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# A unified approach for different concepts of robustness and stochastic programming via nonlinear scalarizing functionals

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**Abstract**

We show that many different concepts of robustness and of stochastic programming can be described as special cases of a general nonlinear scalarization method by choosing the involved parameters and sets appropriately. This leads to a unifying concept which can be used to handle robust and stochastic optimization problems. Furthermore, we introduce multiple objective (deterministic) counterparts for uncertain optimization problems and discuss their relations to well-known scalar robust optimization problems by using the nonlinear scalarization concept. Finally, we mention some relations between robustness and coherent risk measures.

## 1 Introduction

Most real world optimization problems (OPs) are contaminated with uncertain data. One way of dealing with such optimization problems is described in the concept of robustness: Instead of assuming that all data are known, one allows different scenarios for the input parameters and looks for a solution that works well in every uncertain scenario. Robust optimization is an active field of research, we refer to Ben-Tal, L. El Ghaoui and Nemirovski [3] and Kouvelis and Yu [27] for an extensive collection of results and applications for the most prominent concepts. Several other concepts of robustness were introduced more recently, e.g. the concept of light robustness by Fischetti and Monaci [11] or the concept of recovery-robustness in Liebchen et al. [28]. A scenario-based approach is suggested in Goerigk and Schöbel [17]. In all these approaches, the uncertain optimization problem is replaced by a deterministic version, called the *robust counterpart* of the uncertain problem.

Another prominent way of dealing with uncertain optimization is the field of stochastic programming; for an introduction we refer to Birge and Louveaux [7]. Different from robust optimization, stochastic programming assumes some knowledge about the probability distribution of the uncertain data. The objective usually is to find a feasible solution (or a solution that is feasible with a certain probability) that optimizes the expected value of some objective or cost function.

In this paper, we will link the different concepts of robustness and of stochastic programming, that usually have been considered fundamentally different, in a very general and unifying framework. Assuming that the set of scenarios is finite, we will show that all of the

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considered uncertain optimization problems have a (deterministic) counterpart in this framework. Our analysis is based on two closely related concepts: First, we will use *nonlinear scalarizing functionals* to describe the counterpart of uncertain optimization problems, and second, we will relate the solutions of this functional (and likewise of the respective uncertain optimization problems) to the efficient set of a *multiple objective counterpart*. This will lay the ground for a thorough analysis of the interrelations and also the differences between established concepts in robust optimization and stochastic programming. By providing additional trade-off information between alternative efficient solutions, the multiple objective counterpart can facilitate the decision making process when deciding for a most preferred robust solution.

For specific robustness concepts, the connection between uncertain scalar optimization problems and an associated (deterministic) multiple objective counterpart were observed by several authors. Kouvelis and Sayin [26, 32] use this relation to develop efficient solution methods for bi- and multiple objective discrete optimization problems based on algorithms that were originally developed to solve uncertain scalar optimization problems. They focus on two classical robustness concepts that will be referred to as strict robustness and deviation robustness in Sections 3.1 and 3.2 below, see also [27]. Perny et.al. [30] use a multiple objective counterpart to introduce a robustness measure based on the Lorenz dominance rule in the context of minimum spanning tree and shortest path problems. From the stochastic programming perspective, a multiple objective counterpart for a two-stage stochastic programming problem was introduced in Gast [13] and used to interrelate stochastic programming models with the concept of recoverable robustness, see Stiller [36]. A critical analysis is given in Hites et.al. [22] who give a qualitative description of the similarities and differences between the two modeling paradigms. They conclude that from a modeling perspective, a multiple objective counterpart can in general not be used to represent an uncertain scalar optimization problem. However, as will be seen below, there certainly is a strong relation from a theoretical point of view. This reveals interesting properties of alternative solutions of scalar uncertain optimization problems.

In a first step, we will treat uncertain scalar optimization problems as special cases of a nonlinear scalarizing functional. Many methods for scalarization were suggested in the literature that are special cases of a nonlinear scalarization concept introduced by Gerstewitz (Tammer) [14], see also Gerth (Tammer), Weidner [15], Pascoletti, Serafini [29], Göpfert, Tammer, Zălinescu [19] and Göpfert, Riahi, Tammer, Zălinescu [18]. This scalarization method includes, for instance, weighted-sums, Tschebyscheff- and $\epsilon$-constraint-scalarization. We will show that this scalarization method includes a variety of different models from robust optimization and stochastic programming as specifications. So it is possible to get a unified approach for different types of optimization models including uncertainties (see Section 3). Moreover, the well-studied properties of this scalarization method allow the establishment of relations to multiple objective optimization problems (MOPs).

Let $Y$ be a linear topological space, $k \in Y \setminus \{0\}$ and let $\mathcal{F}, B$ be proper subsets of $Y$. We assume that

$$B + [0, \infty) \cdot k \subset B. \tag{1.1}$$

We introduce the functional $z^{B,k} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \bar{\mathbb{R}}$

$$z^{B,k}(y) := \inf\{t \in \mathbb{R} | y \in tk - B\}. \tag{1.2}$$

Now we formulate the problem

$$z^{B,k}(y) \to \inf_{y \in \mathcal{F}}. \tag{$P_{k,B,\mathcal{F}}$}$$

In Section 3 we show that many different concepts of uncertain scalar optimization problems can be described by means of the functional (1.2) and problem ($P_{k,B,\mathcal{F}}$) by choosing $B$, $k$ and $\mathcal{F}$ in different ways.

The scalarizing functional $z^{B,k}$ was used in [15] to prove separation theorems for not necessarily convex sets. Applications of $z^{B,k}$ include coherent risk measures in financial mathematics (see, for instance, [21]). Monotonicity and continuity properties of $z^{B,k}$ were studied by Gerth (Tammer) and Weidner in [15], and later in [39], [18]. Further important properties of the functional $z^{B,k}$, for example the translation property and sublinearity, were shown in [18].

Based on the interpretation of uncertain scalar optimization problems by means of the nonlinear scalarizing functional (1.2), we will in a second step formulate multiple objective counterparts, the efficient sets of which comprise optimal solutions for the considered uncertain scalar optimization problems.

The paper is organized as follows: In Section 2, we introduce the basic concepts and some notation, and summarize the properties of the nonlinear scalarizing functional $z^{B,k}$. In Section 3, we show how several concepts of robustness can be expressed using the nonlinear scalarizing functional $z^{B,k}$. In Section 4, we show that we can develop new concepts for robustness using the nonlinear scalarization approach by varying the parameters. Specifically, we introduce the $\epsilon$-constraint robust counterpart problem. In Section 5, we use the functionals $z^{B,k}$ to establish relations between the multiple objective counterpart and their corresponding uncertain scalar optimization problems. The final Section is concerned with correlations between coherent risk measures and uncertain scalar optimization problems.

## 2 Preliminaries

The goal of this paper is to interpret uncertain scalar optimization problems as scalarizations of deterministic multiple objective optimization problems. We settle this relation by using a nonlinear scalarizing functional. There are some technical preliminaries needed for our approach which are described next. We first recall the definition of efficient solutions as it is used in multiple objective optimization and then discuss the properties of the nonlinear scalarizing functional used in this paper.

**Definition 1.** *Let $Y$ be a linear topological space, $\mathcal{F} \subset Y$, $\mathcal{F} \neq \emptyset$ and $C \subset Y$ a proper pointed closed convex cone. We call an element $y \in \mathcal{F}$ $C$-**efficient** in $\mathcal{F}$, if*

$$\mathcal{F} \cap (y - (C \setminus \{0\})) = \emptyset.$$

*Moreover, if additionally* $\operatorname{int} \mathrm{C} \neq \emptyset$, *$y \in \mathcal{F}$ is **weakly** $C$-**efficient** in $\mathcal{F}$, if*

$$\mathcal{F} \cap (y - \operatorname{int} C) = \emptyset.$$

*We denote the set of all $C$-efficient elements in $\mathcal{F}$ by* $\operatorname{Eff}(\mathcal{F}, C)$, *and the set of all weakly $C$-efficient elements in $\mathcal{F}$ is denoted by* $\operatorname{Eff}_w(\mathcal{F}, C)$.

Note that we obtain the well-known concept of Pareto optimality if $Y = \mathbb{R}^q$, and if the ordering cone $C$ is given by the nonnegative orthant $C = \mathbb{R}^q_+$. We refer to the books of Ehrgott [9] and Jahn [24] for a detailed introduction to multiple objective optimization.

In order to understand the relation between the efficient solutions of a multiple objective programming problem and the minimizers of the nonlinear scalarizing functional in problem ($P_{k,B,\mathcal{F}}$), we need some theoretical properties of the functional $z^{B,k}$.

**Definition 2.** *Let $Y$ be a linear topological space, $D \subset Y$, $D \neq \emptyset$. A functional $z : Y \to \bar{\mathbb{R}}$ is $D$-**monotone**, if for*

$$y_1,\ y_2 \in Y:\ y_1 \in y_2 - D \Rightarrow z(y_1) \leq z(y_2).$$

*Moreover, $z$ is said to be **strictly** $D$-**monotone**, if for*

$$y_1, y_2 \in Y:\ y_1 \in y_2 - D \setminus \{0\} \Rightarrow z(y_1) < z(y_2).$$

*Its **domain** and **epigraph** are denoted by*

$$\mathrm{dom}\ z := \{y \in Y | z(y) < \infty\}, \quad \mathrm{epi}\ z := \{(y,t) \in Y \times \mathbb{R} | z(y) \leq t\}.$$

*The functional $z$ is said to be **proper** if $\mathrm{dom}\ z \neq \emptyset$ and $z$ does not take the value $-\infty$. $z$ is **lower semi-continuous** if $\mathrm{epi}\ z$ is closed. $A \subset Y$ is a **convex** set if $\forall \lambda \in (0,1)$, $\forall a_1, a_2 \in A:\ \lambda a_1 + (1-\lambda)a_2 \in A$. $z$ is **convex** over the convex set $A$ if $\forall \lambda \in (0,1)$, $\forall a_1, a_2 \in A:\ z(\lambda a_1 + (1-\lambda)a_2) \leq \lambda z(a_1) + (1-\lambda)z(a_2)$. $z$ is called **subadditive** if $\forall y_1, y_2 \in Y:\ z(y_1 + y_2) \leq z(y_1) + z(y_2)$. A set $C \subset Y$ is called a **cone** if $\forall \lambda \in \mathbb{R}$, $\lambda > 0$, $\forall y \in C:\ \lambda y \in C$.*

Based on these properties, the following theorem characterizes the relation between multiple objective and scalar optimization problems via the concept of scalarizing functionals. We will apply Theorem 1 in Section 5.

**Theorem 1** ([23, Theorem 2.2],[15, Theorem 3.3])**.** *Let $Y$ be a linear topological space, $C \subset Y$ a proper pointed closed convex cone, and $\mathcal{F}$ a nonempty subset of $Y$. Then it holds:*

*(i) If there exists a strictly $C$-monotone functional $z : Y \to \mathbb{R}$, where $\forall y \in \mathcal{F}:\ z(y^*) \leq z(y)$ holds, then $y^* \in \mathrm{Eff}(\mathcal{F}, C)$.*

*(ii) If there exists a $C$-monotone functional $z : Y \to \mathbb{R}$, where $\forall y \in \mathcal{F} \setminus \{y^*\}:\ z(y^*) < z(y)$, then $y^* \in \mathrm{Eff}(\mathcal{F}, C)$.*

*Additionally, if $\mathrm{int}\, C \neq \emptyset$ and if there exists a strictly $(\mathrm{int}\, C)$-monotone functional $z : Y \to \mathbb{R}$ where $\forall y \in \mathcal{F}:\ z(y^*) \leq z(y)$, then $y^* \in \mathrm{Eff}_w(\mathcal{F}, C)$.*

Part (i) in Theorem 1 can be found in [15, Theorem 3.3]. A proof of the theorem can be found in [23, Theorem 2.2].

Theorem 2 below shows that the nonlinear scalarizing functional $z = z^{B,k}$ introduced in (1.2) satisfies, under quite general assumptions, the properties given in Theorem 1 and thus immediately connects to efficient solutions in multiple objective optimization.

**Theorem 2** ([15, 18])**.** *Let $Y$ be a linear topological space, $B \subset Y$ a closed proper set and $D \subset Y$. Furthermore, let $k \in Y \setminus \{0\}$ be such that (1.1) is satisfied. Then the following properties hold for $z = z^{B,k}$:*

*(a) $z$ is lower semi-continuous.*

*(b) $z$ is convex $\Longleftrightarrow$ $B$ is convex,*
*$[\forall y \in Y,\ \forall\ r > 0:\ z(ry) = rz(y)] \Longleftrightarrow B$ is a cone.*

*(c) $z$ is proper $\Longleftrightarrow$ $B$ does not contain lines parallel to $k$, i.e., $\forall y \in Y\ \exists r \in \mathbb{R}:\ y + rk \notin B$.*

*(d) $z$ is $D$-monotone $\Longleftrightarrow B + D \subset B$.*

*(e) $z$ is subadditive $\Longleftrightarrow B + B \subset B$.*

*(f)* $\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) \le r \iff y \in rk - B$.

*(g)* $\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y + rk) = z(y) + r$ *(translation property).*

*(h)* *$z$ is finite-valued $\iff$ $B$ does not contain lines parallel to $k$ and $\mathbb{R}k - B = Y$.*

*Let furthermore $B + (0, \infty) \cdot k \subset \operatorname{int} B$. Then*

*(i)* *$z$ is continuous.*

*(j)* $\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) = r \iff y \in rk - \operatorname{bd} B$,
$\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) < r \iff y \in rk - \operatorname{int} B$.

*(k)* *If $z$ is proper, then $z$ is $D$-monotone $\iff B + D \subset B \iff \operatorname{bd} B + D \subset B$.*

*(l)* *If $z$ is finite-valued, then $z$ is strictly $D$-monotone $\iff B + (D \setminus \{0\}) \subset \operatorname{int} B \iff \operatorname{bd} B + (D \setminus \{0\}) \subset \operatorname{int} B$.*

*(m)* *Suppose $z$ is proper. Then $z$ is subadditive $\iff B + B \subset B \iff \operatorname{bd} B + \operatorname{bd} B \subset B$.*

For the proof, see [18, Theorem 2.3.1].

**Corollary 1** ([18, Corollary 2.3.5.]). *Let $C$ be a proper closed convex cone and $k \in \operatorname{int} C$. Then $z = z^{C,k}$, defined by (1.2), is a finite-valued continuous sublinear and strictly* ($\operatorname{int} C$)*-monotone functional such that*

$$\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) \le r \iff y \in rk - C,$$

$$\forall y \in Y,\ \forall r \in \mathbb{R}:\ z(y) < r \iff y \in rk - \operatorname{int} C.$$

## 3 Unified approach to uncertain optimization via nonlinear scalarization

Robust optimization has become very popular in the last few years, possibly because of its many applications and closeness to reality. Usually, when solving an optimization problem, all parameters are given. However, in most real-world applications, some parameters are not known. Instead, it might happen that only an estimated value or a set of possible values is given. Throughout this paper we assume that the uncertainty is given by a finite set of scenarios.

In the literature, two main ways of dealing with optimization problems with uncertainty are suggested. In *stochastic optimization*, knowledge about the probability distribution of the uncertain data is assumed. Some authors use a probabilistic approach, for example, to minimize the probability that a solution is - in some sense - not good or not feasible. Other authors use the expected value of some cost function depending on the realization of the uncertain data as the objective function.

While in stochastic optimization a solution is desired which is good on average, a robust solution hedges against the worst case that may happen. Different from stochastic optimization, no distribution is needed in robust optimization. There are different ways of defining robustness in the literature. Here, we focus on several models where uncertainties are assumed to occur in the objective function and in the constraints.

We will now formulate an optimization problem with uncertainties. Throughout this paper, let $\mathcal{U} := \{\xi_1, \ldots, \xi_q\}$ be the uncertainty set, i.e., $\xi \in \mathcal{U}$ can take on $q$ different values.

One could think of $\xi$ being real numbers or real vectors. Furthermore, let $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, $i = 1, \dots, m$. Then an *uncertain scalar optimization problem* (uncertain OP) is defined as a parametrized optimization problem

$$Q(\xi), \xi \in \mathcal{U}, \tag{3.3}$$

where for a given $\xi \in \mathcal{U}$ the optimization problem $(Q(\xi))$ is given by

$$\begin{aligned} & f(x,\xi) \to \min \\ & \text{s.t. } F_i(x,\xi) \le 0, \ i = 1, \dots, m, \\ & x \in \mathbb{R}^n. \end{aligned} \tag{$Q(\xi)$}$$

At the time the uncertain OP (3.3) has to be solved, it is not known which value $\xi \in \mathcal{U}$ is going to be realized. We call $\hat{\xi} \in \mathcal{U}$ the *nominal value*, i.e., the value of $\xi$ that we believe is true today. $(Q(\hat{\xi}))$ is called the *nominal problem.* Throughout this paper, we will assume that the minima of the problems to be proposed in the following sections exist.

### 3.1 Strict robustness

The first formally introduced robustness concept will be called *strict robustness* here. It has been first mentioned by Soyster [35] and then formalized and analyzed by Ben-Tal, El Ghaoui, and Nemirovski in numerous publications, see e.g. [4, 16] for early contributions and [3] for an extensive collection of results. The idea is that the worst possible objective function value is minimized in order to get a solution that is "good enough" even in the worst case scenario. Furthermore, constraints have to be satisfied for every scenario $\xi \in \mathcal{U}$. Then the **strictly robust counterpart** of the uncertain optimization problem $(Q(\xi), \xi \in \mathcal{U})$ is defined by

$$\begin{aligned} & \rho_{RC}(x) = \max_{\xi \in \mathcal{U}} f(x,\xi) \to \min \\ & \text{s.t. } \forall \xi \in \mathcal{U} : \ F_i(x,\xi) \le 0, \ i = 1, \dots, m, \\ & \qquad x \in \mathbb{R}^n. \end{aligned} \tag{$RC$}$$

We call a feasible solution of $(RC)$ *strictly robust.* The set of strictly robust solutions is denoted as

$$\mathfrak{A} := \{x \in \mathbb{R}^n | \forall \xi \in \mathcal{U} : \ F_i(x,\xi) \le 0, \ i = 1, \dots, m\}. \tag{3.4}$$

We will now show how $(RC)$ can be expressed using the nonlinear scalarizing functional $z^{B,k}$ given by (1.2) (cf. [25]).

**Theorem 3.** *Consider*

$$\mathfrak{A}_1 := \mathfrak{A}, \tag{3.5}$$

$$B_1 := \mathbb{R}^q_+, \tag{3.6}$$

$$k_1 = 1_q := (1, \dots, 1)^T, \tag{3.7}$$

$$\mathcal{F}_1 = \{(f(x,\xi_1), \dots, f(x,\xi_q))^T | x \in \mathfrak{A}_1\}. \tag{3.8}$$

*For $k = k_1$, $B = B_1$, condition (1.1) is satisfied and with $\mathcal{F} = \mathcal{F}_1$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to problem $(RC)$ in the following sense:*

$$\begin{aligned} \min\{z^{B_1,k_1}(y) | y \in \mathcal{F}_1\} &= z^{B_1,k_1}(y^*) \\ &= \min\{\rho_{RC}(x) | x \in \mathfrak{A}_1\} \\ &= \rho_{RC}(x^*), \end{aligned}$$

*where $y^* = (f(x^*,\xi_1), \dots, f(x^*,\xi_q))^T$.*

*Proof.* Since $B_1 + [0,\infty)\cdot k_1 = \mathbb{R}^q_+ + [0,\infty)\cdot 1_q \subset \mathbb{R}^q_+ = B_1$, condition (1.1) is satisfied. Since $k_1 \in \operatorname{int}\mathbb{R}^q_+$ and $B_1 = \mathbb{R}^q_+$ is closed, the infimum in the definition of $z^{B_1,k_1}$ is finite and attained such that we can replace the infimum by a minimum:

$$\begin{aligned}
\min_{y\in\mathcal{F}_1} z^{B_1,k_1}(y) &= \min_{y\in\mathcal{F}_1} \min\{t\in\mathbb{R} | y \in tk_1 - B_1\} \\
&= \min_{y\in\mathcal{F}_1} \min\{t\in\mathbb{R} | y - tk_1 \in -B_1\} \\
&= \min_{x\in\mathfrak{A}_1} \min\{t\in\mathbb{R} | (f(x,\xi_1),\dots,f(x,\xi_q))^T - t\cdot(1,\dots,1)^T \leq 0_q\} \\
&= \min_{x\in\mathfrak{A}_1} \min\{t\in\mathbb{R} | (f(x,\xi_1),\dots,f(x,\xi_q))^T \leq t\cdot(1,\dots,1)^T\} \\
&= \min\{\max_{\xi\in\mathcal{U}} f(x,\xi) | x \in \mathfrak{A}_1\} \\
&= \min\{\rho_{RC}(x) | x\in\mathfrak{A}_1\}.
\end{aligned}$$

□

Note that the selection of $k_1 = 1_q$ means that every objective function $f(x,\xi)$, $\xi\in\mathcal{U}$, is treated in the same way, i.e., no objective function is preferred to another one.

*Remark:* Since $B_1$ is a proper closed convex cone and $k_1 \in \operatorname{int} B_1$, the functional $z^{B_1,k_1}$ is continuous, finite-valued, $\mathbb{R}^q_+$-monotone, strictly $(\operatorname{int}\mathbb{R}^q_+)$-monotone and sublinear, taking into account Corollary 1.

*Remark:* The concept of strict robustness is described by the Tschebyscheff scalarization with the origin as reference point as a special case of functional (1.2). Theorem 3 shows that $(RC)$ can be interpreted as a max-ordering problem as defined in multiple objective optimization, see [9]. This relationship was also observed by Kouvelis and Sayin [26, 32] where it was used to determine the nondominated set of discrete bicriteria optimization problems.

### 3.2 Deviation robustness

We will now introduce another prominent robustness concept, called *deviation-robustness* or *min max regret robustness*. In contrast to the concept of strict robustness, the function to be minimized is $\max_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi))$, where $f^*(\xi)\in\mathbb{R}$ is the optimal value of problem $(Q(\xi))$ for the parameter $\xi\in\mathcal{U}$. This robustness concept has a long tradition in many applications such as scheduling or location theory, mostly if no uncertainty in the constraints is present. We refer to [27] for many applications. We formulate the **deviation-robust counterpart** of (3.3) as

$$\begin{aligned}
&\rho_{dRC}(x) = \max_{\xi\in\mathcal{U}}(f(x,\xi) - f^*(\xi)) \to \min \\
&\text{s.t. } \forall \xi \in \mathcal{U}: \ F_i(x,\xi) \leq 0, \ i = 1,\dots,m, \\
&\qquad x\in\mathbb{R}^n.
\end{aligned} \tag{$dRC$}$$

Now let

$$f^* := (f^*(\xi_1),\dots,f^*(\xi_q))^T \tag{3.9}$$

be the vector consisting of the individual minimizers for the respective scenarios which can be interpreted as an *ideal solution vector*. The relation to the ideal point of the multiple objective counterpart of problem (3.3) will be discussed later but can already be noted here. Then we have the following theorem:

**Theorem 4.** *Consider*

$$\mathfrak{A}_2 := \mathfrak{A}, \tag{3.10}$$

$$B_2 := \mathbb{R}^q_+ - f^*, \tag{3.11}$$

$$k_2 := 1_q, \tag{3.12}$$

$$\mathcal{F}_2 = \{(f(x,\xi_1),\dots,f(x,\xi_q))^T | x \in \mathfrak{A}_2\}. \tag{3.13}$$

*For $k = k_2$, $B = B_2$, condition (1.1) is satisfied and with $\mathcal{F} = \mathcal{F}_2$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to problem (dRC) in the following sense:*

$$\begin{aligned}\min\{z^{B_2,k_2}(y)|y\in\mathcal{F}_2\} &= z^{B_2,k_2}(y^*)\\ &= \min\{\rho_{dRC}(x)|x\in\mathfrak{A}_2\}\\ &= \rho_{dRC}(x^*),\end{aligned}$$

*where $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$.*

*Proof.* Since $B_2 + [0,\infty)\cdot k_2 = (\mathbb{R}^q_+ - f^*) + [0,\infty)\cdot 1_q \subset \mathbb{R}^q_+ - f^* = B_2$, condition (1.1) is satisfied. Moreover,

$$\begin{aligned}\min_{y\in\mathcal{F}_2} z^{B_2,k_2}(y) &= \min_{y\in\mathcal{F}_2}\min\{t\in\mathbb{R}|y\in tk_2 - B_2\}\\ &= \min_{x\in\mathfrak{A}_2}\min\{t\in\mathbb{R}|(f(x,\xi_1),\dots,f(x,\xi_q))^T - (f^*(\xi_1),\dots,f^*(\xi_q))^T - t\cdot 1_q \le 0_q\}\\ &= \min_{x\in\mathfrak{A}_2}\min\{t\in\mathbb{R}|(f(x,\xi_1),\dots,f(x,\xi_q))^T - (f^*(\xi_1),\dots,f^*(\xi_q))^T \le t\cdot 1_q\}\\ &= \min\{\max_{\xi\in\mathcal{U}}(f(x,\xi)-f^*(\xi))|x\in\mathfrak{A}_2\}\\ &= \min\{\rho_{dRC}(x)|x\in\mathfrak{A}_2\}.\end{aligned}$$

□

Alternatively, we could have taken $\widetilde{B_2} := B_1 = \mathbb{R}^q_+$ and we could have minimized $z^{B_1,k_1}$ over the set $\widetilde{\mathcal{F}_2} := \{(f(x,\xi_1),\dots,f(x,\xi_q))^T - (f^*(\xi_1),\dots,f^*(\xi_q))^T|x\in\mathfrak{A}_2\}$. Therefore, one can observe that (dRC) is a shifted version of (RC). This means, whenever we have a finite uncertainty set and $f^*$ is known beforehand, the concepts of strict robustness and of deviation robustness can be solved within the same complexity. However, for general uncertainty sets, (dRC) is usually harder to solve than (RC).

*Remark:* Using Theorem 2 and the fact that $B_2 + (0,\infty)\cdot k_2 \subset \operatorname{int} B_2$, we can conclude that the functional $z^{B_2,k_2}$ is continuous, finite-valued, convex, $\mathbb{R}^q_+$-monotone and strictly ($\operatorname{int}\mathbb{R}^q_+$)-monotone. Note that since $B_2$ is not a cone, Corollary 1 cannot be applied.

*Remark:* Similar to the case of strict robustness, the concept of deviation robustness can be described by the Tschebyscheff scalarization, however, not with the origin as reference point but with the ideal point $f^*$ defined in (3.9) as reference point. This shows once again the close relationship between these two robustness concepts, see also Kouvelis and Sayin [26, 32].

### 3.3 Reliable robustness

Sometimes it is difficult to find a point $x$ that satisfies all constraints $F_i(x,\xi) \le 0$ for all $\xi \in \mathcal{U}$, or it is simply not useful for $x$ to satisfy the constraints at the cost of minimality of $f$. We therefore introduce the concept of reliable robustness, where the constraints are allowed to differ from the original problem. Instead of having hard constraints $F_i(x,\xi) \le 0$ for all $\xi \in \mathcal{U}$, we now allow the constraints to satisfy an infeasibility tolerance $\delta_i \in \mathbb{R}_+$ in order to achieve soft constraints $F_i(x,\xi) \le \delta_i$. Nevertheless, the original constraints for the nominal value $\hat{\xi}$ should be fulfilled, i.e., $F_i(x,\hat{\xi}) \le 0,\ i=1,\dots,m$. Then the **reliably robust counterpart** of (3.3) proposed by Ben-Tal and Nemirovski in [5], is defined by

$$\begin{aligned}
&\rho_{rRC}(x) = \max_{\xi\in\mathcal{U}} f(x,\xi) \to \min \\
&\text{s.t. } F_i(x,\hat{\xi}) \le 0,\ i=1,\dots,m, \\
&\quad \forall \xi \in \mathcal{U}:\ F_i(x,\xi) \le \delta_i,\ i=1,\dots,m, \\
&\quad x \in \mathbb{R}^n.
\end{aligned} \tag{$rRC$}$$

A feasible solution of $(rRC)$ is called *reliably robust*. If $\delta_i = 0$ for all $i=1,\dots,m$, the reliably robust OP $(rRC)$ is equivalent to the strictly robust OP $(RC)$. Strict robustness is therefore a special case of reliable robustness.

**Theorem 5.** *Consider*

$$\mathfrak{A}_3 := \{x \in \mathbb{R}^n | F_i(x,\hat{\xi}) \le 0,\ \forall \xi \in \mathcal{U}:\ F_i(x,\xi) \le \delta_i,\ i=1,\dots,m\}, \tag{3.14}$$

$$B_3 := \mathbb{R}^q_+, \tag{3.15}$$

$$k_3 := 1_q, \tag{3.16}$$

$$\mathcal{F}_3 = \{(f(x,\xi_1),\dots,f(x,\xi_q))^T | x \in \mathfrak{A}_3\}. \tag{3.17}$$

*For $k = k_3$, $B = B_3$, condition (1.1) is satisfied and with $\mathcal{F} = \mathcal{F}_3$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to problem $(rRC)$ in the following sense:*

$$\begin{aligned}
&\min\{z^{B_3,k_3}(y) | y \in \mathcal{F}_3\} \\
&= z^{B_3,k_3}(y^*) \\
&= \min\{\rho_{rRC}(x) | x \in \mathfrak{A}_3\} \\
&= \rho_{rRC}(x^*),
\end{aligned}$$

*where $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$.*

Because $\rho_{rRC} = \rho_{RC}$ and only the set $\mathcal{F}_3$ of feasible points differs from the concept of strict robustness, the proof is left out.

*Remark:* Since $z^{B_3,k_3} = z^{B_1,k_1}$, the functional $z^{B_3,k_3}$ is again continuous, finite-valued, $\mathbb{R}^q_+$-monotone, strictly $(\operatorname{int}\mathbb{R}^q_+)$-monotone and sublinear, taking into account Corollary 1.

*Remark:* The concept of reliable robustness is - similar to strict robustness - described by the Tschebyscheff scalarization with the origin as reference point and on the basis of a relaxed feasible set, as a special case of functional (1.2).

### 3.4 Light robustness

When we examine a variation of the constraints $F_i(x,\xi) \le \delta_i$, where $F_i,\ \delta_i,\ i=1,\dots,m$, are defined as in Section 3.3, it could be useful to minimize these tolerances, which the concept of light robustness describes. It was introduced in 2008 by Fischetti and Monaci in [11] for linear programs with the $\Gamma$-uncertainty set introduced by Bertsimas and Sim [6] and generalized to general uncertain robust optimization problems by Schöbel [33]. Let $z^*$ be the optimal value of the nominal problem $(Q(\hat{\xi}))$, which we assume to be positive, i.e., $z^* > 0$. We want the nominal value $f(x,\hat{\xi})$ to be bounded by $(1+\gamma)z^*$, where $\gamma \ge 0$. Then a solution of the **lightly robust counterpart** of (3.3)

$$
\begin{aligned}
&\rho_{lRC}(\delta, x) = \sum_{i=1}^{m} w_i \delta_i \to \min \\
&\text{s.t. } F_i(x, \hat{\xi}) \leq 0,\ i = 1, \dots, m, \\
&\qquad f(x, \hat{\xi}) \leq (1+\gamma) z^*, \\
&\qquad \forall \xi \in \mathcal{U}:\ F_i(x, \xi) \leq \delta_i,\ i = 1, \dots, m, \\
&\qquad x \in \mathbb{R}^n, \\
&\qquad \delta_i \in \mathbb{R}_+,\ i = 1, \dots, m,
\end{aligned}
\tag{$lRC$}
$$

where $w_i \geq 0,\ i = 1, \dots, m,\ \sum_{i=1}^{m} w_i = 1$, is called *lightly robust.*

**Theorem 6.** *Consider*

$$B_4 := \{(\delta_1, \dots, \delta_m)^T | \sum_{i=1}^{m} w_i \delta_i \geq 0,\ \delta_i \in \mathbb{R},\ i = 1, \dots, m\}, \tag{3.18}$$

$$k_4 := 1_m, \tag{3.19}$$

$$
\begin{aligned}
\mathcal{F}_4 = &\{(\delta_1, \dots, \delta_m)^T | \exists x \in \mathbb{R}^n:\ F_i(x, \hat{\xi}) \leq 0,\ f(x, \hat{\xi}) \leq (1+\gamma) z^*, \\
&\forall \xi \in \mathcal{U}:\ F_i(x, \xi) \leq \delta_i,\ \delta_i \in \mathbb{R}_+,\ i = 1, \dots, m\}.
\end{aligned}
\tag{3.20}
$$

*For $k = k_4$, $B = B_4$, condition (1.1) is satisfied and with $\mathcal{F} = \mathcal{F}_4$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to problem $(lRC)$ in the following sense:*

$$
\begin{aligned}
&\min\{z^{B_4,k_4}(y) | y \in \mathcal{F}_4\} \\
&\quad = z^{B_4,k_4}(y^*) \\
&\quad = \min\{\rho_{lRC}(\delta) | \delta \in \mathcal{F}_4\} \\
&\quad = \rho_{lRC}(\delta^*),
\end{aligned}
$$

*where $y^* = \delta^* = (\delta_1^*, \dots, \delta_m^*)^T$.*

*Proof.* In this case, $B_4 + [0, \infty) \cdot k_4 = \{(\delta_1, \dots, \delta_m)^T \in \mathbb{R}^m | \sum_{i=1}^{m} w_i \delta_i \geq 0\} + [0, \infty) \cdot 1_m \subset B_4$, and (1.1) is satisfied in $\mathbb{R}^m$. Moreover,

$$
\begin{aligned}
\min_{y \in \mathcal{F}_4} z^{B_4,k_4}(y) &= \min_{y \in \mathcal{F}_4} \min\{t \in \mathbb{R} | y \in t k_4 - B_4\} \\
&= \min_{y \in \mathcal{F}_4} \min\{t \in \mathbb{R} | y - t k_4 \in -B_4\} \\
&= \min_{\delta \in \mathcal{F}_4} \min\{t \in \mathbb{R} | \sum_{i=1}^{m} w_i(\delta_i - t) \leq 0\} \\
&= \min_{\delta \in \mathcal{F}_4} \min\{t \in \mathbb{R} | \sum_{i=1}^{m} w_i \delta_i \leq t \cdot \sum_{i=1}^{m} w_i\} \\
&= \min\{\sum_{i=1}^{m} w_i \delta_i | \delta \in \mathcal{F}_4\} \\
&= \min\{\rho_{lRC}(\delta) | \delta \in \mathcal{F}_4\}.
\end{aligned}
$$

□

*Remark:* Note that $B_4$ is a proper closed convex cone with $k_4 \in \operatorname{int} B_4$ and Corollary 1 implies that the functional $z^{B_4,k_4}$ is continuous, finite-valued, $\mathbb{R}^m_+$-monotone, strictly $(\operatorname{int} \mathbb{R}^m_+)$-monotone and sublinear.

*Remark:* The concept of light robustness can be interpreted as a weighted sum approach with upper bound constraints in the dimension defined by the number of constraints of the uncertain OP, including the original constraints in the weighted objective function.

### 3.5 Stochastic programming

Stochastic programming models are conceptually different from robust optimization models in the sense that they take information on the probability distribution of the uncertain data into account. For an introduction to stochastic programming we refer to Birge, Louveaux [7] and Shapiro et al. [34]. We focus on two-stage stochastic programming models in the following, see Beale [2], Dantzig [8] and Tintner [38] for early references. Two-stage stochastic programming models allow for a later correction of a solution $x$ selected in stage 1 of the decision process by a recourse action $u$ when the realization of the random data is known. Note that since we assumed that the scenario set $\mathcal{U}$ is finite, each scenario $\xi_k \in \mathcal{U}$ now has an associated probability $p_k \geq 0$, $k = 1, \dots, q$, $\sum_{q=1}^{k} p_k = 1$. In this situation, a **two-stage stochastic counterpart** can be formulated as

$$\begin{aligned} &\rho_{SP}(x) = \mathbb{E}[Q(x,\xi)] = \sum_{k=1}^{q} p_k Q(x,\xi_k) \to \min \\ &\text{s.t. } x \in X. \end{aligned} \tag{3.21}$$

Here, $X$ denotes the feasible set of the first-stage problem which could, for example, be defined based on the nominal scenario as $X = \{x \in \mathbb{R}^n | F_i(x,\hat{\xi}) \leq 0,\, i = 1,\dots,m\}$, or as the set of strictly robust solutions $X = \mathfrak{A}$, see (3.4). The objective is to minimize the expectation of the overall cost $Q(x,\xi)$ that involves, for given $x \in X$ and known $\xi \in \mathcal{U}$, an optimal recourse action $u$, i.e., an optimal solution of the second-stage problem

$$\begin{aligned} &Q(x,\xi) = \min f(x,u,\xi) \\ &\text{s.t. } u \in \mathcal{G}(x,\xi). \end{aligned} \tag{3.22}$$

The second-stage objective function $f(x,u,\xi)$ and the feasible set $\mathcal{G}(x,\xi)$ of the second-stage problem are both parametrized with respect to the stage 1 solution $x \in X$ and the scenario $\xi \in \mathcal{U}$.

In the light of the uncertain optimization problem (3.3), we can assume that the objective function $f$ in (3.3) depends both on the first-stage and the second-stage variables, i.e., on the nominal cost and the cost of the recourse action. We hence consider the following specification of problem (3.21):

$$\begin{aligned} &\rho_{SP}(x,u) = \sum_{k=1}^{q} p_k f(x,u_k,\xi_k) \to \min \\ &\text{s.t. } \forall \xi_k \in \mathcal{U}: \; F_i(x,\xi_k) - \delta_k(u_k) \leq 0, \; i = 1,\dots,m, \\ &\qquad x \in \mathbb{R}^n, \\ &\qquad u_k \in \mathcal{G}(x,\xi_k), \; k = 1,\dots,q, \end{aligned} \tag{$SP$}$$

with compensations $\delta_k : \mathbb{R}^{\bar{n}} \to \mathbb{R}$ that depend on the second-stage decisions $u_k \in \mathbb{R}^{\bar{n}}$, $k = 1, \ldots, q$.

*Remark:* If we set $\mathcal{G}(x, \xi) = \emptyset$ in the two-stage stochastic programming formulation $(SP)$, we obtain a static model as a special case in which the second-stage variables $u \in \mathbb{R}^{\bar{n}q}$ can be omitted. Since this model plays a special role in the comparison of the different robustness and stochastic programming concepts in Section 5 below, we include it here for the sake of completeness.

$$\begin{aligned} &\rho_{sSP}(x) = \sum_{k=1}^{q} p_k f(x, \xi_k) \to \min \\ &\text{s.t. } \forall \xi_k \in \mathcal{U} : \; F_i(x, \xi_k) \leq 0, \; i = 1, \ldots, m, \\ &\qquad x \in \mathbb{R}^n. \end{aligned} \tag{$sSP$}$$

**Theorem 7.** *Let*

$$\begin{aligned} \mathfrak{A}_5 :=\{(x, u) := (x, u_1, \ldots, u_q) \in \mathbb{R}^{n \times \bar{n}q} | \; \forall \xi_k \in \mathcal{U} : \\ F_i(x, \xi_k) - \delta_k(u_k) \leq 0, \; i = 1, \ldots, m\}, \end{aligned} \tag{3.23}$$

$$B_5 := \{(y_1, \ldots, y_q)^T | \sum_{k=1}^{q} p_k y_k \geq 0, \; y_k \in \mathbb{R}, \; k = 1, \ldots, q\}, \tag{3.24}$$

$$k_5 := 1_q, \tag{3.25}$$

$$\mathcal{F}_5 =\{(f(x, u_1, \xi_1), \ldots, f(x, u_q, \xi_q))^T | (x, u) \in \mathfrak{A}_5\}. \tag{3.26}$$

*For $k = k_5$, $B = B_5$, condition (1.1) is satisfied and with $\mathcal{F} = \mathcal{F}_5$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to problem $(SP)$ in the following sense:*

$$\begin{aligned} &\min\{z^{B_5,k_5}(y) | y \in \mathcal{F}_5\} \\ &\quad = z^{B_5,k_5}(y^*) \\ &\quad = \min\{\rho_{SP}(x, u) | (x, u) \in \mathfrak{A}_5\} \\ &\quad = \rho_{SP}(x^*, u^*), \end{aligned}$$

*where $y^* = (f(x^*, u_1^*, \xi_1), \ldots, f(x^*, u_q^*, \xi_q))^T$.*

*Proof.* We have $B_5 + [0, \infty) \cdot k_5 = \{(y_1, \ldots, y_q)^T \in \mathbb{R}^q | \sum_{k=1}^q p_k y_k \geq 0\} + [0, \infty) \cdot 1_m \subset B_5$, thus (1.1) is satisfied. Moreover,

$$\begin{aligned} \min_{y \in \mathcal{F}_5} z^{B_5,k_5}(y) &= \min_{y \in \mathcal{F}_5} \min\{t \in \mathbb{R} | y \in t k_5 - B_5\} \\ &= \min_{y \in \mathcal{F}_5} \min\{t \in \mathbb{R} | y - t k_5 \in -B_5\} \\ &= \min_{y \in \mathcal{F}_5} \min\{t \in \mathbb{R} | \sum_{k=1}^{q} p_k (y_k - t) \leq 0\} \\ &= \min_{y \in \mathcal{F}_5} \min\{t \in \mathbb{R} | \sum_{k=1}^{q} p_k y_k \leq t \cdot \underbrace{\sum_{k=1}^{q} p_k}_{=1}\} \\ &= \min\{\sum_{k=1}^{q} p_k y_k | y \in \mathcal{F}_5\} \\ &= \min\{\rho_{SP}(x, u) | (x, u) \in \mathfrak{A}_5\}. \end{aligned}$$

$\square$

*Remark:* $B_5$ is a proper closed convex cone with $k_5 \in \operatorname{int} B_5$ and Corollary 1 implies that the functional $z^{B_5,k_5}$ is continuous, finite-valued, $\mathbb{R}^q_+$-monotone, strictly $(\operatorname{int}\mathbb{R}^q_+)$-monotone and sublinear.

*Remark:* Similar to the case of light robustness, the above formulated two-stage stochastic programming problem can be interpreted as a weighted sums approach, however, in this case with a relaxed feasible set. This relation was also observed by Gast [13] in the multiple objective context. Note that in the special case of the static model $(sSP)$, the feasible set is in fact identical to the set of strictly robust solutions $\mathfrak{A}$ (and not relaxed), see (3.4).

### 3.6 Summary

The following table summarizes the introduced concepts of robustness and stochastic programming, and the corresponding parameters $B$, $k$, $\mathcal{F}$ in the formulation of the functional $z^{B,k}$. Note that the concept of $\epsilon$-constraints (row 6 in the following table) is introduced in the next section.

| Concept | $B$ | $k$ | $\mathcal{F}$ |
|---|---|---|---|
| Strict R. | $\mathbb{R}^q_+$ | $1_q$ | $\{(f(x,\xi_1),\dots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_1\}$ |
| Deviation R. | $\mathbb{R}^q_+ - f^*$ | $1_q$ | $\{(f(x,\xi_1),\dots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_2\}$ |
| Reliable R. | $\mathbb{R}^q_+$ | $1_q$ | $\{(f(x,\xi_1),\dots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_3\}$ |
| Light R. | $B_4$ | $1_m$ | $\mathcal{F}_4$, see below |
| Stochastic P. | $B_5$ | $1_q$ | $\{(f(x,u_1,\xi_1),\dots,f(x,u_q,\xi_q))^T \mid (x,u) \in \mathfrak{A}_5\}$ |
| $\epsilon$-constraint R. | $\mathbb{R}^q_+ - \bar{b}$ | $k^i = \begin{cases} 1 & \text{for } i=j \\ 0 & \text{for } i \neq j \end{cases}$ | $\{(f(x,\xi_1),\dots,f(x,\xi_q))^T \mid x \in \mathfrak{A}_6\}$ |

$$
\begin{aligned}
f^* &= (f^*(\xi_1),\dots,f^*(\xi_q))^T, \text{ where } f^*(\xi) \in \mathbb{R} \text{ is the optimal value of problem } (Q(\xi)),\\
B_4 &= \{(\delta_1,\dots,\delta_m)^T \in \mathbb{R}^m \mid \sum_{i=1}^m w_i\delta_i \geq 0\},\ w_i \geq 0,\ i=1,\dots,m,\ \sum_{i=1}^m w_i = 1,\\
B_5 &= \{(y_1,\dots,y_q)^T \in \mathbb{R}^q \mid \sum_{k=1}^q p_k y_k \geq 0\},\ p_k \geq 0,\ k=1,\dots,q,\ \sum_{k=1}^q p_k = 1,\\
\bar{b} &= (\bar{b}^1,\dots,\bar{b}^q)^T, \text{ where } \bar{b}^l = \begin{cases} 0 & \text{for } l = j, \\ \epsilon_l & \text{for } l \neq j, \end{cases} \quad \text{(cf. (4.28))}\ ,\\
\mathcal{F}_4 &= \{(\delta_1,\dots,\delta_m)^T \mid \exists x \in \mathbb{R}^n:\ F_i(x,\hat{\xi}) \leq 0,\ f(x,\hat{\xi}) \leq (1+\gamma)z^*,\ \forall \xi \in \mathcal{U}:\ F_i(x,\xi) \leq \delta_i,\\
&\qquad \delta_i \in \mathbb{R}_+,\ i=1,\dots,m\}.
\end{aligned}
$$

We use the following sets $\mathfrak{A}$:

$$
\begin{aligned}
\mathfrak{A}_1 &= \mathfrak{A},\\
\mathfrak{A}_2 &= \mathfrak{A},\\
\mathfrak{A}_3 &= \{x \in \mathbb{R}^n \mid F_i(x,\hat{\xi}) \leq 0,\ \forall \xi \in \mathcal{U}:\ F_i(x,\xi) \leq \delta_i,\ i=1,\dots,m\},\\
\mathfrak{A}_5 &= \{(x,u) := (x,u_1,\dots,u_q) \in \mathbb{R}^{n\times\bar{n}^q} \mid \forall \xi_k \in \mathcal{U}:\ F_i(x,\xi_k) - \delta_k(u_k) \leq 0,\ i=1,\dots,m\},\\
\mathfrak{A}_6 &= \{x \in \mathbb{R}^n \mid x \in \mathfrak{A}, f(x,\xi_l) \leq \epsilon_l,\ l \in \{1,\dots,q\},\ l \neq j\}.
\end{aligned}
$$

Note that $\mathfrak{A}_5 = \mathfrak{A}$ and $\mathcal{F}_5 = \mathcal{F}_1$ in the special case of static stochastic programming $(sSP)$.

The properties of the nonlinear scalarizing functional $z^{B,k}$ defined by (1.2) that were used for the description of the respective robust and stochastic programming counterparts are summarized in the following corollary (compare Theorem 2 and Corollary 1).

**Corollary 2.** *The following properties hold for $i = 1, 2, 3, 5$ ($i = 1$: strict robustness, $i = 2$: deviation robustness, $i = 3$: reliable robustness, $i = 5$: stochastic programming): The corresponding functional $z^{B_i,k_i}$ is continuous, finite-valued, convex, $\mathbb{R}^q_+$-monotone and strictly $(\operatorname{int} \mathbb{R}^q_+)$-monotone, and the following properties hold:*

$$\forall y \in \mathcal{F}_i,\ \forall r \in \mathbb{R} :\ z^{B_i,k_i}(y) \le r \Longleftrightarrow y \in rk_i - B_i, \tag{P1}$$

$$\forall y \in \mathcal{F}_i,\ \forall r \in \mathbb{R} :\ z^{B_i,k_i}(y + rk_i) = z^{B_i,k_i}(y) + r. \tag{P2}$$

$$\forall y \in \mathcal{F}_i,\ \forall r \in \mathbb{R} :\ z^{B_i,k_i}(y) = r \Longleftrightarrow y \in rk_i - \text{ bd } B_i, \tag{P3}$$

$$\forall y \in \mathcal{F}_i,\ \forall r \in \mathbb{R} :\ z^{B_i,k_i}(y) < r \Longleftrightarrow y \in rk_i - \text{ int } B_i. \tag{P4}$$

*For $i = 1, 3, 5$, $z^{B_i,k_i}$ is even sublinear. For $i = 4$ (light robustness), the properties* (P1) - (P4) *are fulfilled, and $z^{B_4,k_4}$ is continuous, sublinear and finite-valued. Additionally, $z^{B_4,k_4}$ is $\mathbb{R}^m_+$-monotone and strictly $(\operatorname{int} \mathbb{R}^m_+)$-monotone.*

Remark: $\mathbb{R}^q_+$-monotonicity, continuity, translation property and convexity of the functional $z^{B_i,k_i}$ were shown for i=1, ..., 5. These properties also occur in the theory of risk measures (see Section 6). For $i = 1, 3, 4, 5$, the functional $z^{B_i,k_i}$ is sublinear.

## 4 New concepts for robustness

The functional (1.2) contains many scalarizations of MOPs as special cases which are well known in the literature (see [37]), for instance the weighted Tschebyscheff scalarization or weighted sum scalarization, see [39] for details. These scalarizations can be regarded in the context of robustness. Specifically, one can develop new concepts for robustness that fit the specific needs of a decision-maker. In order to illustrate this point we will exemplary use the well-known *$\epsilon$-constraint scalarization* [10, 9] given below. This scalarization leads to the $\epsilon$-constraint method in multiple objective optimization. In the following we analyze which type of robust counterpart is defined by this scalarization.

Note that a similar analysis can be done for every variation of the parameters $B$, $k$ and for every feasible set $\mathcal{F}$ in $(P_{k,B,\mathcal{F}})$, i.e., many other possibilities for defining a robust counterpart of an uncertain OP may be derived.

Let us first define the $\epsilon$-constraint scalarization. To this end, let some $j \in \{1, \dots, q\}$ and some real values $\epsilon_l \in \mathbb{R}$, $l = 1, \dots, q, l \neq j$ be given. Then the $\epsilon$-constraint scalarization is given by

$$k_6 = (k_6^1, \dots, k_6^q)^T \text{ where } k_6^l = \begin{cases} 1 & \text{for } l = j, \\ 0 & \text{for } l \neq j, \end{cases} \tag{4.27}$$

$$B_6 := \mathbb{R}^q_+ - \bar{b}, \text{ with } \bar{b} = (\bar{b}^1, \dots, \bar{b}^q)^T, \bar{b}^l = \begin{cases} 0 & \text{for } l = j, \\ \epsilon_l & \text{for } l \neq j, \end{cases} \tag{4.28}$$

$$\mathcal{F}_6 = \{(f(x, \xi_1), \dots, f(x, \xi_q))^T | x \in \mathfrak{A}\}. \tag{4.29}$$

With these parameters the functional $z^{B_6,k_6}$ describes the $\epsilon$-constraint-method (cf. Eichfelder [10] and Haimes, Lasdon, D. A. Wismer [20]). Then the following reformulation holds.

**Theorem 8.** *Let $\epsilon := (\epsilon_1, \dots, \epsilon_q)^T \in \mathbb{R}^q$ and $j \in \{1, \dots, q\}$. Then for $k = k_6,\ B = B_6$,* (1.1) *holds and with $\mathcal{F} = \mathcal{F}_6$, problem $(P_{k,B,\mathcal{F}})$ is equivalent to*

$$\begin{aligned} &\rho_{\epsilon RC}(x) = f(x, \xi_j) \to \min \\ &s.t.\ \forall \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m, \\ &\qquad x \in \mathbb{R}^n, \\ &\qquad f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j. \end{aligned} \tag{$\epsilon RC$}$$

*Proof.* Since $B_6 + [0, \infty) \cdot k_6 \subset B_6$, condition (1.1) is satisfied. Moreover,

$$\begin{aligned} \min_{y \in \mathcal{F}_6} z^{B_6,k_6}(y) &= \min_{y \in \mathcal{F}_6} \min\{t \in \mathbb{R} | y \in tk_6 - B_6\} \\ &= \min_{y \in \mathcal{F}_6} \min\{t \in \mathbb{R} | y - tk_6 \in -B_6\} \\ &= \min_{x \in \mathfrak{A}} \min\{t \in \mathbb{R} | f(x, \xi_j) \leq t,\ f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j\} \\ &= \min\{\rho_{\epsilon RC}(x) | x \in \mathbb{R}^n, \forall \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m, \\ &\qquad\qquad f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j\}. \end{aligned}$$

□

Theorem 8 shows that the nonlinear scalarizing functional $z^{B,k}$ can be formulated as $(\epsilon RC)$. Let us call $(\epsilon RC)$ the **$\epsilon$-constraint robust counterpart** of (3.3). We now discuss its meaning for robust optimization. Contrary to the other robustness concepts, the parameter $k_6$ symbolizes that only a single objective function is minimized. This means, the decision maker picks one particular objective function that is to be minimized subject to the constraints that were also used in strict and deviation-robustness. Additionally, the former objective functions $f(x, \xi_l),\ l \in \{1, \dots, q\},\ l \neq j$, are moved to the constraints. This concept makes sense if a solution is required with a given nominal quality for every scenario $\xi_l, l = 1, \dots, q, l \neq j$ while finding the best possible for the remaining scenarios $j$. Applying this concept, one question immediately arises: How can a decision-maker be sure how to pick the upper bounds $\epsilon_l$ for these constraints? If the bounds $\epsilon_l$ are chosen too small, the set of feasible solutions of $(\epsilon RC)$ may be empty, or the objective function value of $f(x, \xi_j)$ may become very bad. On the other hand, if the bounds $\epsilon_j$ are chosen too large, the quality for the other scenarios decreases.

Note that we could have included the constraint $f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j$ in the set of feasible points $\widetilde{\mathcal{F}_6}$, and we would have obtained $\widetilde{\mathcal{F}_6} = \{(f(x, \xi_1), \dots, f(x, \xi_q))^T | x \in \mathbb{R}^n :\ f(x, \xi_l) \leq \epsilon_l,\ l \in \{1, \dots, q\},\ l \neq j,\ \forall \xi \in \mathcal{U}:\ F_i(x, \xi) \leq 0,\ i = 1, \dots, m\}$. Then we could have used $\widetilde{B_6} = \mathbb{R}^q_+$ instead of $B_6$, but the set of feasible points $\widetilde{\mathcal{F}_6}$ would have been smaller and possibly harder to deal with.

**Corollary 3.** *The functional $z^{B_6,k_6}$ is lower semi-continuous, convex, proper, $\mathbb{R}^q_+$-monotone and strictly* $(\operatorname{int} \mathbb{R}^q_+)$*-monotone, and the properties* (P1) *and* (P2) *from Corollary 2 hold for* $i = 6$.

*Proof.* Since condition (1.1) is satisfied, Theorem 2 implies that $z^{B_6,k_6}$ is lower semi-continuous, convex, proper and the properties (P1) and (P2) hold true. However, in the case of $\epsilon$-constraint robustness we have $B_6 + (0, \infty) \cdot k_6 \not\subset \operatorname{int} B_6$ for $B_6$ given by (4.28). Therefore, we show directly that $z^{B_6,k_6}$ is strictly $(\operatorname{int} \mathbb{R}^q_+)$-monotone: Consider $t \in \mathbb{R},\ y \in tk_6 - \operatorname{int} B_6$. Then $tk_6 - y \in \operatorname{int} B_6$. Consequently, there exists an $s > 0$ such that $tk_6 - y - sk_6 \in \operatorname{int} B_6 \subset B_6$. Using (P1), we deduce $z^{B_6,k_6}(y) \leq t - s < t$, and thus

$$tk_6 - \operatorname{int} B_6 \subset \{y \in \mathbb{R}^q | z^{B_6,k_6}(y) < t\}. \tag{4.30}$$

Furthermore, for $y_1 \in y_2 - \operatorname{int} \mathbb{R}^q_+$, it holds

$$\begin{aligned} y_1 \in y_2 - \operatorname{int} \mathbb{R}^q_+ &\overset{(\mathrm{P1})}{\subset} z^{B_6,k_6}(y_2)k_6 - B_6 - \operatorname{int} \mathbb{R}^q_+ \\ &\subset z^{B_6,k_6}(y_2)k_6 - \operatorname{int} B_6 \\ &\overset{(4.30)}{\subset} \{y \in \mathbb{R}^q | z^{B_6,k_6}(y) < z^{B_6,k_6}(y_2)\}. \end{aligned}$$

We conclude that $z^{B_6,k_6}(y_1) < z^{B_6,k_6}(y_2)$ and thus $z^{B_6,k_6}$ is strictly $(\operatorname{int} \mathbb{R}^q_+)$-monotone. $\square$

# 5 Multiple objective counterpart problems and relations to scalar robust optimization and stochastic programming

In this section we propose a new concept, namely to replace an uncertain (scalar) OP $(Q(\xi), \xi \in \mathcal{U})$, as introduced in (3.3), see Section 3, by its (deterministic) multiple objective counterpart. The idea is that every scenario $\xi_l \in \mathcal{U}, l = 1, \dots, q$ yields its own objective function $h_l(x) := f(x, \xi_l)$, the only exception being the case of light robustness where the roles of objective and constraints are reversed. Following the example of the different robustness concepts discussed above, the multiple objective counterparts formulated below can be distinguished with respect to the solution set $\mathfrak{A}$, i.e., the way in which the (uncertain) constraints are handled. To simplify the following analysis, in the case of stochastic programming we focus on the static model $(sSP)$.

Let

$$h(x) := \begin{pmatrix} h_1(x) \\ \vdots \\ h_q(x) \end{pmatrix} := \begin{pmatrix} f(x, \xi_1) \\ \vdots \\ f(x, \xi_q) \end{pmatrix}. \tag{5.31}$$

Recall from (3.4) (see also Section 3.6) that

$$\mathfrak{A}_1 = \mathfrak{A}_2 = \mathfrak{A}_5 = \mathfrak{A} = \{x \in \mathbb{R}^n | \forall \xi \in \mathcal{U} : \ F_i(x, \xi) \leq 0, \ i = 1, \dots, m\}.$$

Then the **multiple objective strictly robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ is defined by

$$\operatorname{Eff}(h[\mathfrak{A}_1], \mathbb{R}^q_+), \tag{$RC'$}$$

where $h[\mathfrak{A}_1] = \mathcal{F}_1$ (see (3.8)).

Similarly, recall from (3.14) that

$$\mathfrak{A}_3 := \{x \in \mathbb{R}^n | F_i(x, \hat{\xi}) \leq 0, \ \forall \xi \in \mathcal{U} : \ F_i(x, \xi) \leq \delta_i, \ i = 1, \dots, m\}.$$

We propose the **multiple objective reliably robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ as

$$\operatorname{Eff}(h[\mathfrak{A}_3], \mathbb{R}^q_+), \tag{$rRC'$}$$

where $h[\mathfrak{A}_3] = \mathcal{F}_3$ (see formula (3.17)).

Now let us introduce a multiple objective counterpart that corresponds to the lightly robust counterpart $(lRC)$. Let $\mathcal{F}_4$ be defined by (3.20), i.e.,

$$\begin{aligned} \mathcal{F}_4 = &\{(\delta_1, \dots, \delta_m)^T | \exists x \in \mathbb{R}^n : \ F_i(x, \hat{\xi}) \leq 0, \ f(x, \hat{\xi}) \leq (1 + \gamma) z^*, \ \forall \xi \in \mathcal{U} : \ F_i(x, \xi) \leq \delta_i, \\ &\delta_i \in \mathbb{R}, \ i = 1, \dots, m\}. \end{aligned}$$

We propose the **multiple objective lightly robust counterpart** to $(Q(\xi), \xi \in \mathcal{U})$ by

$$\operatorname{Eff}(\mathcal{F}_4, \mathbb{R}^m_+). \tag{$lRC'$}$$

Using Theorem 1 together with Corollaries 2 and 3, we can conclude that problem $(P_{k_i,B_i,\mathcal{F}_i})$, $i = 1,2,5,6$ $((P_{k_3,B_3,\mathcal{F}_3}), (P_{k_4,B_4,\mathcal{F}_4})$, respectively) is a scalarization of the multiple objective counterpart $(RC')$ $((rRC')$, $(lRC')$, respectively), and the following corollary holds due to the monotonicity properties of $z^{B_i,k_i}$, $i = 1,\dots,6$.

**Corollary 4.** *For $i = 1,2,5,6$ ($i = 1$: strict robustness, $i = 2$: deviation robustness, $i = 5$: static stochastic programming, $i = 6$: $\epsilon$-constraint robustness), we have:*

$$[\forall y \in \mathcal{F}_i \setminus \{y^*\}: \ z^{B_i,k_i}(y^*) < z^{B_i,k_i}(y)] \Longrightarrow y^* \in \mathrm{Eff}(h[\mathfrak{A}_1], \mathbb{R}^q_+),$$
$$[\forall y \in \mathcal{F}_i: \ z^{B_i,k_i}(y^*) \le z^{B_i,k_i}(y)] \Longrightarrow y^* \in \mathrm{Eff}_w(h[\mathfrak{A}_1], \mathbb{R}^q_+).$$

*Concerning reliably robustness ($i = 3$), it holds*

$$[\forall y \in \mathcal{F}_3 \setminus \{y^*\}: \ z^{B_3,k_3}(y^*) < z^{B_3,k_3}(y)] \Longrightarrow y^* \in \mathrm{Eff}(h[\mathfrak{A}_3], \mathbb{R}^q_+),$$
$$[\forall y \in \mathcal{F}_3: \ z^{B_3,k_3}(y^*) \le z^{B_3,k_3}(y)] \Longrightarrow y^* \in \mathrm{Eff}_w(h[\mathfrak{A}_3], \mathbb{R}^q_+).$$

*For light robustness ($i = 4$), we conclude*

$$[\forall y \in \mathcal{F}_4 \setminus \{y^*\}: \ z^{B_4,k_4}(y^*) < z^{B_4,k_4}(y)] \Longrightarrow y^* \in \mathrm{Eff}(\mathcal{F}_4, \mathbb{R}^m_+),$$
$$[\forall y \in \mathcal{F}_4: \ z^{B_4,k_4}(y^*) \le z^{B_4,k_4}(y)] \Longrightarrow y^* \in \mathrm{Eff}_w(\mathcal{F}_4, \mathbb{R}^m_+).$$

Because problem $(P_{k_i,B_i,\mathcal{F}_i})$, $i = 1,2,5,6$, with the nonlinear scalarizing objective functional $z^{B_i,k_i}$ is a scalarization of the multiple objective counterpart $(RC')$ and $(P_{k_1,B_1,\mathcal{F}_1})$ $((P_{k_2,B_2,\mathcal{F}_2})$, $(P_{k_5,B_5,\mathcal{F}_5})$, $(P_{k_6,B_6,\mathcal{F}_6})$, respectively) is equivalent to the optimization problem $(RC)$ $((dRC)$, $(sSP)$, $(\epsilon RC)$, respectively), we can conclude: If $x^*$ solves $(RC)$ $((dRC)$, $(sSP)$, $(\epsilon RC)$, respectively), then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is weakly efficient for $(RC')$. If $x^*$ is the unique solution of problem $(RC)$ $((dRC)$, $(sSP)$, $(\epsilon RC)$, respectively), then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is efficient for $(RC')$.

*Remark:* The (weakly) efficient set of the multiple objective strictly robust counterpart $(RC')$ thus comprises optimal solutions of the (scalar) strictly robust counterpart $(RC)$ (which are obtained by Tchebyscheff scalarization with the origin as reference point), the deviation-robust counterpart $(dRC)$ (Tschebyscheff scalarization with the ideal point as reference point), the static stochastic programming equivalent $(sSP)$ (weighted sums scalarization), and the $\varepsilon$-constraint robust counterpart $(\epsilon RC)$ ($\varepsilon$-constraint scalarization).

Analogously, similar results also hold for the other concepts of robustness: $(P_{k_3,B_3,\mathcal{F}_3})$ is equivalent to the reliably robust counterpart $(rRC)$. Therefore, we conclude: If $x^*$ solves $(rRC)$, then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is weakly efficient for $(rRC')$. If $x^*$ is the unique solution of problem $(rRC)$, then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is efficient for $(rRC')$. Concerning light robustness, we can conclude: If $\delta^* = (\delta^*_1,\dots,\delta^*_m)^T$ is a (unique) solution of the lightly robust counterpart $(lRC)$, then $\delta^*$ is (efficient) weakly efficient for the corresponding multiple objective counterpart $(lRC')$.

## 6 Application: Risk Measures and Robustness in Financial Theory

The functional $z^{B,k}$ as defined in Section 1 is an important tool in the field of financial mathematics (compare Heyde [21]). As already mentioned before, it can be used as a coherent risk measure of an investment. For a better understanding of the topic, we will now introduce coherent risk measures and their relation to robustness.

Let $Y$ be a linear space of random variables, and let $\Omega$ be a set of elementary events (a set of all possible states of the future). Then a future payment of an investment is a random variable $y : \Omega \to \mathbb{R}$. Positive payments in the future are wins, negative ones are losses. If no investment is being done, then $y$ takes on the value zero. In order to valuate such an investment, we need to valuate random variables by comparing them. To do that, we introduce an ordering relation that is induced by a set $B \subseteq Y$. Artzner, Delbean, Eber and Heath proposed in [1] axioms for a cone $B \subseteq Y$ of random variables that represent acceptable investments:

(A1) $\{y \in Y | y(\omega) \geq 0 \ (\omega \in \Omega)\} \subset B$, $B \cap \{y \in Y | y(\omega) < 0 \ (\omega \in \Omega)\} = \emptyset$,

(A2) $B + B \subset B$.

In financial terms, the cone property says that every nonnegative multiple of an acceptable investment is again acceptable. Furthermore, axiom (A1) means that every investment with almost sure nonnegative results will be accepted and every investment with almost sure negative results is not acceptable. The convexity property in axiom (A2) means that merging two acceptable investments together results again in an acceptable investment. However, in some applications the cone property of $B$ and axiom (A2) are not useful, especially, if the investor does not want to lose more than a certain amount of money. In this case Föllmer and Schied [12] replace the cone property and axiom (A2) by a convexity axiom.

Cones $B \subset Y$ satisfying the axioms (A1) and (A2) of acceptable investments can be used in order to introduce a preference relation on $Y$. The decision maker prefers $y_1$ to $y_2$ (changing from $y_2$ to $y_1$ is an acceptable risk) if and only if $y_1 - y_2$ is an element of $B$, i.e.,

$$y_1 \geq_B y_2 \Longleftrightarrow y_1 - y_2 \in B.$$

The smallest cone $B$ satisfying the axioms (A1) and (A2) is $B = \{y \in Y | y(\omega) \geq 0 \ (\omega \in \Omega)\}$. A decision maker using this particular cone $B$ of acceptable investments is risk-averse, i.e., he only accepts investments with nonnegative payments.

The functional $z^{B,k}$ can be used to describe risks associated with investments. In [1], Artzner, Delbean, Eber and Heath introduced coherent risk measures, i.e., functionals $\mu : Y \to \mathbb{R} \cup \{+\infty\}$, where $Y$ is the linear space of random variables, that satisfy the following properties:

(P1) $\mu(y + tk) = \mu(y) - t$,

(P2) $\mu(0) = 0$ and $\mu(\lambda y) = \lambda \mu(y)$ for all $y \in Y$ and $\lambda > 0$,

(P3) $\mu(y_1 + y_2) \leq \mu(y_1) + \mu(y_2)$ for all $y_1, y_2 \in Y$,

(P4) $\mu(y_1) \leq \mu(y_2)$ if $y_1 \geq y_2$.

The following interpretation of the properties (P1) - (P4) is to mention: The translation property (P1) means that the risk would be mitigated by an additional safe investment with a corresponding amount, especially, it holds

$$\mu(y + \mu(y)e) = 0.$$

The positive homogeneity of the risk measure in (P2) means that double risk must be secured by double risk capital; the subadditivity in (P3) means that a diversification of risk should be recompensed and finally, the monotonicity of the risk measure in (P4) means that higher risk needs more risk capital.

A risk measure may be negative. In this case it can be interpreted as a maximal amount of cash that could be given away such that the reduced result remains acceptable.

It can be shown that

$$\mu(y) = \inf\{t \in \mathbb{R} | y + tk \in B\}$$

is a coherent risk measure. Obviously, we have (cf. Heyde [21])

$$\mu(y) = z^{B,k}(-y),$$

where $z^{B,k}$ is defined as in Section 1. A risk measure induces a set $B_\mu$ of acceptable risks (dependent on $\mu$)

$$B_\mu = \{y \in Y | \mu(y) \leq 0\}.$$

The following interpretation of coherent risk measures is now possible: If $Y = \mathbb{R}^q$ (there are $q$ states of the future), $B_1 = \mathbb{R}^q_+$ (that is, indeed, the smallest set that satisfies axioms (A1) - (A3)) and $k_1 = 1_q$, then

$$\mu(y) = z^{B_1,k_1}(-y) = \max_{\xi \in \mathcal{U}}(-f(x,\xi)) = -\min_{\xi \in \mathcal{U}} f(x,\xi)$$

is a coherent risk measure. Specifically, the risk measure $\max_{\xi \in \mathcal{U}}(-f(x,\xi))$ is the objective function of the strictly robust counterpart $(RC)$ with negative values of $f$. Because $\mu(y) = -\min_{\xi \in \mathcal{U}} f(x,\xi)$, negative payments $f$ of an investment in the future result in a positive risk measure, and positive payments result in a negative risk measure. This seems very reasonable since negative payments (losses) are riskier than investments with only positive payments (bonds). The above approach can analogously be used for other concepts of robustness (assuming that the cone $B$ satisfies (A1) and (A2)).

Interrelations between robustness and coherent risk measures have also been studied by Quaranta and Zaffaroni in [31]: They minimized the conditional value at risk (which is a coherent risk measure) of a portfolio of shares using concepts of robust optimization.

# 7 Conclusions

In our paper we give a unified approach to well known concepts of robustness in scalar optimization using a general nonlinear scalarization concept. This immediately leads to new concepts of robustness for scalar optimization problems which will be explored in more detail in future work. Furthermore, using the properties of the nonlinear scalarizing functional we studied relations between multiple objective counterpart problems and scalar robust and stochastic optimization. It is ongoing work to use these relations in order to enhance robustness models by using well-known concepts from multiple objective optimization. Finally, we mention some relations between robustness and coherent risk measures. These relations will be studied more detailed in a forthcoming paper.

Furthermore we work on the following two extensions. We will use our results in order to derive necessary conditions for solutions of scalar robust optimization problems as well as for solutions of multiple objective counterpart problems under certain differentiability assumptions. Moreover, we are currently extending the concepts presented in this paper to robust problems with an infinite uncertainty set $\mathcal{U}$.

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## References


[1] P. Artzner, F. Delbean, J.-M. Eber, and D. Heath. Coherent measures of risk. *Math. Finance*, 9:203–228, 1999.

[2] E.M.L. Beale. On minimizing a convex function subject to linear inequalities. *Journal of the Royal Statistical Society, Series B*, 17:173–184, 1955.

[3] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust Optimization.* Princeton University Press, Princeton and Oxford, 2009.

[4] A. Ben-Tal and A. Nemirovski. Robust convex optimization. *Mathematics of Operations Research*, 23(4):769–805, 1998.

[5] A. Ben-Tal and A. Nemirovski. Robust solutions of linear programming problems contaminated with uncertain data. *Mathematical Programming*, 88:411–424, 2000.

[6] D. Bertsimas and M. Sim. The price of robustness. *Operations Research*, 52(1):35–53, 2004.

[7] J.R. Birge and F.V. Louveaux. *Introduction to Stochastic Programming.* Springer Verlag, New York, 1997.

[8] G.B. Dantzig. Linear programming under uncertainty. *Management Science*, 1:197–206, 1955.

[9] M. Ehrgott. *Multicriteria Optimization.* Springer, New York, 2005.

[10] G. Eichfelder. *Adaptive Scalarization Methods in Multiobjective Optimization.* Springer, 2008.

[11] M. Fischetti and M. Monaci. Light robustness. In R. K. Ahuja, R. Moehring, and C. Zaroliagis, editors, *Robust and Online Large-Scale Optimization*, number 5868 in Lecture Notes in Computer Science, pages 61–84. Springer, Berlin Heidelberg, 2009.

[12] H. Föllmer and A. Schied. *Stochastic Finance.* Walter de Gruyter, Berlin, 2004.

[13] S. Gast. *Network Flow Problems with Uncertain Input Data in the Context of Supply Chain Management Applications.* PhD thesis, University of Wuppertal, 2010.

[14] Chr. Gerstewitz (Tammer). Nichtkonvexe Dualitität in der Vektoroptimierung. *Wiss. Zeitschr. TH Leuna-Merseburg*, 25:357–364, 1983.

[15] Chr. Gerth (Tammer) and P. Weidner. Nonconvex separation theorems and some applications in vector optimization. *J. Optim. Theory Appl.*, 67:297–320, 1990.

[16] L. El Ghaoui and H. Lebret. Robust solutions to least-squares problems with uncertain data. *SIAM Journal of Matrix Anal. Appl.*, 18:1034–1064, 1997.

[17] M. Goerigk and A. Schöbel. A scenario-based approach for robust linear optimization. In *Proceedings of the 1st International ICST Conference on Practice and Theory of Algorithms in (Computer) Systems (TAPAS)*, Lecture Notes in Computer Science, pages 139–150. Springer, 2011.

[18] A. Göpfert, H. Riahi, Chr. Tammer, and C. Zălinescu. *Variational Methods in Partially Ordered Spaces.* Springer, 2003.

[19] A. Göpfert, Chr. Tammer, and C. Zălinescu. A new minimal point theorem in product spaces. *Z. Anal. Anwendungen*, 18:767–770, 1999.

[20] Y.Y. Haimes, L. S. Lasdon, and D. A. Wismer. On a bicriterion formulation of the problems of integrated system identification and system optimization. *IEEE Trans. Syst. Man Cybern.*, 1:296–297, 1971.

[21] F. Heyde. *Coherent risk measures and vector optimization.* In K.-H. Küfer (ed.) et al., Multicriteria decision making and fuzzy systems. Theory, methods and applications. Shaker Verlag, Aachen, 3-12, 2006.

[22] R. Hites, Y. De Smet, N. Risse, M. Salazar-Neumann, and P. Vincke. About the applicability of MCDA to some robustness problems. *European Journal of Operational Research*, 174:322–332, 2006.

[23] J. Jahn. Scalarization in vector optimization. *Math. Program.*, 29:203–218, 1984.

[24] J. Jahn. *Vector Optimization - Introduction, Theory, and Extensions.* Springer, 2011. 2nd Edition.

[25] E. Köbis and Chr. Tammer. Relations between strictly robust optimization problems and a nonlinear scalarization method. *AIP Conference Proceedings*, 1479:2371–2374, 2012.

[26] P. Kouvelis and S. Sayin. Algorithm robust for the bicriteria discrete optimization problem. *Annals of Operations Research*, 147:71–85, 2006.

[27] P. Kouvelis and G. Yu. *Robust Discrete Optimization and its Applications.* Kluwer Academic Publishers, 1997.

[28] C. Liebchen, M. Lübbecke, R. H. Möhring, and S. Stiller. The concept of recoverable robustness, linear programming recovery, and railway applications. In R. K. Ahuja, R.H. Möhring, and C.D. Zaroliagis, editors, *Robust and online large-scale optimization*, volume 5868 of *Lecture Note on Computer Science*. Springer, 2009.

[29] A. Pascoletti and P. Serafini. Scalarizing vector optimization problems. *J. Optim. Theory Appl.*, 42:499–524, 1984.

[30] P. Perny, O. Spanjaard, and L.-X. Storme. A decision-theoretic approach to robust optimization. *Annals of Operations Research*, 147:317–341, 2006.

[31] A.G. Quaranta and A. Zaffaroni. Robust optimization of conditional value at risk and portfolio selection. *Journal of Banking & Finance*, 32:2046–2056, 2008.

[32] S. Sayin and P. Kouvelis. The multiobjective discrete optimization problem: A weighted min-max two-stage optimization approach and a bicriteria algorithm. *Management Science*, 51:1572–1581, 2005.

[33] A. Schöbel. Light robustness and the trade-off between robustness and nominal quality. Technical report, Preprint-Reihe, Institut für Numerische und Angewandte Mathematik, Georg-August Universität Göttingen, 2012. submitted.

[34] A. Shapiro, D. Dentcheva, and A. Ruszczyński. *Lectures on Stochastic Programming: Modeling and Theory.* MOS-SIAM Series on Optimization 9, SIAM, 2009.

[35] A.L. Soyster. Convex programming with set-inclusive constraints and applications to inexact linear programming. *Oper. Res.*, 21:1154–1157, 1973.

[36] S. Stiller. *Extending Concepts of Reliability: Network Creation Games, Real-time Scheduling, and Robust Optimization.* PhD thesis, TU Berlin, 2009.

[37] Chr. Tammer and K. Winkler. A new scalarization approach and applications in multicriteria d.c. optimization. *Journal of Nonlinear and Convex Analysis*, 4(3):365–380, 2003.

[38] G. Tintner. Stochastic linear programming with applications to agricultural economics. In H.A. Antosiewicz, editor, *Proceedings of the 2nd Symposium on Linear Programming*, pages 197–228. National Bureau of Standards, 1955.

[39] P. Weidner. *Ein Trennungskonzept und seine Anwendung auf Vektoroptimierungsverfahren.* Martin-Luther-Universität Halle-Wittenberg, 1990 (Dissertation B).

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