Martin–Luther–Universität
Halle–Wittenberg
Institut für Mathematik

Concepts for approximate solutions of vector optimization problems with variable order structure

B. Soleimani and Chr. Tammer

Report No. 01 (2013)

Editors:
Professors of the Institute for Mathematics, Martin-Luther-University Halle-Wittenberg.

Electronic version: see http://www2.mathematik.uni-halle.de/institut/reports/

Concepts for approximate solutions of vector optimization problems with variable order structure

B. Soleimani and Chr. Tammer

Report No. 01 (2013)

B. Soleimani
Chr. Tammer
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany
Email: behnam.soleimani@mathematik.uni-halle.de
christiane.tammer@mathematik.uni-halle.de

Concepts for approximate solutions of vector optimization problems with variable order structure

Behnam Soleimani* and Christiane Tammer**

June 29, 2013

Dedicated to Boris Mordukhovich in honor of his 65th birthday.

Abstract

We introduce concepts for approximate minimal, approximate nondominated solutions and approximate minimizers of vector optimization problems with respect to a variable order structure. The variable order structure is given by set-valued maps $C : Y \rightrightarrows Y$, where we assume that $C(y)$ is a closed set and $0 \in \text{bd}\, C(y)$ for any $y \in Y$. We do not assume that $C(y)$ is convex or $C(y)$ is a cone for $y \in Y$. We illustrate the different concepts for approximate solutions by several examples. Important properties of these three different kinds of approximate solutions of vector optimization problem with respect to the variable order are discussed. Finally, we give some necessary conditions for approximate solutions of vector optimization problems with variable order structure using a vector-valued variant of Ekeland's variational principle.

Key Words: Vector optimization, variable order structure, approximate solution, necessary conditions, Ekeland's variational principle.

Mathematics subject classifications (MSC 2000): 90C29, 90C30,90C48,90C59.

1 Introduction

Many real world optimization problems require the minimization of multiple conflicting objectives, e.g. in finance, the maximization of the expected return versus the minimization of risk in portfolio optimization; in production theory, the minimization of production time versus the minimization of the cost of manufacturing equipment; or the maximization of tumor control versus the minimization of normal tissue complication in radiotherapy treatment design. Such problems can be formulated as multicriteria optimization problems (vector optimization problem). Let

*Institute of Mathematics, Martin-Luther-University Halle-Wittenberg, Theodor-Lieser Str. 5, 06120 Halle, email: behnam.soleimani@mathematik.uni-halle.de

**Institute of Mathematics, Martin-Luther-University Halle-Wittenberg, Theodor-Lieser Str. 5, 06120 Halle, email: christiane.tammer@mathematik.uni-halle.de

Y be a linear topological space, $\Omega \subset Y$ and $C \subseteq Y$ is a proper closed convex and pointed cone in Y. In vector optimization, one says that $\overline{y} \in \Omega$ is a efficient element of Ω with respect to C if the following holds:

$$\Omega \cap (\overline{y} - C \backslash \{0\}) = \emptyset. \tag{1}$$

Recently, vector optimization problems with variable order structure are studied intensively in the literature because they have important applications in economics, engineering design, management science and many other fields (compare Eichfelder [13], Engau [11], Huang, Yang, Chan [21] and Bao, Mordukhovich [2]). The solution concepts in vector optimization with variable order structure are defined with respect to a set-valued map $C : Y \rightrightarrows Y$. The natural way to extend the solution concept for vector optimization problems with fixed ordering cone given by (1) to the case of variable order structure is as follows: We are looking for elements $\overline{y} \in \Omega$ such that

$$(\bar{y} - C(\bar{y}) \backslash \{0\}) \cap \Omega = \emptyset. \tag{2}$$

or

$$\forall y \in \Omega \ : \qquad (\bar{y} - C(y) \backslash \{0\}) \cap \Omega = \emptyset. \tag{3}$$

Solutions $\overline{y} \in \Omega$ with the property (2) are called *minimal points* of Ω with respect to $C(\cdot)$. Minimal points of Ω with respect to $C(\cdot)$ are introduced by Chen, Huang and Yang [5, 6, 7]. Moreover, in this paper we introduce and study a new solution concept given by (3). Solutions with the property by (3) are called *minimizers* of Ω with respect to $C(\cdot)$.

On the other hand, we consider a formulation for efficient elements of Ω with respect to C given by

$$\forall y \in \Omega : \qquad \overline{y} \notin y + C \backslash \{0\}. \tag{4}$$

Of course, in the case of fixed order structure (4) is equivalent to (1), but if we want to generalize this definition to variable ordering structure, then the concept in (4) lead us to *minimal* (see (6)) as well as so called *nondominated points* (see (5)) of Ω with respect to $C(\cdot)$. A minimal point $\bar{y}$ of a set Ω is a candidate element which is not dominated by another point y^1 of Ω with respect to the associated set $C(\bar{y})$ to this candidate point by a set-valued map. In the definition of minimal elements, the ordering set is the associated set to the minimal point but for nondominated elements, an ordering set is an associated set to the another point. Some properties of these points can be found in [4, 5, 6, 7, 12, 13, 39, 40, 41].

If we want to define a solution concept for vector optimization problems with respect to $C : Y \rightrightarrows Y$ in a natural way from (4) we get

$$\forall y \in \Omega : \qquad \bar{y} \notin y + C(y) \backslash \{0\}. \tag{5}$$

or

$$\forall y \in \Omega : \qquad \bar{y} \notin y + C(\bar{y}) \backslash \{0\}. \tag{6}$$

The concept in (5) was introduced by Yu [39] in 1974, the so called *nondominated points*. Furthermore, (6) leads to the definition of *minimal points* considered by Chen, Huang and Yang [5, 6, 7].

However, it is not possible to derive the concept of *minimizers* from (4) because we change in the definition of minimizers the set $C(y)$ independently from the elements belonging to Ω.

For sure, all these points are the same for vector optimization problems with fixed order structure.

It is well known that one needs compactness assumptions in order to show existence results for optimization problems. Such compactness assumption are not fulfilled for many optimization problems. Also, we know that under weak assumptions and without compactness conditions, we have to deal with approximate solutions and we can show several assertions without any compactness assumptions for these solutions. Also, if we apply numerical algorithms for solving optimization problems, then these algorithms usually generate approximate solutions which are closed to the exact solutions. Here, we will introduce approximate solutions of vector optimization problem with variable order structure. Many papers deal with different concepts for approximate solutions with respect to a fixed order structure, see [20, 26, 27, 28, 31, 32, 33, 34, 36, 37, 38] for different definitions, concepts and properties of these elements. Gutiérrez , Jiménez and Novo in [19] introduce a new concept of approximate solution of vector optimization problems and they unified some different concepts of approximate solutions with respect to the fixed order structure. In our paper and [30], we deal with approximate nondominated, approximate minimal solutions and approximate minimizers with respect to the variable order structure.

The paper is organized as follows. First, in Section 3 we define the notion of approximate elements of vector optimization problem with variable order structure. Relations between the set of approximate solutions choosing different parameters ϵ are discussed in Section 4. Sections 5 and 6 are devoted to the presentation of the relations between different concepts of approximate solutions with variable order structure. Necessary conditions for approximate solutions of vector optimization problems with variable order structure are derived in Section 7 using a vector-valued variant of Ekeland's variational principle.

2 Preliminaries

Let Y be a linear topological space. A set $C \subseteq Y$ is called a cone if $\lambda c \in C$ for all $\lambda \geq 0$ and $c \in C$. The set C is said to be pointed if $C \cap (-C) \subseteq \{0\}$. Let Ω be a nonempty subset of linear topological space Y, we denote the topological interior of the set Ω by int Ω, cl Ω denotes the topological closure, bd Ω the topological boundary of Ω, $\operatorname{cone} \Omega$ denotes the cone generated by Ω and conv Ω denotes the convex hull of a set Ω. A nonempty set $C \subset \mathbb{R}^m$ is said to be convex if $\lambda c_1 + (1-\lambda)c_2 \in C$ for all $c_1, c_2 \in C$ and $0 \leq \lambda \leq 1$. A set $C \subset Y$ is said to be solid if $\operatorname{int} C \neq \emptyset$ and a set $C \subset Y$ is a proper set if $\emptyset \neq C \neq Y$. See [10, 17, 18, 22] for basic definitions and concepts for vector optimization. Also, see [14, 15, 29, 35] for some scalarization methods for solving vector optimization with respect to the fix ordering and some properties of these scalarization methods.

Also, suppose that $C : Y \rightrightarrows Y$ is a set valued map where $C(y)$ is a closed set with $0 \in \text{bd } C(y)$ for every $y \in Y$. We define the following three different domination relations: for $y^1, y^2, y^3 \in Y$

$$y^1 \leq_1 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^1)\backslash\{0\}), \tag{7}$$

$$y^1 \leq_2 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^2)\backslash\{0\}), \tag{8}$$

$$y^1 \leq_3 y^2 \quad \text{if} \quad y^2 \in y^1 + (C(y^3)\backslash\{0\}). \tag{9}$$

If $C(y^1) = C(y^2) = C(y^3)$ for all $y^1, y^2, y^3 \in Y$, then these three domination relations are same and the problem reduces to the optimization with standard domination structure.

3 Different concepts of approximate solutions for vector optimization problems with variable order

Suppose that $\epsilon \geq 0$ and $k^0 \in Y\backslash\{0\}$. We define ϵk^0-nondominated, ϵk^0-minimal elements and ϵk^0-minimizers with respect to a variable order structure. Furthermore, we define weakly (strongly) ϵk^0-minimal (nondominated) elements and minimizers. After that, we will have a definition for locally approximate optimal elements. For sure there is no difference between ϵk^0-nondominated, ϵk^0-minimal elements and ϵk^0-minimizers in the case of fixed order structure. This statement is true for weakly (strongly) ϵk^0-optimal elements. In this section, we show that this statement can not be true in variable order structure and all these three definitions define different elements. This will be shown by several examples. In the following, we suppose that Y is a linear topological space.

Assumption (A). Let Y be a linear topological space and $\Omega \subset Y$. Suppose that $C : Y \rightrightarrows Y$ is a set valued map where $C(y)$ is a closed set with $0 \in \text{bd } C(y)$. We assume that $k^0 \in Y\backslash\{0\}$ such that $C(y) + [0, \infty)k^0 \subseteq C(y)$ for all $y \in \Omega$ and $\epsilon \geq 0$.

Definition 3.1. *Let Assumption (A) holds and $y_\epsilon \in \Omega$.*

1. *y_ϵ is said to be an ϵk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if $y \nleq_1 y_\epsilon - \epsilon k^0$ for all $y \in \Omega$, i.e.,*

$$\forall y \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y\} = \emptyset.$$

2. *Suppose that* int $C(y) \neq \emptyset$ for all $y \in \Omega$. *y_ϵ is said to be a weakly ϵk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if*

$$\forall y \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y\} = \emptyset.$$

3. *$y_\epsilon \in \Omega$ is said to be a strongly ϵk^0-nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if*

$$\forall y \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y - C(y).$$

Remark 3.2.

- *We denote the set of all ϵk^0-nondominated elements of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ by ϵk^0-$N(\Omega, C)$.*
- *We denote the set of all weakly ϵk^0-nondominated elements of Ω with respect to the ordering map C by ϵk^0-$WN(\Omega, C)$.*
- *We denote the set of all strongly ϵk^0-nondominated elements of Ω with respect to C by ϵk^0-$SN(\Omega, C)$.*

If $\epsilon = 0$, then all these definitions coincide with the usual definitions of nondominated points, see [13, 39].

Definition 3.3. *Assume that Assumption (A) holds and $y_\epsilon \in \Omega$.*

1. *y_ϵ is said to be an ϵk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if $y \not\leq_2 y_\epsilon - \epsilon k^0$ for all $y \in \Omega$, i.e.*

$$(y_\epsilon - \epsilon k^0 - (C(y_\epsilon)\backslash\{0\})) \cap \Omega = \emptyset.$$

2. *Suppose that $\text{int } C(y_\epsilon) \neq \emptyset$. y_ϵ is said to be a weakly ϵk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if*

$$(y_\epsilon - \epsilon k^0 - \text{int } C(y_\epsilon)) \cap \Omega = \emptyset.$$

3. *y_ϵ is said to be a strongly ϵk^0-minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ if*

$$\forall y \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y - C(y_\epsilon).$$

Remark 3.4.

- *We denote the set of all ϵk^0-minimal elements of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ by ϵk^0-$M(\Omega, C)$.*
- *We denote the set of all weakly ϵk^0-minimal elements of Ω with respect to the ordering map C by ϵk^0-$WM(\Omega, C)$.*
- *We denote the set of all strongly ϵk^0-minimal elements of Ω with respect to C by ϵk^0-$SM(\Omega, C)$.*

If $\epsilon = 0$, then all these definitions coincide with the usual definitions of minimal points, see [13, 21].

Now we introduce ϵk^0- minimizers. Also in the case $\epsilon = 0$, the definition of minimizers is new.

Definition 3.5. *Let Assumption (A) holds and $y_\epsilon \in \Omega$.*

1. y_ϵ *is said to be an* ϵk^0*-minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if* $y^1 \not\leq_3 y_\epsilon - \epsilon k^0$ *for all* $y^1 \in \Omega$*, i.e.,*

$$\forall y, y^1 \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y^1\} = \emptyset.$$

2. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$*.* y_ϵ *is said to be a weak* ϵk^0*- minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*

$$\forall y, y^1 \in \Omega : \qquad (y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} = \emptyset.$$

3. $y_\epsilon \in \Omega$ *is said to be a strong* ϵk^0*-minimizer of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *if*

$$\forall y^1, y^2 \in \Omega : \qquad y_\epsilon - \epsilon k^0 \in y^1 - C(y^2).$$

Remark 3.6.

- *We denote the set of all* ϵk^0*-minimizers of* Ω *with respect to the ordering map* $C : Y \rightrightarrows Y$ *by* ϵk^0*-*$MZ(\Omega, C)$*.*

- *We denote the set of all weak* ϵk^0*-minimizers of* Ω *with respect to the ordering map* C *by* ϵk^0*-*$WMZ(\Omega, C)$*.*

- *We denote the set of all strong* ϵk^0*-minimizers of* Ω *with respect to* C *by* ϵk^0*-*$SMZ(\Omega, C)$*.*

Example 3.7. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 0)$*. Also, suppose that*

$$\Omega = \left\{(y_1, y_2) \in \mathbb{R}^2 \mid \ y_1 + y_2 \geq 2, \ y_1 \geq 0, \ \ 0 \leq y_2 \leq 2, \right\}$$

and

$$C(y_1, y_2) = \begin{cases} \mathbb{R}^2_+ & \text{if } y_2 = 0 \\ \text{cone conv } \{(2,0), (y_1, y_2)\} & \text{otherwise.} \end{cases}$$

It is easy to see that $\text{cl } C(y) + [0, \infty)k^0 \subseteq \text{cl } C(y)$ *for all* $y \in \Omega$*. Then* $\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq 2 + \frac{1}{100}\}$ *are* ϵk^0*-nondominated,* ϵk^0*-minimizer and also* ϵk^0*-minimal elements and the sets of all these points coincide. Furthermore,*

$$\left\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq 2 + \frac{1}{100}\right\} \cup \{(y_1, 0)\}$$

describes the set of weakly ϵk^0*-nondominated,* ϵk^0*-minimizer and also weakly* ϵk^0*-minimal elements (see Figure 1).*

In the case of fixed order structure, (weakly, strongly) ϵk^0-nondominated elements and (weakly, strongly) ϵk^0-minimal elements coincide, but the following examples show that this is not true when we are dealing with variable order structure.

Example 3.8. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 0)$*. Also suppose that*

$$\Omega = \left\{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\right\}.$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid d_1 \geq 0, \ d_2 \leq 0\} & \text{for } (-1, 0) \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

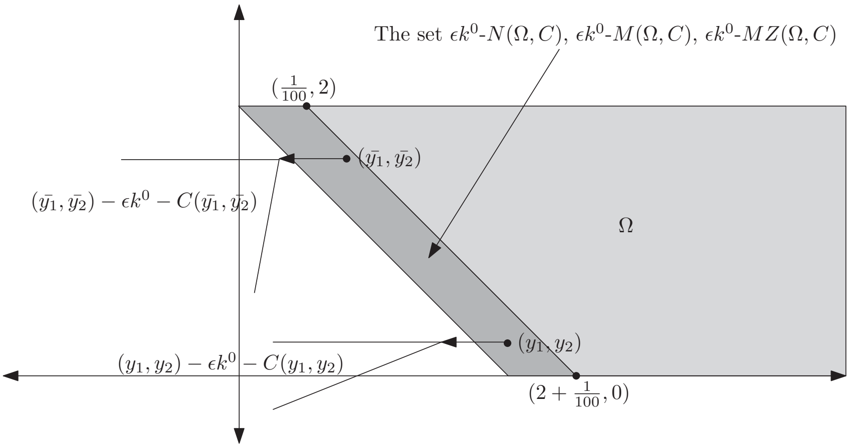

Figure 1: Example 3.7 where the set of ϵk^0-$N(\Omega, C)$, ϵk^0-$MZ(\Omega, C)$ and ϵk^0-$M(\Omega, C)$ of Ω coincide.

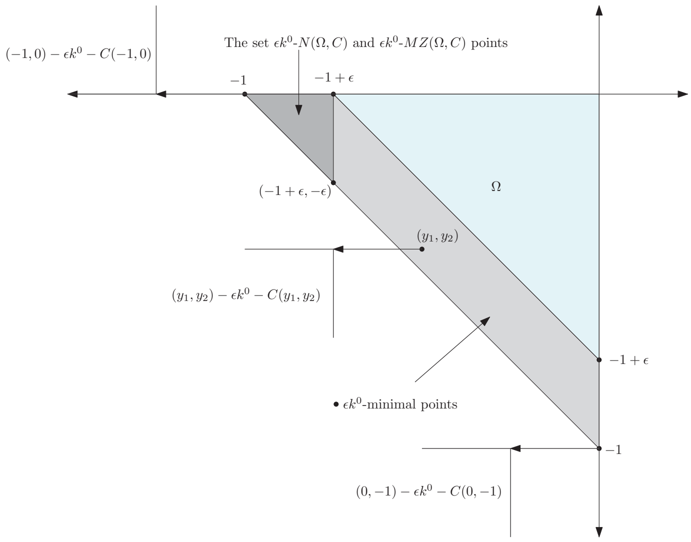

Figure 2: Example 3.8 where there exist ϵk^0-minimal elements of the set Ω which are not ϵk^0-nondominated and not ϵk^0-minimizers .

It is easy to see that $\mathrm{cl}\ C(y) + [0, \infty)k^0 \subseteq \mathrm{cl}\ C(y)$ *for all* $y \in \Omega$. *Then*

$$\{(y_1, y_2) \in \Omega \mid y_1 + y_2 \leq -1 + \epsilon\}$$

are ϵk^0*-minimal but just the elements of the set*

$$\{(y_1, y_2) \in \Omega \mid y_1 < -1 + \epsilon\} \cup \{(-1 + \epsilon, 0)\}$$

are ϵk^0*-nondominated and* ϵk^0*-minimizers (see Figure 2).*

Example 3.9. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1, 1)$. *Also suppose that*

$$\Omega = \Big\{(y_1, y_2) \in \mathbb{R}^2 \mid \ y_1 + y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\Big\}$$

and

$$C(y_1, y_2) = \begin{cases} \{(d_1, d_2) \in \mathbb{R}^2 \mid \ d_2 \geq 0, \ d_1 + d_2 \geq -1\} & \text{for } (y_1, y_2) = (-1, 0) \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

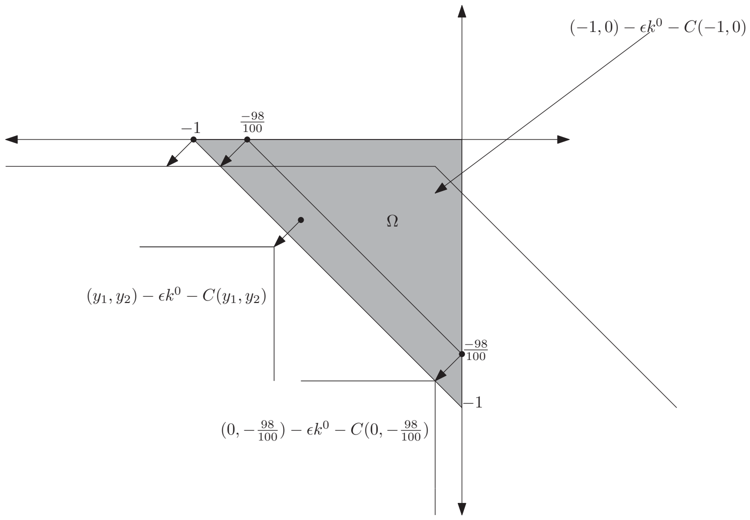

Figure 3: Example 3.9 where $(-1,0)^T$ an ϵk^0-nondominated of the set Ω, but it is neither ϵk^0-minimizer nor ϵk^0-minimal element.

It is easy to see that $\text{cl } C(y) + [0,\infty)k^0 \subseteq \text{cl } C(y)$ *for all* $y \in \Omega$. *Then*

$$\Big\{(y_1,y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{98}{100},\ y_1 \neq -1\Big\}$$

is the set of ϵk^0*-minimal elements, but* $(-1,0)$ *is not minimal. However,* $(-1,0)$ *is an* ϵk^0*-nondominated point and*

$$\Big\{(y_1,y_2) \in \Omega \mid \ y_1 + y_2 \leq -\frac{98}{100}\Big\}$$

is the set of all ϵk^0*-nondominated elements. Obviously,* $(-1,0)$ *is not a* ϵk^0*-minimizer and*

$$\{(y_1,y_2) \in \Omega \mid \ -1+\epsilon < y_2 \leq -1\}$$

is the set of ϵk^0*-minimizers (see Figure 3).*

In the following example, we show that there are some points which are ϵk^0-nondominated and also ϵk^0-minimal but they are not ϵk^0-minimizers.

Example 3.10. *Let* $\epsilon = \frac{1}{100}$ *and* $k^0 = (1,0)$. *Also suppose that*

$$\Omega = \big\{(y_1,y_2) \in \mathbb{R}^2 \mid y_1 + y_2 \geq -1, \quad y_1 \leq 0, \quad y_2 \leq 0\big\}$$

and

$$C(y_1,y_2) = \begin{cases} \{(d_1,d_2) \in \mathbb{R}^2 \mid d_2 \geq 0,\ d_1 + d_2 \geq -1\} & \text{for } (y_1,y_2) = (0,0) \\ \mathbb{R}^2_+ & \text{otherwise.} \end{cases}$$

It is easy to see that $\text{cl } C(y) + [0,\infty)k^0 \subseteq \text{cl } C(y)$ *for all* $y \in \Omega$. *Then*

$$\Big\{\{(y_1,y_2) \in \Omega \mid y_1 + y_2 \leq -\frac{99}{100}\Big\}$$

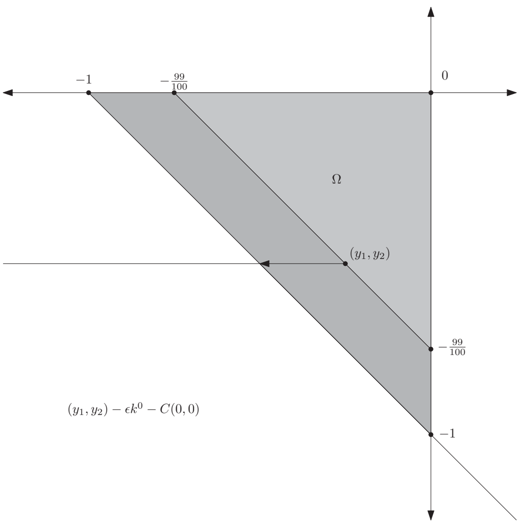

Figure 4: Example 3.10 where there exist an element which is both ϵk^0-nondominated and ϵk^0-minimal but not ϵk^0-minimizer.

is the set of ϵk^0-minimal and ϵk^0-nondominated points. But only points of the set $\left\{(y_1, y_2) \in \Omega|\ y_1 + y_2 < -\frac{99}{100}\right\}$ are ϵk^0-minimizers and the points

$$\left\{(y_1, y_2) \in \Omega|\ y_1 + y_2 = -\frac{99}{100}\right\}$$

are not ϵk^0-minimizers. This shows that there are some points which are both ϵk^0-nondominated and ϵk^0-minimal but not ϵk^0-minimizer (see Figure 4).

Definitions of locally ϵk^0-$N(\Omega, C)$, ϵk^0-$M(\Omega, C)$ and ϵk^0-$MZ(\Omega, C)$ are similar. We just need to substitute Ω with $\Omega \cap U$ in Definitions 3.1, 3.3 and 3.5 where U is a neighborhood of a candidate point.

If Ω is a convex set, then each locally ϵk^0- optimal element is also a globally ϵk^0-optimal element. This is also true for weakly (strongly) ϵk^0-optimal elements.

Note that the sets of minimal elements, weakly minimal and strongly minimal elements of Ω with respect to the ordering map C are special cases of ϵk^0-$M(\Omega, C)$, ϵk^0-$WM(\Omega, C)$ and ϵk^0-$SM(\Omega, C)$, respectively, when $\epsilon = 0$ and we denote them by $M(\Omega, C)$, $WM(\Omega, C)$ and $SM(\Omega, C)$, respectively. Also, the sets of nondominated, weakly nondominated and strongly nondominated elements of the set Ω with respect to the ordering map C can be shown as $N(\Omega, C)$, $WN(\Omega, C)$ and $SN(\Omega, C)$ respectively. Also, we denote the sets of minimizers, weakly minimizers and strongly minimizers of the set Ω with respect to the ordering map C by $MZ(\Omega, C)$, $WMZ(\Omega, C)$ and $SMZ(\Omega, C)$ respectively.

4 Relations of the sets of approximate solutions choosing different parameters ϵ

In this section, we will talk about relations between the sets of approximate nondominated (minimal) elements and minimizers of vector optimization problems with respect to the variable order if we choose different parameters $\epsilon > 0$. In fact, we show that the set of $\epsilon_1 k^0$-nondominated elements is a subset of the set of $\epsilon_2 k^0$-nondominated elements if $\epsilon_1 \leq \epsilon_2$. We show also this property for approximate minimal and approximate minimizers. Furthermore, we will show in this section corresponding relation between weakly ϵk^0-nondominated (minimal, minimizer), strongly ϵk^0-nondominated (minimal, minimizer) and ϵk^0-nondominated (minimal) elements and minimizers. Similar as in the case of fixed order structure, we will show that each strongly approximate nondominated (minimal, minimizer) is an approximate nondominated (minimal, minimizer) and each approximate nondominated (minimal, minimizer) is a weakly approximate nondominated (minimal, minimizer). First, we prove a lemma which will be used for the proof of the next theorems.

Lemma 4.1. *Assume that $C : Y \rightrightarrows Y$ is a set-valued map, $y \in Y$ and $k^0 \in Y \backslash \{0\}$, then $C(y) + [0, \infty)k^0 \subseteq C(y)$ implies* $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$.

Proof. Suppose that there exist an element $c \in \text{int } C(y)$ and $\epsilon > 0$ such that $c + \epsilon k^0 \notin \text{int } C(y)$. Since $C(y) + [0, \infty)k^0 \subseteq C(y)$ and $c + \epsilon k^0 \notin \text{int } C(y)$, then $c + \epsilon k^0 \in \text{bd } C(y)$. By $c \in \text{int } C(y)$ and $c + \epsilon k^0 \in \text{bd } C(y)$, we have the following implication for any $\gamma > 0$:

$$c + \epsilon k^0 + \gamma k^0 \notin C(y) \Rightarrow c + (\epsilon + \gamma)k^0 \notin C(y).$$

But this is a contradiction to $C(y) + [0, \infty)k^0 \subseteq C(y)$. Therefore, we can conclude that $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$. □

The following theorem shows several properties of the approximate nondominated , weakly approximate nondominated and strongly approximate nondominated solutions. This theorem will help us later to show the relations between exact nondominated elements and approximate nondominated elements. In the following we suppose that $C(y)$ is additionally pointed for each $y \in Y$.

Theorem 4.2. *Assume that assumption (A) holds. Additionally, suppose that $C(y)$ is a proper, pointed, closed set for all $y \in \Omega$ and $\epsilon, \epsilon_1, \epsilon_2 \geq 0$.*

1. $N(\Omega, C) \subseteq \epsilon_1 k^0\text{-}N(\Omega, C) \subseteq \epsilon_2 k^0\text{-}N(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
2. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all $y \in \Omega$, then the following holds:*
   $WN(\Omega, C) \subseteq \epsilon_1 k^0\text{-}WN(\Omega, C) \subseteq \epsilon_2 k^0\text{-}WN(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
3. $SN(\Omega, C) \subseteq \epsilon_1 k^0\text{-}SN(\Omega, C) \subseteq \epsilon_2 k^0\text{-}SN(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
4. $\epsilon k^0\text{-}SN(\Omega, C) \subseteq \epsilon k^0\text{-}N(\Omega, C) \subseteq \epsilon k^0\text{-}WN(\Omega, C)$ if $\epsilon \geq 0$.

Proof. 1. We prove that if $\epsilon_1 \geq 0$ then $N(\Omega, C) \subseteq \epsilon_1 k^0\text{-}N(\Omega, C)$. If $\epsilon_1 = 0$, then $N(\Omega, C) = \epsilon_1 k^0\text{-}N(\Omega, C)$. Therefore we assume that $\epsilon_1 > 0$. Suppose that $y_\epsilon \in N(\Omega, C)$ but $y_\epsilon \notin \epsilon_1 k^0\text{-}N(\Omega, C)$. Then there exists $y \in \Omega$ such that $y \in (y_\epsilon - \epsilon_1 k^0 - (C(y) \backslash \{0\}))$, therefore

$$\exists c_1 \in (C(y) \backslash \{0\}) \text{ with } y_\epsilon - \epsilon_1 k^0 - c_1 = y. \tag{10}$$

By $C(y) + [0, \infty)k^0 \subseteq C(y)$, there exists $c_2 \in C(y)$ such that

$$c_1 + \epsilon_1 k^0 = c_2. \tag{11}$$

We prove that $c_2 \neq 0$. Suppose that $c_2 = 0$, then by (11), we have

$$c_1 + \epsilon_1 k^0 = 0 \Rightarrow c_1 = -\epsilon_1 k^0. \tag{12}$$

Since $c_1 \in C(y)$, then $-\epsilon_1 k^0 \in C(y)$. Also by $C(y) + [0, \infty)k^0 \subseteq C(y)$ and $0 \in C(y)$ for all $y \in \Omega$, we know that

$$0 + \epsilon_1 k^0 = \epsilon_1 k^0 \in C(y). \tag{13}$$

Since $\epsilon_1 \neq 0$ and $k^0 \neq 0$, then $\epsilon_1 k^0 \neq 0$ and by (12) and (13), we have $\{\epsilon_1 k^0, -\epsilon_1 k^0\} \in C(y) \cap (-C(y))$, but this is contradiction to the pointedness of $C(y)$. This means that $c_2 \neq 0$ and $c_2 \in (C(y)\backslash\{0\})$. By (10) and (11),

$$y = y_\epsilon - c_2 \Rightarrow (y_\epsilon - (C(y)\backslash\{0\})) \cap \{y\} \neq \emptyset.$$

But this is a contradiction to $y_\epsilon \in N(\Omega, C)$.

Now, suppose that $0 < \epsilon_1 < \epsilon_2$, then $\epsilon_2 = \epsilon_1 + \gamma$ where $\gamma > 0$. Suppose that $y_\epsilon \in \epsilon_1 k^0$-$N(\Omega, C)$ but $y_\epsilon \notin \epsilon_2 k^0$-$N(\Omega, C)$, then there exist $y \in \Omega$ and $c_1 \in (C(y)\backslash\{0\})$ such that

$$y_\epsilon - \epsilon_2 k^0 - c_1 = y. \tag{14}$$

By (14) and $\epsilon_2 = \epsilon_1 + \gamma$, we can write

$$y \in (y_\epsilon - (\epsilon_1 + \gamma)k^0 - (C(y)\backslash\{0\})) \Rightarrow y \in (y_\epsilon - \epsilon_1 k^0 - (\gamma k^0 + (C(y)\backslash\{0\})).$$

This implies that there exists $c_2 \in \gamma k^0 + (C(y)\backslash\{0\})$ with $y = y_\epsilon - \epsilon k^0 - c_2$. Now, because of $C(y) + [0, \infty)k^0 \subseteq C(y)$, we get $c_2 \in \gamma k^0 + (C(y)\backslash\{0\}) \subset C(y) \Rightarrow c_2 \in C(y)$. Similar to above and by the pointedness of $C(y)$ for all $y \in \Omega$, we can write that $c_2 \neq 0$ and $c_2 \in (C(y)\backslash\{0\})$.

Therefore $y_\epsilon - \epsilon_1 k^0 - c_2 = y$ and since $c_2 \neq 0$, then we can write $y \in (y_\epsilon - \epsilon_1 k^0 - (C(y)\backslash\{0\}))$. But this is a contradiction to our assumption $y_\epsilon \in \epsilon k^0$-$N(\Omega, C)$. This means that $\epsilon_1 k^0$-$N(\Omega, C) \subseteq \epsilon_2 k^0$-$N(\Omega, C)$.

2. Suppose that $y_\epsilon \in WN(\Omega, C)$ i.e., y_ϵ is a weakly nondominated point with respect to the ordering map C, then $(y_\epsilon - \text{int } C(y)) \cap \{y\} = \emptyset$ for all $y \in \Omega$. By Lemma 4.1, we have $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$ for all $y \in \Omega$ and therefore for any $\epsilon > 0$, we can write

$$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y\} \subseteq (y_\epsilon - \text{int } C(y)) \cap \{y\} = \emptyset.$$

This means that $y_\epsilon \in \epsilon k^0$-$WN(\Omega, C)$ for all $\epsilon > 0$ and therefore $WN(\Omega, C) \subseteq \epsilon_1 k^0$-$WN(\Omega, C)$.

Now, suppose that $0 < \epsilon_1 < \epsilon_2$, then $\epsilon_2 = \epsilon_1 + \gamma$ where $\gamma > 0$. Suppose that $y_\epsilon \in \epsilon_1 k^0$-$WN(\Omega, C)$ but $y_\epsilon \notin \epsilon_2 k^0$-$WN(\Omega, C)$. There exist an element

$y \in \Omega$ and $c_1 \in \text{int } C(y)$ such that $(y_\epsilon - \epsilon_2 k^0 - \text{int } C(y)) \cap \{y\} \neq \emptyset$ and $y_\epsilon - \epsilon_2 k^0 - c_1 = y$. Therefore, we can write

$$y_\epsilon - (\epsilon_1 + \gamma)k^0 - c_1 = y \Rightarrow y_\epsilon - \epsilon_1 k^0 - (\gamma k^0 + c_1) = y. \tag{15}$$

By $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$, we have $c_2 := c_1 + \gamma k^0 \in \text{int } C(y)$. Because of (15), we can write

$$y_\epsilon - \epsilon_1 k^0 - c_2 = y \Rightarrow (y_\epsilon - \epsilon_1 k^0 - \text{int } C(y)) \cap \{y\} \neq \emptyset.$$

But this is a contradiction to $y_\epsilon \in \epsilon_1 k^0\text{-}WN(\Omega, C)$.

3. Suppose that $\epsilon_1 \geq 0$. We prove that $SN(\Omega, C) \subseteq \epsilon_1 k^0\text{-}SN(\Omega, C)$. For sure, we just need to consider $\epsilon_1 > 0$, otherwise $\epsilon = 0$ and obviously $SN(\Omega, C) = \epsilon_1 k^0\text{-}SN(\Omega, C)$. Suppose that $y_\epsilon \in SN(\Omega, C)$, then for all $\epsilon_1 > 0$,

$$y_\epsilon \in y - (C(y)\backslash\{0\}) \Rightarrow y_\epsilon - \epsilon_1 k^0 \in y - ((C(y)\backslash\{0\}) + \epsilon_1 k^0).$$

By $C(y) + [0, \infty)k^0 \subseteq C(y)$, $\epsilon_1 \neq 0, k^0 \neq 0$ and the pointedness of $C(y)$ for all $y \in \Omega$, we know that $(C(y)\backslash\{0\}) + [0, \infty)k^0 \subseteq (C(y)\backslash\{0\})$. Therefore $y_\epsilon - \epsilon_1 k^0 \in y - (C(y)\backslash\{0\})$ and this means that $y_\epsilon \in \epsilon_1 k^0\text{-}SN(\Omega, C)$.

Now, suppose that $y_\epsilon \in \epsilon_1 k^0\text{-}SN(\Omega, C)$ and $\epsilon_1 < \epsilon_2$. Therefore, there exist $\gamma > 0$ such that $\epsilon_1 + \gamma = \epsilon_2$. Since $y_\epsilon \in \epsilon_1 k^0\text{-}SN(\Omega, C)$, then for all $y \in \Omega$

$$y_\epsilon - \epsilon_2 k^0 = y_\epsilon - \epsilon_1 k^0 - \gamma k^0 \in y - C(y)\backslash\{0\} - \gamma k^0. \tag{16}$$

Similar to the above $(C(y)\backslash\{0\}) + [0, \infty)k^0 \subseteq (C(y)\backslash\{0\})$ for all $y \in \Omega$. By this and (16), we can write

$$y_\epsilon - \epsilon_2 k^0 \in y - \big(C(y)\backslash\{0\} + \gamma k^0\big) \subseteq y - C(y)\backslash\{0\},$$

and this completes the proof.

4. Suppose that y_ϵ is a strongly ϵk^0- nondominated element of Ω with respect to the ordering map C, then for all $y \in \Omega$ we have:

$$y_\epsilon - \epsilon k^0 \in y - C(y) \Rightarrow y_\epsilon - \epsilon k^0 - y \in -C(y). \tag{17}$$

Now, suppose that y_ϵ is not an ϵk^0-nondominated element which means there exist $y \in \Omega$ and $c_1 \in (C(y)\backslash\{0\})$ such that

$$y_\epsilon = y + \epsilon k^0 + c_1 \text{ and } y_\epsilon - \epsilon k^0 - y \in (C(y)\backslash\{0\}). \tag{18}$$

Therefore $y_\epsilon - \epsilon k^0 - y \in (C(y)\backslash\{0\}) \cap -C(y)$. But this is a contradiction because we know that $C(y)$ is a pointed set for all $y \in \Omega$. This means that each strongly ϵk^0-nondominated element is an ϵk^0-nondominated elements and $\epsilon k^0\text{-}SN(\Omega, C) \subseteq \epsilon k^0\text{-}N(\Omega, C)$.

Now, we show that each ϵk^0-nondominated element is a weakly ϵk^0-nondominated element. Since $0 \in \text{bd } C(y)$, then $\text{int } C(y) \subseteq (C(y)\backslash\{0\})$. So, we can write

$$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y\} \subseteq (y_\epsilon - \epsilon k^0 - C(y)\backslash\{0\}) \cap \{y\} = \emptyset.$$

This means that if $y_\epsilon \in \epsilon k^0\text{-}N(\Omega, C)$ and $(y_\epsilon - \epsilon k^0 - C(y)\backslash\{0\}) \cap \{y\} = \emptyset$, then $(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y\} = \emptyset$ and $y_\epsilon \in \epsilon k^0\text{-}WN(\Omega, C)$. Therefore, each ϵk^0-nondominated element is a weakly ϵk^0-nondominated element.

□

We show several properties of the ϵk^0-minimal , weakly ϵk^0-minimal and strongly ϵk^0-minimal elements and their relations in the following theorem. We will use this theorem for proving the relations between the sets of exact minimal elements and ϵk^0-minimal elements.

Theorem 4.3. *Let all the assumptions of Theorem 4.2 be fulfilled. The following properties hold.*

1. $M(\Omega, C) \subseteq \epsilon_1 k^0\text{-}M(\Omega, C) \subseteq \epsilon_2 k^0\text{-}M(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
2. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$, *then the following holds:* $WM(\Omega, C) \subseteq \epsilon_1 k^0\text{-}WM(\Omega, C) \subseteq \epsilon_2 k^0\text{-}WM(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
3. $SM(\Omega, C) \subseteq \epsilon_1 k^0\text{-}SM(\Omega, C) \subseteq \epsilon_2 k^0\text{-}SM(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
4. $\epsilon k^0\text{-}SM(\Omega, C) \subseteq \epsilon k^0\text{-}M(\Omega, C) \subseteq \epsilon k^0\text{-}WM(\Omega, C)$ if $\epsilon \geq 0$.

Proof. Proof is similar to the proof of Theorem 4.2. □

The following theorem shows several properties of approximate minimizers, weakly approximate minimizers and strongly approximate minimizers and their relations to each other. Later, this theorem will help us to see the relations between the sets of exact minimizers and ϵk^0-minimizers.

Theorem 4.4. *Let all the assumptions of Theorem 4.2 be fulfilled. The following properties hold.*

1. $MZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}MZ(\Omega, C) \subseteq \epsilon_2 k^0\text{-}MZ(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
2. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$, *then the following holds:* $WMZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}WMZ(\Omega, C) \subseteq \epsilon_2 k^0\text{-}WMZ(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
3. $SMZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}SMZ(\Omega, C) \subseteq \epsilon_2 k^0\text{-}SMZ(\Omega, C)$ if $0 \leq \epsilon_1 \leq \epsilon_2$.
4. $\epsilon k^0\text{-}SMZ(\Omega, C) \subseteq \epsilon k^0\text{-}MZ(\Omega, C) \subseteq \epsilon k^0\text{-}WMZ(\Omega, C)$ if $\epsilon \geq 0$.

Proof. 1. As supposed, $C(y) + [0, \infty)k^0 \subseteq C(y)$. First, we prove if $\epsilon_1 \geq 0$ then $MZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}MZ(\Omega, C)$. The proof is obvious for $\epsilon_1 = 0$, therefore we suppose that $\epsilon_1 > 0$. Suppose that $y_\epsilon \in MZ(\Omega, C)$ but $y_\epsilon \notin \epsilon_1 k^0\text{-}MZ(\Omega, C)$. Then there exists $y^1, y \in \Omega$ such that $y^1 \in (y_\epsilon - \epsilon_1 k^0 - (C(y)\backslash\{0\})) \cap \Omega$, therefore

$$\exists c_1 \in (C(y)\backslash\{0\}) \text{ with } y_\epsilon - \epsilon_1 k^0 - c_1 = y^1. \tag{19}$$

Because of $C(y) + [0, \infty)k^0 \subseteq C(y)$, there exist $c_2 \in C(y)$ such that

$$c_1 + \epsilon_1 k^0 = c_2. \tag{20}$$

Similar to part 1 of Theorem 4.2, we get $c_2 \neq 0$ and by (19) and (20),

$$y^1 = y_\epsilon - c_2 \Rightarrow (y_\epsilon - (C(y)\backslash\{0\})) \cap \{y^1\} \neq \emptyset.$$

But this is a contradiction to $y_\epsilon \in MZ(\Omega, C)$.

Now, suppose that $0 < \epsilon_1 < \epsilon_2$, then $\epsilon_2 = \epsilon_1 + \gamma$ where $\gamma > 0$. Suppose that $y_\epsilon \in \epsilon_1 k^0\text{-}MZ(\Omega, C)$ but $y_\epsilon \notin \epsilon_2 k^0\text{-}MZ(\Omega, C)$. Then there exist $y^1, y \in \Omega$ and $c_1 \in (C(y)\backslash\{0\})$ such that

$$y_\epsilon - \epsilon_2 k^0 - c_1 = y^1. \tag{21}$$

From above, we can write,

$$y^1 \in (y_\epsilon - (\epsilon_1 + \gamma)k^0 - (C(y)\backslash\{0\})) \Rightarrow y^1 \in (y_\epsilon - \epsilon_1 k^0 - (\gamma k^0 + (C(y)\backslash\{0\})).$$

$C(y) + [0, \infty)k^0 \subseteq C(y)$ implies that there exist $c_2 \in C(y)$ such that $y_\epsilon - \epsilon_1 k^0 - c_2 = y^1$. Also by $C(y) + [0, \infty)k^0 \subseteq C(y)$, $\epsilon_1 \neq 0, k^0 \neq 0$ and pointedness of $C(y)$ for all $y \in \Omega$, we can write $(C(y)\backslash\{0\}) + [0, \infty)k^0 \subseteq (C(y)\backslash\{0\})$. This means that $c_2 \neq 0$ and $c_2 \in (C(y)\backslash\{0\})$. Since $c_2 \neq 0$, then $y^1 \in (y_\epsilon - \epsilon_1 k^0 - (C(y)\backslash\{0\}))$. But this is a contradiction to $y_\epsilon \in \epsilon_1 k^0\text{-}MZ(\Omega, C)$.

2. Suppose that $y_\epsilon \in WMZ(\Omega, C)$ i.e., y_ϵ is a weakly minimizer with respect to the ordering map C, then $(y_\epsilon - \text{int } C(y)) \cap \{y^1\} = \emptyset$ for all $y, y^1 \in \Omega$. Since $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$, then for any $\epsilon > 0$ we can write

$$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} \subseteq (y_\epsilon - \text{int } C(y)) \cap \{y^1\} = \emptyset \qquad \forall y, y^1 \in \Omega.$$

This means that $y_\epsilon \in \epsilon k^0\text{-}WMZ(\Omega, C)$ for all $\epsilon > 0$ and therefore $WMZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}WMZ(\Omega, C)$.

Now, suppose that $\epsilon_1 < \epsilon_2$, then $\epsilon_2 = \epsilon_1 + \gamma$ where $\gamma > 0$. Suppose that $y_\epsilon \in \epsilon_1 k^0\text{-}WMZ(\Omega, C)$ but $y_\epsilon \notin \epsilon_2 k^0\text{-}WMZ(\Omega, C)$. Then there exist elements $y, y^1 \in \Omega$ and $c_1 \in \text{int } C(y)$ such that $y^1 \in (y_\epsilon - \epsilon_2 k^0 - \text{int } C(y))$ and $y_\epsilon - \epsilon_2 k^0 - c_1 = y^1 \in \Omega$. Therefore, we can write

$$y_\epsilon - (\epsilon_1 + \gamma)k^0 - c_1 \in \{y^1\} \Rightarrow y_\epsilon - \epsilon_1 k^0 - (\gamma k^0 + c_1) \in \{y^1\}. \tag{22}$$

By $\text{int } C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$ and (22), we have $c_1 + \gamma k^0 = c_2 \in \text{int } C(y)$ and

$$y_\epsilon - \epsilon_1 k^0 - c_2 \in \{y^1\} \Rightarrow (y_\epsilon - \epsilon_1 k^0 - \text{int } C(y)) \cap \{y^1\} \neq \emptyset.$$

But this is a contradiction to $y_\epsilon \in \epsilon_1 k^0\text{-}WMZ(\Omega, C)$.

3. The proof is similar to the part 3 of Theorem 4.2.

4. The proof is similar to the part 4 of Theorem 4.2.

□

5 Relation between different concepts of approximate solutions

Eichfelder [13] studied relations between exact nondominated and minimal solutions of vector optimization problems with variable order structure. In this section, we show relation between different kinds of approximate optimal elements (ϵk^0-nondominated, ϵk^0-minimal and ϵk^0-minimizers) of vector optimization problems with respect to variable order structure. At the end of this section, it will be obvious to see that the concepts of approximate nondominated, approximate minimal elements and approximate minimizers coincide in the case of fixed order structure. First, we will show relations between (weakly, strongly) ϵk^0-minimizers of the set Ω with respect to ordering map $C : Y \rightrightarrows Y$ and (weakly, strongly) ϵk^0-nondominated elements of Ω with respect to C. These theorems show us that in the case of fixed order structure, there is no difference between the sets of approximate nondominated and approximate minimizers and they do coincide.

In the following theorem we show the relations between ϵk^0-minimizers and ϵk^0-nondominated elements of a set $\Omega \subset Y$.

Theorem 5.1. *Assume that assumption (A) holds and additionally $C(y)$ is a proper, pointed, closed set for all $y \in \Omega$.*

1. *Every ϵk^0-minimizer of Ω with respect to C is also an ϵk^0-nondominated element.*

2. *Every ϵk^0-nondominated element of Ω with respect to C is also an ϵk^0-minimizer if $C(y) = C(y')$ for all $y, y' \in \Omega$. This means that each approximate nondominated element is an approximate minimizer if the ordering set is fixed for all $y \in \Omega$.*

3. *Suppose that* $\operatorname{int} C(y) \neq \emptyset$ *for all $y \in \Omega$, then every weak ϵk^0-minimizer of Ω with respect to C is also a weakly ϵk^0-nondominated element.*

4. *Suppose that* $\operatorname{int} C(y) \neq \emptyset$ *for all $y \in \Omega$, then every weakly ϵk^0-nondominated element of Ω with respect to C is also a weak ϵk^0-minimizer if* $\operatorname{int} C(y) = \operatorname{int} C(y')$ *for all $y, y' \in \Omega$.*

5. *Every Strong ϵk^0-minimizer of Ω with respect to C is also a strongly ϵk^0-nondominated element.*

6. *Every strongly ϵk^0-nondominated element of Ω with respect to C is also a strong ϵk^0-minimizer if $C(y) = C(y')$ for all $y, y' \in \Omega$.*

Proof. 1. This is obvious from the first parts of Definitions 3.1, 3.3.

2. Suppose that y_ϵ is an ϵk^0-nondominated element, then $(y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y\} = \emptyset$ for all $y \in \Omega$. Since $C(y) = C(y^1)$ for all $y, y^1 \in \Omega$, therefore

   $$(y_\epsilon - \epsilon k^0 - C(y^1)\backslash\{0\}) \cap \{y\} = (y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y\} = \emptyset \quad \forall y, y^1 \in \Omega.$$

   This means that $(y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap y^1 = \emptyset$ for all $y, y^1 \in \Omega$ and therefore y_ϵ is an ϵk^0-minimizer.

3. This is obvious from the second parts of Definition 3.1 and Definition 3.5.

4. If int $C(y) =$ int $C(y^1)$ for all $y, y^1 \in \Omega$ and y_ϵ is a weakly ϵk^0-nondominated element, then

   $$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} = (y_\epsilon - \epsilon k^0 - \text{int } C(y^1)) \cap \{y^1\} = \emptyset$$

   for all $y, y^1 \in \Omega$. This means that $(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} = \emptyset$ for all $y, y^1 \in \Omega$ and y_ϵ is a weak ϵk^0-minimizer.

5. This part is obvious from the third parts of Definition 3.1 and Definition 3.5.

6. Suppose that y_ϵ is a strongly ϵk^0-nondominated element, then $y_\epsilon - \epsilon k^0 \in y - (C(y)\backslash\{0\})$ for all $y \in \Omega$. Since $C(y) = C(y^1)$ for all $y, y^1 \in \Omega$, therefore

   $$y_\epsilon - \epsilon k^0 \in y - (C(y)\backslash\{0\}) \Rightarrow y_\epsilon - \epsilon k^0 \in y - C(y^1)\backslash\{0\} \quad \forall y, y^1 \in \Omega.$$

   This means that y_ϵ is a strong ϵk^0-minimizer.

□

The following examples show that the condition $C(y) = C(y')$ for all $y, y' \in \Omega$ in the part 2 of Theorem 5.1 is not a necessary condition and this condition is just sufficient condition.

Example 5.2. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are the same as in Example 3.8. Obviously, $(-1, 0)$ is an ϵk^0-nondominated element and also it is an ϵk^0-minimizer but $C(y') = C(y)$ does not hold for all $y, y' \in \Omega$.*

Example 5.3. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.9. It is easy to see that $(-1, 0)$ is an ϵk^0-nondominated element but not ϵk^0-minimizer. it is obvious that $\{(d_1, d_2) \in \mathbb{R}^2 | d_2 \geq 0,\ d_1 + d_2 \geq -1\}$ is not a subset of $\mathbb{R}^2_+$.*

Now, we discuss the relation between the sets of (weak, strong) ϵk^0-minimizers and (weakly, strongly) ϵk^0-minimal elements with respect to ordering map C. We know that in the case of fixed order structure, these sets coincide. But in the variable order case, Examples 3.8 and 3.10 show that there are some approximate minimal elements which are not approximate minimizers.

In the following theorem we show the relations between ϵk^0-minimal elements and ϵk^0-minimizers of a set $\Omega \subset Y$. Furthermore, this theorem also helps us to show that the sets of ϵk^0-minimal and ϵk^0-minimizers coincide in fixed order structure.

Theorem 5.4. *Let all the assumptions of Theorem 5.1 be fulfilled. The following properties hold.*

1. *Every ϵk^0-minimizer of Ω with respect to C is also an ϵk^0-minimal element.*
2. *Suppose that y_ϵ is an ϵk^0-minimal element of Ω with respect to C, then y_ϵ is also an ϵk^0-minimizer if $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$.*
3. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all $y \in \Omega$, then every weak ϵk^0-minimizer of Ω with respect to C is also a weakly ϵk^0-minimal element.*
4. *Suppose that* $\text{int } C(y) \neq \emptyset$ *for all $y \in \Omega$ and y_ϵ is a weakly ϵk^0-minimal element of Ω with respect to C, then y_ϵ is also a weak ϵk^0-minimizer if* $\text{int } C(y) \subseteq \text{int } C(y_\epsilon)$ *for all $y \in \Omega$.*
5. *Every strong ϵk^0-minimizer of Ω with respect to C is also a strongly ϵk^0-minimal element.*
6. *Suppose that y_ϵ is a strongly ϵk^0-minimal element of Ω with respect to C, then y_ϵ is also a strong ϵk^0-minimizer if $C(y_\epsilon) \subseteq C(y)$ for all $y \in \Omega$.*

Proof.

1. This is obvious from the first parts of Definition 3.3 and Definition 3.5.
2. Suppose that y_ϵ is an ϵk^0-minimal element, then $(y_\epsilon - \epsilon k^0 - C(y_\epsilon)\backslash\{0\}) \cap \Omega = \emptyset$. Since $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$, therefore
$$(y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y^1\} \subseteq (y_\epsilon - \epsilon k^0 - C(y_\epsilon)\backslash\{0\}) \cap \{y^1\} = \emptyset \ \ \forall y, y^1 \in \Omega.$$
This means that y_ϵ is a ϵk^0-minimizer.
3. This is obvious from the second parts of Definition 3.3 and Definition 3.5.
4. If $\text{int } C(y) \subseteq \text{int } C(y_\epsilon)$ and y_ϵ is a weakly ϵk^0-minimal element, then for all $y, y^1 \in \Omega$
$$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} \subseteq (y_\epsilon - \epsilon k^0 - \text{int } C(y_\epsilon)) \cap \{y^1\} = \emptyset.$$
This means that $(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \{y^1\} = \emptyset$ for all $y, y^1 \in \Omega$ and y_ϵ is a weak ϵk^0-minimizer.
5. This part is obvious from the third parts of Definition 3.3 and Definition 3.5.
6. Suppose that y_ϵ is a strongly ϵk^0-minimal element, then $y_\epsilon - \epsilon k^0 \in y - C(y_\epsilon)\backslash\{0\}$ for all $y \in \Omega$. Since $C(y_\epsilon) \subseteq C(y^1)$ for all $y^1 \in \Omega$, then
$$y_\epsilon - \epsilon k^0 \in y - C(y_\epsilon)\backslash\{0\} \Rightarrow y_\epsilon - \epsilon k^0 \in y - C(y^1)\backslash\{0\} \ \forall y, y^1 \in \Omega.$$
This means that y_ϵ is a strongly ϵk^0-minimizer.

□

The following examples show that the condition $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$ in the part 2 of Theorem 5.4 is not a necessary condition.

Example 5.5. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.8. Obviously $(-1, 0)$ is an ϵk^0-minimizer and ϵk^0-minimal element but $C(y) \subseteq C(y_\epsilon)$ does not hold for all $y \in \Omega$.*

Example 5.6. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.8. It is easy to see that $(0, -1)$ is an ϵk^0-minimal element but not ϵk^0-minimizer. It is obvious that $\{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 0,\ d_2 \leq 0\}$ is not a subset of $\mathbb{R}^2_+$.*

We know that in the case of fixed order structure, the sets of all approximate solutions coincide. But in the variable order case, these sets do not coincide and there are some approximate minimal elements which are not approximate nondominated and vice versa, see Examples 3.8 and 3.9. In the following theorem, we show the relation between (weakly, strongly) ϵk^0-nondominated elements of Ω with respect to the ordering map $C : Y \rightrightarrows Y$ and (weakly, strongly) ϵk^0-minimal elements of Ω with respect to C. With this theorem, we can conclude that the sets of ϵk^0-minimal and ϵk^0-nondominated elements do coincide in the case of fixed order structure. Eichfelder [13] studied this for exact solutions of vector optimization with respect to variable order structure.

Theorem 5.7. *Let all the assumptions of Theorem 5.1 be fulfilled. The following properties hold.*

1. *Suppose that y_ϵ is an ϵk^0- minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also an ϵk^0-nondominated element of Ω if $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$.*

2. *Suppose that y_ϵ is an ϵk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also an ϵk^0-minimal element of Ω if $C(y_\epsilon) \subseteq C(y)$ for all $y \in \Omega$.*

3. *Suppose that $\text{int } C(y) \neq \emptyset$ for all $y \in \Omega$ and y_ϵ is a weakly ϵk^0- minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also a weakly ϵk^0-nondominated element of Ω if $\text{int } C(y) \subseteq \text{int } C(y_\epsilon)$ for all $y \in \Omega$.*

4. *Suppose that $\text{int } C(y) \neq \emptyset$ for all $y \in \Omega$ and y_ϵ is a weakly ϵk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also a weakly ϵk^0-minimal element of Ω if $\text{int } C(y_\epsilon) \subseteq \text{int } C(y)$ for all $y \in \Omega$.*

5. *Suppose that y_ϵ is a strongly ϵk^0- minimal element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also a strongly ϵk^0-nondominated element of Ω if $C(y_\epsilon) \subseteq C(y)$ for all $y \in \Omega$.*

6. *Suppose that y_ϵ is a strongly ϵk^0- nondominated element of Ω with respect to the ordering map $C : Y \rightrightarrows Y$, then y_ϵ is also an ϵk^0-minimal element of Ω if $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$.*

Proof. 1. By $C(y) \subseteq C(y_\epsilon)$ and second part of Theorem 5.4, we know that y_ϵ is an ϵk^0-minimizer. Now, from first part of Theorem 5.1, it is obvious that y_ϵ is a ϵk^0-nondominated element.

2. Since y_ϵ is a ϵk^0-nondominated and $C(y_\epsilon) \subseteq C(y)$ for all $y \in \Omega$, then we can write

$$(y_\epsilon - \epsilon k^0 - (C(y)\backslash\{0\})) \cap \{y\} = \emptyset \Rightarrow (y_\epsilon - \epsilon k^0 - C(y_\epsilon)\backslash\{0\}) \cap \{y\} = \emptyset \ \ \forall y \in \Omega.$$

This means that $(y_\epsilon - \epsilon k^0 - C(y_\epsilon)\backslash\{0\}) \cap \Omega = \emptyset$ and y_ϵ is ϵk^0-minimal element.

3. By $C(y) \subseteq C(y_\epsilon)$ and the part 4 of Theorem 5.4, we know that weakly y_ϵ is a weakly ϵk^0-minimizer. Now, by the part 3 of Theorem 5.1, it is obvious that y_ϵ is a weakly ϵk^0-nondominated element.

4. This apart is similar to the part 2 by considering $\text{int } C(y_\epsilon) \subseteq \text{int } C(y)$ for all $y \in \Omega$.

5. By $C(y_\epsilon) \subseteq C(y)$ and the part 6 of Theorem 5.4, we know that strongly y_ϵ is a strongly ϵk^0-minimizer. Now, by the part 5 of Theorem 5.1, it is obvious that y_ϵ is a strongly ϵk^0-nondominated element.

6. Suppose that y_ϵ is a strongly ϵk^0-nondominated element of Ω. By $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$, we can write

$$y_\epsilon - \epsilon k^0 \in y - (C(y)\backslash\{0\}) \Rightarrow y_\epsilon - \epsilon k^0 \in y - C(y_\epsilon)\backslash\{0\} \ \ \forall y \in \Omega.$$

This means that y_ϵ is a strongly ϵk^0-minimal element of Ω.

□

The following examples show that the condition $C(y) \subseteq C(y_\epsilon)$ for all $y \in \Omega$ in the part 1 of Theorem 5.7 is a sufficient condition but it is not a necessary condition.

Example 5.8. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.8. Obviously $(-1, 0)$ is an ϵk^0-minimal element and also it is an ϵk^0-nondominated element but $C(y) \subseteq C(y_\epsilon)$ does not hold for all $y \in \Omega$.*

Example 5.9. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.8. From Example 3.8, we know that $(0, -1)$ is an ϵk^0-minimal element but not ϵk^0-nondominated element. It is easy to see that $\mathbb{R}^2_+$ is not a subset of $\{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 0, \ d_2 \leq 0\}$.*

The following examples show that the condition $C(y_\epsilon) \subseteq C(y)$ for all $y \in \Omega$ in the part 2 of Theorem 5.7 is a sufficient condition but it is not a necessary condition.

Example 5.10. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.8. It is easy to see that $(-1, 0)$ is an ϵk^0-nondominated element and also it is an ϵk^0-minimal element but $C(y_\epsilon) \subseteq C(y)$ does not hold for all $y \in \Omega$.*

Example 5.11. *Suppose ϵ, k^0, Ω and $C(y_1, y_2)$ are same as Example 3.9. From Example 3.9, we know that $(-1, 0)$ is an ϵk^0-nondominated element but not ϵk^0-minimal element. it is obvious that $\{(d_1, d_2) \in \mathbb{R}^2 | d_2 \geq 0,\ d_1 + d_2 \geq -1\}$ is not a subset of $\mathbb{R}^2_+$.*

6 Relations between approximate solutions and exact solutions

In this section, we will show relations between approximate solution and exact solution of vector optimization problem with respect to the variable order. First, we show relations between (weakly) ϵk^0-minimizers of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$ and minimizers of Ω with respect to C. After that, we will show relations between (weakly) ϵk^0-nondominated (minimal) elements of Ω and nondominated (minimal) elements of Ω with respect to C.

Theorem 6.1. *Assume that Assumption (A) holds and additionally $C(y)$ is a proper, pointed, closed set for all $y \in \Omega$.*

1. $\bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C) \subseteq WMZ(\Omega, C)$.
2. $MZ(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$
3. *If* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$, *then* $WMZ(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}WMZ(\Omega, C)$.

   Vice versa holds if $\text{int } C(y) \subseteq \bigcup_{\epsilon>0}(\text{int } C(y) + \epsilon k^0)$ *for all* $y \in \Omega$.

Proof. 1. Suppose that y_ϵ belongs to the set $\bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$ but y_ϵ is not a weakly minimizer of the set Ω.

$$\exists y, y^1 \in \Omega : \ y_\epsilon - \text{int } C(y) \cap \{y^1\} \neq \emptyset \Rightarrow \exists c_1 \in \text{int } C(y) : \ y_\epsilon - c_1 = y^1.$$

Since $c_1 \in \text{int } C(y)$, then there exist $\epsilon_1 > 0$ such that ball $B(c_1, \epsilon) \subseteq C(y)$ and $c_1 - \epsilon_1 k^0 = c_2 \in (C(y) \backslash \{0\}) \Rightarrow c_1 = c_2 + \epsilon_1 k^0$. Therefore we can write

$$(y_\epsilon - c_1) \in \Omega \Rightarrow (y_\epsilon - c_2 - \epsilon_1 k^0) \in \Omega \Rightarrow (y_\epsilon - \epsilon_1 k^0 - (C(y) \backslash \{0\})) \cap \Omega \neq \emptyset.$$

This means that $y_\epsilon \notin \bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$. But this is a contradiction because we supposed that y_ϵ belongs to the set $\bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$.

2. By part 1 of Theorem 4.4, we know that $MZ(\Omega, C) \subseteq \epsilon_1 k^0\text{-}MZ(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $MZ(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$.

3. By part 2 of Theorem 4.4, we know that $WMZ(\Omega, C) \subseteq \epsilon_1 k^0$-$WMZ(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $WMZ(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0$-$WMZ(\Omega, C)$. Now, suppose that $y_\epsilon \in \bigcap_{\epsilon>0} \epsilon k^0$-$WMZ(\Omega, C)$, then for any $\epsilon > 0$ we have

$$(y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \Omega = \emptyset \Rightarrow \bigcup_{\epsilon>0} \big((y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \Omega\big) = \emptyset.$$

By $\text{int } C(y) \subseteq \bigcup_{\epsilon>0}(\text{int } C(y) + \epsilon k^0)$ we can write,

$$(y_\epsilon - \text{int } C(y)) \cap \Omega \subseteq \bigcup_{\epsilon>0} \big((y_\epsilon - \epsilon k^0 - \text{int } C(y)) \cap \Omega\big) = \emptyset.$$

So y_ϵ is a weak minimizer of Ω with respect to the ordering map $C : Y \rightrightarrows Y$.

□

If $C(y)+[0,\infty)k^0 \subseteq \text{int } C(y)$, parts 1 and 2 implies that $WMZ(\Omega, C) = \bigcap_{\epsilon>0} \epsilon k^0$-$MZ(\Omega, C)$.

In the following theorem, we show relations between (weakly) ϵk^0-minimal elements of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$ and minimal elements of Ω with respect to C.

Theorem 6.2. *Let all the assumptions of Theorem 6.1 be fulfilled. We have the following relations.*

1. $\bigcap_{\epsilon>0} \epsilon k^0$-$M(\Omega, C) \subseteq WM(\Omega, C)$.
2. $M(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0$-$M(\Omega, C)$.
3. *If* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$, *then* $WM(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0$-$WM(\Omega, C)$.

   Vice versa holds if $\text{int } C(y) \subseteq \bigcup_{\epsilon>0}(\text{int } C(y) + \epsilon k^0)$ *for all* $y \in \Omega$.

Proof. 1. The proof is similar to the part 1 of Theorem 6.1.

2. By part 1 of Theorem 4.3, we know that $M(\Omega, C) \subseteq \epsilon_1 k^0$-$M(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $M(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0$-$M(\Omega, C)$.

3. By part 2 of Theorem 4.3, we know that $WAM(\Omega, C, 0) \subseteq \epsilon_1 k^0$-$WM(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $WM(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0$-$WM(\Omega, C)$. The proof of reverse implication is similar to the part 7.

□

If $C(y) + [0,\infty)k^0 \subseteq \text{int } C(y)$, parts 1 and 2 implies that $WM(\Omega, C) = \bigcap_{\epsilon>0} \epsilon k^0$-$M(\Omega, C)$.

Now, we show relations between (weakly) ϵk^0-nondominated elements of the set Ω with respect to the ordering map $C : Y \rightrightarrows Y$ and nondominated elements of Ω with respect to C.

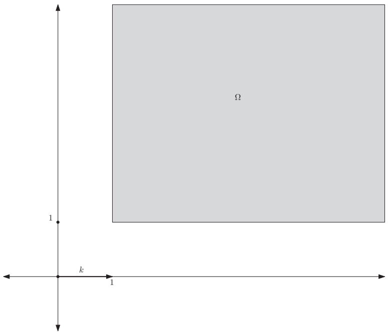

Figure 5: Set Ω, $C(y) = \mathbb{R}^2_+$ for all $y \in \Omega$ and $k^0 = (1, 0)$.

Theorem 6.3. *Suppose that all the assumptions of Theorem 6.1 hold. We have the following relations:*

1. $\bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C) \subseteq WN(\Omega, C)$.
2. $N(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C)$.
3. *If* $\text{int } C(y) \neq \emptyset$ *for all* $y \in \Omega$, *then* $WN(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}WN(\Omega, C)$.

   Vice versa holds if $\text{int } C(y) \subseteq \bigcup_{\epsilon>0}(\text{int } C(y) + \epsilon k^0)$ *for all* $y \in \Omega$.

Proof. 1. The proof is similar to the part 1 of Theorem 6.1.

2. By part 1 of Theorem 4.2, we know that $N(\Omega, C) \subseteq \epsilon_1 k^0\text{-}N(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $N(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C)$.

3. By part 2 of Theorem 4.2, we know that $WN(\Omega, C) \subseteq \epsilon_1 k^0\text{-}WN(\Omega, C)$ for all $\epsilon_1 \geq 0$. Therefore, $WN(\Omega, C) \subseteq \bigcap_{\epsilon>0} \epsilon k^0\text{-}WN(\Omega, C)$. The proof of reverse implication is similar to the part 7.

□

If $C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$, parts 1 and 2 implies that $WN(\Omega, C) = \bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C)$.

Note that the parts 2 of Theorems 6.1, 6.2 and 6.3 do not hold without the assumption $C(y) + [0, \infty)k^0 \subseteq \text{int } C(y)$. In general, the set of weakly nondominated points is not a subset of $\bigcap_{\epsilon>0} AN(\Omega, C, \epsilon k^0)$. This is also true for weakly minimal elements and weak minimizers. Consider Figure 5, where $k^0 = (1, 0)$ and $C(y) = \mathbb{R}^2_+$ for all $y \in \Omega$. Obviously $C(y) + [0, \infty)k^0 \not\subseteq \text{int } C(y)$. It is not difficult to see that $\{\{(1, y_2)\} \cup \{(y_1, 1)\} \cap \Omega\}$ is the set of weakly nondominated points and $\{\{(1, y_2)\} \cap \Omega\}$ is the set of $\bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C)$. Therefore, we can see that the set of weakly nondominated points is not a subset of $\bigcap_{\epsilon>0} \epsilon k^0\text{-}N(\Omega, C)$. This example also shows that the sets of weakly minimal elements and weak minimizers are not subsets of $\bigcap_{\epsilon>0} \epsilon k^0\text{-}M(\Omega, C)$ and $\bigcap_{\epsilon>0} \epsilon k^0\text{-}MZ(\Omega, C)$.

7 Necessary conditions for ϵk^0-minimal elements

In this section we present necessary conditions for approximate solutions of vector optimization problems with variable order structure. Bao and Mordukhovich [2] have shown necessary conditions for nondominated points of sets and nondominated solutions of vector optimization problems with variable ordering structures and general geometric constraints applying methods of variational analysis and generalized differentiation (see Mordukhovich [23] and Mordukhovich, Shao [24]). Necessary conditions for Pareto minimal elements are derived by Bao, Mordukhovich[1]. In the following result we use a generic approach to subdifferentials (compare [9]). We prove the necessary condition for approximate solutions using a vector-valued variant of Ekeland's variational principle (see [16, Corollary 9]).

Let $\mathcal{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces. By an abstract subdifferential ∂ we mean a map which associates to every lsc function $h : X \in \mathcal{X} \to \overline{\mathbb{R}}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subset X^*$. Let $X, Y \in \mathcal{X}$ and denote by $\mathcal{F}(X,Y)$ a class of functions acting between X and Y having the property that by composition at left with a lsc function from Y to $\overline{\mathbb{R}}$ the resulting function is still lsc.

In the following we work with the next properties of the abstract subdifferential ∂:

(H1) If h is convex, then $\partial h(x)$ coincides with the Fenchel subdifferential.

(H2) If x is a local minimum point for h, then $0 \in \partial h(x)$; $\partial h(u) = \emptyset$ if $u \notin \operatorname{Dom} h$.

(H3) If $X \in \mathcal{X}$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz functions and $x \in \operatorname{Dom} h$, then

$$\partial(h+\varphi)(x) \subset \|\cdot\|^* - \limsup_{y \xrightarrow{h} x, z \to x} (\partial h(y) + \partial\varphi(z)),$$

(H4) If $\varphi : Y \to \mathbb{R}$ is locally Lipschitz and $\psi \in \mathcal{F}(X,Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subset \|\cdot\|^* - \limsup_{u \xrightarrow{\psi} x, v \to \psi(x)} \bigcup_{u^* \in \partial\varphi(v)} \partial(u^* \circ \psi)(u),$$

where the following notations are used:

1. $u \xrightarrow{h} x$ means that $u \to x$ and $h(u) \to h(x)$; note that if h is continuous, then $u \xrightarrow{h} x$ is equivalent with $u \to x$.

2. $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ means that for every $\varepsilon > 0$ there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial h(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup_{u \xrightarrow{h} x} \partial h(u)$ has a similar interpretation and it is equivalent with $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ provided that h is continuous.

In the proof of the next result we use the functional $\varphi_k : Y \to \mathbb{R}$ defined for a proper closed convex cone $K \subset Y$ with nonempty interior and $k \in \operatorname{int} K$ by

$$\varphi_k(y) := \inf\{\lambda \in \mathbb{R} \mid \lambda k \in y + K\}. \tag{23}$$

Under the given assumptions this functional is continuous and convex and its subdifferential is given by

$$\partial\varphi_k(u) = \{v^* \in K^* \mid v^*(k) = 1, v^*(u) = \varphi_k(u)\}$$

(see [9, Lemma 2.1]).

In the next theorem we show necessary conditions for approximate minimal elements of a vector optimization problem with variable order structure following the line of the proof of [9, Theorem 5.3].

Theorem 7.1. *Suppose that assumption (A) is fulfilled. Let $X, Y \in \mathcal{X}$, $f \in \mathcal{F}(X, Y)$ be a L-Lipschitz function and S be a closed subset of X. Let $f(x_0) \in f(S)$ be an ϵk^0-minimal element of $f(S)$ in the sense of Definition 3.3, where $C(f(x_0))$ is a closed convex cone with nonempty interior. Then for every $k \in \operatorname{int} C(f(x_0))$ and $\mu > 0$, there exist $u \in B(x_0, \sqrt{\epsilon} + \mu), z \in B(x_0, \sqrt{\epsilon} + \mu/2) \cap S, u^* \in (C(f(x_0)))^*, u^*(k) = 1$, $x^* \in X^*, \|x^*\| \leq 1$ such that*

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\epsilon} u^*(k^0) x^* + N_\partial(S, z) + B(0, \mu),$$

provided that ∂ satisfies (H1), (H2), (H3), (H4). Moreover, for some $x \in B(x_0, \sqrt{\epsilon} + \mu/2)$ and $v \in B(f(x) - f(x_0), L\sqrt{\epsilon} + \mu)$ it holds $u^*(v) = \varphi_k(v)$.

Proof. We consider an ϵk^0-minimal element $f(x_0)$ in the sense of Definition 3.3. Taking into account Definition 3.3 we have

$$(f(x_0) - \epsilon k^0 - (C(f(x_0)) \backslash \{0\})) \cap f(S) = \emptyset.$$

The function f is supposed to be Lipschitz, so it is continuous as well and since S is a closed set in a Banach space it is a complete metric space endowed with the distance induced by the norm. Thus, the assumptions of the vector-valued variant of Ekeland's variational principle given in [16, Corollary 9] are fulfilled. Applying this variational principle we get the existence of an element $\overline{x} \in S$ with $\|\overline{x} - x_0\| < \sqrt{\epsilon}$ and having the property that

$$h(S) \cap (h(\overline{x}) - (C(f(x_0)) \setminus \{0\}) = \emptyset,$$

where

$$h(x) := f(x) + \sqrt{\epsilon} \|x - \overline{x}\| k^0.$$

Let $\mu > 0$. Applying now [9, Theorem 4.2] for a positive number δ with the property that $2\delta < \mu$ and $\sqrt{\epsilon} \|k^0\| \delta/2 + \delta/2 < \mu$. Accordingly, using the functional (23) with $K = C(f(x_0))$ we can find $\overline{u} \in B(\overline{x}, \delta) \subset B(x_0, \sqrt{\epsilon} + \delta), x \in B(\overline{x}, \delta/2) \subset B(x_0, \sqrt{\epsilon} + \delta/2), v \in B(h(x) - h(\overline{x}), \delta/2), z \in B(\overline{x}, \delta/2) \cap S \subset B(x_0, \sqrt{\epsilon} + \delta/2) \cap S, u^* \in \partial\varphi_k(v)$, such that

$$0 \in \partial(u^* \circ h)(\overline{u}) + N_\partial(S, z) + B(0, \delta). \tag{24}$$

Consider the element $\overline{x}^* \in \partial(u^* \circ h)(\overline{u})$ involved in (24). Because of

$$\partial(u^* \circ h)(\overline{u}) = \partial(u^* \circ (f(\cdot) + \sqrt{\epsilon} \|\cdot - \overline{x}\| k^0))(\overline{u}),$$

taking into account (H3) and (H1), there exist $u \in B(\overline{u}, \delta) \subset B(x_0, \sqrt{\epsilon} + 2\delta)$ and $u' \in B(\overline{u}, \delta)$ such that

$$\overline{x}^* \in \partial(u^* \circ f)(u) + \sqrt{\epsilon} u^*(k^0) \partial(\|\cdot - \overline{x}\|)(u') + B(0, \delta). \tag{25}$$

Taking into account the well-known structure of the subdifferential of the norm and combining relations (24) and (25) it follows that there exists $x^* \in X^*$ with $||x^*|| = 1$ such that

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\epsilon} u^*(k^0)x^* + N_\partial(S, z) + B(0, 2\delta).$$

Because of $2\delta < \mu$, it remains only to prove the estimation about the ball which contains v. Then,

$$\begin{aligned} \|v - (f(x) - f(x_0))\| &\leq \|v - (h(x) - h(\overline{x}))\| + \|(h(x) - h(\overline{x})) - (f(x) - f(x_0))\| \\ &\leq \delta/2 + \left\| \sqrt{\epsilon} k^0 \|x - \overline{x}\| - f(\overline{x}) + f(x_0)) \right\| \\ &\leq \delta/2 + \sqrt{\epsilon} \left\| k^0 \right\| \delta/2 + L\sqrt{\epsilon} \\ &< L\sqrt{\epsilon} + \mu, \end{aligned}$$

where the last inequality follows because of the assumptions made on δ. This completes the proof. □

8 Conclusions

Approximate solutions of vector optimization problems with variable order structure play an important role from the theoretical as well as computational point of view. We will use the concepts for approximate solutions of vector optimization problem with variable order structure in order to derive variational principles for problems with variable order structure. Characterization of approximate solutions of vector optimization problem with respect to the variable order structure by means of suitable nonlinear functionals will be discussed in a forthcoming paper. Moreover, we will show necessary conditions for approximate nondominated elements and minimizers of vector optimization problems with variable order structure similar to the approach presented in Section 7.

References

[1] Bao, T.Q., Mordukhovich, B.S.: Relative Pareto minimizers for multiobjective problems: existence and optimality conditions. Math. Program. 122, 301–347 (2010)

[2] Bao, T.Q., Mordukhovich, B.S.: Necessary Nondomination Conditions in Set and Vector Optimization with Variable Ordering Structures. J. Optim. Theory Appl. (2013).

[3] H. Bernau, "Interactive methods for vector optimization. Optimization in mathematical physics, Pap. 11th Conf. Methods Techniques Math. Phys., Oberwolfach/Ger. 1985 ", *Methoden Verfahren Math. Phys.* 34 21–36 (1987).

[4] G.Y. Chen, "Characterizations of variable domination structures via nonlinear scalarization ", *J. Optim. Theory Appl.* 112 97–110 (2002).

[5] G.Y. Chen, “Existence of solutions for a vector variational inequality: an extension of the Hartmann-Stampacchia Theorem ”, *J. Optim. Theory Appl.* 74 445–456 (1992).

[6] G.Y. Chen, “Vector optimization, set-valued and variational analysis ”, *Springer, Berlin.* (2005).

[7] G.Y. Chen and B.D. Craven, “Existence and Continuity of Solutions for Vector Optimization ”, *Optimization theory and applications.* 81(3) 459–468 (1994).

[8] C. Cusano, M. Fini and D. LA Torre, “Characterizations of convex vector functions and optimization ”, *Inequalities in pure and applied mathematics.* 5(4) (2004).

[9] Durea, M., and Tammer, C. *Fuzzy necessary optimality conditions for vector optimization problems.* Optimization, 58, 449–467, (2009).

[10] M. Ehrgott, “Multicriteria Optimization ”, *Springer.* 2005.

[11] A. Engau, ”Variable preference modeling with ideal-symmetric convex cones”, *J. Global. Optim.* 42 295–311 (2008).

[12] G. Eichfelder, “Adaptive Scalarization Methods in Multiobjective Optimization ”, *Springer.* 2008.

[13] G. Eichfelder, “Optimal Elements in Vector Optimization with a variable ordering Structure ”, *J. Optim. Theory Appl.* 151 217–240 (2011).

[14] Chr. Gerstewitz (Tammer), “Nichtkonvexe Dualität in der Vektoroptimierung ”, *Wissensch. Zeitschr. TH Leuna-Merseburg.* 26(3) 357–364 (1983).

[15] Chr. Gerth (Tammer) and P. Weidner, “Nonconvex separation theorems and some applications in vector optimization ”, *Optimization theory and applications.* 67 297–320 (1990).

[16] Göpfert, A., Tammer, Chr. and Zălinescu, C., 2000, On the vectorial Ekeland’s principle and minimal points in product spaces. *Nonlinear Analysis*, **39**, 909–922.

[17] A. Göpfert, H. Riahi, Chr. Tammer and C. Zălinescu, “Variational Methods in Partially Ordered Spaces ”, *Springer.* 2003.

[18] G. Göpfert, T. Riedrich and Chr. Tammer, “Angewandte Funktionalanalysis ”, *Vieweg Teubner, Wiesbaden.* (2009).

[19] C. Gutiérrez, C. Jiménez and V. Novo, “A unified approach and optimality conditions for approximate solutions of vector optimization problems ”, *SIAM J. Optim.* 17 688–710 (2006).

[20] S. Helbig, “On a new concept for ϵ-efficiency, talk at Optimization Days ”, *Montreal.* (1992).

[21] N.J. Huang, X.Q. Yang and W.K. Chan, “Vector complementarity problems with a variable ordering relation ”, *Eur. J. Oper. Res.* 176 15–26 (2007).

[22] J. Jahn,“Vector Optimization, Theory, Applications and Extensions”, *Springer* 2001.

[23] Mordukhovich, B. S., 2006, *Variational Analysis and Generalized Differentiation*, Vol. I: Basic Theory, Vol. II: Applications. (Berlin: Springer, Grundlehren der mathematischen Wissenschaften (A Series of Comprehensive Studies in Mathematics), Vol. 330 and 331).

[24] Mordukhovich, B. S. and Shao, Y., 1996, Nonsmooth sequential analysis in Asplund spaces. *Transactions of American Mathematical Society*, **348**, 1235–1280.

[25] B. Jiménez and V. Novo, “First and second order sufficient conditions for strict minimality in nonsmooth vector optimization ”, *Mathematica analysis and applications.* 284 465–510 (2003).

[26] S.S. Kutateladze, “Convex ϵ-Programming ”, *Soviet Math. Dokl.* 20 391–393 (1979).

[27] P. Loridan, “ϵ-solutions in vector minimization problem ”, *J. Optim. Theory Appl.* 43(2) 265–276 (1984).

[28] A.B. Németh, “A nonconvex vector minimization problem ”, *Nonlinear Anal.* 10 669–678 (1986).

[29] A. Pascoletti and P. Serafini, “Scalarizing vector optimization problems ”, *Optimization Theory Appl.* 42(4) 499–524 (1984).

[30] B. Soleimani and Chr. Tammer, ”Approximate solutions of vector optimization problem with variable ordering structure”, *AIP Conf. Proc.* 1479 2363–2366 (2012).

[31] T. Staib, “On two generalization of Pareto minimality ”, *J. Optim. Theory App.* 59 289–306 (1988).

[32] Chr. Tammer, “A generalization of Ekeland’s variational ”, *Optimization.* 25 129–141 (1992).

[33] Chr. Tammer (Gerth), “Näherungslösungen in der Vektoroptimierung ”, *Seminarbericht der Sektion Mathematik der Humboldt-Universität zu Berlin.* 90 67–76 (1987).

[34] Chr. Tammer, “Stability results for approximately efficient solutions ”, *OR Spectrum.* 16 47–52 (1994).

[35] Chr. Tammer and C. Zălinescu, “Lipschitz properties of the scalarization function and applications ”, *Optimization.* 59(2) 305–319 (2010) .

[36] T. Tanaka, “A new approach to approximation of solutions in vector optimization problems, in Proceedings of APORS, 1994 (eds. M. Fushimi and K. Tone) ”, *World Scientific Publishing, Singapour* 497–504 (1995).

[37] I. Vályi, "Approximate solutions of vector optimization problems, In System Analysis and System Analysis and Simulation, A. Sydow, M. Thoma, and R. Vichnevetsky, eds.", *Akademie-Verlag, Berlin.* 246–250 (1985).

[38] D.J. White, "Epsilon efficiency ", *J. Optim. Theory App.* 49 319–337 (1986).

[39] P.L. Yu, "Cone convexity, cone extreme points, and nondominated solutions in decision problems with multiobjetives ", *Optim. Theory Appl.* 14 319–377 (1974).

[40] P.L. Yu, "Multiple-criteria decision making:concepts, techniques and extensions ", *Plenum Press, New York.* (1985).

[41] P. L. Yu, "The set of all nondominated solutions in decision problems with multiobjectives ", *Syst. Anal. Progr., Working Paper Series F71-32. Univ, of Rochester, New York.* (1971).

Reports of the Institutes 2012

01-12. E. Köbis and Chr. Tammer, *Relations Between Strictly Robust Optimization Problems and a Nonlinear Scalarization Method*

02-12. S. Alzorba and Chr. Günther, *Algorithms for Multicriteria Location Problems*

03-12. K. Klamroth, E. Köbis, A. Schöbel and Chr. Tammer, *A united approach for different concepts of robustness and stochastic programming via nonlinear scalarizing functionals*

Reports are available via WWW: http://www2.mathematik.uni-halle.de/institut/reports/