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Abstract

In this paper, new necessary conditions for Pareto minimal points to sets and Pareto minimizers for constrained multiobjective optimization problems are established without the sequentially normal compactness property and the asymptotical compactness condition imposed on closed and convex ordering cones in [5] and [10], respectively. Our approach is based on a version of the separation theorem for nonconvex sets and the subdifferentials of vector-valued and set-valued mappings. Furthermore, applications in mathematical finance and approximation theory are discussed.

1 Introduction

In the past three decades there has been a growing interest in studying necessary optimality conditions for minimal points of sets and optimal solutions/minimizers of vector-valued/set-valued optimization problems in both finite- and infinite-dimensional spaces. The most classical notion of optimality is *Pareto efficiency* corresponding to a convex ordering cone. To be precise, let Z be a linear space partially ordered by a proper, closed, convex, and pointed ($\Theta \cap (-\Theta) = \{0\}$) cone Θ. Denote this order by " $\leq_\Theta$ ". Its ordering relation is described by

$$z_1 \leq_\Theta z_2 \quad \text{if and only if} \quad z_2 - z_1 \in \Theta \quad \text{for all } z_1, z_2 \in Z. \tag{1.1}$$

Let Ξ be a subset of Z and $\bar{z} \in \Xi$. We consider the following set of *Pareto minimal points* of Ξ with respect to Θ

$$\operatorname{Min}(\Xi; \Theta) := \{\bar{z} \in \Xi \mid \Xi \cap (\bar{z} - \Theta) = \{\bar{z}\}\}. \tag{1.2}$$

Notice that the majority of publications on vector/multiobjective optimization concerns the *weak* ordering relation "$<_\Theta$" defined by replacing the cone Θ in (1.1) by its interior, and the *weak* Pareto minimal points by replacing the optimality condition in (1.2) by its weak counterpart $\Xi \cap (\bar{z} - \operatorname{int}\Theta) = \emptyset$ under the nonempty interiority requirement:

$$\operatorname{int}\Theta \neq \emptyset. \tag{1.3}$$

Denote the set of all weak Pareto points of Ξ with respect to Θ by $\operatorname{WMin}(\Xi, \Theta)$. The most important characteristic of the weak notion is the possibility to study its optimality conditions via numerous powerful scalarization techniques under the additional condition (1.3). However, it is such a serious restriction in infinite dimensional spaces since "the class of ordered topological vector spaces possessing cones with nonempty interiors is not very broad" ([26, page 183]); in particular, the natural ordering cones in the Lebesgue spaces l^p and L^p for $1 \leq p < \infty$ have empty interior.

Obviously, every Pareto minimal point is *weak Pareto* provided that the nonempty interiority condition (1.3) holds; otherwise, the weak concept is no longer defined. In the latter case, the *proper optimality/efficiency* takes the center and plays the role of the weak optimality since it is defined via the Pareto optimality with respect to a bigger convex cone with a nonempty interior. To be specific, we say that $\bar{z} \in \Xi$ is a *proper minimal* point of Ξ with respect to Θ and denote it by $\bar{x} \in \mathrm{PMin}\,(\Xi;\Theta)$ if there exists a convex cone $\widetilde{\Theta}$ satisfying

$$\Theta \setminus \{0\} \subset \mathrm{int}\,\widetilde{\Theta}, \tag{1.4}$$

such that $\bar{z} \in \mathrm{Min}\,(\Xi;\widetilde{\Theta})$. We always have the following relationships

$$\mathrm{PMin}\;(\Xi;\Theta) \subset \mathrm{Min}\,(\Xi;\widetilde{\Theta}) \subset \mathrm{WMin}\,(\Xi;\widetilde{\Theta}),$$

and thus all techniques used for weak optimality work for proper optimality.

Recently, a few works have challenged the lake of the nonempty interiority condition (1.3) imposed on ordering cones. In [9] some necessary conditions were established for the so-called approximate Pareto minimizers via a suitable scalarization scheme. In [10] some Lagrange multiplier rules for Pareto optimal solutions were obtained by using the separation theorem for nonconvex sets (see Lemma 2.1 in Section 2) provided that the ordering cone enjoys the *asymptotical compactness* (AC) condition, i.e., the intersection of it and the closed unit ball is compact which clearly implies that the span of the ordering cone has finite dimensions. This requirement is strict in infinite-dimensional settings. Moreover, the necessary results therein were in fact applied to proper Pareto minimal points since $\mathrm{Min}\,(\Xi;\Theta) = \mathrm{PMin}(\Xi;\Theta)$ under the AC condition and other standard assumptions and since the first set is bigger than the second in general. Differ from this scalarization approach, the variational approach is mainly based on the extremal principle, a variational counterpart of local separation for conconvex sets. It was initiated by Kruger and Mordukhovich in [21, 23], and recently applied to multiobjective optimization in [2, 3, 5, 25]. The advantage of the later approach is the possibility of deriving necessary conditions to Pareto optimal points of a multiobjective optimization problem directly without converting it to a scalar problem in both finite- and infinite-dimensional settings. However, to implement it, the *sequential normal compactness* (SNC) condition of ordering cones is unavoidable due to the natural lack of the compactness property in infinite dimensional spaces which is automatics in finite dimensions. It is known from [24, Theorem 1.21] that a convex cone is fulfilled the SNC condition if it has a nonempty relative interior $\mathrm{ri}\,\Theta \neq \emptyset$ and $\mathrm{cl}\,(\Theta - \Theta)$ has a finite-codimensions. In summary, both approaches can not be applied to the positive ordering cones of the Lebesgue spaces l^p and L^p for $1 < p < \infty$ since they are neither AC nor the SNC at the origin.

Our main aim of this paper is to establish new *subdifferential* necessary optimality conditions for Pareto minimal points of sets and minimizers of constrained multiobjective optimization problems where the ordering cone of a infinite-dimensional Banach space does not necessarily have an empty interior (even relative interior), or a finite-dimensional span. Our approach combines the scalarization scheme in [9, 10] and several advanced tools of variational analysis used in [2, 3, 5]. Precisely, we proceed as follows: Given a Pareto minimal point $\bar{x}$ to a set Ξ with respect to an ordering cone Θ with $\mathrm{int}\,\Theta = \emptyset$. In contrast to [10] in which the existence of a bigger cone satisfying condition (1.4) is required (so that, $\bar{x}$ is proper efficient,) in this paper we will choose an appropriate nonempty interior cone that might not contain in the given cone or be contained in it. Such a cone automatically enjoys the SNC condition, but it might not have the AC property. Then utilizing the strategy in [3, 5] which deals with epigraphical multifunctions, we derive new subdifferential necessary conditions thanks to the full calculus for generalized differentiation in [25, Chapter 3].

The paper is organized as follows: In Section 2 we briefly recall some basic tools of generalized differentiation and preliminary results of variational analysis broadly used in this paper. Besides the fundamentals from the book [24] we also need subdifferential constructions for set-valued mappings with values in partially ordered spaces first introduced in [2] and being important for our main results in this paper. This section also provides the nonconvex scalarization technique of Gerth and Weidner in [29]. Section 3 contains our main results: Necessary optimality conditions for Pareto minimal points of sets, for Pareto optimal solutions of vector optimization problems, and minimizers of set-valued optimization problems without the nonempty interior requirement imposed on ordering cones. In the last Section 4 applications in mathematical finance and in approximation theory are discussed.

2 Preliminaries

Throughout the paper we use the standard notation of variational analysis; cf. the books in [24, 27], and assume that all the spaces under consideration are Asplund unless otherwise stated. Recall that a Banach space is *Asplund* if every convex continuous function $\varphi : U \to \mathbb{R}$ defined on an open convex subset U of X is Fréchet differentiable on a dense subset of U. The class of Asplund spaces is quite broad including every reflexive Banach space and every Banach space with a separable dual; in particular, c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces, but ℓ^1 and ℓ^∞ are not Asplund spaces. It has been comprehensively investigated in geometric theory of Banach spaces, and largely employed in variational analysis; see, e.g., [6, 24, 25]. In the sequel, we present the definitions and properties of the basic generalized differential constructions held in the Asplund space setting and enjoying a full calculus.

Let X be an Asplund space and $\Omega \subset X$ be a subset of X. The *Fréchet normal cone* to Ω at $x \in \Omega$ is defined by

$$\widehat{N}(x;\Omega) := \Big\{ x^* \in X^* \ \Big|\ \limsup_{u \xrightarrow{\Omega} x} \frac{\langle x^*, u - x\rangle}{\|u - x\|} \le 0 \Big\}, \tag{2.1}$$

where $u \xrightarrow{\Omega} x$ means $u \to x$ with $u \in \Omega$. Given $\bar{x} \in \Omega$. Assume that Ω is locally closed around $\bar{x} \in \Omega$, i.e., there is a neighborhood U of $\bar{x}$ such that $\Omega \cap \operatorname{cl} U$ is a closed set. The (basic, limiting, Mordukhovich) *normal cone* to Ω at $\bar{x}$ is defined by

$$\begin{aligned} N(\bar{x};\Omega) &:= \mathop{\rm Lim\,sup}_{x \to \bar{x}} \widehat{N}(x;\Omega) \\ &= \Big\{ x^* \in X^* \ \Big|\ \exists\, x_k \to \bar{x},\ x_k^* \xrightarrow{w^*} x^* \text{ with } x_k^* \in \widehat{N}(x_k;\Omega) \Big\}, \end{aligned} \tag{2.2}$$

where Lim sup stands for the sequential Painlevé-Kuratowski outer limit of Fréchet normal cones to Ω at x as x tends to $\bar{x}$. Note that, in contrast to (2.1), the basic normal cone (2.2) is often *nonconvex* enjoying nevertheless *full calculus*, and that both the cones (2.2) and (2.1) reduce to the normal cone of convex analysis when Ω is convex.

Given a set-valued mapping $F\colon X \rightrightarrows Z$ with its graph

$$\operatorname{gph} F := \big\{ (x,z) \in X \times Z \big|\ z \in F(x) \big\}$$

between Asplund spaces X and Z, the (basic, normal, Mordukhovich) *coderivative* mapping $D^*F(\bar{x},\bar{z})\colon Z^* \rightrightarrows X^*$ of F at $(\bar{x},\bar{z}) \in \operatorname{gph} F$ is defined by

$$D^*F(\bar{x},\bar{z})(z^*) := \big\{x^* \in X^* \big| \; (x^*,-z^*) \in N\big((\bar{x},\bar{z}); \operatorname{gph} F\big)\big\}, \tag{2.3}$$

which is a positively homogeneous function of $z^* \in Z^*$; we omit $\bar{z} = f(\bar{x})$ in (2.3) if $F = f\colon X \to Z$ is single-valued. If $f\colon X \to Y$ happens to be *strictly differentiable* at $\bar{x}$ (which is automatic when it is C^1 around this point), then

$$D^*f(\bar{x})(z^*) = \big\{\nabla f(\bar{x})^* z^*\big\} \;\text{ for all }\; z^* \in Z^*.$$

Let us now consider the case of a mapping $F\colon X \rightrightarrows Z$ between Asplund spaces with the range space Z *partially ordered* by a proper, closed and convex cone $\Theta \subset Z$. The (generalized) *epigraph* of F with respect to the ordering cone Θ is given by

$$\operatorname{epi} F := \big\{(x,z) \in X \times Z \;\big|\; z \in F(x) + \Theta\big\},$$

and the *epigraphical multifunction* of F denoting by $\mathcal{E}_F\colon X \rightrightarrows Z$ is defined by

$$\mathcal{E}_F(x) := \big\{z \in Z \big| \; z \in F(x) + \Theta\big\} \quad \text{with} \quad \operatorname{gph} \mathcal{E}_F = \operatorname{epi} F. \tag{2.4}$$

Let us recall the notions of subdifferentials for set-valued mappings first introduced in [2] and then further developed in [3, 4, 5]; cf. the epi-coderivatives for set-valued mappings in [33] and references therein. Given $(\bar{x},\bar{z}) \in \operatorname{epi} F$, the (limiting, basic, normal) *subdifferential* of F at $(\bar{x},\bar{z})$ in *direction* $z^* \in Z^*$ is defined by

$$\partial F(\bar{x},\bar{z})(z^*) := D^*\mathcal{E}_F(\bar{x},\bar{z})(z^*), \tag{2.5}$$

cf. [2]. As pointed in [5], we always have the following implication

$$D^*\mathcal{E}_F(\bar{x},\bar{z})(z^*) \neq \emptyset \quad \Longrightarrow \quad -z^* \in N(0;\Theta). \tag{2.6}$$

Consequently, the requirement that $-z^* \in N(0;\Theta)$ is redundant for the definitions of subdifferential for set-valued mappings in [2, 3, 4, 5]; in particular, [2, Definition 2.1]. As usual, when $F = f : X \to Z$ is a single-valued vector-valued function, we omit $\bar{z}$ from the notation of subdifferentials. The subdifferential of f at $\bar{x}$ becomes

$$\partial f(\bar{x})(z^*) = D^*\mathcal{E}_f(\bar{x})(z^*),$$

and when $F = f : X \to (-\infty,\infty]$ is a lower semicontinuous extended-real-valued function, the subdifferential (2.5) with $\|z^*\| = 1$ agrees with the (basic, limiting, Mordukhovich) nonconvex subdifferential; see [24]. By [24, Theorem 3.28], the relationship between the coderivative of a vector-valued function and the subdiffential of its scalarization

$$D^*f(\bar{x})(z^*) = \partial\langle z^*, f\rangle(\bar{x}) \quad \text{with} \quad \langle z^*, f\rangle(x) := \big\langle z^*, f(x)\big\rangle \tag{2.7}$$

holds provided that f is *strictly Lipschitzian* at $\bar{x}$, i.e., f is locally Lipschitz continuous at $\bar{x}$ and there is a neighborhood V of the origin in X such that the sequence

$$y_k := \frac{f(x_k + t_k v) - f(x_k)}{t_k}, \quad k \in I\!N,$$

contains a norm convergent subsequence whenever $v \in V$, $x_k \to \bar{x}$, and $t_k \downarrow 0$.

For the sake of compact exposition, we will not recall all the calculus rules for normal cones and coderivatives. However, we will refer to them in Mordukhovich's books [24, 25].

Finally, in this section let us recall a powerful nonlinear scalarization tool from [29] by Tammer and Weidner; cf. [8, 13] which is used in the sequel.

Lemma 2.1. (Scalarization functions for nonconvex sets). Let $Q \subset Z$ be a proper, closed and convex cone with a nonempty interior in a Banach space. Given $e \in \operatorname{int} Q$, the functional $s_e : Z \to \mathbb{R}$ defined by

$$s_e(z) = \inf\{\lambda \in \mathbb{R} \mid \lambda \cdot e \in z + Q\} \tag{2.8}$$

is continuous, sublinear, strictly-int Q-monotone. Moreover, the following relations hold:

(i) $s_e(z + te) = s_e(z) + t \quad \forall z \in Z,\ \forall t \in \mathbb{R}$.

(ii) $\partial s_e(0) = \{z^* \in Q^* \mid \langle z^*, e\rangle = 1\}$ with $Q^* := \{z^* \in Z^* \mid \langle z^*, z\rangle \geq 0,\ \forall z \in Q\} = -N(0; Q)$.

(iii) $\partial s_e(z) = \{z^* \in Q^* \mid \langle z^*, e\rangle = 1, \langle z^*, z\rangle = s_e(z)\}$ for any $z \in Z$.

(iv) Given a nonempty set $\Xi \subset Z$, if $\bar{z} \in \Xi$ is a weak Pareto minimal point of Ξ with respect to Q, then one has

$$s_e(z - \bar{z}) \geq 0 \quad \text{for all} \quad z \in \Xi.$$

This real-valued function plays the role of an utility function.

3 Main Results

In this section we present necessary conditions for Pareto minimal points of sets, for Pareto minimizers of vector-valued optimization problems and for Pareto solutions of set-valued optimization problems in the Asplund setting due to the full calculus for (Mordukhovich) generalized differentiation. It is worth to stressing that our results can be extended to the general Banach setting by using the Ioffe approximate differentiation by imposing stronger assumptions on the given data such that the corresponding calculus rules are applied.

Let Z be an Asplund space, and let $\Theta \subset Z$ be a proper, convex and pointed cone which generates a partial order "$\leq_\Theta$" defined by (1.1). Given a nonempty subset Ξ in Z and $\bar{z} \in \operatorname{Min}(\Xi, \Theta)$. For each $e \in \Theta \setminus \{0\}$ and for each $\varepsilon \in (0, \|e\|)$ we consider the following cone

$$\Theta_{e,\varepsilon} := \operatorname{cone}\big(\mathbb{B}_\varepsilon(e)\big) = \big\{t \cdot z \mid t \geq 0,\ \ z \in \mathbb{B}_\varepsilon(e)\big\}, \tag{3.1}$$

where $\mathbb{B}_\varepsilon(e)$ is a closed ball with the center e and the radius ε. Obviously, it is a proper, closed, convex and pointed cone and its interior is nonempty since $e \in \operatorname{int} \Theta_{e,\varepsilon}$. This cone will take the center in our procedure in formulating necessary conditions for Pareto points with respect to an empty interior ordering cone.

It is worth emphasizing that the approach in [10] involved a specification of proper minimality, the *Henig proper efficiency/optimality*. Recall that Henig minimality is defined in the same way of proper minimality by specifying the cone $\widetilde{\Theta}$ in (1.4) in form of a Henig dilating cone

$$\Theta_{H,\varepsilon} := \bigcup_{e \in \mathbf{B}} \Theta_{e,\varepsilon},$$

where $\mathbf{B}$ is a base of the cone Θ, i.e., $\mathbf{B}$ is a convex set with $0 \notin \operatorname{cl} \mathbf{B}$ and $\Theta = \operatorname{cone} \mathbf{B}$. In this paper, we use just one component of the Henig cone.

The first result is a necessary optimality condition for Pareto minimal points to sets being nonconvex in general.

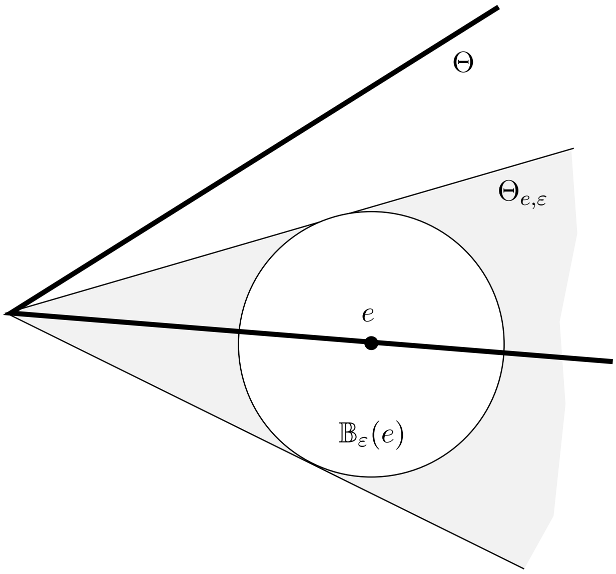

Figure 1: The cone Θ with $\operatorname{int}\Theta=\emptyset$ and the corresponding cone $\Theta_{e,\varepsilon}$ with $\operatorname{int}\Theta_{e,\varepsilon}\neq\emptyset$.

Theorem 3.1. (Necessary conditions for Pareto minimal points of sets). *Let $\bar z\in \operatorname{Min}(\Xi;\Theta)$. Assume that the epigraphical set of Ξ with respect to Θ and $\Xi+\Theta$ are locally closed at $\bar z$. Then for every $e\in\Theta\setminus\{0\}$ satisfying*

$$-e\notin \operatorname{cl}\operatorname{cone}(\Xi+\Theta-\bar z), \tag{3.2}$$

there are a positive number $\varepsilon>0$ and $z^*\in Z^*$ such that

$$-z^*\in N(\bar z;\Xi+\Theta),\quad -z^*\in N(0;\Theta)\cap N(0;\Theta_{e,\varepsilon})\quad\text{and}\quad \langle z^*,e\rangle=1, \tag{3.3}$$

where $\Theta_{e,\varepsilon}$ is given in (3.1).

Proof. First employing [19, Lemma 4.7 (a)], we have the implication

$$\bar z\in\operatorname{Min}(\Xi;\Theta)\quad\Longrightarrow\quad \bar z\in\operatorname{Min}(\Xi+\Theta;\Theta) \tag{3.4}$$

for any proper, convex and pointed ordering cone Θ. Taking into account the Pareto optimality definition, we get from (1.2) that

$$(\Xi+\Theta-\bar z)\cap(-\Theta)=\{0\},\ \text{and thus}\ \operatorname{cone}(\Xi+\Theta-\bar z)\cap(-\Theta)=\{0\}. \tag{3.5}$$

Since $-e\notin\operatorname{cl}\operatorname{cone}(\Xi+\Theta-\bar z)$ by (3.2), there is a positive number $\varepsilon\in(0,\|e\|)$ such that

$$\operatorname{cl}\operatorname{cone}(\Xi+\Theta-\bar z)\cap\big(-\mathbb{B}_\varepsilon(e)\big)=\emptyset,\ \text{and thus}\ \operatorname{cl}\operatorname{cone}(\Xi+\Theta-\bar z)\cap(-\Theta_{e,\varepsilon})=\{0\},$$

which justifies $\bar z\in\operatorname{Min}(\Xi+\Theta;\Theta_{e,\varepsilon})$, where $\Theta_{e,\varepsilon}$ is given in (3.1).

Now we can apply Lemma 2.1 with the cone $Q:=\Theta_{e,\varepsilon}$ and the element $e\in\operatorname{int}\Theta_{e,\varepsilon}$ to get that $\bar z$ is a minimum of the scalarization function s_e defined by (2.8) over $\Xi+\Theta$:

$$\text{minimize}\quad s_e(z-\bar z)\quad\text{subject to}\quad z\in\Xi+\Theta.$$

Since the functional $s_e(\cdot)$ is Lipschitz continuous (in fact, sublinear and continuous) by Lemma 2.1, and since the constraint set $\Xi+\Theta$ is assumed to be locally closed at $\bar z$, all the assumptions of the lower-subdifferential necessary condition for local minima in [25, Proposition 5.3] are satisfied. Employing it to the minimum $\bar{z}$ of the above problem, we have

$$0 \in \partial s_e(\cdot - \bar{z})(\bar{z}) + N(\bar{z}; \Xi + \Theta) = \partial s_e(0) + N(\bar{z}; \Xi + \Theta). \tag{3.6}$$

Taking into account the description of the subdifferential of s_e at the origin in Lemma 2.1 (ii)

$$\partial s_e(0) \quad = \quad \{z^* \in \Theta^*_{e,\varepsilon} \mid \langle z^*, e\rangle = 1\} \quad \text{with} \quad \Theta^*_{e,\varepsilon} = -N(0; \Theta_{e,\varepsilon}),$$

we find from the inclusion (3.6) a subgradient $z^* \in \partial s_e(0)$ with $-z^* \in N(\bar{z}; \Xi + \Theta)$ and $z^* \in -N(0; \Theta_{e,\varepsilon})$, which justifies the necessary conditions in (3.9) provided that $-z^* \in N(0; \Theta)$. To show this, we further elaborate the definition of the normal cone (2.2) to the set $\Xi + \Theta$ at $\bar{z}$. We get from $-z^* \in N(\bar{z}; \Xi + \Theta)$ a sequence $(z_k, z_k^*) \in Z \times Z^*$ satisfying

$$z_k \xrightarrow{\Xi+\Theta} \bar{z} \quad - z_k^* \xrightarrow{w^*} -z^* \quad \text{with} \quad - z_k^* \in \widehat{N}(z_k; \Xi + \Theta).$$

The last inclusion gives

$$\begin{aligned} \limsup_{z \xrightarrow{\Theta} 0} \frac{\langle -z_k^*, z\rangle}{||z||} &= \limsup_{z - z_k \xrightarrow{\Theta} 0} \frac{\langle -z_k^*, z - z_k\rangle}{||z - z_k||} \\ &= \limsup_{z \xrightarrow{\Theta + z_k} z_k} \frac{\langle -z_k^*, z - z_k\rangle}{||z - z_k||} \\ &\leq \limsup_{z \xrightarrow{\Xi+\Theta} z_k} \frac{\langle -z_k^*, z - z_k\rangle}{||z - z_k||} \quad \leq \quad 0, \end{aligned}$$

which verifies $-z_k^* \in \widehat{N}(0; \Theta)$, and thus $-z^* \in N(0; \Theta)$ as $k \to \infty$. The proof is complete. △

Taking into account the definition of the normal cones $N(0; \Theta)$, $N(0; \Theta_{e,\varepsilon})$ and condition (3.9) in Theorem 3.1 we get the following corollary:

Corollary 3.2. (Necessary conditions for Pareto minimal points of sets). *Let $\bar{z} \in \mathrm{Min}\,(\Xi; \Theta)$. Assume that the epigraphical set of Ξ with respect to Θ and $\Xi + \Theta$ are locally closed at $\bar{z}$. Then for every $e \in \Theta \setminus \{0\}$ satisfying (3.2) there are a positive number $\varepsilon > 0$ and $z^* \in \Theta^*$ such that*

$$-z^* \in N(\bar{z}; \Xi + \Theta), \quad \langle z^*, e\rangle \geq \varepsilon ||z^*|| \quad \text{and} \quad \langle z^*, e\rangle = 1, \tag{3.7}$$

where $\Theta_{e,\varepsilon}$ is given in (3.1).

Proof. Because of $-z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon})$ in (3.9) in Theorem 3.1 and the properties of the included normal cones we get $z^* \in -N(0; \Theta) = \Theta^*$ (cf. Lemma 2.1 (ii)). Furthermore, $-z^* \in N(0; \Theta_{e,\varepsilon})$ implies $z^* \in \Theta^*_{e,\varepsilon}$ and so we have

$$\forall \lambda > 0,\ \forall z \in Z \text{ with } ||z|| = 1 : \qquad \langle z^*, \lambda(e + \varepsilon z)\rangle \geq 0.$$

This yields for $z^* \neq 0$ and for all $z \in Z$ with $||z|| = 1$

$$\langle \frac{z^*}{||z^*||}, e\rangle \geq -\varepsilon \langle \frac{z^*}{||z^*||}, z\rangle,$$

and so

$$\langle \frac{z^*}{||z^*||}, e\rangle \geq \sup_{||z||=1} -\varepsilon \langle \frac{z^*}{||z^*||}, z\rangle = \varepsilon,$$

i.e., $\langle z^*, e\rangle \geq \varepsilon ||z^*||$. △

Remark 3.3. *Note that the condition (3.2) is fulfilled if the cone* $\operatorname{cone}(\Xi + \Theta - \bar{z})$ *happens to be closed. Furthermore, (3.2) is satisfied for any* $e \in \operatorname{int}\Theta$ *provided that* $\operatorname{int}\Theta \neq \emptyset$*.*

Note that the condition (3.2) is essential since we do not assume that the ordering cone has a nonempty interior. Analyzing the proof of Theorem 3.1, the existence of a vector e satisfying (3.2) is ensured provided that

$$-\Theta \cap \operatorname{bd}\operatorname{cone}(\Xi + \Theta - \bar{z}) \neq \emptyset.$$

Remark 3.4. *It is worth emphasizing that if the cone* $\operatorname{cone}(\Xi + \Theta - \bar{z})$ *is closed at the origin, then every Pareto minimal point* $\bar{z}$ *of* Ξ *with respect to* Θ *is a* **Benson minimal point** *in the sense that*

$$\operatorname{cl}\operatorname{cone}(\Xi + \Theta - \bar{z}) \cap (-\Theta) = \{0\}. \tag{3.8}$$

To the best of our knowledge, there are no necessary results directly addressing Benson efficiency in terms of subdifferentials and/or coderivatives. In the book chapter by Ha [14] necessary conditions for Benson efficiency are (indirectly) established under the condition that the cone $Q := Z \setminus -\operatorname{cl}\operatorname{cone}(\Xi + \Theta - \bar{z})$ *has a nonempty interior. Ha used the concept of Benson efficiency to obtain the sufficient conditions.*

Remark 3.5. *Theorem 3.1 can be applied to Pareto efficiency and not Benson efficiency. Take* Ξ *be a subset of* $\mathbb{R}^3$ *defined by*

$$\Xi := \{(x, y, z) \mid x^2 + y^2 + (z-1)^2 = 1\},$$

where $\mathbb{R}^3$ *is partially ordered by the non-solid cone* $\Theta := \{0\} \times \mathbb{R}^2_+$*, i.e.,* $\operatorname{int}\Theta = \emptyset$*. Obviously, the origin is a Pareto point of* Ξ *with respect to* Θ*, but it is not a Benson since* $\operatorname{cl}\operatorname{cone}(\Xi + \Theta - 0) = \{(x, y, z) \mid z \geq 0\}$ *is a closed half-space of* $\mathbb{R}^3$ *and* $\{(0,0)\} \times \mathbb{R}_- = \operatorname{cl}\operatorname{cone}(\Xi + \Theta - 0) \cap (-\Theta)$*. Moreover, every vector* $e = (0, y, z)$ *with* $z > 0$ *satisfies the condition (3.2). Therefore, the necessary conditions in Theorem 3.1 can be applied to this example.*

The necessary conditions in Theorem 3.1 does not hold true without the condition (3.2). Consider a closed set in $\mathbb{R}^2$ *given by* $\Xi := \operatorname{epi}(-\sqrt{x})$*, where* $\mathbb{R}^2$ *is partially ordered by the cone* $\Theta = \{0\} \times \mathbb{R}^+$*. Obviously, the origin is a Pareto minimal point but not a Benson efficient point. It is easy to check that*

$$N(0; \Xi + \Theta) = N(0; \Xi) = \mathbb{R}_- \times 0.$$

There are only the unique choice of $z^* = (-1, 0) \in N(0; \Xi + \Theta)$ *and* $e = (0, 1) \in \Theta$*. We can easy check that* $-z^* \notin N(0; \Theta_{e,\varepsilon})$ *for every* $\varepsilon > 0$*, and thus the necessary conditions in Theorem 3.1 is not applicable to this example.*

Next, we will establish a revised version of Theorem 3.1

Theorem 3.6. (Refined necessary conditions for Pareto minimal points of sets). *Let* $\bar{z} \in \operatorname{Min}(\Xi; \Theta)$*. Assume that* $\Xi + \Theta$ *is locally closed at* $\bar{z}$ *and* $\operatorname{cone}(\Xi + \Theta - \bar{z})$ *is closed. Then for every* $e \in \Theta \setminus \{0\}$*, there are a positive number* $\varepsilon > 0$ *and* $z^* \in Z^*$ *such that*

$$-z^* \in N(\bar{z}; \Xi + \Theta), \quad -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}) \quad \textit{and} \quad \langle z^*, e \rangle = 1, \tag{3.9}$$

where $\Theta_{e,\varepsilon}$ *is given in (3.1).*

Proof. The assertion follows by the proof of Theorem 3.1. Since the cone $\operatorname{cone}(\Xi + \Theta - \bar{z})$ is closed, we get from (3.5) that for every $e \in \Theta \setminus \{0\}$, the condition (3.2) is fulfilled, and thus the necessary condition is proved. △

Note that the third condition in (3.9) implies the nontriviality condition $z^* \neq 0$ in the Lagrange-type necessary conditions while the second provides the range of multipliers. Note also that the necessary conditions in Theorem 3.6 is new when the ordering cone is not SNC at the origin; otherwise, they are weaker than the existing necessary results due to the additional closedness assumption imposed on $\mathrm{cone}\,(\Xi + \Theta - \bar{z})$.

Remark 3.7. (Alternative necessary conditions).

(1) It is important to note that the closedness of the cone $\mathrm{cone}\,(\Xi + \Theta - \bar{z})$ and the set itself are indifferent. Indeed, in $\mathbb{R}^2$ equipped with the usual Pareto order $\Theta = \mathbb{R}^2_+$, the set $\Xi := \mathrm{gph}\,(x^2)$ has its epigraphical set $(\Xi + \Theta - \mathbf{0})$ is closed, but its cone is not at the origin, while the set $\Xi := \{(x, y) \mid x > (y-1)^2 - 1 \quad \text{and} \quad y \geq x\} \cup \{\mathbf{0}\}$ has the epigraphical set is not closed, but its cone is closed at the origin.

(2) Observe that the closedness assumption of the set $(\Xi + \Theta - \bar{z})$ in Theorem 3.6 can be dropped. In such a case, i.e., $\Xi + \Theta$ is not locally closed at $\bar{z}$, the first condition in (3.9) will be formulated by

$$-z^* \in N\big(\bar{z}; \mathrm{cl}\,(\Xi + \Theta)\big).$$

Observe also that the closedness assumption of the cone $\mathrm{cone}\,(\Xi+\Theta-\bar{z})$ is essential. Indeed, if Ξ is the graph of the cardioid $r(\theta) := 1+\sin(\theta)$ in $\mathbb{R}^2$, $\Theta := \{(0, t) \in \mathbb{R}^2 \mid t \geq 0\}$, and $\bar{z} = 0$, then 0 is a Pareto minimal point to Ξ. Since $\mathrm{cone}\,(\Xi + \Theta - \bar{z}) = \mathbb{R}^2 \setminus (-\Theta)$, and since the only vector $e \in \Theta$ with $\|e\| = 1$ is $e = (0, 1)$, there is no $\varepsilon > 0$ such that $0 \in \mathrm{Min}\,(\Xi; \Theta_{e,\varepsilon})$. Therefore, the arguments in the proof are no longer valid.

(3) The first condition in (3.9) formulated with the epigraphical set $(\Xi+\Theta)$ is much better than the conventional condition with the set itself

$$-z^* \in N(\bar{z}; \Xi)$$

provided that $\Xi + \Theta$ is *order continuous* at $\bar{z}$ since we have the following inclusion

$$N(\bar{z}; \Xi + \Theta) \subset N(\bar{z}; \Xi). \tag{3.10}$$

To justify this, let us first recall that Ξ is *order continuous* at $\bar{z}$ if for any sequence $\{z_k\} \subset \Xi + \Theta$ converging to $\bar{z}$, there is a sequence $\{v_k\} \subset \Xi$ with $v_k \leq_\Theta z_k$ and $v_k \to \bar{z}$; cf. [4, Definition 4.1] for the order continuity of set-valued mappings. In fact, a set Ξ is order continuous if the constant mapping $F : \mathbb{R} \rightrightarrows Z$ with $F(x) \equiv \Xi$ is order continuous at $(0, \bar{z})$. The reader is referred to [4] for several efficient conditions ensuring this property.

To justify inclusion (3.10), we mimic the proof of [4, Proposition 4.3]. Take $z^* \in N(\bar{z}; \Omega+\Theta)$, then find a sequence $(z_k, z_k^*) \in Z \times Z^*$ such that

$$z_k \stackrel{\Xi+\Theta}{\longrightarrow} \bar{z}, \quad z_k^* \stackrel{w^*}{\to} z^* \quad \text{with} \quad z_k^* \in \widehat{N}(z_k; \Xi + \Theta).$$

Fix k and write z_k in the form $z_k = v_k + t_k$ with $v_k \in \Xi$ and $t_k \in \Theta$. For every $\gamma > 0$ we get from the definition of Féchet normal of $x_k^* \in \widehat{N}(z_k; \Xi + \Theta)$ that

$$\langle z_k^*, z - z_k \rangle \leq \gamma \|z - z_k\|$$

for all $z \in (\Xi + \Theta) \cap (z_k + \eta\mathbb{B})$ for sufficiently small $\eta > 0$. Taking now a neighborhood of v_k in form $v_k + \eta\mathbb{B}$, for every $\tilde{z} \in \Xi \cap (v_k + \eta\mathbb{B})$ we have $z = \tilde{z} + t_k \in (\Xi + \Theta) \cap (z_k + \eta\mathbb{B})$, and thus

$$\langle z_k^*, \tilde{z} - v_k \rangle = \langle z_k^*, (z + t_k) - (v_k + t_k) \rangle = \langle z_k^*, z - z_k \rangle \leq \gamma \|z - z_k\| = \gamma \|\tilde{z} - v_k\|,$$

which justifies that $z_k^* \in \widehat{N}(v_k; \Xi)$. This together with the convergent sequence $v_k \to \bar{z}$ as $k \to \infty$ derived from the order continuity assumption implies $z^* \in N(\bar{z}; \Xi)$. We have proved the inclusion (3.10) under the order continuity assumption. Note that the normal cones to a set and its epigraphical set are incomparable in general. To illustrate this, consider the set

$$\Xi := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 2 \quad \text{and} \quad x + y \geq 0\}$$

in $\mathbb{R}^2$ equipped with the usual ordering cone $\Theta := \mathbb{R}^2_+$, then it is easy to check that

$$N((1, 1); \Xi) := \text{span}\,\{(1, 1)\} \quad \text{and} \quad N((1, 1); \Xi + \Theta) := \text{bd}\,\mathbb{R}^2_-.$$

Considering now a vector-valued optimization problem with a geometric constraint:

$$\text{minimize} \quad f(x) \quad \text{subject to} \quad x \in \Omega, \tag{3.11}$$

where $f : X \to Z$ is a vector-valued function between Asplund spaces, Ω is a subset of X, and "minimization" is understood with respect to a partial order $\leq_\Theta$ defined in (1.1). We say that a point $\bar{x} \in \Omega$ is a *Pareto minimizer of f over Ω with respect to* Θ or a Pareto minimizer to the problem (3.11) if $\bar{z} := f(\bar{x})$ is a Pareto minimal point of the image set $f(\Omega) := \{f(x) \in Z \mid x \in \Omega\}$ with respect to Θ, i.e., $\bar{z} \in \text{Min}\,\big(f(\Omega); \Theta\big)$.

The next result provides a refined version of Lagrange-type necessary conditions for ordering cones with neither the SNC condition imposed in [5] nor the AC property needed in [10].

Theorem 3.8. (Necessary conditions for minimizers in vector optimization). *Let $\bar{x} \in \Omega$ be a Pareto minimizer of the problem (3.11) and $\bar{z} = f(\bar{x})$. Assume that the cost function f is locally Lipschitz at $\bar{x}$ and the constraint set Ω is locally closed around this point. Furthermore, suppose that* cone $\big(f(\Omega) + \Theta - \bar{z}\big)$ *is closed. Then for every $e \in \Theta \setminus \{0\}$, there are a positive number $\varepsilon > 0$ and a dual element $z^* \in Z^*$ such that*

$$0 \in \partial f(\bar{x})(z^*) + N(\bar{x}; \Omega), \quad -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}), \quad \text{and} \quad \langle z^*, e\rangle = 1. \tag{3.12}$$

Moreover, if f is strictly Lipschitzian at $\bar{x}$, then the first condition in (3.12) reduces to

$$0 \in \partial\langle z^*, f\rangle(\bar{x}) + N(\bar{x}; \Omega), \tag{3.13}$$

where the scalarization $\langle z^*, f\rangle$ of f with respect to z^* is defined in (2.7).

Proof. Given the Pareto minimizer $\bar{x}$ of the problem (3.11), we get by using the same arguments in the proof of Theorem 3.6 under the closedness assumption of cone $\big(f(\Omega) + \Theta - \bar{z}\big)$ that for every $e \in \Theta \setminus \{0\}$, there exists a positive number $0 < \varepsilon < \|e\|$ such that $\bar{z} = f(\bar{x}) \in \text{Min}\,\big(f(\Omega) + \Theta; \Theta_{e,\varepsilon}\big)$, where $\Theta_{e,\varepsilon}$ is given by (3.1).

Applying Lemma 2.1(iv) with $Q = \Theta_{e,\varepsilon}$ to this Pareto minimal point $\bar{z}$, we have

$$s_e\big(f(x) + \theta - f(\bar{x})\big) \geq 0 \quad \text{for all } x \in \Omega \text{ and } \theta \in \Theta, \tag{3.14}$$

where s_e is defined in (2.8). Set $g(x) := f(x) - f(\bar{x})$, and $G(x) \equiv \mathcal{E}_g(x) := g(x) + \Theta$ the epigraphical mapping of g. Define the (generalized) composition $s_e \circ G : X \rightrightarrows \mathbb{R}$ by

$$(s_e \circ G)(x) := s_e(G(x)) = \bigcup\{s_e(y) \mid y \in G(x)\},$$

then (3.14) reduces to

$$s_e(z) \geq 0 \quad \text{for all} \quad x \in \Omega \text{ and } z \in G(x),$$

i.e., $(\bar{x}, 0)$ with $0 = (s_e \circ g)(\bar{x}) \in (s_e \circ G)(\bar{x})$ is an optimal solution of the following set-valued (and real-valued) optimization problem:

$$\text{minimize} \quad \big(s_e \circ G\big)(x) \quad \text{subject to} \quad x \in \Omega. \tag{3.15}$$

Next, we will apply the coderivative necessary condition from [5, Theorem 5.3] to the optimal solution $(\bar{x}, 0)$ of the problem (3.15). First, we check the fulfillment of the assumptions therein:

- It is easy to check that $\operatorname{gph} G$ is locally closed around $(\bar{x}, \bar{z})$. Indeed, for any sequence $\{(x_k, z_k)\} \subset \operatorname{gph} G$ converging to (u, v) around $(\bar{x}, \bar{z})$, we have $z_k - g(x_k) \in \Theta$ by the construction of G. Since the sequence $g(x_k) = f(x_k) - f(\bar{x})$ converges to $g(u) = f(u) - f(\bar{x})$ due to the continuity of f around $\bar{x}$, we have $z_k - g(x_k) \to v - g(u) \in \Theta$, which implies $(u, v) \in \operatorname{gph} G$, and justifies the locally closedness property of G around $(\bar{x}, \bar{z})$. Taking into account the continuity of s_e, the graph of the cost mapping $\operatorname{gph}(s_e \circ G)$ is closed around $(\bar{x}, 0)$.
- The composition $(s_e \circ G)$ is locally Lipschitz-like around $(\bar{x}, 0)$ by [24, Corollary 3.15], coderivative chain rules for Lipschitz-like mappings since the single-valued outer function s_e is Lipschitz continuous at 0 and the inner mapping G is Lipschitz-like at $(\bar{x}, 0)$. (Note that if a single-valued function g is Lipschitz continuous at $\bar{x}$, then its epigraphical multifunction is Lipschitz-like at $(\bar{x}, g(\bar{x}))$.)

Applying now [5, Theorem 5.3] to the optimal solution $(\bar{x}, 0)$ of the problem (3.15), we have

$$0 \in D^*\big(s_e \circ G\big)(\bar{x}, 0)(1) + N(\bar{x}; \Omega). \tag{3.16}$$

Next, we will further elaborate the coderivative of the composition. The chain rule in [24, Theorem 3.13] for the single-valued Lipschitz continuous outer function s_e and the inner Lipschitz-like mapping G gives

$$D^*\big(s_e \circ G\big)(\bar{x}, 0)(1) \subset D^*G(\bar{x}, 0) \circ D^*s_e(0)(1).$$

By [24, Theorem 1.80], we have

$$D^*s_e(0)(1) = \partial s_e(0)$$

since s_e is continuous at the origin. Furthermore, we have from the definition of subdifferentials (2.5) that $\partial^* f(\bar{x})(z^*) = D^*G(\bar{x}, 0)(z^*)$ since

$$\begin{aligned}
x^* \in \partial^* f(\bar{x})(z^*) \quad &\Longleftrightarrow \quad (x^*, -z^*) \in N\big((\bar{x}, \bar{z}); \operatorname{epi} f\big) \\
&\Longleftrightarrow \quad (x^*, -z^*) \in N\big((\bar{x}, \bar{z} - \bar{z}); \operatorname{epi}(f - \bar{z})\big) = N\big((\bar{x}, 0); \operatorname{epi}(f - f(\bar{x}))\big) \\
&\Longleftrightarrow \quad (x^*, -z^*) \in N\big((\bar{x}, 0); \operatorname{epi} g\big) = N\big((\bar{x}, 0); \operatorname{gph} G\big) \\
&\Longleftrightarrow \quad x^* \in D^*G(\bar{x}, 0)(z^*).
\end{aligned}$$

Substituting these estimates into (3.16) and taking into account the subdifferential of s_e in Lemma 2.1 (ii):

$$\partial s_e(0) = \{z^* \in -N(0; \Theta_{e;\varepsilon}) \mid \langle z^*, e \rangle = 1\},$$

we get

$$0 \in \partial^* f(\bar{x})(z^*) + N(\bar{x}; \Omega)$$

for some $z^* \in -N(0; \Theta_{e;\varepsilon})$ with $\langle z^*, e \rangle = 1$, which justifies the necessary conditions (3.12) to the problem (3.11) at the Pareto minimizer $\bar{x}$ except $z^* \in -N(0; \Theta)$. The proof of the first part in this theorem is complete since the remaining condition automatically holds thanks to the implication (2.6).

To complete the whole theorem, we only need to show that

$$\partial^* f(\bar{x})(z^*) \subset \partial\langle z^*, f\rangle(\bar{x})$$

when f is strictly Lipschitz continuous at $\bar{x}$. However, this is done since $\partial^* f(\bar{x})(z^*) \subset D^* f(\bar{x})(z^*)$ for any continuous function by [4, Proposition 4.2] while $D^* f(\bar{x})(z^*) = \partial\langle z^*, f\rangle(\bar{x})$ for any strictly Lipschitz continuous function by [24, Theorem 3.28]. △

Remark 3.9. (Comparisons to known necessary conditions). Note that since

$$\partial^* f(\bar{x})(z^*) \subset D^* f(\bar{x})(z^*)$$

for single-valued functions being continuous around $\bar{x}$, our subdifferential necessary conditions in Theorem 3.8 is better than its coderivative counterpart.

Note also that when the ordering cone is not SNC at the origin, our necessary results are completely new. When it is SNC, the known necessary conditions in [3, Theorem 4.1], [5, Theorem 4.1], and [33, Theorem 4.1 (a)]: there is a nonzero dual element $z^* \in -N(0;\Theta)$ satisfying

$$0 \in \partial f(\bar{x})(z^*) + N(\bar{x};\Omega)$$

seems to be more efficient than ours since the latter do not imposed the closedness assumption on the cone $\operatorname{cone}(f(\Omega) + \Theta - \bar{z})$. However, it is important to emphasize that this assumption provides a better range of the dual element $-z^* \in N(0;\Theta) \cap N(0;\Theta_{e,\varepsilon})$.

Note finally that Theorem 3.8 improves [11, Theorem 3.1] in both finite- and infinite-dimensional settings.

In the rest of this section we extend the new necessary conditions in Theorem 3.8 to set-valued cost mappings. Consider a set-valued optimization problem with a general geometric constraint:

$$\text{minimize} \quad F(x) \quad \text{subject to} \quad x \in \Omega, \tag{3.17}$$

where the cost mapping $F : X \rightrightarrows Z$ is a set-valued mapping, Ω is a subset of X, and "minimization" is understood with respect to a partial order $\leq_\Theta$ defined in (1.1). Differ from single-valued functions, for every $\bar{x} \in \operatorname{dom} F$ there are many distinct values $\bar{z} \in Z$ such that $\bar{z} \in F(\bar{x})$. Hence, when studying minimizers of a set-valued mapping, we fix one element in $\bar{z} \in F(\bar{x})$, and say that $(\bar{x}, \bar{z})$ is a minimizer of the problem (3.17) if $\bar{z} \in \operatorname{Min}\big(F(\Omega);\Theta\big)$ with $F(\Omega) := \bigcup\big\{F(x) \mid x \in \Omega\big\}$. In the sequel, we need an extension of Lipschitzian continuity. A set-valued mapping F is *epigraphically Lipschitz-like* (ELL) around $(\bar{x}, \bar{z})$ with modulus $l \geq 0$ if there are neighborhoods U of $\bar{x}$ and V of $\bar{z}$ such that

$$\mathcal{E}_F(x) \cap V \subset \mathcal{E}_F(u) + l\|x - u\|\mathbb{B} \quad \text{for all} \quad x, u \in U,$$

where $\mathbb{B}$ stands for the closed unit ball of Z. In other words, F is ELL at $(\bar{x}, \bar{z})$ if its epigraphical multifunction $\mathcal{E}_F$ is Lipschitz-like at that point. This property is known also as the Aubin property, the pseudo Lipschitzian property; cf. [24, 27]. It agrees with the classical local Lipschitzian behavior in the case of single-valued functions, and reduces to the standard Hausdorff of local Lipschitzian property for set-valued mappings when $V = Z$.

Theorem 3.10. (Necessary conditions for minimizers in set-valued optimization). *Let* $(\bar{x}, \bar{z}) \in \operatorname{gph} F$ *with* $\bar{x} \in \Omega$ *be a minimizer of the problem* (3.17). *Assume that* F *is ELL around* $(\bar{x}, \bar{z})$, $\operatorname{epi} F$ *is locally closed around this point, and* Ω *is locally closed around* $\bar{x}$. *Assume furthermore that* $\operatorname{cone}\big(F(\Omega) + \Theta - \bar{z}\big)$ *is closed. Then for every* $e \in \Theta \setminus \{0\}$, *there are a positive number* $\varepsilon > 0$ *and a dual element* $z^* \in Z^*$ *such that*

$$0 \in \partial F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \Omega), \quad -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}) \quad \text{and} \quad \langle z^*, e\rangle = 1. \tag{3.18}$$

Proof. Since $\operatorname{cone}\big(F(\Omega) + \Theta - \bar{z}\big)$ is assumed to be closed and $\operatorname{epi} F$ is locally closed around $(\bar{x}, \bar{z})$ (this yields that $\big(F(\Omega) + \Theta - \bar{z}\big)$ is locally closed at the origin), we can show that for every $e \in \Theta\backslash\{0\}$, there exists a positive number $\varepsilon \in (0, \|e\|)$ such that $\bar{z} \in \operatorname{Min}\big(F(\Omega) + \Theta; \Theta_{e,\varepsilon}\big)$. Then, we get from Lemma 2.1 (iv) with $Q = \Theta_{e,\varepsilon}$ that $(\bar{x}, \bar{z})$ is a minimum of the problem

$$\text{minimize} \quad \varphi(x, z) + \delta\big((x, z); \Xi\big),$$

where the cost $\varphi : X \times Z \to \mathbb{R}$ is given by

$$\varphi(x, z) := s_e(z - \bar{z}),$$

and the indicator function $\delta\big(\cdot; \Xi\big)$ has the value 0 for all elements of Ξ and the value $+\infty$ for all elements of $X \times Z$ not in Ξ, and $\Xi := \operatorname{epi} F_\Omega$. Recall that F_Ω is the restriction of the mapping F on Ω given by

$$F_\Omega(x) := F(x) + \triangle(x; \Omega) \quad \text{with} \quad \triangle(x; \Omega) := \begin{cases} \{0\}(\subset Z) & \text{if} \quad x \in \Omega, \\ \emptyset & \text{otherwise.} \end{cases}$$

Since $\Xi = \operatorname{epi} F \cap (\Omega \times Z)$ is locally closed at $(\bar{x}, \bar{z})$, the indicator function $\delta\big(\cdot; \Xi\big)$ is lower semicontinuous at that point. Since the function s_e is Lipschitz continuous, so is the cost function φ. Employing first the Fermat rule to this auxiliary scalar optimization problem and then the sum rule to the sum of one Lipschitz continuous function and one lower semicontinuous one, we have

$$\begin{aligned} 0 &\in \partial\big(\varphi + N(\cdot; \Xi)\big)(\bar{x}, \bar{z}) \\ &\subset \partial\varphi(\bar{x}, \bar{z}) + N\big((\bar{x}, \bar{z}); \Xi\big) \\ &= \{0\} \times \partial s_e(0) + N\big((\bar{x}, \bar{z}); \operatorname{epi} F \times (\Omega \times Z)\big), \end{aligned}$$

where the equality holds due to the definitions of the functions.

Taking into account now the subdifferential of s_e (see Lemma 2.1 (ii)) and the fact that $\Xi = \operatorname{epi} F_\Omega = \operatorname{gph} \mathcal{E}_{F_\Omega}$ we have the existence of $z^* \in -N(0; \Theta_{e,\varepsilon})$ with $\langle z^*, e\rangle = 1$ such that

$$(0, -z^*) \in N\big((\bar{x}, \bar{z}); \Xi\big) = N\big((\bar{x}, \bar{z}); \operatorname{epi} F_\Omega\big),$$

which is equivalent, by definition, to the following inclusion

$$0 \in D^*\mathcal{E}_{F_\Omega}(\bar{x}, \bar{z})(z^*) = D^*\big(\mathcal{E}_F + \triangle(\cdot; \Omega)\big)(\bar{x}, \bar{z})(z^*). \tag{3.19}$$

Note that the graphs of both set-valued mappings $\mathcal{E}_F$ and $\triangle(\cdot; \Omega)$ are locally closed at $(\bar{x}, \bar{z})$ under the closedness assumptions imposed on F and Ω. Note also that the imposed ELL property of F at $(\bar{x}, \bar{z})$ is nothing but the Lipschitz-like property of the epigraphical multifunction $\mathcal{E}_F$ at the same point. Hence, the pair of mappings $\mathcal{E}_F$ and $\triangle(\cdot; \Omega)$ meets all the assumptions of the coderivative sum rule from [24, Proposition 3.12], and thus we get from this rule that

$$D^*\big(\mathcal{E}_F + \triangle(\cdot; \Omega)\big)(\bar{x}, \bar{z})(z^*) \subset D^*\mathcal{E}_F(\bar{x}, \bar{z})(z^*) + N(\bar{x}; \Omega). \tag{3.20}$$

Substituting (3.20) into (3.19) and taking into account the definition of subdifferential (2.5) and the implication $\big[\ D^*\mathcal{E}_F(\bar{x},\bar{z})(z^*) \neq \emptyset \implies -z^* \in N(0;\Theta)\ \big]$ in (2.6), we arrive at the necessary condition (3.18)

$$0 \in \partial F(\bar{x},\bar{z})(z^*) + N(\bar{x};\Omega)$$

and complete the proof. △

Note that if the cost mapping $F = f : X \to Z$ in Theorem 3.10 happens to be single-valued, the necessary conditions in Theorem 3.10 reduce to those in Theorem 3.8 while the reduced assumption that $\operatorname{epi} f$ is locally closed around $(\bar{x}, f(\bar{x}))$ is abundant since it is automatics from the Lipschitz continuity of f. However, the ELL property of a set-valued mapping does not imply the closedness of the epigraph of F in general. Consider the set-valued mapping $F : \mathbb{R} \rightrightarrows \mathbb{R}$ defined by

$$F(x) := \begin{cases} (-x,\infty) & \text{if} \quad x < 0, \\ [0,\infty) & \text{if} \quad x = 0, \\ (x,\infty) & \text{if} \quad x > 0, \end{cases}$$

where the usual order on real numbers is used, i.e., $\Theta := [0,\infty)$. It is easy to check that F is ELL at $(0,0)$, but its epigraph $\operatorname{epi} F = \operatorname{gph} F$ is not locally closed at this point.

Note also that the difference in the two proofs of Theorem 3.8 for vector-valued cost functions and Theorem 3.10 for set-valued cost mappings is the domain space of the scalar-valued cost function. In comparison with the existing necessary results in [2, 3, 5, 10, 9, 11, 24, 33] and the references therein, ours in Theorem 3.6, Theorem 3.8, and Theorem 3.10 are *new* when the ordering cone has an empty interior, and enjoys neither AC nor SNC conditions.

Remark 3.11. (Necessary conditions under regularity assumptions). Mimicking the penalization scheme used in [10, Theorem 4.4], we are able to obtain a refined necessary condition for the problem (3.17) under the *metric regularity* assumption, i.e., there is $\gamma > 0$ and a neighborhood U of $(\bar{x},\bar{z}) \in \operatorname{epi} F \cap (\Omega \times Z)$ such that

$$\operatorname{dist}\big((x,z);\operatorname{epi} F \cap (\Omega \times Z)\big) \leq \gamma\big(\operatorname{dist}(x;\Omega) + \operatorname{dist}((x,z);\operatorname{epi} F)\big), \quad \text{for all} \quad (x,z) \in U. \tag{3.21}$$

Necessary conditions with regularity properties. Let $(\bar{x},\bar{z}) \in \operatorname{gph} F$ with $\bar{x} \in \Omega$ be a Pareto minimizer of the problem (3.17), where the image space Z is partially ordered by a closed, convex and pointed cone Θ whose interior might be empty. Assume that Ω is locally closed around $\bar{x}$, $\big(F(\Omega) + \Theta - \bar{z}\big)$ and $\operatorname{cone}\big(F(\Omega) + \Theta - \bar{z}\big)$ are locally closed around the origin. Assume also that the regularity condition (3.21) is fulfilled. Then for every $e \in \Theta \setminus \{0\}$, there is a positive number $\varepsilon > 0$ and $z^* \in Z^*$ with $-z^* \in N(0;\Theta) \cap N(0;\Theta_{e,\varepsilon})$ and $\langle z^*, e\rangle = 1$ satisfying

$$\big(-\partial \operatorname{dist}(\bar{x};\Omega) \times \{-z^*\}\big) \cap \partial \operatorname{dist}\big((\bar{x},\bar{z});\operatorname{epi} F\big) \neq \emptyset, \tag{3.22}$$

which surely implies the relationships in (3.18).

Remark 3.12. (variants of necessary optimality conditions in Banach spaces). Although the results in this paper were established in Asplund spaces, they are easily extended to general Banach spaces by using other types of generalized differentiation which have the calculus rules used in the proofs of Theorem 3.6, Theorem 3.8, and Theorem 3.10; for example, Clarke's generalized differentiations in [7], and the Ioffe's approximate constructions in [16, 17, 18]. We use the Mordukhovich's since it has a full calculus for normal cones to sets and coderivatives for set-valued mappings under the SNC conditions and the Mordukhovich qualification conditions; they are satisfied under the Lipschitz-like properties, and since is is smaller or at least not bigger than the other generalized differentiation constructions.

If we want to have necessary conditions in terms of Ioffe's normal cones and coderivatives, we need the corresponding conditions for the fulfillment of the calculus rules such as the sum rule [18, Theorem 5.6], the chain rule [18, Theorem 6.2], the chain rule for coderivatives [20, Theorem 3.1], and the sum rule for coderivatives [20, Theorem 3.4]. Loosely speaking, instead of the Lipchitz-like assumptions needed for the the Mordukhovich's calculus, the Ioffe calculus requires the strongly compactness Lipschitzian continuity property whose definition is quite complicated even for vector-valued functions. Recall that a function $f : X \to Z$ is said to be strongly compactness Lipschitzian at $\bar{x}$ for if there exist a multifunction $R : X \rightrightarrows \mathrm{Comp}(Z)$, where $\mathrm{Comp}(Z)$ is the collection of all nonempty $\|\cdot\|$-compact subsets of Z, and a function $r : X \times X \to \mathbb{R}_+$ satisfying the following properties:

(i) $\lim\limits_{\substack{x\to\bar{x}\\ h\to 0}} r(x,h) = 0.$

(ii) There is $\mu > 0$ such that for all $h \in \mu\mathbb{B}_X$, $x \in \bar{x} + \mu\mathbb{B}_X$, and all $t \in (0,\mu)$, one has

$$t^{-1}\big(f(x+th) - f(x)\big) \in R(h) + \|h\| r(x,th)\mathbb{B}_X.$$

(iii) $R(0) = \{0\}$ and R is upper semicontinuous.

It agrees with the usual Lipschitzian continuity when Z is finite dimensions.

4 Applications

4.1 Applications in Mathematical Finance

In a wide range of financial applications decision problems concern the minimization of *risk* within the available circumstances. We study certain problems in risk management, formulate a corresponding vector optimization problem and develop optimality conditions using our results in Section 3. It will be shown that the relations between the scalarizing functional (2.8) and risk measures have an important role in such conditions, and that aspects of risk acceptability can require attention as well.

Let us consider a set W of elementary events (i.e., of possible states) and suppose it to be supplied with a probability measure P and a field Σ of measurable sets. The result of an investment in the future can be considered as a random variable, i.e., a function $z : W \to \mathbb{R}$ which assigns to every possible state the result of the investment (the difference between incoming and outcoming payments, may be discounted) in this state. If no investment will be done, then the random variable is equal to zero, positive outcomes are wins and negative ones are losses. The space of these random variables is denoted by Z.

Especially, consider the probability space (W,Σ,P) and the space $Z = L^2(W,\Sigma,P) =: L^2$ of random variables z, i.e., the space of measurable functions for which the mean and the variance

$$Ez = \int_W z(w)\ \mathrm{d}\,P(w),$$

$$E[z - Ez]^2 = \int_W [z(w) - Ez]^2\,\mathrm{d}\,P(w),$$

exist (i.e., these integrals are well defined). The inner product in $Z = L^2(W,\Sigma,P)$ between any $z^1, z^2 \in Z = L^2(W,\Sigma,P)$ is $\langle z^1, z^2\rangle = E[z^1 z^2] = \int_W z^1(w)z^2(w)\ \mathrm{d}\,P(w)$. Obviously, the usual ordering cone in L^2 has an empty interior. However, it is possible to apply our methods because they are working without assuming the nonemptyness of the interior of the ordering cone.

Now, we consider a space Z of random variables and a subset Ξ of Z, consisting of the random variables z deemed to be feasible possibilities. We study the **vector optimization problem** (P_{invest}) (with the image space Z) to find Pareto minimal points of the set Ξ of feasible risky investments with respect to a certain preference relation on Z (cf. Heyde [15]). In this vector optimization problem the valuation of risky investments means valuation of random variables. In order to formulate a suitable preference relation in the space Z of random variables (such that one can compare two investments and to decide whether the risk is acceptable) we can use the concept of *acceptable sets* by Artzner, Delbean, Eber and Heath [1] (cf. Heyde [15]). They introduced axioms to be satisfied for a set $\Theta \subset Z$ of random variables corresponding to acceptable investments:

(A1) $\{z \in Z \mid z(w) \geq 0 \; \forall \, w \in W\} \subset \Theta$, $\Theta \cap \{z \in Z \mid z(w) < 0 \; \forall \, w \in W\} = \emptyset$,

(A2) Θ is a cone,

(A3) $\Theta + \Theta \subset \Theta$.

In the case $Z = L^p(W, \Sigma, P)$ for some probability space (W, Σ, P) and $p \geq 1$ in the above axioms "$\forall w \in W$" is to understood as "for P-almost all $w \in W$".

Sets $\Theta \subset Z$ satisfying the axioms (A1) - (A3) of acceptable investments can be used in order to introduce a preference relation on Z. The decision maker prefers z^1 to z^2 (changing from z^2 to z^1 is an acceptable risk) iff $z^1 - z^2$ is an element of Θ, i.e.,

$$z^1 \geq_\Theta z^2 \Longleftrightarrow z^1 - z^2 \in \Theta.$$

In financial terms, Axiom (A1) means that every investment with almost sure nonnegative results will be accepted and every investment with almost sure negative results is not acceptable. Furthermore, the cone property in Axiom (A2) says that every nonnegative multiple of an acceptable investment is again acceptable. The convexity property in Axiom (A3) means that merging two acceptable investments together results again in an acceptable investment. However, in some applications axioms (A2) and (A3) are not useful, especially, if the investor does not want to loose more than a certain amount of money. In this case Föllmer and Schied [12] replace the axioms (A2) and (A3) by a convexity axiom.

Risk measures play an important role in Mathematical Finance, where *risk measures* are to understand as a certain amount of money (capital that secures almost surely the solvency of an enterprise in the case of losses). Analogously, the reserves of insurance companies that save them the survival in the case of a large number of insurance events can be understood as risk measure.

An important question for decision makers in Mathematical Finance is: How to find a quantification of an existing risk such that the solvency is almost sure on the one side and on the other hand, that not too much capital is engaged?

One has to find a function $\mu : Z \to \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$ which assign to every investment $z \in Z$ a real number or the values $-\infty$, $+\infty$. A very natural way to express risk measures is the description by means of functions of type (2.8).

A function s_e in (2.8) is used in financial mathematics in order to express a *risk measure* (for example a valuation of risky investments) with respect to an acceptance set $\Theta \subset Z$. The element $e \in \Theta \setminus \{0\}$ is a fixed financial parameter given by the decision maker. Artzner, Delbean, Eber and Heath [1] introduced coherent risk measures. In the papers by Artzner, Delbean, Eber and Heath [1] and Rockafellar, Uryasev and Zabarankin [28] the following properties for *coherent risk measures* μ are supposed:

(P1) $\mu(z + te) = \mu(z) - t$,

(P2) $\mu(0) = 0$ and $\mu(\lambda z) = \lambda\mu(z)$ for all $z \in Z$ and $\lambda > 0$,

(P3) $\mu(z^1 + z^2) \leq \mu(z^1) + \mu(z^2)$ for all $z^1, z^2 \in Z$,

(P4) $\mu(z^1) \leq \mu(z^2)$ if $z^1 \geq z^2$.

The following interpretation of the properties (P1)–(P4) is to mention: The translation property (P1) means that the risk would be mitigated by an additional safe investment with a corresponding amount, especially, it holds

$$\mu(z + \mu(z)e) = 0.$$

The positive homogeneity of the risk measure in (P2) means that double risk must be secured by double risk capital; the subadditivity in (P3) means that a diversification of risk should be recompensed and finally, the monotonicity of the risk measure in (P4) means that higher risk needs more risk capital.

A risk measure may be negative. In this case it can be interpreted as a maximal amount of cash that could be given away such that the reduced result remains acceptable.

Examples for coherent risk measures are the **conditional value at risk** (cf. [12], Section 4.4, Definition 4.43) and the **worst-case risk measure**.

Example: (**Value at Risk**) Let W be a fixed set of scenarios. A financial position is described by a mapping $z : W \to \mathbb{R}$ and z belongs to a given class $\mathcal{X}$ of financial positions. Assume that $\mathcal{X}$ is the linear space of bounded measurable functions containing the constants on some measurable space (W, Σ). Furthermore, let P be a probability measure on (W, Σ).

A position z is considered to be acceptable if the probability of a loss is bounded by a given level $\lambda \in (0, 1)$, i.e., if $P[z < 0] \leq \lambda$. The corresponding monetary risk measure $V@R_\lambda$, defined by

$$V@R_\lambda(z) := \inf\{m \in \mathbb{R} \mid P(m + z < 0) \leq \lambda\}$$

is called *Value at Risk*. $V@R_\lambda$ is the smallest amount of capital which, if added to z and invested in the risk-free asset, keeps the probability of a negative outcome below the level λ.

$V@R_\lambda$ is positively homogeneous (i.e., (P2) is fulfilled) but in general it is not convex (i.e., (P3) does n't hold), cf. Föllmer and Schied [12], Example 4.11.

Example: (**Worst-case risk measure**) Consider the *worst-case risk measure* ρ_{max} defined by

$$\rho_{max}(z) := -\inf_{w \in W} z(w) \text{ for all } z \in \mathcal{X},$$

where W is a fixed set of scenarios, $z : W \to \mathbb{R}$ and z belongs to a given class $\mathcal{X}$ of financial positions. Assume that $\mathcal{X}$ is the linear space of bounded measurable functions containing the constants on some measurable space (W, Σ). The value $\rho_{max}(z)$ is the least upper bound for the potential loss which occur in any scenario. ρ_{max} is a coherent risk measure, i.e., (P1)–(P4) hold (cf. Föllmer and Schied [12], Example 4.8).

The sublevel set $L_\mu(0) =: \Theta$ of μ to the level 0 is a convex cone and corresponds to the acceptance set. This is the reason that a coherent risk measure admits a representation as

$$\mu(z) = \inf\{t \in \mathbb{R} \mid z + te \in \Theta\}, \tag{4.23}$$

where $e \in \Theta \setminus \{0\}$. In financial terms, the function μ represents the minimal amount of capital that one has to add such that the whole investment is acceptable.

It is easy to see that a coherent risk measure can be identified with the functional $s_e(-z)$ (see (2.8)) by

$$s_e(z) = \mu(-z). \tag{4.24}$$

Optimality conditions relying on the special features of risk envelopes are shown by Rockafellar, Uryasev, Zabarankin in [28, Theorem 5]. In the special case of coherent risk measures we get from the properties of elements belonging to ∂s_e (see Lemma 2.1) the corresponding properties as in the dual description of coherent risk measures given by Rockafellar, Uryasev, Zabarankin [28] and Heyde [15]:

Consider the Hilbert space $Z = L^2(W, \Sigma, P)$ with the inner product $\langle z^*, z\rangle = E[z^* z]$ for $z^*, z \in L^2(W, \Sigma, P)$. A (continuous, convex and positive homogeneous) coherent risk measure can be considered as a support functional to some closed convex set $M \subset (L^2)^* = L^2(W, \Sigma, P)$, i.e.,

$$\mu(z) = \sigma_M(z) = \sup\{E[z^* z] \mid z^* \in M\}$$

or

$$M = \{z^* \in L^2(W, \Sigma, P) \mid E[z^* z] \leq \mu(z) \quad \forall z \in L^2(W, \Sigma, P)\}.$$

Putting $\mathcal{M} := -M$ one gets

$$\mu(z) = \sup\{-E[z^* z] \mid z^* \in \mathcal{M}\}$$

and

$$\mathcal{M} = \{v^* \in L^2(W, \Sigma, P) \mid -E[v^* z] \leq \mu(z) \quad \forall z \in L^2(W, \Sigma, P)\}.$$

Taking into account these properties Rockafellar, Uryasev, Zabarankin [28] have shown the following properties in the dual description of coherent risk measures:

(i) $Ez^* = 1$ for all $z^* \in \mathcal{M}$,

(ii) $z^* \geq 0$ for all $z^* \in \mathcal{M}$.

These properties **(i)** and **(ii)** correspond to the properties of elements belonging to the subdifferential of the functional s_e (see Lemma 2.1 (ii)) which play an important role for deriving necessary condition for Pareto minimizers of the set of feasible risky investments in an Asplund space Z from Theorem 3.6:

Corollary 4.1. *Let Ξ be a subset of feasible possibilities in an Asplund space of random variables Z equipped with a partial order generated by a proper, convex and pointed cone Θ (a set $\Theta \subset Z$ of random variables corresponding to acceptable investments), and let $\bar{z} \in \mathrm{Min}\,(\Xi; \Theta)$. Assume that $\Xi + \Theta$ is locally closed at $\bar{z}$ and* cone $(\Xi + \Theta - \bar{z})$ *is closed at the origin. Then for every $e \in \Theta \setminus \{0\}$, there are a positive number $\varepsilon > 0$ and $z^* \in Z^*$ such that*

$$-z^* \in N(\bar{z}; \Xi + \Theta), \quad -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}) \quad \textit{and} \quad \langle z^*, e\rangle = 1, \tag{4.25}$$

where $\Theta_{e,\varepsilon}$ is given by (3.1).

Remark 4.2. *For the special case $Z = L^2(W, \Sigma, P) =: L^2$ the conditions $-z^* \in N(0; \Theta)$ and $\langle z^*, e\rangle = 1$ in Corollary 4.1 correspond to results by Rockafellar, Uryasev, Zabarankin [28, Theorem 5] taking into account (4.24). The elements z^* in (4.25) are elements belonging to the subdifferential of the coherent risk measure $\mu(-z) = s_e(z)$. The condition $-z^* \in N(0; \Theta)$ means $-z^* \in \Theta^*$ (see Lemma 2.1, (ii)). Furthermore, $-z^* \in N(\bar{z}; \Xi + \Theta)$ follows from $0 \in \partial s_e(\bar{z}) + N(\bar{z}; \Xi + \Theta)$.*

4.2 Application in approximation theory

As a second application of our results we present some optimality conditions for vector-valued (not necessary convex) approximation problems having a practical importance described in [13].

We assume that X, Y and Z are real Banach spaces; as in the preceding, $\Theta \subset Z$ is a proper pointed closed convex cone. In order to formulate the vector control approximation problem, we envisaged, let us introduce a vector-valued norm (see [19]) as an application $|||\cdot||| : Y \to \Theta$ which for all $y, y_1, y_2 \in Y$ and for all $\lambda \in \mathbb{R}$ satisfies:

(1) $|||y||| = 0 \Longleftrightarrow y = 0$;

(2) $|||\lambda\, y\, ||| = |\, \lambda \,|\ \ |||y|||$;

(3) $|||y_1 + y_2|||\ \in\ |||y_1||| + |||y_2||| -\ \Theta.$

A subdifferential (denoted $\partial^{\leq}$) for vector-valued functions was proposed by Jahn in [19] and for the particular case of above vector-valued norm $|||\cdot|||$ it has the following form:

$$\partial^{\leq} |||\cdot|||(y_0) = \{T \in L(Y, Z) \mid T(y_0) = |||y_0|||,\ |||y||| - T(y) \in \Theta\ \forall\, y \in Y\}, \tag{4.26}$$

where $L(Y, Z)$ denotes the space of linear continuous operators from Y into Z.

Furthermore, we assume that $\partial^{\leq} |||\cdot||| \neq \emptyset$. Sufficient conditions for $\partial^{\leq} |||\cdot||| \neq \emptyset$ are given by Jahn [19] (for instance that $|||\cdot|||$ is continuous and Θ has the Daniell property which means that every decreasing net (i.e., $i \leq j$ implies $x_j \leq x_i$) having a lower bound converges to its infimum).

Note that for every $z^* \in \Theta^*$, the mapping $\langle z^*, |||\cdot|||\rangle$ is convex, hence its subdifferential is understood in the sense of convex analysis. Note also that the epigraphical multifunction of $|||\cdot|||$, denoted as usual by $\mathcal{E}_{|||\cdot|||}$, is graph-convex. Indeed, for any $t \in (0, 1)$ and for any (x, z) and (u, v) in $\operatorname{gph} \mathcal{E}_{|||\cdot|||}$ we can find $\theta, ; \zeta \in \Theta$ such that $z = |||x||| + \theta$ and $u = |||u||| + \zeta$. Taking into account the properties (2) and (3) of the function $|||\cdot|||$ and $\Theta + \Theta = \Theta$, we have

$$\begin{aligned} tz + (1-t)v &= |||tx||| + |||(1-t)u||| + \big(t\theta + (1-t)\zeta\big) \\ &\in |||tx + (1-t)u||| + \Theta + \Theta \\ &= |||tx + (1-t)u||| + \Theta \in \mathcal{E}_{|||\cdot|||}\big(tx + (1-t)u\big), \end{aligned}$$

which clearly implies that $t(u, z) + (1-t)(u, v) \in \operatorname{gph} \mathcal{E}_{|||\cdot|||}$, and thus the convexity of $\operatorname{gph} \mathcal{E}_{|||\cdot|||}$.

Assuming additionally that Θ has a weakly compact base we adapt a result by Valadier [31] which is useful in the sequel.

Theorem 4.3. *([31]) Let $(Y, \|\cdot\|)$ and $(Z, \|\cdot\|)$ be real reflexive Banach spaces and $\Theta \subset Z$ a proper convex Daniell cone with a weakly compact base. Suppose that the vector-valued norm $|||\cdot|||$ is continuous. Then for every $z \in Z$ and $z^* \in \Theta^*$ one has*

$$\langle z^*, \partial^{\leq} |||\cdot|||\rangle(z) = \partial\langle z^*, |||\cdot|||)\rangle(z).$$

Suppose now that $g : X \to Z$ is a locally Lipschitz cost function, $A_i \in L(X, Y)$ and $\alpha_i \geq 0$ $(i = 1, \ldots, n)$. In the following A_i^* denotes the adjoint operator to A_i. Then we consider for $x \in \Omega \subset X$ and $a^i \in Y$ $(i = 1, \ldots, n)$ the vector-valued approximation problem

$$\text{minimize } f(x) := g(x) + \sum_{i=1}^{n} \alpha_i |||A_i(x) - a^i||| \text{ subject to } x \in \Omega, \tag{4.27}$$

where "minimization" is understood with respect to the partial order generated by a proper, closed and convex cone $\Theta \subset Z$ in (1.1) and $\Omega \subset X$ is closed.

Theorem 4.4. *Suppose that X, Y, Z are reflexive Banach spaces, $\Theta \subset Z$ a proper pointed closed convex Daniell cone with a weakly compact base and Ω is a closed subset of X. Let $\bar{x} \in \Omega$ be a Pareto minimizer to (4.27) and $\bar{z} = f(\bar{x})$. Assume that g is strictly Lipschitzian at $\bar{x}$ and $\||\cdot\||$ is locally Lipschitz around $\bar{x}$. Suppose that* cone $(f(\Omega)+\Theta-\bar{z})$ *is closed. Then for every $e \in \Theta \setminus \{0\}$, there is a positive number $\varepsilon > 0$ and $z^* \in Z^*$ with $\langle z^*, e\rangle = 1$ such that*

$$0 \in \partial\langle z^*, g(\bar{x})\rangle + \sum_{i=1}^{n} \alpha_i A_i^* z^* T_i + N(\bar{x}; \Omega), \quad -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}),$$

where $T_i \in L(Y, Z)$ and

$$T_i(A_i(\bar{x}) - a^i) = \||A_i(\bar{x}) - a^i\||,$$
$$\||v\|| - T_i(v) \in \Theta \qquad \forall\, v \in Y \quad (i = 1, \ldots, n).$$

Proof. Because X, Y, Z are reflexive Banach spaces they are Asplund spaces too. Assume $\bar{x} \in \Omega$ is a Pareto minimizer to (4.27). Then from Theorem 3.8 we get the existence of $z^* \in Z^*$ with $\langle z^*, e\rangle = 1$ and

$$0 \in \partial f(\bar{x})(z^*) + N(\bar{x}, \Omega), \ -z^* \in N(0; \Theta) \cap N(0; \Theta_{e,\varepsilon}). \tag{4.28}$$

Taking into account the subdifferential definition (2.5) for vector-valued functions, the definition of epigraphical multifunction (2.4) and the convexity of Θ (in fact, $\Theta + \Theta = \Theta$) we have

$$\mathcal{E}_f(\cdot) = \mathcal{E}_g(\cdot) + \sum_{i=1}^{n} \alpha_i \mathcal{E}_{\||\cdot\||} \circ \big(A_i(\cdot) - a^i\big), \tag{4.29}$$

and thus

$$\partial f(\bar{x})(z^*) = D^*\mathcal{E}_f(\bar{x}, \bar{z})(z^*) = D^*\Big(\mathcal{E}_g(\cdot) + \sum_{i=1}^{n} \alpha_i \mathcal{E}_{\||\cdot\||} \circ \big(A_i(\cdot) - a^i\big)\Big)(\bar{x}, \bar{z})(z^*).$$

Applying now the sum rule for coderivative from Corollary 3.11 to the sum of Lipschitz-like mappings in (4.29) we, by setting $\bar{u}_i = A_i(\bar{x}) - a^i$ for $i = 1, \ldots, n$, have

$$\partial f(\bar{x})(z^*) \subset D^*\mathcal{E}_g\big(\bar{x}, g(\bar{x})\big)(z^*) + \sum_{i=1}^{n} \alpha_i D^*\Big(\mathcal{E}_{\||\cdot\||} \circ \big(A_i(\cdot) - a^i\big)\Big)\big(\bar{x}, \||\bar{u}_i\||\big)(z^*). \tag{4.30}$$

Note that

$$D^*\mathcal{E}_g\big(\bar{x}, g(\bar{x})\big)(z^*) \subset D^*g\big(\bar{x}, g(\bar{x})\big)(z^*) = \partial\langle z^*, g\rangle(\bar{x}) \tag{4.31}$$

where the first inclusion holds due to [4, Proposition 4.3] and the equality holds thanks to [24, Theorem 3.28]. Note also that

$$\begin{aligned} D^*\Big(\mathcal{E}_{\||\cdot\||} \circ \big(A_i(\cdot) - a^i\big)\Big)\big(\bar{x}, \||\bar{u}_i\||\big)(z^*) &= D^*\big(A_i(\cdot) - a^i\big)(\bar{x}) \circ D^*\mathcal{E}_{\||\cdot\||}\big(\bar{x}, \||\bar{u}_i\||\big)(z^*) \\ &\subset A_i^* \circ \partial\langle z^*, \||\cdot\||\rangle(\bar{x}, \||\bar{u}_i\||)(z^*), \end{aligned} \tag{4.32}$$

where the equality holds by the chain rule for coderivatives in [24, Theorem 3.13(iii)] applied to the graph-convex outer mapping $\mathcal{E}_{\||\cdot\||}$ and the linear inner function $A_i(\cdot) - a^i$, and the inclusion holds since there is no difference between the basic and mixed coderivatives of the convex-graph multifunction $\mathcal{E}_{\||\cdot\||}$, i.e., $D^*\mathcal{E}_{\||\cdot\||}(\cdot)(z^*) = D_N^*\mathcal{E}_{\||\cdot\||}(\cdot)(z^*) = D_M^*\mathcal{E}_{\||\cdot\||}(\cdot)(z^*)$, then [4, Proposition 4.3] gives $D_M^*\mathcal{E}_{\||\cdot\||}(\cdot)(z^*) \subset D_M^*\||\cdot\||(\cdot)(z^*)$, then finally [24, Theorem 1.90] provides $D_M^*\||\cdot\||(z^*) = \partial\langle z^*, \||\cdot\||\rangle$; see the book by Mordukhovich [24] for the mixed coderivative.

Substituting now (4.32) and (4.31) into (4.30) and using Theorem 4.3 and [19, Example 2.22], we obtain an upper bound of $\partial f(\bar{x})(z^*)$ as

$$\begin{aligned}\partial f(\bar{x})(z^*) &\subseteq \partial\langle z^*, g\rangle(\bar{x}) + \sum_{i=1}^{n} \alpha_i A_i^* \langle z^*, \partial^{\leq} |\!|\!| \bar{u}_i |\!|\!| \rangle \mid_{\bar{u}_i = A_i(\bar{x}) - a^i} \\ &= \partial\langle z^*, g\rangle(\bar{x}) + \Big\{ \sum_{i=1}^{n} \alpha_i A_i^* z^* T_i \mid T_i \in L(Y,Z), T_i(A_i(\bar{x}) - a^i) = |\!|\!| A_i(\bar{x}) - a^i |\!|\!|, \\ &\qquad |\!|\!| v |\!|\!| - T_i(v) \in \Theta \ \forall\, v \in Y \ (i = 1, \dots, n) \Big\}.\end{aligned}$$

Then we get together with (4.28) the desired relation. △

Remark 4.5. *In the papers [11] and [8] Lagrange multiplier rules are shown for weak Pareto minimizers. In comparison with the corresponding results in [8, Theorem 5.2] we show our results for Pareto minimizers and we do not need the assumption that int* $\Theta \neq \emptyset$.

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