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Algorithms for Multicriteria Location Problems

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Abstract. We study a multicriteria location problem where several other criteria (for instance cost functions) are added to the locational criteria, then we investigate the reducibility of it and introduce a decomposition algorithm for solving such extended multicriteria location problems. Furthermore, we derive a corresponding computer program in MATLAB.

Keywords: Locational Analysis, Multicriteria Optimization, Pareto Reducibility, Decomposition, Software
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INTRODUCTION

The search for a new suitable location has a huge importance in industry, technology and in many other applications. For instance, the suitable choice of a new central store can save a lot of cost and time in the transport industry.

In this paper we study a multicriteria location problem with distances between p existing facilities $a^1,\cdots,a^p \in I\!R^2$ and a new facility $x \in I\!R^2$ extended by adding one or several linear cost functions $C_{p+i}: X \to I\!R$ (for $i = 1,\cdots,m$), where X is a nonempty, bounded, closed and convex subset of $I\!R^2$. We consider a multicriteria problem (P) with the vector-valued objective function

$$F(x) = \begin{pmatrix} \|x-a^1\| \\ \cdots \\ \|x-a^p\| \\ C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix}. \tag{1}$$

In the multicriteria extended problem (P) a new facility $x \in X$ is to be determined such that $F(x)$ is a *weakly efficient* or an *efficient element* of F over X with respect to $I\!R^{p+m}_+$. The set of weakly efficient elements of F over X with respect to $I\!R^{p+m}_+$ is denoted by $\text{WEff}(F[X], I\!R^{p+m}_+)$ and the set of efficient elements of F over X with respect to $I\!R^{p+m}_+$ is denoted by $\text{Eff}(F[X], I\!R^{p+m}_+)$, where

$$\begin{aligned} \text{WEff}(F[X], I\!R^{p+m}_+) &:= \{y \in F[X] \mid F[X] \cap (y - \text{int} I\!R^{p+m}_+) = \emptyset\}, \\ \text{Eff}(F[X], I\!R^{p+m}_+) &:= \{y \in F[X] \mid F[X] \cap (y - (I\!R^{p+m}_+ \setminus \{0\})) = \emptyset\}. \end{aligned}$$

The distances (norms) between the new facility $x = (x_1, x_2) \in X \subset I\!R^2$ and the given facilities $a^i = (a^i_1, a^i_2) \in I\!R^2, i = 1,\ldots,p$, can be chosen in different ways. With the choice of the norm the distance is determined, for instance as air-way in the case of the Euclidean norm. From now on we are going to use either the maximum or Lebesgue norm in the formulation of the problem (P).

In this paper we study Pareto reducibility of (P) and introduces a decomposition algorithm for solving it and finally we show an application of this algorithm with a MATLAB program.
Next section investigates a very important subproblem of (P) which is a multicriteria location problem.

MULTICRITERIA LOCATION PROBLEMS

We consider a multicriteria location problem (P_L) with the objective function $f = (f_1, \cdots, f_p) : I\!R^2 \to I\!R^p$, where $f_i(x) = \|x - a^i\|$ for $i = 1, \ldots, p$. Actually (P_L) is a subproblem of (P) by taking the first p objectives in (1).
In the formulation of the location problem (P_L) we will use the maximum norm ($\|x\|_\infty = \max\{|x_1|, |x_2|\}$).
Now we introduce an algorithm for solving (P_L), for that we consider the following sets with respect to the given facilities $a^i \in I\!R^2$ $(i = 1, \ldots, p)$ which are related to the structure of the subdifferential of the maximum norm:

$$\begin{aligned} s_1(a^i) &= \{x \in I\!R^2 \mid a_2^i - x_2 > |a_1^i - x_1|\}, \\ s_2(a^i) &= \{x \in I\!R^2 \mid x_2 - a_2^i > |a_1^i - x_1|\}, \\ s_3(a^i) &= \{x \in I\!R^2 \mid a_1^i - x_1 > |a_2^i - x_2|\}, \\ s_4(a^i) &= \{x \in I\!R^2 \mid x_1 - a_1^i > |a_2^i - x_2|\}. \end{aligned}$$

Moreover, we introduce the sets

$$S_r := \{x \in N \mid \exists i \in \{1, \ldots, p\} \quad \text{and} \quad x \in s_r(a^i)\} \qquad (r = 1, 2, 3, 4),$$

where N denotes the smallest level set of the dual norm to the maximum norm (Lebesgue norm) containing the points a^i $(i = 1, \ldots, p)$.
Now we are able to describe the following formula for computing the set $X_{\text{Eff}}(f)$ of all elements $x \in I\!R^2$ with $f(x) \in \text{Eff}(f[I\!R^2], I\!R^p_+)$ (cf. Gerth (Tammer), Pöhler [1]):

$$X_{\text{Eff}}(f) := \big\{(cl\, S_1 \cap cl S_2) \cup [(N \setminus S_1) \cap (N \setminus S_2)]\big\} \cap \big\{(cl\, S_3 \cap cl\, S_4) \cup [(N \setminus S_3) \cap (N \setminus S_4)]\big\}.$$

This algorithm is called geometrical algorithm for solving multicriteria location problems with maximum norm, which can be analogously formulated for Lebesgue norm (see [1]).

PARETO REDUCIBILITY OF (P)

In order to study the structure of the different solution sets of the problem (P) we use the concept of *Pareto Reducible Multicriteria Optimization Problems* introduced by Popovici [2].

Suppose that $I_{p+m} = \{1,\cdots,p+m\}$, for $1 \le k \le p+m$ let $I = \{1,\cdots,k\} \subseteq I_{p+m}$. Then for any $I \subseteq I_{p+m}$ we get a subproblem (P_I) of (P). For $|I| = k < p+m$ we get a new multicriteria problem:

$$\begin{cases} \text{minimize} & F_I(x) \\ \text{subject to} & x \in X, \end{cases} \tag{2}$$

which is a subproblem of (P) and its objective function F_I is defined by means of the unique increasing bijection $\sigma_I : I \to I$, as

$$F_I := (F_{\sigma_I(1)},\cdots,F_{\sigma_I(k)}) : X \to I\!R^k.$$

Using the results in Popovici in [2] we get that the problem (P) is **Pareto reducible**. This means that the set of weakly efficient solutions of (P) are actually efficient solution either for problem (P) itself or for some subproblem of type (2).

It is clear that the multicriteria location problem (P_L) is Pareto reducible as well.

DECOMPOSITION ALGORITHM FOR SOLVING (P)

We introduce a decomposition algorithm in order to solve the problem (P) by dividing it into a multicriteria location subproblem and a multicriteria linear subproblem. The multicriteria location subproblem can be solved through the geometric algorithm described in the second section, which produces the set of efficient solutions $X_{\text{Eff}}(f)$. Then we minimize the multicriteria linear objective function over the set of efficient solutions $X_{\text{Eff}}(f)$. In general this set is not convex, so we divide it into convex subsets. Finally, we minimize the multicriteria linear objective function over each of these convex subsets.

Next we describe shortly an algorithm for dividing the set of efficient solutions of (P_L) with maximum norm. Note also that the algorithm can be analogously formulated for Lebesgue norm.

Algorithm 1.

1. Consider the set N as the smallest level set of the dual norm to the maximum norm (which is the Lebesgue norm), which contains the existing points $a^1,\cdots,a^p$.
2. Divide the set N through the lines

$$\begin{aligned} S_\alpha(a^i) &:= \left\{x \in I\!R^2 \mid x_1 + x_2 = a_1^i + a_2^i\right\}, \\ S_\beta(a^i) &:= \left\{x \in I\!R^2 \mid x_1 - x_2 = a_1^i - a_2^i\right\}. \end{aligned}$$

3. The convex partition consists of such rectangles and line segments from N with the property that they are totally included in the efficiency set $X_{\text{Eff}}(f)$.
4. Finally, we eliminate the repeated rectangles and line segments.

Now we can introduce in more details the algorithm for solving the problem (P) through the above described decomposition method, but for the feasibility of this algorithm we suppose that $N \subset X$.

Algorithm 2.

1. Decompose (P) into two problems:
   The first problem is a multicriteria location problem $(P_L):\ \mathrm{Eff}\,(f[X], I\!R^p_+)$ with the multiobjective function:
   $$f(x) = \begin{pmatrix} \|x-a^1\| \\ \cdots \\ \|x-a^p\| \end{pmatrix}.$$
   The second problem is a multicriteria linear problem $(P_C):\ \mathrm{Eff}\,(C[X], I\!R^m_+)$, where
   $$C(x) = \begin{pmatrix} C_{p+1}(x) \\ \cdots \\ C_{p+m}(x) \end{pmatrix}.$$
2. Compute the solution set of (P_L) represented in the set $X_{\mathrm{Eff}}\,(f)$ through the geometric algorithm from the second section (according to this algorithm the set $X_{\mathrm{Eff}}\,(f)$ is always contained in N).
3. Divide $X_{\mathrm{Eff}}\,(f)$ through Algorithm 1 into nonempty, closed and convex pieces $X_1,\ldots,X_l \subset X_{\mathrm{Eff}}\,(f)$, so we get
   $$X_{\mathrm{Eff}}\,(f) = \bigcup_{i=1}^{l} X_i.$$
4. Minimize the multi objective function $C(x)$ of the problem (P_C) over the sets X_i for $i = 1,\cdots,l$. In other words, we get l efficiency sets by solving each of the l subproblems $\mathrm{Eff}\,(C[X_i], I\!R^m_+)$.
5. Compute the following set:
   $$\mathrm{Eff}\left(\bigcup_{i=1}^{l} \mathrm{Eff}\,(C[X_i], I\!R^m_+), I\!R^m_+\right). \tag{3}$$

We want now to show what happens in the case of adding one more linear objective to the objective function in the problem (P_L). So for $m = 1$ let $x^1,\ldots,x^l$ be respectively solutions from the solution sets of the problems $\min\limits_{x\in X_i} C_{p+1}(x)$ for $i = 1,\ldots,l$ and let

$$C_{p+1}(x^0) := \min_{i\in\{1,\ldots,l\}} C_{p+1}(x^i).$$

Actually the solution x^0 is a weakly efficient solution of the entire problem (P), that is because of the reducibility of (P) like we saw before. Furthermore, the next result shows that if x^0 minimizes $C_{p+1}(x)$ for $x^0 \in X_{\mathrm{Eff}}\,(f)$, then x^0 is also an efficient solution of the entire problem (P).

Theorem 1. Considering for $m = 1$ the problems $(P),(P_L)$ and (P_C). Then the solution of the problem (P_C) according to the decomposition method and like described in (3) is an efficient solution for the problem (P).

APPLICATIONS WITH MATLAB

Motivated by the importance of the visualization of many location algorithms Christian Günther has developed a graphical user surface in MATLAB to compute and plot the solutions of many types of location optimization problems.

According to the current version of the program we can see in the main window a map with a coordinate system where we can locate the existing facilities (see Figure 1). This can be also made directly on a loaded real map by choosing the corresponding option.

So, after locating the given facilities, the program can compute the set of efficient solutions of (P_L) with the maximum norm and Lebesgue norm. The solution set is shown on the map in the main window, like we can see in Figure 1. The intersection of both solution sets of (P_L) with maximum norm and Lebesgue norm is useful in order to reduce the set of efficient elements taking into account the preferences of the decision maker.

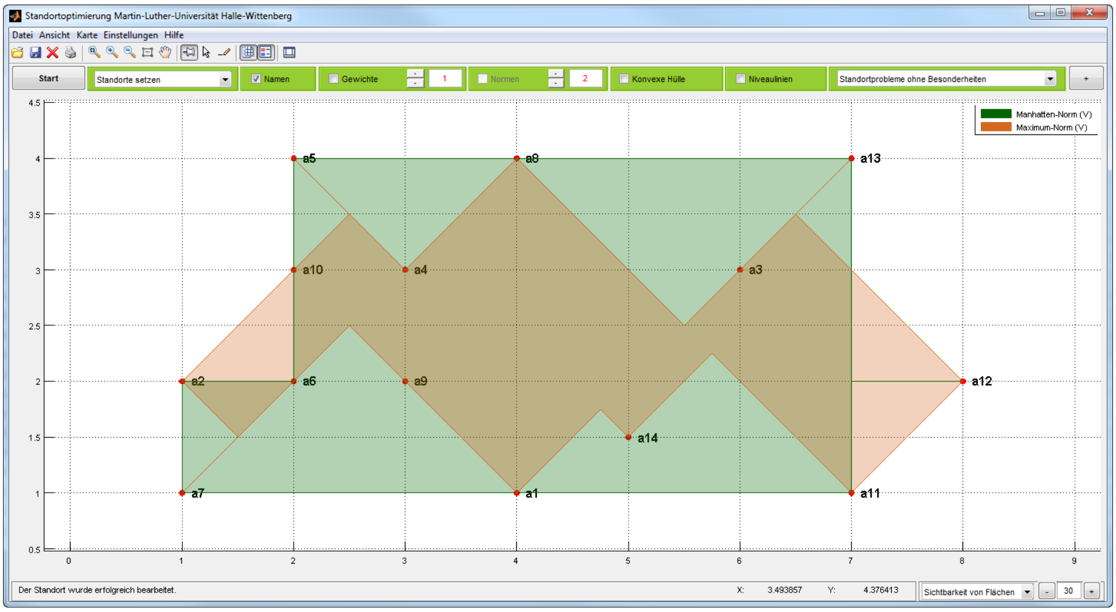

FIGURE 1. The set of efficient solutions of (P_L) with maximum norm and Lebesgue norm.

This approach and many other locational algorithms including Algorithm 1 are implemented in this program. With the program it is possible to compute the solution of the extended multicriteria location problem (for $m = 1$) according to the decomposition method after giving a linear function in the model window, which is shown on the right of Figure 2.

Furthermore, Figure 2 shows how a set of efficient solutions of (P_L) with maximum norm is divided into convex subsets. The line in Figure 2 represents the optimal level line for the linear cost functional, and the point a^1 on this line is the solution of the extended problem (P) according to the decomposition method. This computed solution is not only weakly efficient solution (through the reducibility of (P)), but also efficient solution (see Theorem 1).

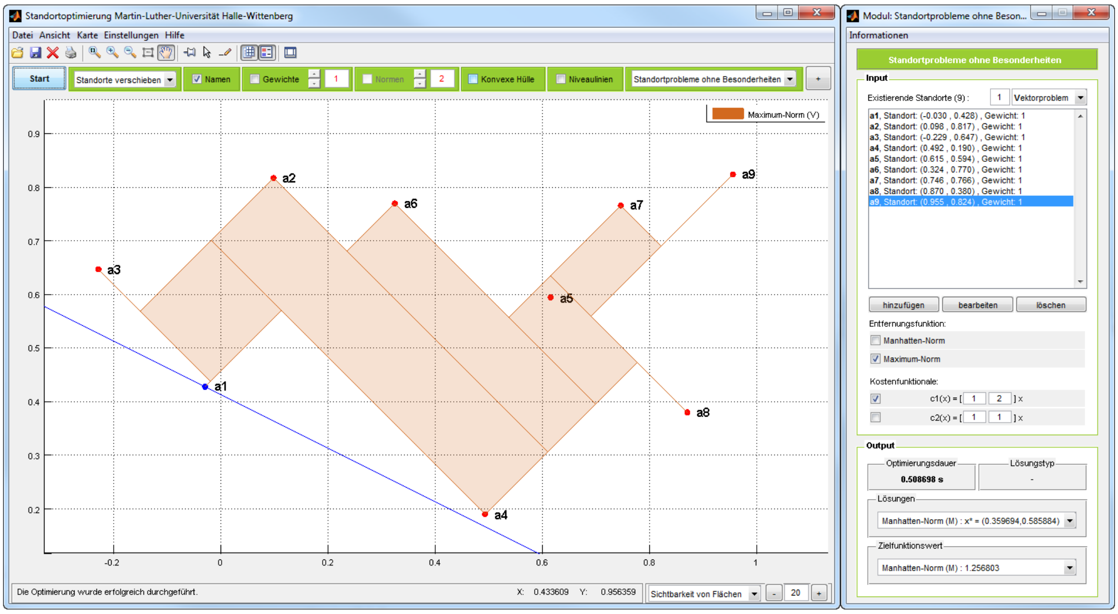

FIGURE 2. An example of solving the extended multicriteria location problem with the MATLAB program.

CONCLUSION

The extended multicriteria location problem (P) was solved through a decomposition algorithm. Also we discussed the reducibility of (P) and the structure of the elements generated by the decomposition algorithm. Finally, we introduced briefly a MATLAB program and showed some examples of the studied topics and the presented algorithms.

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