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Second-Order Optimality Conditions in Set-valued Optimization via Asymptotic Derivatives

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Abstract

In this paper we give new second order optimality conditions in set-valued optimization. We use the second order asymptotic tangent cones to define second order asymptotic derivatives and employ them to give the optimality conditions. We extend the well-known Dubovitskii-Milutin approach to set-valued optimization to express the optimality conditions given as an empty intersection of certain cones in the objective space. We also use some duality arguments to give new multiplier rules. By following the more commonly adopted direct approach we also give optimality conditions in terms of a disjunction of certain cones in the image space. Several particular cases are discussed.

Keywords. Set-valued optimization, second-order asymptotic derivatives, second-order contingent derivatives, second-order contingent sets, contingent cones, adjacent cones, interiorly tangent cones, Aubin property, optimality conditions.

2000 Mathematics Subject Classification: 90C26, 90C29, 90C30.

1 Introduction

Let X and Y be real normed spaces, let Q be a nonempty subset of X, let $C \subset Y$ be a proper, pointed, closed, and convex cone, and let $F : X \rightrightarrows Y$ be a given set-valued map. Our primary objective in this work is to study optimization problems that can be expressed in the form

$$(P) \qquad \text{minimize}\, F(x) \quad \text{subject to} \quad x \in Q.$$

Since the set-valued map F takes values in the normed space Y, the minimum will be defined with respect to the ordering induced by the cone $C \subset Y$. Of course, this specification naturally

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gives rise to a wide range of possibilities for the sought minimizer. In this work, we will give new optimality conditions for weak optimality for *set-valued optimization* problems of type (P).

Set-valued optimization is a vibrant and expanding branch of applied mathematics that deals with optimization problems where the objective map and/or the constraints maps are set-valued maps acting between abstract spaces. Since set-valued maps subsumes single valued maps, set-valued optimization provides an important generalization and unification of the scalar as well as the vector optimization problems. Therefore this relatively new discipline has justifiably attracted a great deal of attention in the recent years, see the excellent monograph [14, 25, 26] and the cited references therein. Set-valued optimization benefits enormously from the recently developed techniques of nonsmooth and variational analysis, and it not only provides elegant proofs of known results in scalar, vector, and nonsmooth optimization, but it also gives a new spectrum of powerful techniques to these vital branches of applied mathematics. Even more important is the fact that there are many research domains that directly lead to the optimization problems which can most satisfactorily be studied in the unified framework provided by set-valued optimization. For instance, the duality principles in vector optimization, gap functions for vector variational inequalities, inverse problems for partial differential equations and variational inequalities, fuzzy optimization, image processing, viability theory, mathematical economics, etc. all lead to optimization problems that can conveniently be cast as set-valued optimization problems. Moreover, since the set-valued maps appear naturally in many branches of pure and applied mathematics, set-valued optimization has the evident potential to remain as an important active research topic in the near future.

This work is devoted to optimality conditions of second-order in set-valued optimization. We recall that primarily there are two approaches for studying the optimality conditions in set-valued optimization. The first approach uses suitable derivatives of the involved set-valued maps whereas the second approach yields optimality conditions by means of alternative-type theorems. Although interesting results have been obtained by means of alternative-type theorems, in this work we focus entirely on obtaining optimality conditions by using some suitable derivatives of the involved set-valued maps.

Before describing the approach adopted in this work, let us first recall some of the major developments in the context of derivative based optimality conditions in set-valued optimization. We recall that the foundation of set-valued optimization was laid by the interesting paper by Corley [5] where the contingent derivatives and the circatangent derivative were employed to give general optimality conditions. Since the theory of variational analysis has been enriched by various notions of tangent cones, many extensions of Corley's results have been given by means of other approximating cones, for instance by the radial cone (see [8, 20]). Some authors have considered more general optimization problems but still used the contingent derivatives (see [23]). Starting from Corley's work, many subsequent contributions revolved around the graphs of the involved set-valued maps. Although this idea gives suitable necessary optimality conditions, the sufficient optimality conditions demand that graph of the involved set-valued map to be convex. This, however, is a quite stringent assumption. Of course, a quick remedy for this situation is to work with the profile map of the set-valued maps. Indeed Corley [5] used this idea. However, since the profile map does not seem a natural candidate for giving optimality conditions, another useful approach was proposed by Jahn and Rauh [18] by means of the contingent epiderivatives, where the authors employed the certain boundary parts of the epigraphs of the involved set-valued maps. Since then this approach has been rigorously pursued in [4, 15], among others. Another fruitful approach in set-valued optimization is based on the notion of Mordukhovich coderivatives and has attracted a great deal of attention in recent years (see [2, 7, 36]). All of the aforementioned results deal with first-order optimality conditions. Although many new refinements and interesting ideas related to the first-order optimality conditions in set-valued optimization are still in making, recent developments in non-smooth scalar and vector optimization have shown an acute interest in the development of second-order (or general higher-order) optimality conditions. This remains particularly true for vector optimization problems, see the interesting work [19] and the cited references therein.

Motivated by these developments, in [17], the second order contingent epiderivatives were introduced and employed to give new second-order optimality conditions in set-valued optimization. These results were further refined in [21] where the second order asymptotic derivatives were used. Some extensions of these and related results for higher-order optimality conditions are given in [34].

In this work, we present an extension of the well-known Dubovitski-Milutin approach (see [6]) to set-valued optimization and obtain new second-order optimality conditions for several notions of optimality. During the last several decades Dubovitski-Milutin approach has been used to study various optimal control and optimization problems with ordinary or partial differential equations as constraints. Several authors have also worked on extending this approach to nonsmooth optimization problems. Much recently, in [12], an extension of the Dubovitski-Milutin approach to set-valued optimization problems was proposed. However, the developments there were strictly limited to first-order necessary optimality conditions. In this contribution, we obtain new second-order optimality conditions in set-valued optimization.

The contents of this paper are organized into seven sections. In Section 2 we collect some definitions and concepts to be used in the later part of the paper. In Section 3, we recall the optimality notion and give some examples of cones with nonempty interior. In Section 4, we give some new second order optimality conditions. A multiplier rule is given in Section 5. Section 6 studies a direct approach to give optimality conditions. The paper concludes with some remarks concerning the approach and some of its possibly extensions.

2 Preliminaries

We begin by recalling the notion of the first order tangent cones and their second order analogues (see [1, 30] for details). In the following we use the notion: $P := \{t \in \mathbb{R} \mid t > 0\}$.

Definition 2.1. *Let Z be a real normed space, let $S \subset Z$ be nonempty and let $w \in Z$.*

1. *The second order contingent set $T^2(S, \bar{z}, w)$ of S at $\bar{z} \in \mathrm{cl}(S)$ (closure of S) in the direction $w \in Z$ is the set of all $z \in Z$ such that there are a sequence $(z_n) \subset Z$ with $z_n \to z$ and a*

sequence $(\lambda_n) \subset P$ *with* $\lambda_n \downarrow 0$ *so that* $\bar{z} + \lambda_n w + (\lambda_n^2/2) z_n \in S$.

2. *The second order adjacent set* $K^2(S, \bar{z}, w)$ *of* S *at* $\bar{z} \in \mathrm{cl}(S)$ *in the direction* $w \in Z$ *is the set of all* $z \in Z$ *such that for every sequence* $(\lambda_n) \subset P$ *with* $\lambda_n \downarrow 0$ *there exists* $(z_n) \subset Z$ *with* $z_n \to z$ *so that* $\bar{z} + \lambda_n w + (\lambda_n^2/2) z_n \in S$.

3. *The second order asymptotic tangent cone* $\widetilde{T}^2(S, \bar{z}, w)$ *of* S *at* $\bar{z} \in \mathrm{cl}(S)$ *in the direction* $w \in S$ *is the set of all* $z \in S$ *such that there are a sequence* $(z_n) \subset Z$ *with* $z_n \to z$ *and a sequence* $(r_n, t_n) \subset P \times P$ *with* $(r_n, t_n) \downarrow (0,0)$ *and* $r_n/t_n \to 0$ *so that* $\bar{z} + r_n w + r_n t_n z_n \in S$.

4. *The second order asymptotic adjacent cone* $\widetilde{K}^2(S, \bar{z}, w)$ *of* S *at* $\bar{z} \in \mathrm{cl}(S)$ *in the direction* $w \in S$ *is the set of all* $z \in S$ *such that for every sequence* $(r_n, t_n) \subset P \times P$ *with* $(r_n, t_n) \downarrow (0,0)$ *and* $r_n/t_n \to 0$ *there exists a sequence* $(z_n) \subset Z$ *with* $z_n \to z$ *and* $\bar{z} + r_n w + r_n t_n z_n \in S$.

5. *The interior second order adjacent cone* $\widetilde{IT}^2(S, \bar{z}, w)$ *of* S *at* $\bar{z} \in \mathrm{cl}(S)$ *in the direction* $w \in Z$ *is the set of all* $z \in Z$ *such that for every sequence* $(z_n) \subset Z$ *with* $z_n \to z$ *and for every sequence* $(r_n, t_n) \subset P \times P$ *with* $(r_n, t_n) \downarrow (0,0)$ *and* $r_n/t_n \to 0$ *we have* $\bar{z} + r_n w + r_n t_n z_n \in S$, *for sufficiently large* n.

6. *The contingent cone* $T(S, \bar{z})$ *of* S *at* $\bar{z} \in cl(S)$ *is the set of all* $z \in Z$ *such that there are a sequence* $(z_n) \subset Z$ *with* $z_n \to z$ *and a sequence* $(\lambda_n) \subset P$ *with* $\lambda_n \downarrow 0$ *so that* $\bar{z} + \lambda_n z_n \in S$.

7. *The interiorly contingent cone* $IT(S, \bar{z})$ *of* S *at* $\bar{z}$ *is the set of all* $v \in Z$ *such that for any sequences* $(\lambda_n) \subset \mathbb{P}$ *and* $(v_n) \subset Z$ *with* $\lambda_n \downarrow 0$ *and* $v_n \to v$, *there exists an integer* $m \in \mathbb{N}$ *such that* $\bar{z} + \lambda_n v_n \in S$ *for all* $n \geq m$.

Remark 2.1. It is known that the contingent cone $T(S, \bar{z})$ is a nonempty closed cone. However, $T^2(S, \bar{z}, w)$ is only a closed set (possibly empty), non-connected in general, and it may be nonempty only if $w \in T(S, \bar{z})$. On the other hand the interiorly contingent cone $IT(S, \bar{z})$ is an open cone. As concern the relationship between $T(S, \bar{z})$ and $IT(S, \bar{z})$, we have $IT(S, \bar{z}) = Z \backslash T(Z \backslash S, \bar{z})$. For any $S \subset Z$, the identities $T(S, \bar{z}) = T(\mathrm{cl}(S), \bar{z})$ and $IT(S, \bar{z}) = IT(\mathrm{int}(S), \bar{z})$ hold. Moreover, for a convex solid set S, we have $\mathrm{cl}(IT(S, \bar{z})) = T(S, \bar{z})$ and $\mathrm{int}(T(S, \bar{z})) = IT(S, \bar{z})$. In contrast, $\widetilde{T}^2(S, \bar{z}, w)$ and $\widetilde{K}^2(S, \bar{z}, w)$ are closed cones, and $\widetilde{IT}^2(S, \bar{z}, w)$ is an open cone. Some details and examples of these cone are given in [1, 3, 14, 27, 30]. In particular, the reader is referred to a timely survey by Giorgi, Jimenez, and Novo [9] that contains significant details of the asymptotic cones mentioned above (see also [28]).

Definition 2.2. *The set* $S \subset Z$ *is second order asymptotic derivable at* $(\bar{z}, \bar{w}) \in \mathrm{cl}(S) \times S$ *if the second order asymptotic tangent cone* $\widetilde{T}(S, \bar{z}, \bar{w})$ *coincides with the second order asymptotic adjacent cone* $\widetilde{K}(S, \bar{z}, \bar{w})$. *Moreover, the set* $S \subset Z$ *is second order derivable at* $(\bar{z}, \bar{w}) \in \mathrm{cl}(S) \times S$ *if the second order tangent cone* $T(S, \bar{z}, \bar{w})$ *coincides with the second order adjacent cone* $K(S, \bar{z}, \bar{w})$.

Let X and Y be real normed spaces and let $F : X \rightrightarrows Y$ be a set-valued map. The effective domain and the graph of F are given by

$$\begin{aligned} \mathrm{dom}(F) &:= \{x \in X \mid F(x) \neq \emptyset\}, \\ \mathrm{gph}(F) &:= \{(x,y) \in X \times Y \mid y \in F(x)\}. \end{aligned}$$

Furthermore, if $C \subset Y$ is a proper, convex, and pointed cone, the epigraph of $F : X \rightrightarrows Y$ is given by

$$\mathrm{epi}(F) := \{(x,y) \in X \times Y \mid y \in F(x) + C\}.$$

Given a proper, convex, and pointed cone $C \subset Y$, the profile map $F_+ : X \rightrightarrows Y$ is given by:

$$F_+(x) := F(x) + C, \quad \text{for every } x \in \mathrm{dom}(F).$$

Of course, the relationship $\mathrm{epi}(F) = \mathrm{graph}(F_+)$ holds trivially. We define the *weak-inverse image* $F[S]^-$ of F with respect to any set $S \subseteq Y$ by

$$F[S]^- := \{x \in X \mid F(x) \cap S \neq \emptyset\}.$$

The map F is called *convex*, if $\mathrm{gph}(F)$ is a convex set and C-*convex*, if $\mathrm{epi}(F)$ is convex set. We shall say that F is *strict* if $\mathrm{dom}(F) = X$. Furthermore, the map F is called regular if it is strict and convex. Throughout this work, by B_Y we represent the unit ball of the space Y.

We next collect the notion of derivatives that will be used in this work. To give the definition of generalized derivatives, we recall that an element $y \in D \subset Y$ is said to a minimal point of the set D, if $D \cap (\{y\} - C) = \{y\}$. The set of all minimal points of D with respect to the ordering cone C is denoted by $\mathrm{Min}(D,C)$.

Definition 2.3. *Let $F : X \rightrightarrows Y$ be set-valued, let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$, and let $(\bar{u},\bar{v}) \in X \times Y$.*

(i) *The second order contingent derivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ is the set-valued map $D^2_c F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by*

$$D^2_c F(\bar{x},\bar{y},\bar{u},\bar{v})(x) := \left\{ y \in Y \mid \quad (x,y) \in T^2(\mathrm{gph}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})) \right\}.$$

(ii) *The second order asymptotic derivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ is the set-valued map $D^2 F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by*

$$D^2 F(\bar{x},\bar{y},\bar{u},\bar{v})(x) := \left\{ y \in Y \mid \quad (x,y) \in \widetilde{T}^2(\mathrm{gph}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})) \right\}.$$

(iii) *The second order generalized asymptotic epiderivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ is the set-valued map $D^2_g F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \rightrightarrows Y$ defined by*

$$D^2_g F(\bar{x},\bar{y},\bar{u},\bar{v})(x) = \mathrm{Min}(D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})(x),C) \;\; x \in \mathrm{dom}(D^2(F+C)(\bar{x},\bar{y},\bar{u},\bar{v})).$$

(iv) *The second order asymptotic epiderivative of F at $(\bar{x},\bar{y})$ in the direction $(\bar{u},\bar{v})$ is the single-valued map $D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v}) : X \to Y$ defined by the condition*

$$\mathrm{epi}(D_e^2F(\bar{x},\bar{y},\bar{u},\bar{v})) = \widetilde{T}^2(\mathrm{epi}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v}).$$

It is clear that if $(\bar{u},\bar{v}) = (0_X, 0_Y)$ in the above definition, where 0_X and 0_Y are the zero elements in X and Y, we recover the contingent derivative $DF(\bar{x},\bar{y})$ of F at $(\bar{x},\bar{y})$ (cf. [1]). Furthermore, by replacing the asymptotic cone by contingent set we can recover the notion of second order contingent epiderivatives and second order generalized contingent epiderivatives. In particular, if $F : X \to Y$ is a single valued map which is twice continuously Fréchet differentiable around $\bar{x} \in S \subset X$, then the second order contingent derivative of the restriction F_0 of F to S at $\bar{x}$ in a direction $\bar{u}$ is given by the formula (see [1, p.215]):

$$D^2F_0(\bar{x},F(\bar{x}),\bar{u},F'(\bar{x})(\bar{u}))(x) = F'(\bar{x})(x) + F''(\bar{x})(\bar{u},\bar{u}) \text{ for } x \in T^2(S),\bar{x},\bar{u}). \tag{1}$$

It is empty when $x \notin T^2(S,\bar{x},\bar{u})$.

We conclude this section by recalling another important concept. Let B_Y be the unit ball of the space Y. The map F is said to have the *Aubin property around* $(u,v) \in \mathrm{gph}(F)$, if there are a constant $L \geq 0$ and neighborhoods U of u and V of v so that

$$F(x_1) \cap V \subseteq F(x_2) + L \parallel x_1 - x_2 \parallel B_Y \quad \text{for all } x_1, x_2 \in U \cap \mathrm{dom}(F).$$

This concept is due to J. P. Aubin. For several useful features of this notion, see [24], [30].

3 Optimality Conditions

We recall that Y is a real normed space which is partially ordered by a proper, pointed, and convex cone $C \subset Y$. We additionally assume that C is solid, that is, it has a nonempty interior $\mathrm{int}(C)$. For some $S \subset Y$, let $y \in S$ be arbitrary. The element y is said to be a weakly minimal point of S, if

$$S \cap (\{y\} - \mathrm{int}(C)) = \emptyset.$$

The set of all weakly minimal points of S with respect to C will be denoted by $\mathrm{WMin}(S,C)$.

Let X, Y, and Z be normed spaces, and let the spaces Y and Z be partially ordered by nontrivial pointed, solid, closed, and convex cones $C \subset Y$ and $D \subset Z$. Let $Q_0 \subset X$ be nonempty. Let $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Z$ be given set-valued maps.

We are concerned with the following set-valued optimization problems:

$$(P_0) \quad \mathrm{Min}\, F(x) \quad \text{subject to} \quad x \in Q_0.$$
$$(P_1) \quad \mathrm{Min}\, F(x) \quad \text{subject to} \quad x \in Q_1 := \{x \in Q_0 |\ G(x) \cap -D \neq \emptyset\}.$$

To define various optimality notions, we set

$$F(Q_1) := \cup_{x \in Q_1} F(x).$$

The following definition collect some of the most commonly used optimality notions for (P_1) :

Definition 3.1. *A pair* $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ *is called a* weak minimizer *of* (P_1) *if* $\bar{y} \in \mathrm{WMin}(F(Q_1),C)$.

In view of the definition of the weakly minimal points, $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ is a weak minimizer of (P_1), if and only if, $F(Q_1) \cap (\bar{y} - \mathrm{int}(C)) = \emptyset$. Notice that (P_1) reduces to (P_0) if $G(x) = 0_Z$ for all $x \in Q_0$. In this case the set of constraints Q_0 is not explicitly specified. If additionally we have $Q_0 = X$, then (P_1) is an unconstrained set-valued optimization problem. The optimality notion given in the above definition is a global one, that is, the whole set $F(Q_1)$ has been taken into account. Its local versions is defined as follows: The point $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ is said to be a local weak minimizer of (P_1), if there exists a neighborhood U of $\bar{x}$ such that $\bar{y} \in \mathrm{WMin}(F(Q_1 \cap U),C)$.

The notion of weak-minimality require that the ordering cone has a nonempty interior which is a quite stringent requirement. Nonetheless, many important cones have nonempty interior, as shown in the following example (see [13] for more details):

Example 3.1. (1). In the n-dimensional Euclidean space $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$, the cone $\mathbb{R}^n_+ = \{x = (x_1,x_2,\ldots,x_n) |\ x_1 \geq 0, x_2 \geq 0, \ldots, x_n \geq 0\}$ has a nonempty interior.

(2). Consider the space of continuous functions $C([a,b],\mathbb{R})$ with the norm $\|x\| = \sup\{\|x(t)\| |\ t \in [a,b]\}$. Then the cone $K = \{x \in C([a,b],\mathbb{R}) |\ x(t) \geq 0$ for any $t \in [a,b]\}$ has a nonempty interior.

(3). Consider the space $\ell^2(\mathbb{N},\mathbb{R})$ with the well-known structure of a Hilbert space. The convex cone $K = \{x = \{x_i\}_{i\geq 0} |\ x_0 \geq 0,\ \Sigma_{i=1}^n x_i^2 \leq x_0^2\}$ has a nonempty interior given by $\mathrm{int}(K) = \{x = \{x_i\}_{i\geq 0} |\ x_0 > 0,\ \Sigma_{i=1}^n x_i^2 < x_0^2\}$.

(4). Let ℓ^∞ be the space of bounded sequences of real numbers, equipped with the norm $\|x\| = \sup_{n\in N}\{\|x_n\|\}$. The convex cone $K = \{x = \{x_n\}_{n\in\mathbb{N}} |\ \ x_n \geq 0,$ for any $n \in \mathbb{N}\}$ has a nonempty interior.

(5). Consider the space $C^1([a,b],\mathbb{R})$ of real continuously differentiable functions equipped with the norm $\|f\|_1 = \{\int_a^b (f(t))^2 dt + \int_a^b (f'(t))^2 dt\}\}^{1/2}$ for any $t \in C^1([a,b],\mathbb{R})$. It is known that the cone $K = \{f \in C^1([a,b],\mathbb{R}) |\ f(t) \geq 0,$ for any $t \in [a,b]\}$ has a nonempty interior.

(6) Let $(X, \|\cdot\|)$ be a normed vector space and X^* be the topological dual of X^*. Let $f \in X^*$, and let $0 < \varepsilon < 1$. The convex cone $K_{f,\varepsilon} = \{x \in X |\ f(x) \geq \varepsilon\|x\|\}$ (Bishop-Phelps cone) has a nonempty interior given by $\mathrm{int}(K_{f,\varepsilon}) = \{x \in X |\ f(x) > \varepsilon\|x\|\}$.

Finally we conclude this section by recalling the following important result (see [6]) that we will use for deriving Lagrange multiplier rules in Section 5.

Lemma 3.1. *(Dubovitski, Milyutin [6]) Let* $C_0, C_1, \ldots, C_n$ *be non-empty convex cones in a normed space X and let* C_i*, for* $i \in I := \{1,2,\ldots,n\}$*, be open.*
Then $\cap_{i=0}^n C_i = \emptyset$ *if and only if there exist* $f_j \in C_j^\star$, $j \in \{0\} \cup I$*, not all zero, such that*

$$f_0 + f_1 + \cdots + f_n = 0.$$

4 Optimality Conditions by the Dubovitski-Milyutin Approach

We begin with the following necessary optimality condition for Problem (P_1):

Theorem 4.1. *Let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be a local weak minimizer of (P_1) and let $\bar{z} \in G(\bar{x})$. Let $\bar{u} \in \mathrm{dom}(DF(\bar{x},\bar{y})) \cap \mathrm{dom}(DG(\bar{x},\bar{z}))$ and $(\bar{v},\bar{w}) \in (DF(\bar{x},\bar{y}), DG(\bar{x},\bar{z})(\bar{u}) \cap -C \times T(-D,\bar{z})$ be arbitrary. Assume that* epi(F) *is asymptotic derivable at $(\bar{x},\bar{y},\bar{u},\bar{v})$ and possesses the Aubin property around $(\bar{x},\bar{y})$. Then*

$$D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^- \cap \widetilde{IT}^2(G[-D]^-,\bar{x},\bar{u}) \cap \widetilde{T}^2(Q_0,\bar{x},\bar{u}) = \emptyset. \tag{2}$$

Proof. Since $(\bar{x},\bar{y})$ is a local weak-minimizer of (P_1), there exists a neighborhood U_1 of $\bar{x}$ such that $\bar{y} \in \mathrm{WMin}(F(Q_1 \cap U_1), C)$. Therefore,

$$F(Q_1 \cap U_1) \cap (\{\bar{y}\} - \mathrm{int}(C)) = \emptyset. \tag{3}$$

We will show that if (2) fails then a feasible u can be constructed in a suitable vicinity of $\bar{x}$ such that $F(u) \cap (\bar{y} - \mathrm{int}(C)) \neq \emptyset$, and hence violating the local weak minimality of $(\bar{x},\bar{y})$ (see (3)).

For the sake of argument, we assume that there exists an $x \in X$ that violates (2), that is

$$x \in D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^- \cap \widetilde{IT}^2(G[-D]^-,\bar{x},\bar{u}) \cap \widetilde{T}^2(Q_0,\bar{x},\bar{u}). \tag{4}$$

Selecting the containment $x \in \widetilde{T}^2(Q_0,\bar{x},\bar{u})$ from (4), and using the definition of the second order asymptotic tangent cone, we ensure that there are a sequence $(x_n) \subset X$ with $x_n \to x$ and a sequence $(s_n,t_n) \subset P \times P$ with $(s_n,t_n) \downarrow (0,0)$ and $s_n/t_n \to 0$ so that for every $n \in \mathbb{N}$, we have

$$\bar{x} + s_n\bar{u} + s_nt_nx_n \in Q_0.$$

Next we notice that due to the containment $x \in \widetilde{IT}^2(G[D]^-,\bar{x},\bar{u})$, and the facts that $x_n \to x$, $(s_n,t_n) \downarrow (0,0)$ and $s_n/t_n \to 0$, we ensure the existence of $n_1 \in \mathbb{N}$ such that

$$\bar{x} + s_n\bar{u} + s_nt_nx_n \in G[-D]^-, \quad \text{for every } n \geq n_1,$$

or equivalently

$$G(\bar{x} + s_n\bar{u} + s_nt_nx_n) \cap -D \neq \emptyset, \quad \text{for every } n \geq n_1. \tag{5}$$

Furthermore, since $x \in D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^-$, there exists $y \in IT(-C,\bar{u})$ such that

$$(x,y) \in \widetilde{T}(\mathrm{epi}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v})).$$

Consequently, there are a sequence $((\hat{x}_n,\hat{y}_n)) \subset X \times Y$ with $(\hat{x}_n,\hat{y}_n) \to (x,y)$ and a sequence $(p_n,q_n) \subset P \times P$ with $(p_n,q_n) \downarrow (0,0)$ and $p_n/q_n \to 0$ so that for every $n \in \mathbb{N}$, we have

$$\bar{y} + p_n\bar{v} + p_nq_n\hat{y}_n \in F(\bar{x} + p_n\bar{u} + p_nq_n\hat{x}_n) + C.$$

By using the second-order asymptotic derivability of epi(F), we can set $p_n = s_n$ and $q_n = t_n$.

Let us now define

$$\begin{aligned} u_n &:= (\bar{x}+s_n\bar{u}+s_nt_nx_n), \\ \hat{u}_n &:= (\bar{x}+s_n\bar{u}+s_nt_n\hat{x}_n). \end{aligned}$$

Since both (u_n) and $(\hat{u}_n)$ converge to $\bar{x}$, there exists $n_2 \in \mathbb{N}$ such that $u_n, \hat{u}_n \in U := U_1 \cap U_2$ for $n \geq n_2$ where U_1 is the neighborhood of $\bar{x}$ described above and U_2 is a neighborhood of $\bar{x}$ which exists, along with a neighborhood V of $\bar{y}$, as a consequence of the Aubin property. Moreover, since $(\bar{y}+s_n\bar{v}+s_nt_n\hat{y}_n) \to \bar{y}$, there exists $n_3 \in \mathbb{N}$ so that $\bar{y}+s_n\bar{v}+s_nt_n\hat{y}_n \in V$ for all $n \geq n_3$.

By employing the Aubin property of F_+ at $(\bar{x},\bar{y})$, we get

$$\begin{aligned} \bar{y}+s_n\bar{v}+s_nt_n\hat{y}_n &\in [F(\bar{x}+s_n\bar{u}+s_nt_n\hat{x}_n)+C]\cap V \quad (\text{for } n \geq n_3) \\ &\subseteq F(\bar{x}+s_n\bar{u}+s_nt_nx_n)+C+Ls_nt_n \parallel x_n-\hat{x}_n \parallel B_Y \quad (\text{for } n \geq \max\{n_1,n_2,n_3\}). \end{aligned}$$

In view of the above inclusion, we can choose a sequence $b_n \in B_Y$ such that for $n \geq \max\{n_1,n_2\}$, we have

$$\bar{y}+s_n\bar{v}+s_nt_ny_n \in F(\bar{x}+s_n\bar{u}+s_nt_nx_n)+C,$$

where

$$y_n = (\hat{y}_n - Lb_n \parallel x_n-\hat{x}_n \parallel) \to y.$$

Since $y \in IT(-\text{int}(C),\bar{v})$, $t_n \downarrow 0$ and $y_n \to y$, it follows from the definition of the interior tangent cones that there exists $n_4 \in \mathbb{N}$ such that for $n \geq n_4$, we have

$$\bar{v}+t_ny_n \in -\text{int}(C).$$

Using the fact that $s_n > 0$, we obtain that

$$\bar{y}+s_n\bar{v}+s_nt_ny_n \in \bar{y}-\text{int}(C). \tag{6}$$

Next we choose $w_n \in F(u_n)$ such that $\bar{y}+s_n\bar{v}+s_nt_ny_n \in w_n+C$ for every $n \in \mathbb{N}$. We have

$$w_n \in \bar{y}+s_n\bar{v}+s_nt_ny_n-C \subseteq \bar{y}-\text{int}(C)-C \subset \bar{y}-\text{int}(C),$$

and consequently

$$w_n \in F(u_n)\cap(\bar{y}-\text{int}(C)), \quad \text{for } n \geq \max\{n_1,n_2,n_3\}.$$

Therefore, we have shown that for every $n \geq \max\{n_1,n_2,n_3,n_4\}$ there are $u_n \in Q_1 \cap U_1$ such that

$$F(u_n)\cap(\bar{y}-\text{int}(C)) \neq \emptyset.$$

This however contradicts the weak optimality of $(\bar{x},\bar{y})$. The proof is complete. □

Before any further advancement, let us formulate the following necessary optimality condition for Problem (P_0):

Theorem 4.2. *Let $(\bar{x},\bar{y})\in \operatorname{gph}(F)$ be a local weak minimizer of (P_0). Let $\bar{u}\in \operatorname{dom}(DF(\bar{x},\bar{y}))$ and let $\bar{v}\in DF(\bar{x},\bar{y})(\bar{u})\cap -C$ be arbitrary. Assume that* $\operatorname{epi}(F)$ *is asymptotic derivable at $(\bar{x},\bar{y},\bar{u},\bar{v})$. Then*

$$D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^- \cap \widetilde{T}^2(Q_0,\bar{x},\bar{u}) = \emptyset. \tag{7}$$

Clearly Theorem 4.1 and Theorem 4.2 remains valid if the asymptotic derivatives and cones are replaced by second order contingent derivatives.

Notice that Theorem 4.1 does not involve any derivative of the constraint map G. Therefore, in order to give multiplier rules it becomes necessary to either impose some constraint qualifications that connect some asymptotic derivatives of the map G to the cone $\widetilde{IT}^2(G[-D]^-,\bar{x},\bar{u})$. Another possibility that servers this purpose is furnished in the following:

Theorem 4.3. *Let $(\bar{x},\bar{z})\in \operatorname{gph}(G)$, $\bar{u}\in \operatorname{dom}(DG(\bar{x},\bar{z}))$ and let $\bar{w}\in DG(\bar{x},\bar{z})(\bar{u})$. Assume that* $\operatorname{epi}(G)$ *is asymptotic derivable at $(\bar{x},\bar{z},\bar{u},\bar{w})$ and possesses the Aubin property around $(\bar{x},\bar{z})$. Then the following inclusion holds:*

$$D^2G(\bar{x},\bar{z},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^- \subset \widetilde{IT}^2(G[-D]^-,\bar{x},\bar{u}) \tag{8}$$

Proof. Let $x\in D^2G(\bar{x},\bar{z},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^-$ be arbitrary. Then there exists $z\in Z$ such that

$$z\in D^2G(\bar{x},\bar{y},\bar{u},\bar{w})(x)\cap \widetilde{IT}^2(-D,\bar{z},\bar{u}).$$

Since

$$(x,z)\in \widetilde{T}(\operatorname{gph}(G),(\bar{x},\bar{z}),(\bar{u},\bar{w})),$$

there are a sequence $((\hat{x}_n,\hat{z}_n))\subset X\times Z$ with $(\hat{x}_n,\hat{z}_n)\to(x,z)$ and a sequence $(p_n,q_n)\subset P\times P$ with $(p_n,q_n)\downarrow(0,0)$ and $p_n/q_n\to 0$ so that for every $n\in\mathbb{N}$, we have

$$\bar{z}+p_n\bar{w}+p_nq_n\hat{z}_n\in G(\bar{x}+p_n\bar{u}+p_nq_n\hat{x}_n)+D.$$

Choose arbitrary sequences $(x_n)\subset X$ with $x_n\to x$, and $(s_n,t_n)\subset P\times P$ with $(s_n,t_n)\downarrow(0,0)$ and $s_n/t_n\to 0$. To show that $x\in\widetilde{IT}^2(G[-D]^-,\bar{x},\bar{u})$ it suffices to show that there exists $m\in\mathbb{N}$ such that

$$\bar{x}+s_n\bar{u}+s_nt_nx_n\in G[-D]^-,\quad \text{for every } n\geq m,$$

or equivalently,

$$G(\bar{x}+s_n\bar{u}+s_nt_nx_n)\cap -D\neq\emptyset,\quad \text{for every } n\geq m.$$

In view of the derivability, we can set $p_n=s_n$ and $q_n=t_n$. Furthermore, by following the same line of arguments as in the proof of Theorem 4.1, by using the Aubin's property, we can show that there exists $n_1\in\mathbb{N}$ such that for every $n\geq n_1$, we have

$$\bar{z}+s_n\bar{w}+s_nt_nz_n\in G(\bar{x}+s_n\bar{u}+s_nt_nx_n)+D,$$

where for some $b_n\in B_Y$,

$$z_n=(\hat{z}_n-Lb_n\parallel x_n-\hat{x}_n\parallel)\to z.$$

We choose $w_n \in G(\bar{x}+s_n\bar{u}+s_nt_nx_n)$ so that $\bar{z}+s_n\bar{w}+s_nt_nz_n \in G(w_n)+D$ implying that $w_n \in -D$ for $n \geq n_1$. Therefore, for sufficiently large n, we have

$$\bar{x}+s_n\bar{u}+s_nt_nx_n \in G[-D]^-.$$

This, however, conforms that $x \in T^2(G[-D]^-,\bar{x},\bar{u})$. The proof is complete. $\square$

A direct consequence of the above result, is the following analogue of Theorem 4.1:

Theorem 4.4. *Let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be a local weak minimizer of (P_1) and let $\bar{z} \in G(\bar{x})$. Let $\bar{u} \in \mathrm{dom}(DF(\bar{x},\bar{y})) \cap \mathrm{dom}(DG(\bar{x},\bar{z}))$ and $(\bar{v},\bar{w}) \in (DF(\bar{x},\bar{y}), DG(\bar{x},\bar{z})(\bar{u}) \cap -C \times T(-D,\bar{z})$ be arbitrary. Assume that $\mathrm{epi}(F)$ is asymptotic derivable at $(\bar{x},\bar{y},\bar{u},\bar{v})$ and possesses the Aubin property around $(\bar{x},\bar{y})$. Assume that $\mathrm{epi}(G)$ is asymptotic derivable at $(\bar{x},\bar{z},\bar{u},\bar{w})$ and possesses the Aubin property around $(\bar{x},\bar{z})$. Then*

$$D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^- \cap D^2G(\bar{x},\bar{y},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^- \cap \widetilde{T}^2(Q_0,\bar{x},\bar{u}) = \emptyset. \quad (9)$$

5 Lagrange Multipliers Rule

We begin by recalling the following result which was used in [12] and generalizes a result by Rigby [29] given for single valued convex maps. For the sake of completeness and the importance of this result in our approach, we sketch a proof here.

Lemma 5.1. [12] *Let X and Y be normed spaces, let $M \subseteq X$ be convex and let $A \subset Y$ be a solid closed convex cone. Let $T : M \rightrightarrows Y$ be a A-convex set-valued map. If $T[-\mathrm{int}(A)]^- \neq \emptyset$, then for every $\ell \in L^\star$ where $L := T[-A]^-$, there exists $t \in A^*$ such that*

$$t \circ T(x) \geq \ell(x) \quad \textit{for every} \quad x \in M.$$

If $T[-\mathrm{int}(A)]^- = \emptyset$, then there exists $t \in A^* \backslash \{0_{Y^*}\}$ such that

$$t \circ T(x) \geq 0 \quad \textit{for every} \quad x \in M.$$

Proof. Let us begin with the case when the set $T[-\mathrm{int}(A)]^-$ is nonempty. Then the (negative) dual $L^\star$ of $L := T[-A]^-$ is nonempty as well. We choose $\ell \in L^\star$ arbitrarily and define a set

$$E := \{(y,\ell(x)) \in Y \times \mathbb{R} \,|\, y \in T(x)+A,\ x \in M\}.$$

Using the facts that M is convex, T is A-convex and $\ell \in Y^*$, we deduce that E is a convex set. Indeed, let $(y_1,z_1),(y_2,z_2) \in E$ be arbitrary. Then by the definition of E, for $i = 1,2$, there exists $x_i \in X$ with $z_i = \ell(x_i)$ and $y_i \in T(x_i)+A$. For $\lambda \in (0,1]$, we have $\lambda z_1 + (1-\lambda)z_2 = \ell(\lambda x_1 + (1-\lambda)x_2)$. This, in view of the A-convexity of T, ensures that $\lambda y_1 + (1-\lambda)y_2 \in \lambda T(x_1) + (1-\lambda)T(x_2) + A \subseteq T(\lambda x_1 + (1-\lambda)x_2) + A$. Since the set M is convex, we at once obtain that $\lambda(y_1,z_1) + (1-\lambda)(y_2,z_2) \in E$.

Having established convexity of the set E, we claim that

$$E \cap (-\mathrm{int}(A) \times \mathbb{P}) = \emptyset.$$

In fact, if this is not the case, then there exists $(x,y) \in X \times Y$ such that $y \in (T(x)+A) \cap (-\mathrm{int}(A))$ and $\ell(x) > 0$. Let $w \in T(x)$ be such that $y \in w+A$. Then $w \in y - A \subset -\mathrm{int}(A) - A = -\mathrm{int}(A)$. This however contradicts that $\ell \in L^\star$. Therefore $E \cap (-\mathrm{int}(A) \times \mathbb{P}) = \emptyset$ and hence by a separation theorem, we get the existence of $(f,g) \in Y^* \times \mathbb{R} \backslash \{0_{Y^*}, 0\}$ and a real number α such that we have

$$\begin{aligned} f(u)+g(v) &\geq \alpha \quad \text{for every } \ (u,v) \in E && \text{(10a)} \\ f(c)+g(d) &< \alpha \quad \text{for every } \ \ (c,d) \in -\mathrm{int}(A) \times \mathbb{P}. && \text{(10b)} \end{aligned}$$

Since A is a cone, we can set $\alpha = 0$ in (10a) and (10b). By taking $d \in \mathbb{P}$ arbitrary close to 0 and $c \in -\mathrm{int}(A)$ arbitrary close to 0_Y, we obtain $f \in A^*$ and $g \leq 0$, respectively. We claim that $g < 0$. Indeed, if $g = 0$, we get $f(c) < 0$ for every $c \in -\mathrm{int}(A)$ and $f(u) \geq 0$ for every $u \in T(D)+A$. This, however is impossible because we have $(T(M)+A) \cap (-\mathrm{int}(A)) \neq \emptyset$. Therefore $g < 0$. Moreover, from (10a), for every $x \in M$ we have $f \circ (T+A)(x) \geq -(g \cdot \ell)(x)$. Therefore, by setting $t = (-f/g) \in A^*$ and noticing that $0_Y \in A$, we finish the proof of the first part.

For the second part, we notice that if $T(-\mathrm{int}(A)) = \emptyset$, we have $T(D) \cap -\mathrm{int}(A) = \emptyset$ and hence by the arguments similar to those given above we can prove the existence of $t \in A^* \backslash \{0_Y\}$ such that $t \circ T(x) \geq 0$ for every $x \in D$. $\square$

The following result is the Lagrange multiplier rule. For notational simplicity, in the following result we set

$$\begin{aligned} L &:= IT(C, -\bar{u}); \\ N &:= \widetilde{IT}^2(D, -\bar{z}, -\bar{u}); \\ P &:= D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^-; \\ Q &:= D^2G(\bar{x},\bar{y},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^-. \end{aligned}$$

Theorem 5.1. *Let $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ be a local weak minimizer of (P_1) and let $\bar{z} \in G(\bar{x})$. Let $\bar{u} \in \mathrm{dom}(DF(\bar{x},\bar{y})) \cap \mathrm{dom}(DG(\bar{x},\bar{z}))$ and $(\bar{v},\bar{w}) \in (DF(\bar{x},\bar{y}), DG(\bar{x},\bar{z})(\bar{u}) \cap -C \times T(-D,\bar{z})$ be arbitrary. Assume that $\mathrm{epi}(F)$ is asymptotic derivable at $(\bar{x},\bar{y},\bar{u},\bar{v})$ and possesses the Aubin property around $(\bar{x},\bar{y})$. Assume that $\mathrm{epi}(G)$ is asymptotic derivable at $(\bar{x},\bar{z},\bar{u},\bar{w})$ and possesses the Aubin property around $(\bar{x},\bar{z})$. Assume that P and Q are proper, open, convex cones. Let there exist a nonempty, closed, and convex cone $M \subseteq \widetilde{T}(Q_0,\bar{x},\bar{u})$. Then there exist functionals $p \in M^*$, $t \in L^*$, $s \in N^*$, not all zero, such that the following inequality holds for every $x \in X$:*

$$t \circ D^2F_+(\bar{x},\bar{y},\bar{u},\bar{v})(x) + s \circ D^2G_+(\bar{x},\bar{z},\bar{u},\bar{z})(x) \geq p(x). \tag{11}$$

Proof. We will prove the theorem by analyzing the following three possibilities:

(i) $P = \emptyset$;

(ii) $Q = \emptyset$;

(iii) $P \neq \emptyset$ and $Q \neq \emptyset$.

Let us begin with case (i) and assume that $P = \emptyset$, that is

$$D^2 F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^- = \emptyset.$$

Then it follows from Lemma 5.1 that there exists $t \in L^* \backslash \{0_{Y^*}\}$ such that

$$t \circ D^2 F_+(\bar{x},\bar{y},\bar{u},\bar{v}) \geq 0, \quad \text{for every } x \in X.$$

We choose $s = 0_{Z^*}$ and $p = 0_{X^*}$, we obtain the desired result.

For case (ii), let us begin by assuming that $Q = \emptyset$, that is,

$$D^2 G(\bar{x},\bar{y},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^- = \emptyset.$$

By invoking Lemma 5.1 once again, we obtain $s \in N^\star \backslash \{0_{Z^*}\}$ such that

$$s \circ D^2 G(\bar{x},\bar{y},\bar{u},\bar{w})(x) \geq 0, \quad \text{for every } x \in X.$$

By setting $t = 0_{X^*}$, and $p = 0_{X^*}$, we obtain the desired result.

Finally, we consider the case (iii). Since $(\bar{x},\bar{y})$ is a weak minimizer of (P_1), we have

$$P \cap Q \cap M = \emptyset.$$

Since P, Q, and M are all nonempty, we can apply Lemma 3.1 to assure the existence of

$$\begin{array}{rcl} l & \in & M^\star, \\ l_0 & \in & \left(D^2 F_+(\bar{x},\bar{y},\bar{u},\bar{v})[IT(-C,\bar{u})]^-\right)^* \\ l_1 & \in & \left(D^2 G(\bar{x},\bar{y},\bar{u},\bar{w})[\widetilde{IT}^2(-D,\bar{z},\bar{u})]^-\right)^*, \end{array}$$

such that

$$l + l_0 + l_1 = 0. \tag{12}$$

Now, in view of Lemma 5.1, we get the existence of functionals $t \in L^*$ and $s \in N^*$ such that for all $x \in X$, the following inequalities hold

$$\begin{array}{rcl} (t \circ D^2 F_+(\bar{x},\bar{y},\bar{u},\bar{v}))(x) & \geq & l_0(x); \\ (s \circ D^2 G(\bar{x},\bar{y},\bar{u},\bar{w}))(x) & \geq & l_1(x). \end{array}$$

Combining of the above inequalities with (12) and setting $p = -l$ yield (11). $\square$

6 Optimality Conditions by the Direct Approach

In this section, we give some optimality conditions by using the direct approach. The main objective here is that we want to shed some light on the approach which is more commonly used in set-valued optimization and contrast it with the Dubovitskii-Milyutin approach. We begin with the following:

Theorem 6.1. *Let* $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ *be a local weak minimizer of* (P_1) *and let* $\bar{z} \in G(\bar{x})$*. Assume that* $\mathrm{dom}(F) = \mathrm{dom}(G) = Q_0$*. Let* $\bar{u} \in \mathrm{dom}(DF(\bar{x},\bar{y})) \cap \mathrm{dom}(DG(\bar{x},\bar{z}))$ *and let*

$$(\bar{v},\bar{w}) \in (DF(\bar{x},\bar{y}), DG(\bar{x},\bar{z}))(\bar{u}) \cap -C \times T(D,-\bar{z})$$

be arbitrary. Assume that $\mathrm{epi}(F)$ *is asymptotic derivable at* $(\bar{x},\bar{y},\bar{u},\bar{v})$ *and possesses the Aubin property around* $(\bar{x},\bar{y})$*. Then for every* $x \in \mathrm{dom}((D^2F(\bar{x},\bar{y},\bar{u},\bar{v}), D^2G(\bar{x},\bar{z},\bar{u},\bar{w}))$*, we have*

$$(D^2F(\bar{x},\bar{y},\bar{u},\bar{v}), D^2G(\bar{x},\bar{z},\bar{u},\bar{w})(x) \cap IT(-C,\bar{u}) \times \widetilde{IT}^2(-D,\bar{z},\bar{w}) = \emptyset. \tag{13}$$

Proof. Assume that (13) is false. Then there exists $x \in \mathrm{dom}((D^2F(\bar{x},\bar{y},\bar{u},\bar{v}), D^2G(\bar{x},\bar{z},\bar{u},\bar{w}))$ such that

$$(y,z) \in (D^2F(\bar{x},\bar{y},\bar{u},\bar{v}), D^2G(\bar{x},\bar{z},\bar{u},\bar{w}))(x) \cap IT(-C,\bar{u}) \times \widetilde{IT}^2(-D,\bar{z},\bar{w}).$$

Consequently, $(x,y) \in \widetilde{T}^2(\mathrm{gph}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v}))$ and $(x,z) \in \widetilde{T}^2(\mathrm{gph}(G),(\bar{x},\bar{z}),(\bar{u},\bar{w}))$. From $(x,y) \in \widetilde{T}^2(\mathrm{gph}(F),(\bar{x},\bar{y}),(\bar{u},\bar{v}))$ and the definition of the second order asymptotic contingent set ensures that there are a sequence $((\hat{x}_n,\hat{y}_n)) \subset X \times Y$ with $(\hat{x}_n,\hat{y}_n) \to (x,y)$ and a sequence $(p_n,q_n) \subset P \times P$ with $(p_n,q_n) \downarrow (0,0)$ and $p_n/q_n \to 0$ so that for every $n \in \mathbb{N}$, we have

$$\bar{y} + p_n\bar{v} + p_nq_n\hat{y}_n \in F(\bar{x} + p_n\bar{u} + p_nq_n\hat{x}_n).$$

From $(x,z) \in \widetilde{T}^2(\mathrm{gph}(G),(\bar{x},\bar{z}),(\bar{u},\bar{w}))$, we deduce that there are sequences $((\hat{x}_n,\hat{y}_n)) \subset X \times Y$ with $(\bar{x}_n,\bar{y}_n) \to (x,y)$ and a sequence $(s_n,t_n) \subset P \times P$ with $(s_n,t_n) \downarrow (0,0)$ and $s_n/t_n \to 0$ so that for every $n \in \mathbb{N}$, we have

$$\bar{y} + s_n\bar{v} + s_nt_n\hat{y}_n \in F(\bar{x} + s_n\bar{u} + s_nt_n\hat{x}_n).$$

Using the asymptotic derivability of $\mathrm{gph}(G)$, we can set $s_n = p_n$ and $t_n = q_n$.

Furthermore, repeating the arguments given in the proof of Theorem 4.1, we can show that for sufficiently large n there exists a feasible sequence a_n that violates the weak minimality of $(\bar{x},\bar{y})$ is proper minimizer. □

Within the absence of the Aubin property and the asymptotic derivability assumptions a weaker disjunction can be proved which is stated in the following:

Theorem 6.2. *Let* $(\bar{x},\bar{y}) \in \mathrm{gph}(F)$ *be a local weak minimizer of* (P_1) *and let* $\bar{z} \in G(\bar{x})$*. Let* $\mathrm{dom}(F) = \mathrm{dom}(G) = Q_0$*. Let* $\bar{u} \in \mathrm{dom}(D(FG)(\bar{x},\bar{y},\bar{z}))$ *and* $(\bar{v},\bar{w}) \in (D(FG)(\bar{x},\bar{y},\bar{z})(\bar{u}) \cap -C \times T(D,-\bar{z})$ *be arbitrary.*
Then the following disjunction holds for for every $x \in \mathrm{dom}((D^2(FG)(\bar{x},\bar{y},\bar{z}))$:

$$D^2(FG)(\bar{x},\bar{y},\bar{z},\bar{u},\bar{v},\bar{w})(x) \cap IT(-C,\bar{u}) \times \widetilde{IT}^2(-D,\bar{z},\bar{w}) = \emptyset. \tag{14}$$

7 Concluding Remarks

We have given new second-order optimality conditions by employing second-order asymptotic derivatives. Here we have focused on necessary optimality condition only but in the near future, we intend to give some suitable higher-order sufficient optimality conditions in set-valued optimization. Several extensions of our results are possible. It would be interesting to pay special attention to include explicit equality constraints, and to the case when the ordering cone D is not solid. It seems also possible to introduce a notion of second order subdifferentials by using the second order derivatives and epiderivatives (see [14, 12]).

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