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Relations Between Strictly Robust Optimization Problems and a Nonlinear Scalarization Method

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Abstract. We study a strictly robust optimization problem in the context of a nonlinear scalarization method. We introduce a strictly robust multicriteria optimization problem and discuss its relation to a well-known scalar strictly robust optimization problem by using the nonlinear scalarization concept. Furthermore, we propose an unrestricted multicriteria optimization problem and note that its set of weakly Pareto optimal solutions contains all solutions of the scalar strictly robust optimization problem.

Keywords: Multicriteria Optimization, Nonlinear Scalarization, Robust Optimization
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INTRODUCTION

In this talk, we study strictly robust optimization problems (OPs) as proposed by Ben-Tal and Nemirovski in [1] and characterize this strictly robust OP by a nonlinear scalarization method. Using the monotonicity properties of the scalarizing functional, we derive relations to multicriteria optimization. Especially, we extend the concept of strict robustness to multicriteria OPs, i.e., we propose strictly robust multicriteria optimization problems (MOPs).

Robust optimization problems have become very popular for the past years and were intensely studied in [1, 2, 3]. In most OPs, parameters are fixed in order to get an OP that is easy to solve. In practice, however, those parameters are often uncertain and they are only known to belong to a certain set. Ben-Tal and Nemirovski observed in [3] that solutions of OPs can show high sensitivity to perturbations of the parameters which often results in infeasibility and/or suboptimality of the solution. Therefore, it seems reasonable to include uncertainty in the OP in order to get a solution that works well in an uncertain scenario.

Let $\mathscr{U} := \{\xi_1, \ldots, \xi_q\}$ be the uncertainty set, where $\xi_l \in I\!R^p,\ p \geq 1,\ l = 1, \ldots, q$, and $f : I\!R^n \times \mathscr{U} \to I\!R,\ F_i : I\!R^n \times \mathscr{U} \to I\!R,\ i = 1, \ldots, m$. We study an uncertain OP

$$\begin{aligned} & f(x,\xi) \to \min \\ \text{s.t. } & F_i(x,\xi) \leq 0,\ i = 1, \ldots, m, \\ & x \in I\!R^n, \end{aligned} \qquad (Q(\xi))$$

where $\xi \in \mathcal{U}$ is the uncertain parameter. At the time the OP $(Q(\xi))$ is solved, it is not known which value ξ is going to be realized. Robust optimization deals with such kinds of uncertainty in OPs.

STRICTLY ROBUST OPTIMIZATION PROBLEMS

Many concepts for robustness were studied in literature (see [4, 2, 3]). Here, we will concentrate on strictly robust OPs as introduced in [1]. A "strictly robust" solution of $(Q(\xi))$ is defined to be feasible for every uncertain scenario $\xi \in \mathcal{U}$, i.e., the constraints in $(Q(\xi))$ must be satisfied for each uncertain parameter ξ. In practice, this seems very reasonable - one can think of many examples where constraints must be fulfilled in every scenario, for instance in transportation: At an airport, these constraints could describe the time that an airplane is allowed to park in between flights. Of course the parking space is limited and even small changes in the constraints can cause severe problems for the airport. Many more examples, for instance in finance (see [5]) are described in literature. If the uncertain parameter ξ affects the objective function as well (which we will assume here), it has to be decided which of the possible objective functions (depending on ξ) is minimized. We will follow Ben-Tal and Nemirovski [1], who choose the worst-case scenario for the objective function. The *strictly robust* counterpart of problem $(Q(\xi))$ is introduced as

$$\begin{array}{c} \rho_{RC}(x) = \max_{\xi \in \mathcal{U}} f(x,\xi) \to \min \\ \text{s.t. } \forall \xi \in \mathcal{U} : \; F_i(x,\xi) \le 0, \; i = 1,\ldots,m, \\ x \in I\!R^n. \end{array} \tag{RC}$$

We call a solution x^* of (RC) *strictly robust*.

Ben-Tal and Nemirovski studied this problem in [1] for f and F_i, $i = 1,\ldots,m$, being linear functions. They proposed several uncertainty sets that differ from $\mathcal{U}$ as given above (for instance, they assume $\mathcal{U}$ to be an ellipsoid) and studied the problem's tractability. Furthermore, they studied the dual of strictly robust linear programming problems, see [6].

We show that (RC) can be expressed using a nonlinear functional that was originally introduced by Gerstewitz (Tammer) in [7] for scalarizing vector OPs. Properties of this nonlinear scalarizing functional have also been studied by Gerth (Tammer), Weidner in [8] and Pascoletti, Serafini in [9]. Furthermore, we propose a strictly robust MOP that corresponds to (RC) in the sense that the set of feasible points of both problems is the same. Furthermore, we propose a strictly robust MOP associated to (RC) and show that (RC) corresponds to a scalarization of the strictly robust MOP by means of a nonlinear scalarizing functional. Then, by using the nonlinear scalarizing functional, we show relations between (*RC*) and the strictly robust MOP.

CHARACTERIZATION BY MEANS OF A NONLINEAR SCALARIZING FUNCTIONAL

Let Y be a linear topological space, $k \in Y \setminus \{0\}$, $\mathscr{F}, C \subset Y$ and

$$C + [0,\infty) \cdot k \subset C.$$

We introduce the functional $z^{C,k} : Y \to I\!R \cup \{+\infty\} \cup \{-\infty\} =: \bar{I\!R}$

$$z^{C,k}(y) := \inf\{t \in I\!R | y \in tk - C\}. \tag{1}$$

Now we formulate the problem

$$z^{C,k}(y) \to \inf_{y \in \mathscr{F}}. \tag{$P_{k,C,\mathscr{F}}$}$$

The scalarizing functional $z^{C,k}$ was used in [8] to prove nonconvex separation theorems. Applications of $z^{C,k}$ include coherent risk measures in financial mathematics (see, for instance, [10]).

Theorem 1 *For*

$$C_1 = I\!R^q_+,\ k_1 = 1_q := (1,\dots,1)^T,$$

$y = (f(x,\xi_1),\dots,f(x,\xi_q))^T$, $x \in I\!R^n$, $\xi_l \in \mathscr{U}$, $l = 1,\dots,q$, *and*

$$\mathscr{F}_1 = \{(f(x,\xi_1),\dots,f(x,\xi_q))^T | x \in I\!R^n : \forall \xi \in \mathscr{U} : F_i(x,\xi) \le 0,\ i = 1,\dots,m\}, \tag{2}$$

$(P_{k_1,C_1,\mathscr{F}_1})$ *is equivalent to (RC) in the following sense:*

$$\begin{aligned}
\min\{z^{C_1,k_1}(y) | y \in \mathscr{F}_1\} &= z^{C_1,k_1}(y^*) \\
&= \min\{\rho_{RC}(x) | x \in I\!R^n : \forall \xi \in \mathscr{U} : F_i(x,\xi) \le 0,\ i = 1,\dots,m\} \\
&= \rho_{RC}(x^*),
\end{aligned}$$

where $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$, *assuming that the minimum* x^* *exists.*

Proof.

$$\begin{aligned}
\min_{y \in \mathscr{F}_1} z^{C_1,k_1}(y) &= \min_{y \in \mathscr{F}_1} \min\{t \in I\!R | y \in tk_1 - C_1\} \\
&= \min_{y \in \mathscr{F}_1} \min\{t \in I\!R | y - tk_1 \in -C_1\} \\
&= \min_{y \in \mathscr{F}_1} \min\{t \in I\!R | (f(x,\xi_1),\dots,f(x,\xi_q))^T - t \cdot (1,\dots,1)^T \le 0_q\}
\end{aligned}$$

$$
\begin{aligned}
&= \min_{y\in\mathscr{F}_1} \min\{t\in I\!R \,|\, (f(x,\xi_1),\ldots,f(x,\xi_q))^T \le t\cdot(1,\ldots,1)^T\} \\
&= \min\{\max_{\xi\in\mathscr{U}} f(x,\xi) \,|\, x\in I\!R^n:\ \forall\xi\in\mathscr{U}:\ F_i(x,\xi)\le 0,\ i=1,\ldots,m\} \\
&= \min\{\rho_{RC}(x) \,|\, x\in I\!R^n:\ \forall\xi\in\mathscr{U}:\ F_i(x,\xi)\le 0,\ i=1,\ldots,m\}.
\end{aligned}
$$

□

$k_1 = 1_q$ means that every objective function $f(x,\xi),\ \xi\in\mathscr{U}$, is treated in the same way, i.e., no objective function is preferred to another one.

Additionally, we get the following properties of the nonlinear scalarizing functional z^{C_1,k_1} that can be used in order to describe (RC):

Proposition 1 *The functional z^{C_1,k_1} is continuous, convex, proper, finite-valued, sublinear, C_1-monotone and strictly (int C_1)-monotone, and the following properties hold:*

- $\forall y\in\mathscr{F}_1,\ \forall r\in I\!R:\ z^{C_1,k_1}(y)\le r \Longleftrightarrow y\in rk_1 - C_1,$
- $\forall y\in\mathscr{F}_1,\ \forall r\in I\!R:\ z^{C_1,k_1}(y)= r \Longleftrightarrow y\in rk_1 - \operatorname{bd} C_1,$
- $\forall y\in\mathscr{F}_1,\ \forall r\in I\!R:\ z^{C_1,k_1}(y)< r \Longleftrightarrow y\in rk_1 - \operatorname{int} C_1,$
- $\forall y\in\mathscr{F}_1,\ \forall r\in I\!R:\ z^{C_1,k_1}(y+rk_1) = z^{C_1,k_1}(y)+r.$

Now we propose a **strictly robust MOP** by

$$
\begin{gathered}
\begin{pmatrix} f(x,\xi_1) \\ \vdots \\ f(x,\xi_q) \end{pmatrix} \to v-\min \\
\text{s.t. } \forall\xi\in\mathscr{U}:\ F_i(x,\xi)\le 0,\ i=1,\ldots,m, \\
x\in I\!R^n.
\end{gathered}
\tag{RC'}
$$

We call $y^* = (f(x^*,\xi_1),\ldots,f(x^*,\xi_q))^T$ a *strictly robust efficient* solution of (RC') if $\mathscr{F}_1\cap(y^* - (C_1\setminus\{0\})) = \emptyset$, where $\mathscr{F}_1$ is given by (2). y^* is called a *strictly robust weakly efficient* solution of (RC') if $\mathscr{F}_1\cap(y^* - \operatorname{int} C_1) = \emptyset$. The set of all strictly robust (weakly) efficient solutions of (RC') is called $Eff(\mathscr{F}_1,C_1)$ ($Eff_w(\mathscr{F}_1,C_1)$ respectively).

Obviously, $(P_{k_1,C_1,\mathscr{F}_1})$ is a scalarization of (RC'). Because z^{C_1,k_1} is C_1-monotone and strictly (int C_1)-monotone, we achieve the following result:

Proposition 2

$$
\left(\forall y\in\mathscr{F}_1\setminus\{y^*\}:\ z^{C_1,k_1}(y^*) < z^{C_1,k_1}(y)\right) \implies y^*\in Eff(\mathscr{F}_1,C_1),
$$

$$\left(\forall y \in \mathscr{F}_1 :\; z^{C_1,k_1}(y^*) \leq z^{C_1,k_1}(y)\right) \implies y^* \in Eff_w(\mathscr{F}_1, C_1).$$

To put it another way: If x^* is the unique solution to (*RC*), then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is strictly robust efficient for $(RC^{'})$. If x^* solves (*RC*) (and x^* is not necessarily a unique solution), then $y^* = (f(x^*,\xi_1),\dots,f(x^*,\xi_q))^T$ is strictly robust weakly efficient for $(RC^{'})$.

Furthermore, we propose a MOP without restrictions. Let

$$\begin{pmatrix} f(x,\xi_1) \\ \vdots \\ f(x,\xi_q) \\ F_1(x,\xi_1) \\ \vdots \\ F_m(x,\xi_q) \end{pmatrix} \to v-\min \qquad\qquad (RC^{'}_{\text{unrestricted}})$$
$$\text{s.t. } x \in I\!R^n.$$

Examining the set of weakly Pareto optimal solutions of $(RC^{'}_{\text{unrestricted}})$ gives the following result: Solutions of (*RC*) are always contained in the set of weakly Pareto optimal solutions of $(RC^{'}_{\text{unrestricted}})$. We are able to show that solutions of other robust problems as introduced in [4, 2, 3] also belong to this set. This seems very useful for the decision maker because now he only has to solve one MOP (which is even unrestricted) and he gets different robust solutions at once. Of course, he has to pick the solutions that satisfy the constraints in the required way.

CONCLUSIONS

We are able to establish relations between well-known strictly robust OPs and a nonlinear scalarizing method. Using the nonlinear scalarizing functional $z^{C,k}$ defined by (1), we are able to use the functional's properties to determine relations to a strictly robust MOP. As already mentioned above, there are several other ways of defining robustness in literature. In fact, many of those kinds of robustness can be expressed by using the nonlinear scalarizing method.

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