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Set-Valued Optimization**

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# Generalized Dubovitskii-Milyutin Approach in Set-Valued Optimization

Akhtar A. Khan\*

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Dedicated to the 60th birthday of Professor Phan Quoc Khanh

## Abstract

The primary objective of this paper is to employ the Dubovitskii-Milyutin approach to give first and second-order optimality conditions for set-valued optimization problems with more general set-valued constraints. In the particular case when some constraints are single-valued, we recover the case of set-valued optimization problems with many set-valued inequality constraints and many single-valued equality constraints. It is known that the classical Dubovitskii-Milyutin approach is not suitable for optimization problems with multi-equality constraints. The main reason for this deficiency is the fact that the separation arguments used in the classical Dubovitskii-Milyutin approach are applicable to an empty intersection of cones in which at most one cone can be closed. However, a proper formulation of multi-equality constraints leads to an empty intersection with more than one closed cones. To study optimization problems with multi-equality constraints, a generalized Dubovitskii-Milyutin theory has been developed. In this work we present an extension of the generalized Dubovitskii-Milyutin theory to the set-valued optimization problems. In this process, we also obtain new applications of this theory to nonsmooth optimization and to more general vector optimization problems. New second-order asymptotic derivatives of set-valued maps are introduced and used to give the optimality conditions.

**Keywords.** Set-valued optimization, lower Dini derivative, second-order lower Dini asymptotic derivatives, asymptotic contingent cones, interiorly tangent cones, contingent cones, optimality conditions.

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# 1 Introduction

Let  $I := \{1, 2, \dots, m\}$  and  $J := \{1, 2, \dots, n\}$  be two index sets. Let  $X$  and  $Y$  be normed spaces, where the space  $Y$  is partially ordered by a pointed, closed, and convex cone  $K$  with a nonempty interior, let  $Q \subset X$  be nonempty, and let  $F : X \rightrightarrows Y$  be a given set-valued map. For  $i \in I$ , let  $Z_i$  be a normed space, let  $C_i \subset Z_i$  be a pointed, closed, and convex cone with nonempty interior, and let  $G_i : X \rightrightarrows Z_i$  be a given set-valued map. Analogously, for  $j \in J$ , let  $W_j$  be a normed space, let  $D_j \subset W_j$  be closed, and convex set, and let  $H_j : X \rightrightarrows W_j$  be a given set-valued map.

In this work, we focus on the following set-valued optimization problem  $(P)$ :

$$\begin{aligned} & \text{minimize } F(x) \\ & \text{subject to } x \in S := \{x \in Q \mid G_i(x) \cap -C_i \neq \emptyset, \forall i \in I, H_j(x) \cap -D_j \neq \emptyset, \forall j \in J\}. \end{aligned}$$

Our objective is to give optimality conditions for a local weak-minimizer. However, our approach can readily be extended to other kinds of optimality as well. We recall that a point  $(\bar{x}, \bar{y}) \in X \times Y$  is a local weak-minimizer of  $(P)$ , if there is a neighborhood  $U$  of  $\bar{x}$  such that

$$F(S \cap U) \cap (\{\bar{y}\} - \text{int}(K)) = \emptyset,$$

where  $\bar{y} \in F(\bar{x})$ , the notion  $\text{int}(K)$  designate the interior of the cone  $K$ , and

$$F(S \cap U) := \cup_{x \in S \cap U} F(x).$$

By appropriately adjusting the data, and by choosing the maps  $G_i$  and  $H_j$  to be single valued, we recover the following set-valued optimization problem with single-valued constraints:

$$\begin{aligned} & \text{minimize } F(x) \\ & \text{subject to } x \in S := \{x \in Q \mid G_i(x) \geq 0, \forall i \in I, H_j(x) = 0, \forall j \in J\}. \end{aligned}$$

We remark that the above optimization problem with multi-equality constraints can not be tackled by the classical Dubovitskii-Milyutin approach (see [6]). This is mainly due to the fact that the separation arguments used in the classical Dubovitskii-Milyutin approach are applicable to an empty intersection of cones in which at most one cone can be closed. On the other hand, a proper formulation of multi-equality constraints leads to an empty intersection with several closed cones. To handle optimization problems with multi-equality constraints, a generalized Dubovitskii-Milyutin theory has been developed (see [14]). This theory is enriched by the notion of the cones of the same sense and the cones of the opposite sense introduced by Walczak [22]. Although many interesting contributions have appeared in this direction, the generalized Dubovitskii-Milyutin theory so far has been only used to study vector optimization problems with differentiable data. Motivated by this, the primary objective of this paper is to present an extension of the generalized Dubovitskii-Milyutin theory to the set-valued optimization problems. In this process, we also obtain new applications of this theory to nonsmooth optimization and to more general vector optimization problems.

This paper is organized into four sections. In Section 2, we collect some concepts and results to be used in the rest of the paper. In Section 3, we give first-order optimality conditions for set-valued optimization problems. Section 4 presents some second-order optimality conditions by using a new notion of second-order lower Dini asymptotic derivative.

## 2 Preliminaries

We begin with by recalling the notions of some tangent cones. We set  $\mathbb{P} := \{t \in \mathbb{R} \mid t > 0\}$ .

**Definition 2.1.** Let  $\mathfrak{E}$  be a normed space, let  $\Omega \subset \mathfrak{E}$  and let  $\bar{z} \in \text{cl}(\Omega)$  (closure of  $\Omega$ ).

- (a) The contingent cone  $T(\Omega, \bar{z})$  of  $\Omega$  at  $\bar{z}$  is the set of all  $z \in \mathfrak{E}$  such that there are sequences  $(\lambda_n) \subset \mathbb{P}$  and  $(z_n) \subset \mathfrak{E}$  with  $\lambda_n \downarrow 0$  and  $z_n \rightarrow z$  such that  $\bar{z} + \lambda_n z_n \in \Omega$  for every  $n \in \mathbb{N}$ .
- (b) The interiorly contingent cone  $IT(\Omega, \bar{z})$  of  $\Omega$  at  $\bar{z}$  is the set of all  $z \in \mathfrak{E}$  such that for any sequences  $(\lambda_n) \subset \mathbb{P}$  and  $(z_n) \subset \mathfrak{E}$  with  $\lambda_n \downarrow 0$  and  $z_n \rightarrow z$ , there exists an integer  $m \in \mathbb{N}$  such that  $\bar{z} + \lambda_n z_n \in \Omega$  for every  $n \geq m$ .

**Remark 2.1.** It is known that  $T(\Omega, \bar{z})$  is a closed cone possessing the isotony property, that is, for subsets  $\Omega_1$  and  $\Omega_2$  such that  $\Omega_1 \subset \Omega_2$ , the inclusion  $T(\Omega_1, \bar{z}) \subset T(\Omega_2, \bar{z})$  holds for every  $\bar{z} \in \text{cl}(\Omega_1) \cap \text{cl}(\Omega_2)$ . On the other hand the interiorly contingent cone  $IT(\Omega, \bar{z})$  is an isotone open cone. As concern the relationship between  $T(\Omega, \bar{x})$  and  $IT(\Omega, \bar{z})$ , we have  $IT(\Omega, \bar{z}) = \mathfrak{E} \setminus T(\mathfrak{E} \setminus \Omega, \bar{z})$ . As a useful implication of this relationship, the cones  $T(\Omega, \bar{z})$  and  $IT(\Omega, \bar{z})$  form an admissible pair, that is, for every pair of sets  $\Omega_1, \Omega_2 \subset \mathfrak{E}$  with  $\Omega_1 \cap \Omega_2 = \emptyset$ , we have  $T(\Omega_1, \bar{z}) \cap IT(\Omega_2, \bar{z}) = \emptyset$  for every  $\bar{z} \in \mathfrak{E}$ . Also for arbitrary sets  $\Omega_1, \Omega_2 \subset \mathfrak{E}$  we have  $IT(\Omega_1 \cap \Omega_2, \bar{z}) = IT(\Omega_1, \bar{z}) \cap IT(\Omega_2, \bar{z})$  for every  $\bar{z} \in \Omega_1 \cap \Omega_2$ . In general, this important property is not shared by the contingent cones. However, for arbitrary sets  $\Omega_1, \Omega_2 \subset \mathfrak{E}$ , we have  $T(\Omega_1 \cap \Omega_2, \bar{z}) \subset T(\Omega_1, \bar{z}) \cap T(\Omega_2, \bar{z})$  for every  $\bar{z} \in \Omega_1 \cap \Omega_2$ . For some  $\Omega \subset \mathfrak{E}$ , the identities  $T(\Omega, \bar{z}) = T(\text{cl}(\Omega), \bar{z})$  and  $IT(\Omega, \bar{z}) = IT(\text{int}(\Omega), \bar{z})$  hold. Moreover, for a convex solid set  $\Omega$ , we have  $\text{cl}(IT(\Omega, \bar{z})) = T(\Omega, \bar{z})$  and  $\text{int}(T(\Omega, \bar{z})) = IT(\Omega, \bar{z})$  (see [2, 7] for details).

Next we collect some notions for set-valued maps. Given the normed spaces  $X$  and  $Y$ , let  $F : X \rightrightarrows Y$  be a set-valued map. The (effective) domain and the graph of  $F$  are defined by

$$\begin{aligned} \text{dom}(F) &:= \{x \in X \mid F(x) \neq \emptyset\} \\ \text{graph}(F) &:= \{(x, y) \in X \times Y \mid y \in F(x)\}. \end{aligned}$$

We shall say that  $F$  is *strict* if  $\text{dom}(F) = X$ . Given a convex cone  $C \subset Y$ , which induces a partial ordering in  $Y$ , the profile map  $F_+ : X \rightrightarrows Y$  is given by:  $F_+(x) := F(x) + C$  for every  $x \in \text{dom}(F)$ . Now the epigraph of  $F$  can be defined as the graph of  $F_+$ , that is,  $\text{epi}(F) = \text{graph}(F_+)$ . The map  $F$  is called convex, if  $\text{graph}(F)$  is a convex set and  $C$ -convex, if  $\text{epi}(F)$  is a convex set. Finally, we define the *weak-inverse image*  $F[\Theta]^-$  of  $F$  with respect to a set  $\Theta \in Y$  as

$$F[\Theta]^- := \{x \in X \mid F(x) \cap \Theta \neq \emptyset\}.$$

Now, let  $X^*$  be the dual of  $X$  and let  $L \subset X$  be arbitrary. The negative dual of  $L$ , denoted by  $L^\diamond$ , is a subset of  $X^*$  defined by:

$$L^\diamond = \{\ell \in X^* : \ell(x) \leq 0 \text{ for every } x \in L\}.$$

It is known that if  $L_1 \subseteq L_2$  then  $L_2^\diamond \subseteq L_1^\diamond$ . Additionally,  $L \subset (L^\diamond)^\diamond$  with equality if and only if  $L$  is a closed, and convex cone. The positive dual then is the set defined by  $L^* = -L^\diamond$ . Both the positive and the negative duals are closed, and convex cones. Moreover, the properties just mentioned for the negative dual hold for the positive dual as well.

Given a set-valued map  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in \text{graph}(F)$ , the contingent derivative of  $F$  at  $(\bar{x}, \bar{y})$  is the set-valued map  $D_c F(\bar{x}, \bar{y}) : X \rightrightarrows Y$  defined by:

$$D_c F(\bar{x}, \bar{y})(x) := \{y \in Y \mid (x, y) \in T(\text{graph}(F), (\bar{x}, \bar{y}))\}.$$

The contingent derivative and related contingent epiderivative and generalized contingent epiderivative are commonly used derivative notions in set-valued optimization (see[4, 5, 9, 10]).

The derivative for set-valued maps that turns out to be of great importance in the present approach is the so-called lower Dini derivative introduced by Penot [17].

We recall that given a set-valued map  $F : X \rightrightarrows Y$  and a point  $(\bar{x}, \bar{y}) \in \text{graph}(F)$ , the lower Dini derivative of  $F$  at  $(\bar{x}, \bar{y})$  is the set-valued map  $D_l F(\bar{x}, \bar{y}) : X \rightrightarrows Y$  defined by:

$$D_l F(\bar{x}, \bar{y})(x) := \liminf_{(t, z) \rightarrow (0_+, x)} \frac{F(\bar{x} + tz) - \bar{y}}{t}.$$

Equivalently  $y \in D_l F(\bar{x}, \bar{y})(x)$  if and only if for every  $(\lambda_n) \subset \mathbb{P}$  and for every  $(x_n) \subset X$  with  $\lambda_n \downarrow 0$  and  $x_n \rightarrow x$  there are a sequence  $(y_n) \subset Y$  with  $y_n \rightarrow y$  and an integer  $m \in \mathbb{N}$  such that  $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n)$  for every  $n \geq m$ .

We also need to recall the following important notion of certain cones:

**Definition 2.2.** [22] *Let  $\Xi$  be a normed space, let  $\{K_i\}_{i=1}^k$  be a system of cones in  $\Xi$ , and let  $B_\varepsilon$  be a ball with center 0 and radius  $\varepsilon > 0$  in the space  $\Xi$ .*

(a) *The cones  $\{K_i\}_{i=1}^k$  are of the same sense, if for any  $\varepsilon > 0$  there exists  $\varepsilon_1, \dots, \varepsilon_k > 0$  such that for any  $x \in B_\varepsilon \cap (K_1 + \dots + K_k)$  where  $x = x_1 + \dots + x_k$  with  $x_i \in K_i$ , we have  $x_i \in B_{\varepsilon_i} \cap K_i$  for  $i = 1, \dots, k$ .*

(b) *The cones  $\{K_i\}_{i=1}^k$  are of the opposite sense, if there exists nontrivial vectors  $\{x_i\}_i^n$ , that is  $(x_1, \dots, x_k) \neq (0, \dots, 0)$ ,  $x_i \in K_i$ , such that  $x_1 + x_2 + \dots + x_k = 0$ .*

**Remark 2.2.** *It follows from the above definitions that the cones of same sense and the cones of opposite sense do not intersect. Furthermore, any subsystem of a system of the cones the same sense is of the same sense, and if a subsystem is of the opposite sense, then the whole system is also of the opposite sense. Moreover, if cones  $K_1$  and  $K_2$  are subspaces, and if  $K_1 \cap K_2$  contains a nonzero element, then these cones are of the opposite sense.*

We also need to formulate the following notion:

**Definition 2.3.** A system of sets  $\{\Omega_1, \Omega_2, \dots, \Omega_n\}$  is called *optimally positioned*, if for any  $z \in \Omega_1 \cap \Omega_2 \cap \dots \cap \Omega_n$ , we have

$$T(\Omega_1 \cap \Omega_2 \cap \dots \cap \Omega_n, z) = T(\Omega_1, z) \cap T(\Omega_2, z) \cap \dots \cap T(\Omega_n, z). \quad (1)$$

Conditions ensuring (1) are available in [15] (see also [2]).

Finally, we conclude this section by recalling the following important result.

**Theorem 2.1.** [14] Let  $\Xi$  be a normed space, let  $\{K_i\}_{i=1}^k$ , be a system of cones in  $\Xi$ . Assume that the following conditions holds:

- (a) The cones  $K_1, \dots, K_p \subset \Xi$  are open and convex with vertices at 0.
- (b) The cones  $K_{p+1}, \dots, K_k \subset \Xi$  are closed and convex with vertices at 0.
- (c) The (positive) dual cones  $K_{p+1}^*, \dots, K_k^*$  to  $K_{p+1}, \dots, K_k$  are either of the same sense or the opposite sense.

Then the following two statement are equivalent:

$$(i) \bigcap_{i=1}^k K_i = \emptyset.$$

- (ii) There exists linear, continuous functionals  $f_i \in K_i^*$  for  $i = 1, \dots, k$ , not all simultaneously zero, such that

$$f_1 + \dots + f_k = 0.$$

### 3 First-Order Generalized Dubovitskii-Milyutin Approach

We begin by introducing the following notions.

**Definition 3.1.** Let  $X$  and  $Y$  be normed spaces and let  $R : X \rightrightarrows Y$  be a set-valued map. The map  $R$  is called *locally convex* at  $(\bar{x}, \bar{y}) \in \text{graph}(R)$ , if the lower Dini derivative  $D_l R(\bar{x}, \bar{y})$  of  $R$  at  $(\bar{x}, \bar{y})$  is a convex set-valued map. The map  $R$  is called *regular* at  $(\bar{x}, \bar{y}) \in \text{graph}(R)$ , if additionally  $R$  is strict and the weak-inverse image  $D_l R(\bar{x}, \bar{y})$  with respect to an open cone is an open cone.

Given  $A \subset \mathbb{R}$  and  $b \in \mathbb{R}$ , by the inequality  $A \geq b$  we understand that  $a \geq b$  for every  $a \in A$ . With this convention in mind we are ready to give the promised multiplier rule.

**Theorem 3.1.** Let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be a local weak-minimizer of  $(P)$  and let  $\bar{z}_i \in G_i(\bar{x}) \cap (-C_i)$  where  $i \in I := \{1, 2, \dots, m\}$ . Let there exist an open convex cone  $L \subseteq IT(Q, \bar{x})$ . Assume that the sets  $\{H_j[-D_j]^- \}_{j=1}^n$  are optimally positioned. Assume that  $M_j := T(H_j[-D_j]^-, \bar{x})$  for  $j \in J$  are convex and  $\{M_j^*\}_{j=1}^n$  are either of the same sense or the opposite sense. Let  $F$  be regular

at  $(\bar{x}, \bar{y})$  and let  $G_i$  be regular at  $(\bar{x}, \bar{z}_i)$  for  $i \in I$ . Then there exist functionals  $s \in L^*$ ,  $t \in C^*$ ,  $u_i \in (T(C_i, -\bar{z}_i))^*$ ,  $v_j \in M_j^*$ , not all zero, such that  $u_i(\bar{z}_i) = 0$ . Moreover, the following inequality holds for every  $x \in X$ :

$$t \circ D_l F(\bar{x}, \bar{y})(x) + u_1 \circ D_l G_1(\bar{x}, \bar{z}_1)(x) + \cdots + u_m \circ D_l G_m(\bar{x}, \bar{z}_m)(x) \geq s(x) + v_1(x) + \cdots + v_n(x). \quad (2)$$

We shall divide the proof in several lemmas. Although some of them are based on our previous works (see [8, 11]), for the sake of completeness we give shorter proofs here. We begin with the following:

**Lemma 3.1.** *Let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be a local weak-minimizer to (P). Then:*

$$U \cap Q \cap F[\bar{y} - \text{int}(K)]^- \bigcap_{i=1}^m G_i[-C_i]^- \bigcap_{j=1}^n H_j[-D_j]^- = \emptyset, \quad (3)$$

where  $U$  is a neighborhood of  $\bar{x}$  corresponding to the definition of the local weak-minimality.

*Proof.* Assume that there exists

$$x \in U \cap Q \cap F[\bar{y} - \text{int}(K)]^- \bigcap_{i=1}^m G_i[-C_i]^- \bigcap_{j=1}^n H_j[-D_j]^-.$$

From the containment

$$x \in Q \bigcap_{i=1}^m G_i[-C_i]^- \bigcap_{i=1}^n H_i[-D_i]^-,$$

we notice that  $x \in Q$  and for all  $i \in I$  and all  $j \in J$ , we have

$$\begin{aligned} G_i(x) \cap -C_i &\neq \emptyset, \\ H_j(x) \cap -D_j &\neq \emptyset, \end{aligned}$$

which ensures that  $x$  is a feasible point. Furthermore, from the containment

$$x \in U \cap F[\bar{y} - \text{int}(K)]^-,$$

we obtain that there exists  $x \in U$ , a neighborhood of  $\bar{x}$ , such that

$$F(x) \cap (\bar{y} - \text{int}(K)) \neq \emptyset.$$

This, however, in view of the feasibility of  $x$ , contradicts the local weak-minimality of  $(\bar{x}, \bar{y})$ .  $\square$

We can employ the above notion to give the following:

**Lemma 3.2.** *Let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be a local minimizer to (P). Assume that the sets  $H_j[-D_j]^-$  for  $j \in J$ , are optimally positioned. Then:*

$$IT(Q, \bar{x}) \cap IT(F[\bar{y} - \text{int}(K)]^-, \bar{x}) \bigcap_{i=1}^m IT(G_i[-C_i]^- , \bar{x}) \bigcap_{j=1}^n T(H_j[-D_j]^- , \bar{x}) = \emptyset. \quad (4)$$

*Proof.* In view of Lemma 3.1 and the properties of the interiorly contingent cones and the contingent cones stated in Remark 2.1, we obtain

$$IT(Q, \bar{x}) \cap IT(F[\bar{y} - \text{int}(K)]^-, \bar{x}) \bigcap_{i=1}^m IT(G_i[-C_i]^-, \bar{x}) \cap T\left(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}\right) = \emptyset.$$

The assertion then follows by the assumption that the sets  $H_j[-D_j]^-$  for  $j \in J$  are properly positioned and hence

$$T\left(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}\right) = \bigcap_{j=1}^n T(H_j[-D_j]^-, \bar{x}).$$

□

**Lemma 3.3.** [8, 11] *Let  $X$  and  $Y$  be normed spaces, let  $F : X \rightrightarrows Y$  be a set valued map and let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$ . Let  $K$  be an open and convex cone. Then:*

$$D_l F(\bar{x}, \bar{y})[-\text{int}(K)]^- \subseteq IT(F[\bar{y} - \text{int}(K)]^-, \bar{x}).$$

*Proof.* Let  $x \in D_l F(\bar{x}, \bar{y})[-\text{int}(K)]^-$  be arbitrary. Therefore, there exists  $y \in D_l F(\bar{x}, \bar{y})(x) \cap -\text{int}(K)$ . Let  $(x_n) \subset X$  and  $(\lambda_n) \subset \mathbb{P}$  be arbitrary sequences such that  $x_n \rightarrow x$  and  $\lambda_n \downarrow 0$ . It suffices to show that there exists  $m \in \mathbb{N}$  such that  $\bar{x} + \lambda_n x_n \in F[\bar{y} - \text{int}(K)]^-$  for every  $n \geq m$ . By the definition of  $D_l F(\bar{x}, \bar{y})(\cdot)$ , there exist  $(y_n) \subset Y$  with  $y_n \rightarrow y$  and  $n_1 \in \mathbb{N}$  such that  $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n)$  for every  $n \geq n_1$ . Since  $y \in -\text{int}(K)$  and  $y_n \rightarrow y$ , there exists  $n_2 \in \mathbb{N}$  such that  $\lambda_n y_n \in -\text{int}(K)$  for every  $n \geq n_2$ . This implies that  $\bar{y} + \lambda_n y_n \in F(\bar{x} + \lambda_n x_n) \cap (\bar{y} - \text{int}(K))$  for  $n \geq m := \max\{n_1, n_2\}$ . Hence for the sequences  $(x_n)$  and  $(\lambda_n)$  we have  $\bar{x} + \lambda_n x_n \in F[\bar{y} - \text{int}(K)]^-$  for  $n \geq m$ . This is equivalent to saying that  $x \in IT(F[\bar{y} - \text{int}(K)]^-, \bar{x})$ . The proof is complete. □

**Lemma 3.4.** [8, 11] *Let  $X$  and  $Z$  be normed spaces, let  $G : X \rightrightarrows Z$  be a set valued map and let  $(\bar{x}, \bar{z}) \in \text{graph}(G)$ . Let  $A \subset Z$  with  $\text{int}(A) \neq \emptyset$ . Then the following holds:*

$$D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^- \subseteq IT(G[-A]^- , \bar{x}).$$

*Proof.* Let  $u \in D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^-$  be arbitrary. Let  $(u_n) \subset X$  and  $(\lambda_n) \subset \mathbb{P}$  be arbitrary sequences such that  $u_n \rightarrow u$  and  $\lambda_n \downarrow 0$ . It suffices to show that there exists  $m \in \mathbb{N}$  such that  $\bar{x} + \lambda_n u_n \in G[-A]^-$  for every  $n \geq m$ . Since  $u \in D_l G(\bar{x}, \bar{z})[IT(-A, \bar{z})]^-$ , there exists  $v \in D_l G(\bar{x}, \bar{z})(u) \cap IT(-A, \bar{z})$ . Therefore, there are a sequence  $(v_n) \subset Z$  and an integer  $n_1 \in \mathbb{N}$  such that  $v_n \rightarrow v$  and  $\bar{z} + \lambda_n v_n \in G(\bar{x} + \lambda_n u_n)$  for every  $n \geq n_1$ . Because of the containment  $v \in IT(-A, \bar{z})$  there exists  $n_2 \in \mathbb{N}$  such that  $\bar{z} + \lambda_n v_n \in -A$  for every  $n \geq n_2$ . Therefore we have  $\bar{z} + \lambda_n v_n \in G(\bar{x} + \lambda_n u_n) \cap (-A)$  for every  $n \geq m := \max\{n_1, n_2\}$ . Consequently  $u \in IT(G[-A]^- , \bar{x})$ . □

**Lemma 3.5.** [8, 11] *Let  $X$  and  $Y$  be normed spaces, let  $D \subseteq X$  be convex and let  $A \subset Y$  be a solid closed convex cone. Let  $T : D \rightrightarrows Y$  be a  $A$ -convex set-valued map. If  $T[-\text{int}(A)]^- \neq \emptyset$ , then for every  $p \in P^\diamond$  where  $P := T[-A]^-$ , there exists  $t \in A^*$  such that*

$$t \circ T(x) \geq p(x) \quad \text{for every } x \in D.$$

If  $T[-\text{int}(A)]^- = \emptyset$ , then there exists  $t \in A^* \setminus \{0_{Y^*}\}$  such that

$$t \circ T(x) \geq 0 \quad \text{for every } x \in D.$$

*Proof.* We begin with the case when the set  $T[-\text{int}(A)]^-$  is nonempty. Then the (negative) dual  $P^\circ$  of  $P := T[-A]^-$  is also nonempty. We choose  $p \in P^\circ$  arbitrarily and define a set  $E := \{(y, p(x)) \in Y \times \mathbb{R} \mid y \in T(x) + A, x \in D\}$ . In view of the assumptions that  $D$  is convex,  $T$  is  $A$ -convex and  $p \in Y^*$ , we deduce that  $E$  is a convex set. Indeed, let  $(y_1, z_1), (y_2, z_2) \in E$  be arbitrary. Then by the definition of  $E$ , for  $i = 1, 2$ , there exists  $x_i \in X$  with  $z_i = p(x_i)$  and  $y_i \in T(x_i) + A$ . For  $\lambda \in (0, 1]$ , we have  $\lambda z_1 + (1 - \lambda)z_2 = p(\lambda x_1 + (1 - \lambda)x_2)$ . Further, in view of the  $A$ -convexity of  $T$ , we have  $\lambda y_1 + (1 - \lambda)y_2 \in \lambda T(x_1) + (1 - \lambda)T(x_2) + A \subseteq T(\lambda x_1 + (1 - \lambda)x_2) + A$ . This, in view of the convexity of the set  $D$ , implies that  $\lambda(y_1, z_1) + (1 - \lambda)(y_2, z_2) \in E$ .

Next, we claim that  $E \cap (-\text{int}(A) \times \mathbb{P}) = \emptyset$ . In fact, if this is not the case, then there exists  $(x, y) \in X \times Y$  such that  $y \in (T(x) + A) \cap (-\text{int}(A))$  and  $p(x) > 0$ . Let  $w \in T(x)$  be such that  $y \in w + A$ . Then  $w \in y - A \subset -\text{int}(A) - A = -\text{int}(A)$ . This however contradicts that  $p \in P^\circ$ . Therefore  $E \cap (-\text{int}(A) \times \mathbb{P}) = \emptyset$  and hence by a separation theorem, we get the existence of  $(f, g) \in Y^* \times \mathbb{R} \setminus \{0_{Y^*}, 0\}$  and a real number  $\alpha$  such that we have

$$f(u) + g(v) \geq \alpha \quad \text{for every } (u, v) \in E \tag{5}$$

$$f(c) + g(d) < \alpha \quad \text{for every } (c, d) \in -\text{int}(A) \times \mathbb{P}. \tag{6}$$

Since  $A$  is a cone, we can set  $\alpha = 0$  in (5) and (6). By taking  $d \in \mathbb{P}$  arbitrary close to 0 and  $c \in -\text{int}(A)$  arbitrary close to  $0_Y$ , we obtain  $f \in A^*$  and  $g \leq 0$ , respectively. We claim that  $g < 0$ . Indeed, if  $g = 0$ , we get  $f(c) < 0$  for every  $c \in -\text{int}(A)$  and  $f(u) \geq 0$  for every  $u \in T(D) + A$ . This, however is impossible because we have  $(T(D) + A) \cap (-\text{int}(A)) \neq \emptyset$ . Therefore  $g < 0$ . Moreover, from (5), for every  $x \in D$  we have  $f \circ (T + A)(x) \geq -(g \cdot p)(x)$ . By setting  $t = (-f/g) \in A^*$  and noticing that  $0_Y \in A$ , we finish the proof of the first part.

For the second part, we notice that if  $T(-\text{int}(A)) = \emptyset$ , we have  $T(D) \cap -\text{int}(A) = \emptyset$  and hence by the arguments similar to those given above we can prove the existence of  $t \in A^* \setminus \{0_Y\}$  such that  $t \circ T(x) \geq 0$  for every  $x \in D$ .  $\square$

**Proof of Theorem 3.1.** Notice that the equation

$$IT(Q, \bar{x}) \cap IT(F[\bar{y} - \text{int}(K)]^-, \bar{x}) \bigcap_{i=1}^m IT(G_i[-C_i]^- , \bar{x}) \bigcap_{j=1}^n T(H_j[-D_j]^- , \bar{x}) = \emptyset,$$

in view of the inclusions

$$\begin{aligned} D_l F(\bar{x}, \bar{y})[-\text{int}(K)]^- &\subseteq IT(F[\bar{y} - \text{int}(K)]^-, \bar{x}) \\ D_l G_i(\bar{x}, \bar{z})[IT(-C_i, \bar{z}_i)]^- &\subseteq IT(G[-C_i]^- , \bar{x}), \end{aligned}$$

that follow from Lemma 3.3 and Lemma 3.4 (by choosing  $A = C_i$  and  $G = G_i$ ) ensure that

$$IT(Q, \bar{x}) \bigcap D_l F(\bar{x}, \bar{y}) [-\text{int}(K)]^- \bigcap_{i=1}^m D_l G_i(\bar{x}, \bar{z}_i) [IT(-C_i, \bar{z}_i)]^- \bigcap_{j=1}^n T(H_j[-D_j]^-, \bar{x}) = \emptyset.$$

We define

$$\begin{aligned} \Phi &:= D_l F(\bar{x}, \bar{y}) [-\text{int}(K)]^- \\ \Psi_i &:= D_l G_i(\bar{x}, \bar{z}_i) [IT(-C_i, \bar{z}_i)]^-. \end{aligned}$$

We shall prove the theorem by analyzing the three possibilities, namely:

- (i)  $\Phi = \emptyset$ ;
- (ii)  $\Psi_i = \emptyset$  for some  $i \in I$ ;
- (iii)  $\Phi \neq \emptyset$  and  $\Psi_i \neq \emptyset$  for every  $i \in I$ .

We begin with case (i). Since  $\Phi = \emptyset$ , we can apply Lemma 3.5 for

$$T := D_l F(\bar{x}, \bar{y}) : X \rightrightarrows Y, \quad D := X, \quad A := K$$

such that there exists  $t \in K^* \setminus \{0_{Y^*}\}$  with

$$t \circ D_l F(\bar{x}, \bar{y})(x) \geq 0, \quad \text{for every } x \in X.$$

By choosing  $s = 0_{X^*}$ ,  $u_i = 0_{Z_i^*}$ , for every  $i \in I$ , and  $v_j = 0_{W_j^*}$ , for every  $j \in J$ , we obtain the desired result.

For case (ii), let there exist  $i \in I$  such that  $\Psi_i = \emptyset$ . Then again by invoking Lemma 3.5 with

$$T := D_l G_i(\bar{x}, \bar{z}_i) : X \rightrightarrows Z_i \quad (i \in I), \quad D := X, \quad A := T(C_i, -\bar{z}_i),$$

we obtain  $u_i \in (T(C_i, -\bar{z}_i))^* \setminus \{0_{Z_i^*}\}$  such that

$$(u_i \circ D_l G_i(\bar{x}, \bar{z}_i))(x) \geq 0, \quad \text{for every } x \in X.$$

By setting  $s = 0_{X^*}$ ,  $v_j = 0_{W_j^*}$  for every  $j \in J$ , and  $u_j = 0_{Z_j^*}$ ,  $i \neq j \in I$ , we obtain (2). For  $u_i(C_i + \bar{z}_i) \geq 0$ , it suffices to notice that in view of the convexity of  $C_i$ , we have  $T(C_i, -\bar{z}_i) \supseteq C_i + \bar{z}_i$  and hence  $u_i(z + \bar{z}_i) \geq 0$  for every  $z \in C_i$ .

Finally, we consider the case (iii). Since  $(\bar{x}, \bar{y})$  is a local-minimizer of  $(P)$ , it follows from Lemma 3.2 and the imposed conditions that we have

$$L \bigcap_{i=1}^m \Phi \bigcap_{j=1}^n \Psi_j \bigcap M_j = \emptyset.$$

Since  $L, \Phi, \Psi_i, (i \in I)$  and  $M_j, (j \in J)$  are all nonempty, we can apply Theorem 2.1 to assure the existence of

$$\begin{aligned} \ell &\in L^* \\ \ell_0 &\in (D_l F(\bar{x}, \bar{y})[-\text{int}(K)]^-)^* \\ \ell_i &\in (D_l G_i(\bar{x}, \bar{z}_i)[IT(-C_i, \bar{z}_i)]^-)^*, \quad i \in I \\ \ell_{m+j} &\in M_j^*, \quad j \in J \end{aligned}$$

such that

$$-\ell - \ell_0 - \ell_1 - \ell_2 - \cdots - \ell_m - \ell_{m+1} - \cdots - \ell_{m+n} = 0. \quad (7)$$

Now, in the case  $\Phi \neq \emptyset$  in view of Lemma 3.5 for

$$T := D_l F(\bar{x}, \bar{y}) : X \rightrightarrows Y, \quad D := X, \quad A := K$$

we get the existence of functionals  $t \in C^*$  such that for all  $x \in X$  and  $-\ell_0 \in (D_l F(\bar{x}, \bar{y})[-Q])^- \diamond$  the following inequality holds

$$(t \circ D_l F(\bar{x}, \bar{y}))(x) \geq -\ell_0(x). \quad (8)$$

Analogously, in the case  $\Psi \neq \emptyset$  with Lemma 3.5 for

$$T := D_l G_i(\bar{x}, \bar{y}) : X \rightrightarrows Z_i \quad (i \in I), \quad D := X, \quad A := T(C_i, -\bar{z}_i)$$

we get the existence of functionals  $u_i \in (T(C_i, -\bar{z}_i))^*$  such that for all  $x \in X$  and  $-\ell_i \in (D_l G_i(\bar{x}, \bar{z}_i)[T(-C_i, \bar{z}_i)]^-) \diamond$  the inequality

$$(u_i \circ D_l G_i(\bar{x}, \bar{z}_i))(x) \geq -\ell_i(x), \quad i \in I \quad (9)$$

holds.

Combining of the above inequalities (8) and (9) with (7) and setting  $s = \ell$  and  $v_j = \ell_{m+j}$  with  $j \in J$  yield (2). This completes the proof.  $\square$

**Remark 3.1.** It is of interest to obtain optimality conditions that involve derivatives of the set-valued map  $H_j$  with  $j \in J$ . In this regard, when  $H_j$  are suitable single-valued maps, then some variants of the well-known Lyusternik theorem can be used.

## 4 Second-Order Generalized Dubovitskii-Milyutin Approach

We now discuss the second-order analogues of the results obtained in the previous section. We begin with by recalling some second-order tangent cones.

**Definition 4.1.** Let  $\Xi$  be a real normed space, let  $\Omega \subset \Xi$  be nonempty and let  $w \in \Xi$ .

1. The second order asymptotic tangent cone  $\widetilde{T}^2(\Omega, \bar{z}, w)$  of  $\Omega$  at  $\bar{z} \in \text{cl}(\Omega)$  in the direction  $w \in \Omega$  is the set of all  $z \in \Omega$  such that there are a sequence  $(z_n) \subset \Xi$  with  $z_n \rightarrow z$  and a sequence  $(s_n, t_n) \subset P \times P$  with  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$  so that  $\bar{z} + s_n w + s_n t_n z_n \in \Omega$ .
2. The second order asymptotic adjacent cone  $\widetilde{K}^2(\Omega, \bar{z}, w)$  of  $\Omega$  at  $\bar{z} \in \text{cl}(\Omega)$  in the direction  $w \in \Omega$  is the set of all  $z \in \Omega$  such that for every sequence  $(s_n, t_n) \subset P \times P$  with  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$  there exists a sequence  $(z_n) \subset \Xi$  with  $z_n \rightarrow z$  and  $\bar{z} + s_n w + s_n t_n z_n \in \Omega$ .
3. The interior second order adjacent cone  $\widetilde{IT}^2(\Omega, \bar{z}, w)$  of  $\Omega$  at  $\bar{z} \in \text{cl}(\Omega)$  in the direction  $w \in \Xi$  is the set of all  $z \in \Xi$  such that for every sequence  $(z_n) \subset \Xi$  with  $z_n \rightarrow z$  and for every sequence  $(s_n, t_n) \subset P \times P$  with  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$  we have  $\bar{z} + s_n w + s_n t_n z_n \in \Omega$ , for sufficiently large  $n$ .

**Remark 4.1.** We notice that  $\widetilde{T}^2(S, \bar{z}, w)$  and  $\widetilde{K}^2(S, \bar{z}, w)$  are closed cones, whereas  $\widetilde{IT}^2(S, \bar{z}, w)$  is an open cone. We refer the reader to an interesting and timely survey by Giorgi, Jimenez, and Novo [7] that contains significant details of the asymptotic cones mentioned above ( see also [19]). We also remark that second-order contingent sets and second-order adjacent sets (see [7]) are more commonly used objects in set-valued and nonsmooth optimization. However, their asymptotic analogues, being cones, are more suitable for our approach

In the following definition, we recall the notion of the second-order asymptotic derivative, and being inspired by the lower Dini derivative and its usefulness in the preset approach, we also introduce the second-order lower Dini asymptotic derivative.

**Definition 4.2.** Let  $F : X \rightrightarrows Y$  be set-valued, let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$ , and let  $(\bar{u}, \bar{v}) \in X \times Y$ .

- (i) The second order asymptotic derivative of  $F$  at  $(\bar{x}, \bar{y})$  in the direction  $(\bar{u}, \bar{v})$  is the set-valued map  $D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$  defined by

$$D^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})(x) := \left\{ y \in Y \mid (x, y) \in \widetilde{T}^2(\text{graph}(F), (\bar{x}, \bar{y}), (\bar{u}, \bar{v})) \right\}.$$

- (ii) The second-order lower Dini asymptotic derivative of  $F$  at  $(\bar{x}, \bar{y})$  in the direction  $(\bar{u}, \bar{v})$  is the set-valued map  $D_l^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) : X \rightrightarrows Y$  such that  $(x, y) \in \text{graph}(D_l^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v}))$  if and only if for every  $(x_n) \subset X$  with  $x_n \rightarrow x$  and for every sequence  $(s_n, t_n) \in \mathbb{P} \times \mathbb{P}$  with  $(s_n, t_n) \rightarrow 0$ , and  $s_n/t_n \rightarrow 0$ , there exists a sequence  $(y_n) \subset Y$  with  $y_n \rightarrow y$  and integer  $m \in \mathbb{N}$  such that  $\bar{y} + s_n \bar{v} + s_n t_n y_n \in F(\bar{x} + s_n \bar{u} + s_n t_n x_n)$  for every  $n \geq m$ .

We begin with the following necessary optimality condition for Problem (P):

**Theorem 4.1.** Let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be a local weak minimizer of (P). Let  $(\bar{u}, \bar{v}) \in \text{graph}(DF(\bar{x}, \bar{y}))$ . Then

$$\widetilde{IT}^2(Q, \bar{x}, \bar{u}) \cap D_l^2F(\bar{x}, \bar{y}, \bar{u}, \bar{v})[IT(-K, \bar{v})]^- \cap \bigcap_{i=1}^m \widetilde{IT}^2(G_i[-C_i]^-, \bar{x}, \bar{u}) \cap \widetilde{T}^2\left(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}, \bar{u}\right) = \emptyset. \quad (10)$$

*Proof.* Since  $(\bar{x}, \bar{y})$  is a local weak-minimizer of  $(P)$ , there exists a neighborhood  $U$  of  $\bar{x}$  such that

$$F(S \cap U) \cap (\{\bar{y}\} - \text{int}(K)) = \emptyset. \quad (11)$$

We will show that if (10) fails then the above criteria for the weak-minimality will be violated.

For the sake of argument, we assume that there exists an  $x \in X$  such that

$$x \in \widetilde{IT}^2(Q, \bar{x}, \bar{u}) \cap D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) [IT(-K, \bar{v})]^- \bigcap_{i=1}^m \widetilde{IT}^2(G_i[-C_i]^-, \bar{x}, \bar{u}) \cap \widetilde{T}^2(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}, \bar{u}).$$

In view of the containment  $x \in \widetilde{T}^2(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}, \bar{u})$ , and the definition of the second order asymptotic tangent cone, we ensure that there are a sequence  $(x_n) \subset X$  with  $x_n \rightarrow x$  and a sequence  $(s_n, t_n) \subset \mathbb{P} \times \mathbb{P}$  with  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$  so that for every  $n \in \mathbb{N}$ , we have

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in \bigcap_{j=1}^n H_j[-D_j]^-,$$

implying that

$$H_j(\bar{x} + s_n \bar{u} + s_n t_n x_n) \cap -D_j \neq \emptyset, \quad \forall j \in J.$$

Furthermore, due to the containment  $x \in \widetilde{IT}^2(G_i[-C_i]^-, \bar{x}, \bar{u})$ , and the facts that  $x_n \rightarrow x$ ,  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$ , we ensure the existence of  $n_1 \in \mathbb{N}$  such that

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in G_i[-C_i]^-, \quad \text{for every } n \geq n_1, \text{ for every } i \in I,$$

or equivalently

$$G_i(\bar{x} + s_n \bar{u} + s_n t_n x_n) \cap -C_i \neq \emptyset, \quad \text{for every } n \geq n_1, \text{ for every } i \in I. \quad (12)$$

Moreover, since  $x \in D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) [IT(-K, \bar{v})]^-$ , there exists  $y \in IT(-K, \bar{v})$  such that

$$(x, y) \in \text{graph}(D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})).$$

This, in view of the definition of the derivative  $D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v})$  and the sequences  $(s_n, t_n)$ , ensure that there are a sequence  $(y_n) \subset Y$  and an integer  $n_2 \in \mathbb{N}$  such that

$$\bar{y} + s_n \bar{v} + s_n t_n y_n \in F(\bar{x} + s_n \bar{u} + s_n t_n x_n).$$

Since  $y \in IT(-\text{int}(K), \bar{v})$ ,  $t_n \downarrow 0$  and  $y_n \rightarrow y$ , it follows from the definition of the interior tangent cones that there exists  $n_3 \in \mathbb{N}$  such that for  $n \geq n_3$ , we have

$$\bar{v} + t_n y_n \in -\text{int}(K).$$

Using the fact that  $s_n > 0$ , we obtain that

$$\bar{y} + s_n \bar{v} + s_n t_n y_n \in \bar{y} - \text{int}(K). \quad (13)$$

Finally, from the containment  $x \in \widetilde{IT}^2(Q, \bar{x}, \bar{u})$  and the definition of sequences  $(x_n)$  and  $(s_n, t_n)$ , we infer that there exists an integer  $n_4 \in \mathbb{N}$  such that for every  $n > n_4$ , we have

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in U \cap Q.$$

Therefore, we have shown that for every  $n \geq \max\{n_1, n_2, n_3, n_4\}$  there are

$$u_n := \bar{x} + s_n \bar{u} + s_n t_n x_n \in Q \cap U,$$

satisfying the constraints such that

$$F(u_n) \cap (\bar{y} - \text{int}(K)) \neq \emptyset.$$

This however contradicts the weak optimality of  $(\bar{x}, \bar{y})$ . The proof is complete.  $\square$

To obtain a variant of the above optimality condition that involves the derivative of the maps  $G_i$  for  $i \in I$ , as well, we need the following:

**Theorem 4.2.** *Let  $X$  and  $Y$  be normed spaces, let  $G : X \rightarrow Z$  be a set-valued map, let  $(\bar{x}, \bar{z}) \in \text{graph}(G)$  and let  $(\bar{u}, \bar{w}) \in \text{graph}(D_I G(\bar{x}, \bar{z}))$ . Let  $A \subset Z$  with  $\text{int}(A) \neq \emptyset$ . Then the following inclusion holds:*

$$D_I^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^- \subset \widetilde{IT}^2(G[-A]^- , \bar{x}, \bar{u}) \quad (14)$$

*Proof.* Let  $x \in D_I^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^-$  be arbitrary. Choose arbitrary sequences  $(x_n) \subset X$  with  $x_n \rightarrow x$ , and  $(s_n, t_n) \subset P \times P$  with  $(s_n, t_n) \downarrow (0, 0)$  and  $s_n/t_n \rightarrow 0$ . It suffices to show that there exists  $m \in \mathbb{N}$  such that

$$\bar{x} + s_n \bar{u} + s_n t_n x_n \in G[-A]^-$$

for every  $n \geq m$ . Since  $x \in D_I^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})[\widetilde{IT}^2(-A, \bar{z}, \bar{w})]^-$ , there exists

$$z \in D_I^2 G(\bar{x}, \bar{z}, \bar{u}, \bar{w})(x) \cap \widetilde{IT}^2(-A, \bar{z}, \bar{w}).$$

Therefore, there are a sequence  $(z_n) \subset Z$  and an integer  $n_1 \in \mathbb{N}$  such that  $z_n \rightarrow z$  and

$$\bar{z} + s_n \bar{w} + s_n t_n z_n \in G(\bar{x} + s_n \bar{u} + s_n t_n x_n)$$

for every  $n \geq n_1$ . Because of the containment  $z \in \widetilde{IT}^2(-A, \bar{z}, \bar{w})$  there exists  $n_2 \in \mathbb{N}$  such that  $\bar{z} + s_n \bar{w} + s_n t_n z_n \in -A$  for every  $n \geq n_2$ . Therefore we have

$$\bar{z} + s_n \bar{w} + s_n t_n z_n \in G(\bar{x} + s_n \bar{u} + s_n t_n x_n) \cap (-A)$$

for every  $n \geq m := \max\{n_1, n_2\}$ . Consequently  $x \in \widetilde{IT}^2(G[-A]^- , \bar{x}, \bar{u})$ .  $\square$

Combining the above two results, we obtain the following optimality condition:

**Theorem 4.3.** Let  $(\bar{x}, \bar{y}) \in \text{graph}(F)$  be a local weak minimizer of  $(P)$ . Let  $(\bar{u}, \bar{v}) \in \text{graph}(DF(\bar{x}, \bar{y}))$ . Let  $(\bar{x}, \bar{z}_i) \in \text{graph}(G_i)$ , let  $\bar{u} \in \bigcap_{i=1}^m \text{dom}(D_l G_i(\bar{x}, \bar{z}_i))$ , and let  $\bar{w}_i \in D_l G_i(\bar{x}, \bar{z}_i)(\bar{u})$ . Then

$$\widetilde{IT}^2(Q, \bar{x}, \bar{u}) \cap D_l^2 F(\bar{x}, \bar{y}, \bar{u}, \bar{v}) [IT(-K, \bar{u})]^- \bigcap_{i=1}^m D_l^2 G_i(\bar{x}, \bar{z}_i, \bar{u}, \bar{w}_i) [\widetilde{IT}^2(-C_i, \bar{z}, \bar{w}_i)]^- \cap \widetilde{T}^2\left(\bigcap_{j=1}^n H_j[-D_j]^-, \bar{x}, \bar{u}\right) = \emptyset. \quad (15)$$

**Remark 4.2.** The above theorem can be combined with the techniques of the previous section to obtain second-order multiplier rules. A high-order generalization of the Lyusternik theorem can be used to include the derivatives of  $H_j$  for some specific case (see [16]).

## References

- [1] N.L.H. Anh, P.Q. Khanh, L.T. Tung, *Variational sets: Calculus and application to nonsmooth vector optimization*, *Nonlinear Analysis*, 74 (2011) 2358–2379.
- [2] J. P. Aubin and H. Frankowska, *Set valued analysis*, Birkhäuser, Boston (1990).
- [3] T.Q. Bao and B.S. Mordukhovich, *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, *Math. Program.*, 122 (2010), 301347.
- [4] G. Q. Chen and J. Jahn, *Optimality conditions for set-valued optimization problems*, *Math. Meth. Oper. Res.*, 48, 187-200 (1998).
- [5] H.W. Corley, *Optimality conditions for maximization of set-valued functions*, *J. Optim. Theory Appl.*, 58, 1-10, (1988).
- [6] A.Y. Dubovitskii and A.A. Milyutin, *Extremal problems in presence of constraints*, *Comp. Math. Math. Phys.*, 5, 395-453 (1965).
- [7] G. Giorgi, B. Jimenez, and V. Novo, *An overview of second order tangent sets and their application to vector optimization*, *Bol. Soc. Esp. Mat. Apl.*, 52 (2010), 7396.
- [8] G. Isac and A. A. Khan, *Dubovitski-Milutin approach in set-valued optimization*, *SIAM Journal on Control and Optimization*, 47 (2008), 144-162.
- [9] J. Jahn and A.A. Khan, *Generalized contingent epiderivatives in set-valued optimization*, *Numer. Func. Anal. Optim.*, 27, 807-831 (2002).
- [10] J. Jahn and R. Rauh, *Contingent epiderivatives and set-valued optimization*, *Math. Meth. Oper. Res.*, 46, 193-211 (1997).
- [11] A.A. Khan and F. Raciti, *A multiplier rule in set-valued optimization*, *Bull. Austral. Math. Soc.*, 68 (2003), 93-100.

- [12] P.Q. Khanh and N.D. Tuan, *Variational sets of multi-valued mappings and a unified study of optimality condition*, J. Optim. Theory Appl. 139 (2008) 45–67.
- [13] P.Q. Khanh and N.D. Tuan, *Higher-order variational sets and higher-order optimality conditions for proper efficiency in set-valued nonsmooth vector optimization*, J. Optim. Theory Appl. 139 (2008) 243–261.
- [14] W. Kotarski, *Characterization of Pareto optimal points in problems with multi-equality constraints*, Optimization 20 (1989), 93–106.
- [15] V. Krivan, *On the Intersection of Contingent Cones*, J. Optim. Theory Appl., 70 (1991), 397–404.
- [16] U. Ledzewicz and H. Schattler, *A high-order generalization of the Lyusternik theorem*, Nonlinear Anal., 34 (1998), 93–815.
- [17] J.P. Penot, *Differentiability of relations and differential stability of perturbed optimization problems*, SIAM J. Control Optimization 22, 529-551 (1984).
- [18] J. P. Penot, *Variations on the theme of nonsmooth analysis: Another subdifferential*, Non-differentiable optimization, Lect. Notes Econ. Math. Syst., 255, 41-54 (1985).
- [19] J.P. Penot, *Second-order conditions for optimization problems with constraints*, SIAM J. Control Optim., 37 (1998), 303–318.
- [20] L. Rigby, *Contribution to Dubovitskiy and Milyutin’s optimization formalism*, Optimization Techniques, Lect. Notes Comp. Sci., 41, 438-453 (1976).
- [21] M. Sama, *The role of directional compactness in the existence and computation of contingent epiderivatives*, J. Math. Anal. Appl., 372 (2010), 262272.
- [22] S. Walczak, *Some properties of cones in normed spaces and their application to investigating extremal problems*, J. Optim. Theory Appl. 42 (1984), 561–582.



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