$ for every $n \in \mathbb{N}^*$. From Definition 2.2, there exist $x_n, y_n \in B(x, \frac{1}{n})$ such that

$$||f(x_n) - f(y_n)||_Y > n||x_n - y_n||_X.$$

Because of $x_n, y_n \in B(x, \frac{1}{n})$, we have that the sequences $\{x_n\}, \{y_n\}$ converge to x.
Setting $z_n := \frac{f(x_n) - f(y_n)}{||x_n - y_n||_X} \in Y$, we have $||z_n||_Y \geq n$ for all n.
For every $y^* \in Y^*$, $y^* \circ f$ is Lipschitz around x. This means that there exists $\theta = \theta_{y^*} > 0$ such that $y^* \circ f$ is Lipschitz on $B(x, \theta)$. Hence there exists $L_{y^*} > 0$ such that

$$|y^* \circ f(x) - y^* \circ f(x')| \leq L_{y^*} ||x - x'||_X, \quad \forall x, x' \in B(x, \theta).$$

Since $x_n, y_n \to x$, there exists n_{y^*} such that $x_n, y_n \in B(x, \theta)$ for every $n > n_{y^*}$, and so $|y^* \circ f(x_n) - y^* \circ f(y_n)| \leq L_{y^*}||x_n - y_n||_X$, hence $|y^*(z_n)| \leq L_{y^*}$ for every $n \geq n_{y^*}$. It follows that there exists L'_{y^*}, such that

$$|y^*(z_n)| \leq L'_{y^*}, \qquad \forall y^* \in Y^*, \forall n \geq 1.$$

This means that the assumptions of [27, Corollary 3.18] are fulfilled, so applying it to the normed space Y, we have that $\{||z_n||_Y | n \geq 1\}$ is bounded. This contradicts the fact that $||z_n||_Y \geq n$ for all n. □

In the following, we will extend Theorem 2.14 for a w-normal cone C. Of course, if C is normal then C is w-normal.

Theorem 2.16 *Let X, Y be two normed vector spaces, C is a w-normal cone, and let $f : X \to Y^\bullet$ be C-convex,* $\operatorname{int}(\operatorname{dom} f) \neq \emptyset$*. Suppose that f is C-bounded from above on a neighborhood U of $x_0 \in \operatorname{dom} f$, i.e., $f(U) \subset \mu U_Y - C$ for a real number $\mu > 0$, then f is Lipschitz around x_0.*

Proof. Set $f_0 := f|_{\operatorname{int}(\operatorname{dom} f)} : \operatorname{int}(\operatorname{dom} f) \to Y$, then f_0 also has C-convexity, C-boundedness properties like f.
Let $y^* \in Y^*$ be arbitrary, since C is w-normal, by Lemma 2.8(i), we have $Y^* = C^+ - C^+$, so we can represent $y^* = y_1^* - y_2^*$. Then

$$y^* \circ f_0 = y_1^* \circ f_0 - y_2^* \circ f_0 \text{ with } y_1^*, y_2^* \in C^+. \tag{2}$$

So, by the C-convexity of the function f_0, and according to [17, Proposition 1.6.2], $y_1^* \circ f_0$, and $y_2^* \circ f_0$ are scalar convex functions.
Since f_0 is C-bounded on a neighborhood U of x_0, $f_0[U] \subset \mu U_Y - C$ for some $\mu > 0$; hence for every $x \in U$, there exist $y \in U_Y, c \in C$ such that $f_0(x) = \mu y - c$. For every $z^* \in C^+$, we have that $z^*(\mu y') \leq \mu \|z^*\| \cdot \|y'\| \leq \mu \|z^*\| =: \mu'$ for all $y' \in U_Y$. It follows

$$z^* \circ f_0(x) \leq z^*(\mu y) \leq \mu', \quad \forall x \in U.$$

so $z^* \circ f_0$ is bounded from above on a neighborhood of x_0 for all $z^* \in C^+$.
Hence, from [25, Theorem B], $y_1^* \circ f_0$ and $y_2^* \circ f_0$ are Lipschitz around x_0. From (2), we can assert that $y^* \circ f_0$ is also Lipschitz around x_0 for every $y^* \in Y^*$. Applying Proposition 2.15, f_0 is Lipschitz around x_0, which completes the proof. □

Using Proposition 2.11 and Theorem 2.16 we deduce the following result.

Theorem 2.17 *Let X, Y, C and f be as in Theorem 2.16. In addition, assume that f is C-bounded from above on a neighborhood of one point of* $\operatorname{int}(\operatorname{dom} f)$*; then f is locally Lipschitz on* $\operatorname{int}(\operatorname{dom} f)$.

Remark 2.18 *For relationships among different kinds of cones look at [10, Sections 2.2], compare the following table, where we use the notations for properties of cones given at [10, Sections 2.1 and 2.2].*

$$
\begin{array}{ccccc}
 & & C \text{ has compact base} & \Longrightarrow & C \text{ is Daniell} \\
 & & \Downarrow \ \Uparrow {\scriptstyle C = \operatorname{cl} C,\ Y = \mathbb{R}^n} & & \\
C \text{ has angle property} & \Longleftrightarrow & C \text{ is well-based} & \Longleftrightarrow & \operatorname{int} C^+ \neq \emptyset \\
 & & \Downarrow & & \\
\exists \text{ proper cone } K\text{: } C \setminus \{0\} \subset \operatorname{int} K & \Longleftrightarrow & C \text{ is based} & \Longleftrightarrow & C^\# \neq \emptyset \\
 & & \Downarrow \ \Uparrow {\scriptstyle C = \operatorname{cl} C,\ Y \text{ separable}} & & \\
C \text{ well-based} & \overset{C = \operatorname{cl} C,\ Y = \mathbb{R}^n}{\Longleftarrow} & C \text{ pointed} & \Longleftarrow & C^+ - C^+ = Y^* \\
\Downarrow & & \Uparrow & & \Updownarrow \\
\operatorname{cl} C \text{ is normal} & \Longleftrightarrow & C \text{ is normal} & \Longrightarrow & C \text{ is } w\text{-normal.}
\end{array}
$$

Observe that the convex cone $C \subset \mathbb{R}^n$ is normal if and only if $\operatorname{cl} C$ *is pointed (see [10, Corollary 2.2.11]). So, from Theorem 2.17 one deduces the assertion of Lemma 2.5 (compare [16, Theorem 3.1]).*

We say that a convex cone C is well-based if there exists a bounded convex set B such that $C = \mathbb{R}^+ B$ and $0 \notin \operatorname{cl} B$. So a cone C having a convex weakly compact base is *well-based*. From the Figure above, we know that a proper convex cone C which is *well-based* ($\Leftrightarrow \operatorname{int} C^+ \neq \emptyset$), is also normal. Therefore, we get the following Corollary.

Corollary 2.19 *Let X, Y be normed vector spaces, and let $f : X \to Y^\bullet$ be C-convex. If the cone C is well-based ($\Leftrightarrow$ $\operatorname{int} C^+ \neq \emptyset$), or C has a weakly compact base, and f is C-bounded from above on a neighborhood of one point of* $\operatorname{int}(\operatorname{dom} f)$*, then f is locally Lipschitz on* $\operatorname{int}(\operatorname{dom} f)$.

3 The Lipschitzianity of convex set-valued mappings

In [18], Minh and Tan already studied the C-Lipschitzianity of C-convex set-valued mappings $F : X \rightrightarrows Y$, where X is a finite dimensional spaces, and Y is a Banach space. In this section, we derive corresponding results in general normed spaces. In what follows we assume that X and Y are normed spaces, and C is a proper convex cone in Y. $F : X \rightrightarrows Y$ is a set-valued mapping from X into Y with $\operatorname{dom} F = \{x \in X | F(x) \neq \emptyset\}$. We define the graph of the set-valued mapping F by

$$\operatorname{gph} F := \big\{(x, y) \in X \times Y \big|\ y \in F(x)\big\}.$$

The *epigraph* of F with respect to the cone C is given by

$$\operatorname{epi} F := \big\{(x, y) \in X \times Y \ \big|\ y \in F(x) + C\big\}. \tag{3}$$

The ***epigraphical multifunction*** of $F : X \rightrightarrows Y$, $\mathcal{E}_F(x) : X \rightrightarrows Y$ is defined by:

$$\mathcal{E}_F(x) := F(x) + C, \tag{4}$$

hence, $\operatorname{gph} \mathcal{E}_F = \operatorname{epi} F$.

Remark 3.1 *For a vector-valued function $f : X \to Y^\bullet$ we associate the set-valued function $F : X \rightrightarrows Y$ given by*

$$F(x) := \begin{cases} \{f(x)\}, & x \in \operatorname{dom} f, \\ \emptyset, & \textit{otherwise}; \end{cases} \tag{5}$$

hence $\operatorname{dom} F = \operatorname{dom} f$.
We say that F is at most single-valued. Inversely, for each at most single-valued function $F : X \rightrightarrows Y$, we associate the corresponding vector-valued function $f : X \to Y^\bullet$ given by

$$f(x) := \begin{cases} y, & \textit{if } x \in \operatorname{dom} F, \textit{ and } F(x) = \{y\}, \\ +\infty, & \textit{if } x \notin \operatorname{dom} F. \end{cases} \tag{6}$$

At first, we introduce the definition of Lipschitz properties of set-valued mappings following the book by Mordukhovich [20, Section 1.2.2].

Definition 3.2 ***(Lipschitz properties of set-valued mappings).*** *Let $F : X \rightrightarrows Y$ with* $\operatorname{dom} F \neq \emptyset$.

*(i) Given nonempty sets $U \subset X$ and $V \subset Y$, we say that F is **Lipschitz-like** on U relative to V if there is $l \geq 0$ such that*

$$F(x) \cap V \subset F(u) + l||x - u||_X U_Y, \quad \forall x, u \in U. \tag{7}$$

Hence, if $F(U) \cap V \neq \emptyset$, then $U \subset \operatorname{dom} F$.

*(ii) Given $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, we say that F is **Lipschitz-like** around $(\bar{x}, \bar{y})$ with modulus $l \geq 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that (7) holds, hence necessarily $\bar{x} \in \operatorname{int}(\operatorname{dom} F)$. The infimum of all such moduli l is called the exact Lipschitz bound of F around $(\bar{x}, \bar{y})$ and is denoted by $\operatorname{lip} F(\bar{x}, \bar{y})$.*

*(iii) F is **Lipschitz continuous** on U if (7) holds with $V = Y$. The infimum of $l \geq 0$ for which (7) holds is denoted by $\operatorname{lip} F(\bar{x})$. Furthermore, F is Lipschitz around $\bar{x}$ with exact bound $\operatorname{lip} F(\bar{x})$ if $V = Y$ in (ii).*

*(iv) F is **epigraphically Lipschitz-like (ELL)** around a given point $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ if its epigraphical multifunction $\mathcal{E}_F$ is Lipschitz-like around that point.*

Of course, if $F : X \rightrightarrows Y$ is at most single-valued, then:

(i) if $V = Y$, Definition 3.2(i) reduces to Definition 2.2(i) for the corresponding vector-valued mapping f defined in (6),

(ii) if $\bar{x} \in \operatorname{dom} F$, $F(\bar{x}) = \{\bar{y}\}$, and $V = Y$ Definition 3.2(ii) reduces to Definition 2.2(ii).

We continue to introduce the definition of C-Lipschitzianity.

Definition 3.3 *Let $F : X \rightrightarrows Y$ be a set-valued mapping with $\operatorname{dom} F \neq \emptyset$, and $C \subset Y$ be a proper convex cone.*

*(i) F is said to be **C-Lipschitz around** $x_0 \in X$ if there is a neighborhood U of x_0, and a constant $l \geq 0$ such that*

$$F(x) \subset F(x') + l||x - x'||_X U_Y + C, \quad \forall x, x' \in U. \tag{8}$$

*(ii) F is said to be **locally C-Lipschitz** on $D \subset X$ if it is C-Lipschitz around any point of D.*

Remark 3.4 *(i) In Definition 3.3(i), if F is C-Lipschitz around x_0, then $x_0 \notin \operatorname{cl}(\operatorname{dom} F)$ or $x_0 \in \operatorname{int}(\operatorname{dom} F)$. Moreover if $U \cap \operatorname{dom} F \neq \emptyset$, then $U \subset \operatorname{dom} F$, and so $x_0 \in \operatorname{int}(\operatorname{dom} F)$.*

(ii) Obviously, if F is C-Lipschitz around $\bar{x} \in \operatorname{dom} F$, from (8), we have that $\mathcal{E}_F$ is Lipschitz-like continuous around $(\bar{x}, \bar{y})$, for all $\bar{y} \in F(\bar{x})$, so F is (ELL) around $(\bar{x}, \bar{y})$, for all $\bar{y} \in F(\bar{x})$.

Definition 3.5 *Let $F : X \rightrightarrows Y$ with $\operatorname{dom} F \neq \emptyset$, and C be a proper convex cone; F is said to be **C-convex** if*

$$\alpha F(x) + (1 - \alpha)F(y) \subset F(\alpha x + (1 - \alpha)y) + C,$$

holds for all $x, y \in \operatorname{dom} F$ and $\alpha \in (0, 1)$ (hence $\operatorname{dom} F$ is convex).

Remark 3.6 *(i) If $F : X \rightrightarrows Y$ is at most single-valued, then F is C-convex in the sense of Definition 3.5 if and only if the corresponding vector-valued function $f : X \to Y^\bullet$ defined in (6) is C-convex in the sense of Definition 2.1.*

(ii) Obviously, $F(x) + C$ is a convex set for all $x \in \operatorname{dom} F$ if F is C-convex, .

In order to study properties of set-valued mappings, Minh and Tan [19] used a scalarization method for set-valued mappings. For given $F : X \rightrightarrows Y$ and $y^* \in C^+$ we define a function $g_{y^*} : X \to \overline{\mathbb{R}}$ by

$$g_{y^*}(x) := \inf_{y \in F(x)} y^*(y), \quad x \in X, \tag{9}$$

with the convention $\inf \emptyset := +\infty$. Obviously, $\operatorname{dom} g_{y^*} = \operatorname{dom} F$ and (for $y^* = 0$) $g_0 = \iota_{\operatorname{dom} F}$, where ι_A is the indicator function of A defined by $\iota_A(x) = 0$ if $x \in A$, and $\iota_A(x) = +\infty$ otherwise.

We will recall some properties of the scalar function g corresponding to properties of F (see [18, 19]). The following proposition is stated in [19, Proposition 2.2] without proof. For convenience of the reader we give a proof of this proposition.

Proposition 3.7 *(i) If F is a C-convex mapping, then g_{y^*} is convex for all $y^* \in C^+$.*

(ii) Conversely, if $F(x) + C$ is closed and convex for all $x \in \operatorname{dom} F \neq \emptyset$, and g_{y^*} is convex for all $y^* \in C^+$, then F is C-convex.

Proof. (i) Let F be C-convex, and $y^* \in C^+$ be chosen arbitrarily; for every $\lambda \in (0,1), x_1, x_2 \in \operatorname{dom} g_{y^*} = \operatorname{dom} F$, we have

$$\begin{aligned} g_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)} y^*(y) = \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)+C} y^*(y) \\ &\leq \inf_{y \in \lambda F(x_1) + (1-\lambda)F(x_2)} y^*(y) \\ &= \inf_{y \in \lambda F(x_1)} y^*(y) + \inf_{y \in (1-\lambda)F(x_2)} y^*(y) \\ &= \lambda \inf_{y \in F(x_1)} y^*(y) + (1-\lambda) \inf_{y \in F(x_2)} y^*(y) \\ &= \lambda g_{y^*}(x_1) + (1-\lambda) g_{y^*}(x_2). \end{aligned}$$

Therefore, g_{y^*} is convex for all $y^* \in C^+$.

(ii) Because $g_0 = \iota_{\operatorname{dom} F}$ is convex, $\operatorname{dom} F$ is convex. Suppose by contradiction that F is not C-convex, so there exist $x_1, x_2 \in \operatorname{dom} F$ and $\lambda \in (0,1)$ such that

$$\lambda F(x_1) + (1-\lambda)F(x_2) \nsubseteq F(\lambda x_1 + (1-\lambda)x_2) + C.$$

One can take $\bar{y} \in \lambda F(x_1) + (1-\lambda)F(x_2)$ such that $\bar{y} \notin F(\lambda x_1 + (1-\lambda)x_2) + C \neq \emptyset$. Because $F(\lambda x_1 + (1-\lambda)x_2) + C$ is closed and convex, there exists $y^* \in Y^*$ such that

$$y^*(\bar{y}) < \inf\{y^*(y) \big| y \in F(\lambda x_1 + (1-\lambda)x_2) + C\}.$$

It follows that $y^* \in C^+ \setminus \{0\}$ and

$$\begin{aligned} g_{y^*}(\lambda x_1 + (1-\lambda)x_2) &= \inf_{y \in F(\lambda x_1 + (1-\lambda)x_2)+C} y^*(y) > y^*(\bar{y}) \\ &\geq \inf_{y \in \lambda F(x_1) + (1-\lambda)F(x_2)} y^*(y) \quad (\text{as } \bar{y} \in \lambda F(x_1) + (1-\lambda)F(x_2)) \\ &= \lambda g_{y^*}(x_1) + (1-\lambda) g_{y^*}(x_2). \end{aligned}$$

This contradicts our assumption on the convexity of g_{y^*}. □

Definition 3.8 *Let $\{f_\alpha\}_{\alpha \in I}$ be a family of functions $f_\alpha : X \to \overline{\mathbb{R}}$, where I is a nonempty parameter set. We say that the family is **equi-Lipschitz** around $x_0 \in X$ if there are a neighborhood U of x_0 and a real number $L > 0$ such that for every $\alpha \in I$, f_α is finite and Lipschitz on U with the same Lipschitz constant L, i.e.,*

$$|f_\alpha(x) - f_\alpha(y)| \leq L||x-y||_X, \quad \forall x, y \in U, \alpha \in I.$$

Proposition 3.9 *Let X, Y be two normed spaces, $F : X \rightrightarrows Y$, and $F(x)+C$ is convex for all $x \in X$. Let $x_0 \in \operatorname{int}(\operatorname{dom} F)$ be such that $F(x_0)$ is C-bounded from below. Then F is C-Lipschitz around x_0 if and only if the family $\{g_{y^*} | y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0.*

Proof. As F is C-Lipschitz around x_0, there exist a neighborhood $U \subset \operatorname{dom} F$ of x_0 and a real number $l > 0$ such that

$$F(x) \subset F(x') + l||x - x'||_X U_Y + C, \quad \forall x, x' \in U. \tag{10}$$

As $F(x_0)$ is C-bounded from below, we assume that $F(x_0) \subset \mu U_Y + C$ for some $\mu > 0$ (see Definition 2.10). From (10) we get

$$F(x) \subset (\mu + l||x - x'||_X) U_Y + C, \quad \forall x \in U. \tag{11}$$

Hence, for all $x \in U, y^* \in C^+, ||y^*||_* = 1$, we get

$$g_{y^*}(x) = \inf_{y \in F(x)} y^*(y) \geq -(\mu + l||x - x'||_X) > -\infty.$$

Hence g_{y^*} is finite on U, for all $y^* \in C^+$, $\|y^*\|_* = 1$.
Taking into account (10), we have for all $x, x' \in U \subset \operatorname{dom} F$

$$g_{y^*}(x) = \inf_{y \in F(x)} y^*(y) \geq \inf_{y \in F(x')} y^*(y) - l||x - x'||_X = g_{y^*}(x') - l||x - x'||_X.$$

Hence

$$g_{y^*}(x') - g_{y^*}(x) \leq l||x - x'||_X, \quad \forall x, x' \in U, y^* \in C^+, ||y^*||_* = 1.$$

By interchanging x and x', we get

$$|g_{y^*}(x) - g_{y^*}(x')| \leq l||x - x'||_X, \quad \forall x, x' \in U, y^* \in C^+, ||y^*||_* = 1.$$

This shows that the family $\{g_{y^*} | y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0.

We prove the converse implication by contradiction: if the family $\{g_{y^*} | y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0 then F is C-Lipschitz around x_0. Assuming that F is not C-Lipschitz around x_0, it follows that for any $n \in \mathbb{N}^*$, there are $x_n, x'_n \in B(x_0, \frac{1}{n})$ with

$$F(x_n) \not\subseteq F(x'_n) + n||x_n - x'_n||_X U_Y + C;$$

hence $x_n \neq x'_n$ for all $n \in \mathbb{N}^*$.
Since $x_0 \in \operatorname{int}(\operatorname{dom} F)$, for n large enough, $B(x_0, \frac{1}{n}) \subset \operatorname{dom} F$, and we can take $y_n \in F(x_n)$ such that

$$y_n \notin B_n := F(x'_n) + n||x_n - x'_n||_X U_Y + C.$$

Since the set B_n is convex and $\operatorname{int} B_n \neq \emptyset$, one can find $y^*_n \in Y^*$ with $||y^*_n||_* = 1$ such that

$$y^*_n(y_n) \leq y^*_n(v) \quad \forall v \in B_n.$$

Hence

$$y_n^*(y_n) \le \inf y_n^*(B_n) = \inf y_n^*(F(x_n')) - n||x_n - x_n'||_X + \inf y_n^*(C).$$

It follows that $y_n^* \in C^+$ for large $n \in \mathbb{N}$ and

$$g_{y_n^*}(x_n) \le g_{y_n^*}(x_n') - n||x_n - x_n'||_X,$$

that is

$$n||x_n - x_n'||_X \le g_{y_n^*}(x_n') - g_{y_n^*}(x_n) \le l||x_n - x_n'||_X,$$

hence $n \le l$, a contradiction for large n. □

Remark 3.10 *Proposition 3.9 is stated in [18, Theorem 2.5] without the assumption that $F(x_0)$ is C-bounded from below. Taking $F(x) = Y$ $\forall x \in X$, it is clear that F is C-Lipschitz, but $\{g_{y^*} \big| y^* \in C^*, \|y^*\| = 1\}$ is not equi-Lipschitz.*

Definition 3.11 *We say that $F : X \rightrightarrows Y$ is **weakly C-upper bounded** on a set $A \subseteq X$ if there exists $\mu' > 0$ such that $F(x) \cap (\mu' U_Y - C) \neq \emptyset$ for all $x \in A$.*

Theorem 3.12 *Let X, Y be two normed spaces, C be a proper convex cone, and $F : X \rightrightarrows Y$ be C-convex. If F is C-bounded from below and weakly C-upper bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$, then F is C-Lipschitz around x_0.*

Proof. Without loss of generality we suppose that $x_0 = 0$ and $0 \in F(0)$. As F is C-bounded from below and weakly C-upper bounded on a neighborhood $U = \theta U_X \subset \operatorname{dom} F$ of 0 ($\theta > 0$), taking into account Definition 2.10 and Definition 3.11 there exist a real number $\mu > 0$ such that $F(U) \subset \mu U_Y + C$ and $F(x) \cap (\mu U_Y - C) \neq \emptyset$ for all $x \in U$.
Take $y^* \in C^+$ with $||y^*||_* = 1$. Let $\bar{x} \in U$ be arbitrary, take $\bar{y} \in F(\bar{x})$, $c \in C$, and $y' \in \mu U_Y$ such that $\bar{y} = y' - c$. We have

$$\begin{aligned} g_{y^*}(\bar{x}) &= \inf_{y \in F(\bar{x})} y^*(y) \le y^*(\bar{y}) = y^*(y' - c) = y^*(y') - y^*(c) \\ &\le y^*(y') \le \|y^*\|_* \|y'\|_Y = \|y'\|_Y \le \mu, \quad \forall \bar{x} \in U. \end{aligned}$$

Analogously, from $F(U) \subset \mu U_Y + C$, we get $g_{y^*}(x) = \inf_{y \in F(x)} y^*(y) \ge -\mu$ for every $x \in U$.
It follows that g_{y^*} is finite on U and

$$g_{y^*}(x) \le g_{y^*}(0) + 2\mu, \quad \forall x \in U = \theta U_X.$$

By Proposition 3.7, g_{y^*} is convex. Applying Lemma 2.4 for the convex function g_{y^*} and $\theta' \in (0, \theta)$, we get

$$|g_{y^*}(x) - g_{y^*}(x')| \le L\|x - x'\|_X, \quad \forall x, x' \in \theta' U_X,$$

where $L := 2\mu(\theta + \theta')/[\theta(\theta - \theta')]$; clearly L does not depend on y^*.
So $\{g_{y^*} \mid y^* \in C^+, ||y^*||_* = 1\}$ is equi-Lipschitz around x_0 with the Lipschitz constant L.
Because of the convexity of F, $F(x) + C$ is convex for all $x \in X$. Applying Proposition 3.9, we have F is C-Lipschitz around x_0. □

Theorem 3.13 *Let X, Y be two normed spaces, $C \subset Y$ be a proper convex cone, and let $F : X \rightrightarrows Y$ be C-convex. If F is C-bounded from below and weakly C-upper bounded on a neighborhood of some point $x \in \operatorname{int}(\operatorname{dom} F)$, then F is locally C-Lipschitz on $\operatorname{int}(\operatorname{dom} F)$.*

Remark 3.14 *1. It is clear that the assumptions in Theorem 3.13 are much weaker than those of [18, Theorem 2.9]; the assumption X be finite dimensional space is not necessary. We even don't need any more conditions concerning the cone C like $C^+ = \operatorname{cone}(\operatorname{conv}\{y_1^*, \dots, y_n^*\})$ for some $y_1^*, \dots, y_n^* \in Y^*$ and $0 \notin \operatorname{conv}\{y_1^*, \dots, y_n^*\}$ in [18].*

2. By [10, Proposition 2.6.2], if F is C-bounded from below on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$ then F is C-bounded from below on a neighborhood of x for every $x \in \operatorname{dom} F$.

3. If F is weakly C-upper bounded on a neighborhood of $x_0 \in \operatorname{int}(\operatorname{dom} F)$ then F is weakly C-upper bounded on a neighborhood of x for every $x \in \operatorname{int}(\operatorname{dom} F)$. Indeed, assume that $F(x) \cap (\mu U_Y - C) \neq \emptyset\ \forall x \in B(x_0, r)$. Fix $\bar{x} \in \operatorname{int}(\operatorname{dom} F)$. There exist $x_1 \in \operatorname{dom} F$, $\lambda \in (0,1)$ such that $\bar{x} = \lambda x_0 + (1-\lambda)x_1$. Fix $y_1 \in F(x_1)$. Take $u \in rU_X$. Then $\bar{x}+\lambda u = \lambda(x_0+u)+(1-\lambda)x_1$, and there exists $y_u \in F(x_0+u) \cap (\mu U_Y - C)$; hence $y_u = \mu v_u - c_u$ with $\|v_u\| \le 1$, $c_u \in C$. Then $\lambda(\mu v_u - c_u) + (1-\lambda)y_1 \in F(\bar{x}+\lambda u) + C$, whence $\exists c_u' \in C$ and $\bar{y} \in F(\bar{x}+\lambda u)$ such that $\bar{y} = \lambda(\mu v_u - c_u) + (1-\lambda)y_1 - c_u' \in \bar{\mu} U_Y - C$ with $\bar{\mu} = \lambda\mu + (1-\lambda)\|y_1\|$.

Especially, when $F : X \rightrightarrows Y$ is an at most single-valued mapping, we have a result similar to Theorem 3.12 for the corresponding vector-valued mapping $f : X \to Y^\bullet$ (see Remark 3.1):

Theorem 3.15 *Under the hypotheses of Theorem 3.12, if $F : X \rightrightarrows Y$ is an at most single-valued mapping, C a normal cone, then $f : X \to Y^\bullet$ (defined in (6)) is Lipschitz around x_0.*

Proof. Theorem 3.12 shows that f is C-Lipschitz around $x_0 \in \operatorname{int}(\operatorname{dom} F)$, so there is a neighborhood $U \subset \operatorname{dom} F$ of x_0 and a constant l such that

$$F(x) \subset F(x') + l||x - x'||_X U_Y + C, \quad \forall x, x' \in U,$$

or equivalently

$$f(x) \in f(x') + l||x - x'||_X U_Y + C, \quad \forall x, x' \in U. \tag{12}$$

Because C is normal, there is $\rho > 0$ such that

$$(\rho U_Y + C) \cap (\rho U_Y - C) \subset U_Y.$$

From (12), we have

$$\frac{\rho(f(x) - f(x'))}{l||x - x'||_X} \subset \rho U_Y + C, \quad \forall x, x' \in U, x \neq x'.$$

By interchanging x and x', we also have

$$\frac{\rho(f(x)-f(x'))}{l||x-x'||_X} \subset \rho U_Y - C, \quad \forall x, x' \in U, x \neq x'.$$

Therefore,

$$\frac{\rho(f(x)-f(x'))}{l||x-x'||_X} \subset (\rho U_Y + C) \cap (\rho U_Y - C) \subset U_Y, \quad \forall x, x' \in U, x \neq x'.$$

This shows that f is Lipschitzian around x_0. □

When C is normal, by Proposition 2.11 the C-boundedness from below and weakly C-upper boundedness of f can replace by C-boundedness from above in Theorem 3.15, so we get the assertions of Theorem 2.14.

4 Applications

4.1 Variational analysis and generalized differentiation

This section is devoted to present the definitions and properties of the basic generalized differential constructions held in ***Asplund spaces***, i.e., a Banach space X is Asplund if every convex continuous function on a nonempty open convex subset D of X is generically Fréchet differentiable at each point of some nonempty dense G_δ subset of D. The class of Asplund spaces is quite broad, for example every reflexive Banach space, every Banach space with a separable dual; in particular, c_0 and $l_p, L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces, but l_1 and l_∞ are not Asplund spaces.
We will not recall all the calculus for normal cones and coderivative, the readers can see them in Mordukhovich's books [20, 21].
Consider a set-valued mapping $F : X \rightrightarrows X^*$ between an Asplund space and its dual, and a subset Ω of X. We define the Painlevé-Kuratowski outer limits of F at $\bar{x}$ with respect to the norm topology of X and the weak* topology of X^* by

$$\limsup_{x\to\bar{x}} F(x) := \{x^* \in X^* \big| \exists \text{ sequence } x_k \to \bar{x}, \text{ and } x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in F(x_k) \text{ for all } k \in \mathbb{N}\}. \tag{13}$$

In this section, we use the notation $x' \xrightarrow{\Omega} x$ for $x' \to x$ with $x' \in \Omega$. We define the ***Fréchet normal cone*** to Ω at $x \in \Omega$ by

$$\hat{N}_L(x,\Omega) := \left\{ x^* \in X^* \Big| \quad \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u-x\rangle}{||u-x||} \leq 0 \right\}, \tag{14}$$

Now, assume that Ω is closed around a given point $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap U$ is a closed set. The ***(basic, limiting, or Mordukhovich) normal***

cone to Ω at $\bar{x}$ is defined by

$$N_L(\bar{x};\Omega) := \limsup_{x\to\bar{x}} \hat{N}_L(x;\Omega) \tag{15}$$
$$= \{x^* \in X^* \big| \exists x_k \to \bar{x}, x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \hat{N}(x_k;\Omega)\},$$

If Ω is a convex set, then both the cones (14) and (15) reduce to the normal cone of convex analysis:

$$\hat{N}_L(\bar{x};\Omega) = N_L(\bar{x};\Omega) = \big\{x^* \in X^* \big| \quad x^*(x-\bar{x}) \le 0 \quad \forall x \in \Omega\big\} \tag{16}$$

(see [20, Proposition 1.5]).

The (basic, normal, Mordukhovich) *coderivative* mapping $D^*F(\bar{x},\bar{y})\colon Y^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{y}) \in \operatorname{gph} F$ is defined by

$$D^*F(\bar{x},\bar{y})(y^*) := \big\{x^* \in X^* \big|\ (x^*,-y^*) \in N_L\big((\bar{x},\bar{y});\operatorname{gph} F\big)\big\}, \tag{17}$$

which is a positively homogeneous multifunction of $y^* \in Y^*$; we omit $\bar{y} = f(\bar{x})$ in (17) if $f\colon X \to Y$ is single-valued, and the set-valued mapping $F(x) = \{f(x)\}$ for $x \in X$.

The ***(basic, limiting, Mordukhovich) subdifferential*** for functions $f : X \to \overline{\mathbb{R}}$ at $\bar{x} \in X$ with $|f(\bar{x})| < \infty$ (see Mordukhovich [20]) is denoted by

$$\partial_L f(\bar{x}) := \big\{x^* \in X^* \big|\ (x^*,-1) \in N_L\big((\bar{x},f(\bar{x}));\operatorname{epi} f\big)\big\}. \tag{18}$$

We put $\partial_L f(\bar{x}) := \emptyset$ if $|f(\bar{x})| = \infty$.

Note that the limiting subdifferential agrees with the subdifferential of convex analysis when f is convex (see [20, Theorem 1.93]).

Let us recall the notion of *strict Lipschitzianity* introduced in [20, Theorem 3.25]. Consider a function $f : X \to Y$ and a point $\bar{x} \in X$. f is called ***strictly Lipschitzian*** at $\bar{x}$ if f is Lipschitz around $\bar{x}$ and the sequence

$$y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in \mathbb{N},$$

contains a norm convergent subsequence whenever $v \in X$, $x_k \to \bar{x}$ and $t_k \downarrow 0$. If f is strictly Lipschitzian, the relationship between the coderivative of a vector function and the subdifferential of its scalarization is given by [20, Theorem 3.28]:

$$D^*f(\bar{x})(y^*) = \partial_L(y^* \circ f)(\bar{x}). \tag{19}$$

4.2 Vector and set-valued optimization problems

This section will be concerned with nonconvex constrained optimization problems given by:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D, \tag{20}$$

where X, Y are two Asplund spaces, the feasible set $D \subset X$ is not supposed to be convex, C is a proper closed convex pointed cone in Y. If $F : X \rightrightarrows Y$ is a set-valued mapping,

(20) is a problem of ***set-valued optimization***, and "minimization" is to be understood in the sense of Definition 4.2 below, while the term ***vector optimization*** is used when $F = f : X \to Y$ is a single-valued mapping, and "minimization" is to be understood in the sense of Definition 4.1.
The advantage of using optimality conditions in Asplund spaces lies in the fact that the calculus for Mordukhovich coderivatives and subdifferentials are rather complete. In Section 4.1, we summerized the relevant formula on generalized differentiation in Asplund spaces.
In order to formulate solution concepts for the vector-valued problem (20), we introduce the well-known notions of (weak) Pareto optimal point or Pareto efficient point.

Definition 4.1 *Let A be a nonempty subset of Y.*

*(i) We define the set of **Pareto minimal points** of A with respect to C by*

$$Min(A; C) := \{\bar{y} \in A | A \cap (\bar{y} - C) = \{\bar{y}\}\}.$$

*(ii) The set of **weakly Pareto minimal points** of A with respect to C (with $\operatorname{int} C \neq \emptyset$) is given by*

$$WMin(A; C) := \{\bar{y} \in A | A \cap (\bar{y} - \operatorname{int} C) = \emptyset\}.$$

By choosing $A := F(D)$ in Definition 4.1, where $F : X \rightrightarrows Y$ and $F(D) := \bigcup_{x \in D} F(x)$, we obtain the following solution concept for a set-valued optimization problem.

Definition 4.2 *Let X, Y be two Banach spaces, D be a nonempty subset of X. Let $F : X \rightrightarrows Y$ be a set-valued mapping.*

*(i) A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{x} \in D$ is said to be a **minimizer** of the problem (20) for the set-valued mapping F with respect to C if $\bar{y} \in \operatorname{Min}(F(D); C)$.*

*(ii) A pair $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{x} \in D$ is said to be a **weak minimizer** of the problem (20) for the set-valued mapping F with respect to C if $\bar{y} \in WMin(F(D); C)$.*

In the single-valued case that $f : X \to Y$, $A := f(D)$.

*(iii) A point $\bar{x} \in D$ with $f(\bar{x}) \in \operatorname{Min}(f(D); C)$ is called **Pareto efficient solution**.*

*(iv) A point $\bar{x} \in D$ with $f(\bar{x}) \in \operatorname{WMin}(f(D); C)$ is called **weakly Pareto efficient solution**.*

Our goal is to use scalarizations of the problem (20), so we will introduce a powerful tool that will be needed throughout this section. We will recall its properties (see [10, Theorem 2.3.1] and [7, Lemma 2.4]) in terms of Fenchel classical subdifferential ∂ in the sense of convex analysis.

Lemma 4.3 *Let $C \subset Y$ be a proper, closed and convex cone with nonempty interior in a Banach space. Given $e \in \operatorname{int} C$, the function $s_e : Y \to \mathbb{R}$ defined by*

$$s_e(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda \cdot e \in y + C\} \tag{21}$$

is continuous, convex, sublinear, and strictly-$\operatorname{int} C$*-monotone. Moreover, the following relations hold:*

(i) $s_e(y + te) = s_e(y) + t \quad \forall y \in Y,\ \forall t \in \mathbb{R}.$

(ii) $\partial s_e(0) = \{y^* \in C^+ \mid y^*(e) = 1\}$

(iii) $\partial s_e(y) = \{y^* \in C^+ \mid y^*(e) = 1, y^*(y) = s_e(y)\}$ *for any* $y \in Y$.

(iv) s_e *is* $d(e, \operatorname{bd}(C))^{-1}$*-Lipschitz and for every* $y \in Y$ *and* $y^* \in \partial s_e(y)$ *one has* $||e||^{-1} \leq ||y^*|| \leq d(e, \operatorname{bd}(C))^{-1}$.

(v) Given a nonempty subset A of Y, one has that $\bar{y} \in A$ is a weakly Pareto minimal point of A with respect to C if and only if

$$s_e(y - \bar{y}) \geq 0 \qquad \forall y \in A.$$

We will apply Lemma 4.3 to scalarize the vector optimization problem in Proposition 4.4, also to find the necessary optimality conditions for a weakly Pareto efficient solution of vector optimization problems in Theorems 4.5 and 4.6.

4.2.1 Necessary optimality conditions for vector optimization problems

In this section, we consider the vector optimization problem:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in D, \tag{VP}$$

where X, Y are Asplund spaces, $f : X \to Y$ is a single-valued mapping, $D \subset X$ is not supposed to be convex, and C is a proper closed convex pointed cone in Y. Throughout the section, let $\bar{x} \in D$ and $\bar{y} = f(\bar{x})$ (or $(\bar{x}, \bar{y}) \in \operatorname{gph} f$).
In the case $Y = \mathbb{R}$ and $C = \mathbb{R}_+$, vector optimization reduces to scalar optimization.

First, we study the case $\operatorname{int} C \neq \emptyset$, and take $e \in \operatorname{int} C$.
We recall the epigraphical multifunction of $f : X \to Y$, $\mathcal{E}_f : X \rightrightarrows Y$ given by (compare (4) for set-valued mappings F)

$$\mathcal{E}_f(x) := f(x) + C.$$

Set

$$g(x) := f(x) - f(\bar{x}), \text{ and } G(x) \equiv \mathcal{E}_g(x) := g(x) + C. \tag{22}$$

We use the scalarization scheme s_e given by (21) to convert the vector optimization problem into an equivalent set-valued optimization problem including a suitable scalarization

(see [1, Theorem 3.8]).
We define the composition function $s_e \circ G : X \rightrightarrows \mathbb{R}$ by

$$(s_e \circ G)(x) := s_e(G(x)) = \{s_e(y) \mid y \in G(x)\} = s_e(g(x)) + \mathbb{R}^+, \tag{23}$$

where s_e is given by (21).

In the following proposition we study the relationship between weakly Pareto efficient solutions of (VP) and minimizers of the corresponding set-valued optimization problem with a set-valued objective function defined by (23) without the closedness and convexity assumptions concerning D.

Proposition 4.4 *Let C be a proper closed convex pointed cone in Y with nonempty interior, let D be a nonempty subset of X, and $f : X \to Y$. If $\bar{x} \in D$ is a weakly Pareto efficient solution of* (VP) *and $G : X \rightrightarrows Y$ is given by* (22)*, then $(\bar{x}, 0)$ is a minimizer in the sense of Definition 4.2 of the following set-valued optimization problem:*

$$\textit{minimize} \quad \big(s_e \circ G\big)(x) \quad \textit{subject to} \quad x \in D, \tag{24}$$

where $e \in \operatorname{int} C$ is arbitrary and the set-valued objective function $s_e \circ G : X \rightrightarrows \mathbb{R}$ is given by (23)*.*

Proof. Suppose that $\bar{x} \in X$ is the weakly Pareto efficient solution of (VP). Applying [14, Lemma 4.13 (a)], $f(\bar{x}) \in \operatorname{WMin}\big(f(D); C\big)$ implies that $f(\bar{x}) \in \operatorname{WMin}\big(f(D) + C; C\big)$ under the given assumption concerning C.
Take an arbitrary point $e \in \operatorname{int} C$, and consider the function s_e as in (21), and apply Lemma 4.3(v) with proper closed cone C to the weakly Pareto minimal point $f(\bar{x})$, we have

$$s_e\big(f(x) + c - f(\bar{x})\big) \geq 0 \qquad \forall x \in D \text{ and } \ c \in C,$$

we thus get

$$s_e(y) \geq 0 \quad \forall x \in D \text{ and } y \in G(x).$$

Obviously, $\big(s_e \circ G\big)(\bar{x}) = 0$, so we conclude that $(\bar{x}, 0)$ is minimizer of (24). □

By using the technique like in [1, Theorem 3.8], we will derive Lagrangian necessary conditions for weakly Pareto efficient solution of vector optimization problems without convexity assumptions concerning the constraint set on the basis of Mordukhovich coderivative, limiting subdifferential and classical Fenchel subdifferential in infinite dimensional spaces. However, we do not ask the closedness of $\operatorname{cone}(f(D) + C - \bar{y})$ from the hypotheses, and suppose the stronger condition that $\operatorname{int} C \neq \emptyset$.

Theorem 4.5 *Consider the vector optimization problem* (VP)*. Let C, D be as in Proposition 4.4, and a function $f : X \to Y$. Let $\bar{x} \in D$ be a weakly Pareto efficient solution of* (VP)*. If f is Lipschitz around $\bar{x}$, and the constraint set D is closed around this point, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in D^*\mathcal{E}_f(\bar{x}, \bar{y})(y^*) + N_L(\bar{x}; D), \qquad \textit{with } \bar{y} = f(\bar{x}). \tag{25}$$

Further, if f is strictly Lipschitzian at $\bar{x}$, then (25) reduces to

$$0 \in \partial_L(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D). \tag{26}$$

Proof. We take $e \in \operatorname{int} C$, and consider the function s_e defined in (21). Because of Lemma 4.3(iv), s_e is Lipschitz, hence Lipschitz around $0 \in Y$, so the composition $(s_e \circ G)$ is Lipschitz-like around $(\bar{x}, 0)$ by [20, Corollary 3.15].
Because of the construction of G, it is not difficult to see that $\operatorname{gph} G$ is closed around $(\bar{x}, \bar{y})$, therefore $\operatorname{gph}(s_e \circ G)$ is closed around $(\bar{x}, 0)$ (as s_e is continuous).
By Proposition 4.4, $(\bar{x}, 0)$ is a minimizer of the problem (24), and the assumptions of the coderivative necessary condition in [4, Theorem 5.3] are fulfilled, applying it, we have

$$0 \in D^*\big(s_e \circ G\big)(\bar{x}, 0)(1) + N_L(\bar{x}; D). \tag{27}$$

Using the chain rule of coderivative of the composition in [20, Theorem 3.13] for the single-valued Lipschitz continuous outer function s_e, and the inner Lipschitz-like mapping G gives

$$D^*(s_e \circ G)(\bar{x}, 0)(1) \subset D^*G(\bar{x}, 0) \circ D^*s_e(0)(1). \tag{28}$$

From the definition of coderivatives (17), we get

$$x^* \in D^*G(\bar{x}, 0)(y^*) \Longleftrightarrow (x^*, -y^*) \in N_L((\bar{x}, 0); \operatorname{gph} G) \tag{29}$$

since

$$\begin{aligned} N_L((\bar{x}, 0); \operatorname{gph} G) &= N_L((\bar{x}, 0); \operatorname{epi} g) = N_L((\bar{x}, 0); \operatorname{epi}(f - f(\bar{x}))) \\ &= \hat{N}_L((\bar{x}, \bar{y} - \bar{y}); \operatorname{epi}(f - \bar{y})) = N_L((\bar{x}, \bar{y}); \operatorname{epi} f) \\ &= N_L((\bar{x}, \bar{y}); \operatorname{gph} \mathcal{E}_f). \end{aligned} \tag{30}$$

Substituting (30) into (29) we obtain

$$\begin{aligned} x^* \in D^*G(\bar{x}, 0)(y^*) &\Longleftrightarrow (x^*, -y^*) \in N_L((\bar{x}, \bar{y}); \operatorname{gph} \mathcal{E}_f) \\ &\Longleftrightarrow x^* \in D^*\mathcal{E}_f(\bar{x}, \bar{y})(y^*), \end{aligned}$$

we deduce that

$$D^*G(\bar{x}, 0)(y^*) = D^*\mathcal{E}_f(\bar{x}, \bar{y})(y^*) \tag{31}$$

By [20, Theorem 1.80], the definition of s_e in (21), and Lemma 4.3(ii), we have

$$D^*s_e(0)(1) = \partial s_e(0) = \{y^* \in C^+ | y^*(e) = 1\}.$$

Substituting (28) and (31) into (27), we see that (25) holds for some $y^* \in C^+ \setminus \{0\}$, we obtain the first assertion of the theorem.

Now, if we assume additionally that f is strictly Lipschitzian at $\bar{x}$, using [20, Theorem 3.28] we can show that $D^*f(\bar{x})(y^*) = \partial_L(y^* \circ f)(\bar{x})$. By the definition of coderivative, it is easily seen that $D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) \subseteq D^*f(\bar{x})(y^*)$. Thus $D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) \subseteq \partial_L(y^* \circ f)(\bar{x})$, which completes the proof. □

To obtain necessary optimality conditions for vector optimization problems, Dutta and Tammer [8] used the Mordukhovich's subdifferential when X is an Asplund space, Y is finite dimensional (see [8, Theorem 3.2]), and Ioffe's approximate subdifferential in general Banach spaces (see [8, Theorem 3.1]). Obviously, from Theorem 4.5 one deduces the assertion of [8, Theorem 3.2]. In Durea and Tammer [7], the authors enlarged the framework of article [8] to the concepts of abstract subdifferentials satisfying certain axioms, and considered not only "exact calculus rules" (see [7, Theorem 3.1]) but also "fuzzy calculus rules" (see [7, Theorem 4.1]).

We consider now the problem (VP) with a feasible set not necessarily convex, and an objective function $f : X \to Y$ being C-convex. We will apply results from Section 2 that a C-convex, locally C-bounded function is locally Lipschitz with the assumption about the normality of the cone C. Hence all the calculus rules of coderivative and generalized differentiation for locally Lipschitz mappings in [20, 21] are fulfilled for the class of C-convex mappings.

In our next theorem, under the assumption that f is C-convex, we will establish the Lagrangian necessary condition in form of (26) using the Fenchel subdifferential in the sense of convex analysis. We suppose the C-convexity of f such that we can apply [20, Proposition 1.5] for deriving a scalarization result in the proof of the following theorem without assuming that f is strictly Lipschitz continuous. Furthermore, we do not assume that D is convex.

Theorem 4.6 *Let C be a proper closed w-normal cone in Y with nonempty interior, let D be a nonempty subset of X, and $f : X \to Y$ be C-convex. Let $\bar{x} \in D$ be a weakly Pareto efficient solution of* (VP). *If f is C-bounded from above on a neighborhood U of $\bar{x}$, and the constraint set D is closed around this point, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in \partial(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D), \tag{32}$$

Proof. If C is w-normal, then the cone C satisfies the assumptions of Theorem 2.16, and so f is C-convex and C-bounded from above around $\bar{x}$, thus f is Lipschitz around $\bar{x}$. Hence all the calculus rules of coderivative and generalized differentiation for locally Lipschitz mappings in [20, Chapter 1] are fulfilled for the class of C-convex mappings. Assumptions of Theorem 4.5 are fulfilled, therefore there exists $y^* \in C^+ \setminus \{0\}$ such that (25) holds.
We have that

$$x^* \in D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) \Longleftrightarrow (x^*, -y^*) \in N_L((\bar{x},\bar{y}); \operatorname{epi} f) \tag{33}$$

As f is C-convex, it follows that $\operatorname{epi} f$ is convex. Because of the convexity of $\operatorname{epi} f$, [20, Proposition 1.5] can be applied such that we get $N_L((\bar{x},\bar{y}); \operatorname{epi} f) = \hat{N}_L((\bar{x},\bar{y}); \operatorname{epi} f)$, i.e., representation (16) holds. We can rewrite (33) as

$$\begin{aligned} x^* \in D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) &\Longleftrightarrow (x^*,-y^*) \in \hat{N}_L((\bar{x},\bar{y}); \operatorname{epi} f) \\ &\Longleftrightarrow x^*(x-\bar{x}) - y^*(y-\bar{y}) \le 0 \quad \forall (x,y) \in \operatorname{epi} f. \end{aligned}$$

Note that $y^* \in C^+$, hence we have:

$$\begin{aligned} x^* \in D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) &\Longleftrightarrow x^*(x-\bar{x}) - y^*(f(x)-f(\bar{x})) \le 0 \quad \forall x \in X. \\ &\Longleftrightarrow x^* \in \partial(y^* \circ f)(\bar{x}). \end{aligned}$$

It follows that $D^*\mathcal{E}_f(\bar{x},\bar{y})(y^*) = \partial(y^* \circ f)(\bar{x})$, this gives (32) when substituted in (25), and the proof is complete. □

If we remove the local Lipschitzianity and the strict Lipschitzianity assumptions in [1, Theorem 3.8], and if we additionally assume that C is w-normal, $f : X \to Y$ is C-convex and C-bounded from above around $\bar{x}$, then we also obtain a result as Theorem 4.6 for Pareto efficient solution in case $\operatorname{int} C = \emptyset$.
Our next goal is to find Lagrangian necessary conditions for Pareto efficient solutions of the problem (VP) in the case $\operatorname{int} C = \emptyset$, which is much harder than previous one. In order to overcome the difficulties of this case, we refer the reader to [1, 6] for more references and discussions. In [6], the authors discussed about three possibilities to deal with this case, however in this section we only mention the result of Bao and Tammer [1]. In the following theorem we consider the case $\operatorname{int} C = \emptyset$ and we don't need the Lipschitzianity assumptions like in [1, Theorem 3.8], if we suppose the C-convexity and C-boundedness of the objective function f, then we will get the following result which is similar to Theorem 4.6.

Theorem 4.7 *Let C be a proper closed w-normal cone in Y with empty interior, let D be a nonempty subset of X, and $f : X \to Y$ be C-convex. Let $\bar{x} \in D$ be a Pareto efficient solution of* (VP) *and $\bar{y} = f(\bar{x})$. Assume that f is C-bounded from above around $\bar{x} \in D$, and the constraint set D is closed around this point. Furthermore, suppose that* $\operatorname{cone}(f(D) + C - \bar{y})$ *is closed, then for every $e \in C \setminus \{0\}$, there exists $y^* \in C^+$ with $y^*(e) = 1$ such that*

$$0 \in \partial(y^* \circ f)(\bar{x}) + N_L(\bar{x}; D). \tag{34}$$

Proof. Taking into account Theorem 2.16, a C-convex, C-bounded function is locally Lipschitz. We now follow the proof of Theorem 4.6 and the results of [1, Theorem 3.8] and [1, Corollary 3.2] to obtain assertion (34). □

4.2.2 Necessary optimality condition for set-valued optimization problems

We are now going to establish necessary conditions for minimizers of the following set-valued optimization problem:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in D, \tag{SP}$$

where X, Y are Asplund spaces, $F : X \rightrightarrows Y$ is a set-valued mapping, C is a proper closed convex pointed cone in Y, $D \subset X$ is not necessarily convex.

In this section, we mention again the concept epigraphically Lipschitz-like (ELL) given in Definition 3.2(iv). A set-valued $F : X \rightrightarrows Y$ is (ELL) around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with modulus $l \geq 0$ if there exist neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\mathcal{E}_F(x) \cap V \subset \mathcal{E}_F(u) + l||x - u||U_Y \quad \forall x, u \in U.$$

We will provide necessary optimality conditions for minimizers of C-convex set-valued mappings without assuming that the constraint set D is convex, using a scalarization by means of the function s_e defined in (21).

Theorem 4.8 *Let X, Y be Asplund spaces, C be a proper closed convex pointed cone in Y, and D be a nonempty subset of X. Let $F : X \rightrightarrows Y$ be C-convex, $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ be a minimizer of* (SP)*. Suppose that F is C-bounded from below and weakly C-upper bounded on a neighborhood of $\bar{x}$,* $\operatorname{epi} F$ *is closed around $(\bar{x}, \bar{y})$, and the constraint set D is closed around $\bar{x}$. Assume furthermore that* $\operatorname{cone}\big(F(D) + C - \bar{y}\big)$ *is closed; then for every $e \in C \setminus \{0\}$, there exists a dual element $y^* \in C^+$ with $y^*(e) = 1$ such that*

$$0 \in D^*\mathcal{E}_F(\bar{x}, \bar{y})(y^*) + N_L(\bar{x}; D). \tag{35}$$

Proof. It follows from Theorem 3.12 and Remark 3.4(ii) that if F is C-convex, C-bounded from below and weakly C-upper bounded on a neighborhood of $\bar{x}$, then F is (ELL) at $(\bar{x}, \bar{y}) \in \operatorname{gph} F$ with $\bar{y} \in F(\bar{x})$. Hence we get the assertion (35) as a corollary of [1, Theorem 3.10]. □

Note that when D is closed around $\bar{x}$ and $\operatorname{epi} f$ is closed around $(\bar{x}, f(\bar{x}))$, then the indicator functions of the sets D and $\operatorname{epi} f$ are lower-semicontinuous around $\bar{x}$ and $(\bar{x}, f(\bar{x}))$, respectively. Thus, the local closedness assumptions are essential in all Theorems of Section 4 to apply the necessary conditions of [4, Theorem 5.3], in comparison with ones in [1, 2, 3, 11, 12]. We know that if $F = f : X \to Y$ is at most single-valued, then the necessary condition in Theorem 4.8 reduces to that in Theorem 4.7.

4.3 Necessary optimality conditions for approximation problems

Finally, we will introduce an example of Theorem 4.6 in vector control approximation problems which play an important role in optimization theory, and many pratical problems can be described as approximation problems. Let X, Y, Z be real reflexive Banach spaces, $C \subset Y$ be a proper closed pointed convex cone. We denote the set of linear continuous mappings from X to Y by $L(X, Y)$.
We define $|||\cdot||| : Z \to C$ a **vector-valued norm** if for all $z, z_1, z_2 \in Z$ and $\lambda \in \mathbb{R}$,

1. $|||z||| = 0 \Longleftrightarrow z = 0$;
2. $|||\lambda z||| = |\lambda| \quad |||z|||$;

3. $\|z_1 + z_2\| \in \|z_1\| + \|z_2\| - C$.

We recall the subdifferential for vector-valued functions (denoted by $\partial^{\leq}$) (see Jahn [14])

$$\partial^{\leq} f(z_0) = \{T \in L(Z,Y) \mid f(z) - f(z_0) \in T(z) - T(z_0) + C \;\forall\, z \in Z\}.$$

It follows that

$$\partial^{\leq}\|\cdot\|(0) = \{T \in L(Z,Y) \mid \|z\| - T(z) \in C \;\forall\, z \in Z\},$$

and

$$\partial^{\leq}\|\cdot\|(z) = \{T \in \partial^{\leq}\|\cdot\|(0) \mid T(z) = \|z\|\} \quad \forall\, z \in Z. \tag{36}$$

Moreover, if $\|\cdot\|$ is continuous, and C has the Daniell property which means that every decreasing net (i.e., $i \leq j$ implies $x_j \leq x_i$) having a lower bound converges to its infimum, then $\partial^{\leq}\|\cdot\| \neq \emptyset$ (see Jahn [14]).

Furthermore, if we assume additionally that C has a weakly compact base, we get the well-known result of Valadier [29] which is useful in the sequel.

Theorem 4.9 *(Valadier [29]) Let $(Y, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be real reflexive Banach spaces, and $C \subset Y$ a proper convex Daniell cone with a weakly compact base. If $f : Z \to Y$ is a C-convex mapping, continuous at some point of its domain, then for every $z \in Z$ and $y^* \in C^+$ one has*

$$y^* \circ \partial^{\leq} f(z) = \partial(y^* \circ f)(z).$$

We consider now the **vector control approximation problem** with the concept of weakly Pareto efficient solution introduced in Definition 4.2:

$$\mathrm{WMin}(f(D); C), \tag{37}$$

where $C \subset Y$ is a proper closed pointed convex cone, $D \subset X$ is closed and not supposed to be convex, and $f : X \to Y$ given by

$$f(x) := f_1(x) + \sum_{i=1}^{n} \alpha_i \|A_i(x) - a^i\|,$$

with $f_1 : X \to Y, A_i \in L(X,Z)$, $a_i \in Z$.

Lemma 4.10 *We assume that C is a proper w-normal cone. If the vector-valued norm $\|\cdot\| : Z \to C$ is continuous around a given point $z \in Z$, then $\|\cdot\|$ is Lipschitz.*

Proof. From the continuity assumption, $\|\cdot\|$ is bounded around z, therefore $\|\cdot\|$ is C-bounded from above around z. Obviously, $\|\cdot\|$ is C-convex because of definition of vector-valued norm. Applying Theorem 2.17, then $\|\cdot\|$ is locally Lipschitz. Hence there exist $r, l > 0$ such that

$$\big|\big| \|z_1\| - \|z_2\| \big|\big|_Y \leq l \big|\big| z_1 - z_2 \big|\big|_Z \quad \forall z_1, z_2 \in rU_Z.$$

Take $z_1, z_2 \in Z$. There exists $\alpha > 0$ such that $\alpha z_1, \alpha z_2 \in rU_Z$. Then

$$\big|\big|\ \|\alpha z_1\| - \|\alpha z_2\|\ \big|\big|_Y \leq l \big|\big|\alpha z_1 - \alpha z_2\big|\big|_Z,$$

whence

$$\big|\big|\ \|z_1\| - \|z_2\|\ \big|\big|_Y \leq l \big|\big|z_1 - z_2\big|\big|_Z.$$

□

Remark 4.11 *Obviously, $\|A_i(\cdot) - a^i\|$ is also C-convex, hence with the assumptions in Lemma 4.10, $\|\cdot\|$ is continuous and C is w-normal, then $\|A_i(\cdot) - a^i\|$ is also continuous, hence $\|A_i(\cdot) - a^i\|$ is bounded by $||.||_Y$ around a given point $\bar{x} \in X$. Employing Theorem 2.16, we obtain that $\|A_i(\cdot) - a^i\|$ is Lipschitz around $\bar{x}$.*

In [8, Theorem 4.1], Dutta and Tammer showed Lagrange multiplier rules for the vector control approximation problems for weakly Pareto efficient solution; they considered objective function from an Asplund space X to a finite dimensional space Y, in which case Lipschitzian continuity, strictly Lipschitz continuity and strongly compactly Lipschitz continuity are equivalent. A similar result was proved in [7, Theorem 5.2] in terms of the abstract subdifferential (a subdifferential satisfying certain axioms). However, if we assume additionally C-boundedness and C-convexity of f_1, the strictly Lipschitz continuity is not necessary, we can get the following results in infinite dimensional spaces.

Theorem 4.12 *Suppose that X, Y, Z are reflexive Banach spaces, D is a closed subset of X, $C \subset Y$ is a proper pointed closed convex Daniell cone with a weakly compact base and nonempty interior, and f_1 is C-convex. Let $\bar{y} = f(\bar{x})$ with $\bar{x} \in D$ be a weakly Pareto efficient solution of (37). If f_1 is C-bounded from above around $\bar{x}$, and $\|\cdot\|$ is continuous, then there exists $y^* \in C^+ \setminus \{0\}$ such that*

$$0 \in y^* \circ \partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n} \alpha_i A_i^* y^* T_i + N(\bar{x}; D), \tag{38}$$

where $T_i \in L(Z, Y)$ and

$$T_i \in \partial^{\leq} \|\cdot\| (A_i(\bar{x}) - a^i), \quad i = 1, \ldots, n.$$

Proof. By the assumption on C, it is easily seen that C is normal (see [10], Section 2.2). Remark 4.11 shows that $\|A_i(\cdot) - a^i\|$ is C-convex and C-bounded from above around $\bar{x}$; so is f, hence the assumptions of Theorem 4.6 are fulfilled. Consequently, for every $e \in \operatorname{int} C$, we get the existence of $y^* \in C^+$ with $y^*(e) = 1$ such that

$$0 \in \partial(y^* \circ f)(\bar{x}) + N(\bar{x}; D), \tag{39}$$

The sum rule for subdifferentials of convex continuous functions (see [23, Theorem 3.23]) and Theorem 4.9 yield the relation

$$\begin{aligned}
\partial(y^* \circ f)(\bar{x}) &= \partial \left(y^*(f_1(\cdot)) + \sum_{i=1}^{n} \alpha_i y^* \|A_i(\cdot) - a^i\| \right)(\bar{x}) \\
&= y^* \partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n} \alpha_i y^* \partial^{\leq} \|A_i(\cdot) - a^i\|(\bar{x}) \\
&= y^* \partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n} \alpha_i A_i^* y^* \partial^{\leq}(\|u\|)|_{u=A_i(\bar{x})-a^i} \\
&= y^* \circ \partial^{\leq} f_1(\bar{x}) + \sum_{i=1}^{n} \alpha_i A_i^* y^* T_i, \qquad (40)
\end{aligned}$$

where $T_i \in L(Z,Y)$ and $T_i \in \partial^{\leq}\| \cdot \|(A_i(\bar{x}) - a^i), \quad i = 1, \ldots, n.$
Substituting now (40) into (39) we get the desired inclusion. □

To our knowledge, the result above is completely new for weakly Pareto efficient solution in class of C-convex functions. In comparison with the corresponding results in [1, Theorem 4.4], we do not need the Lipschitzianity assumption of $\| \cdot \|$, the vector-valued norm $\| \cdot \|$ only needs to be continuous.

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