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Regularization of Quasi Variational Inequalities

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Abstract

An ill-posed quasi variational inequality with contaminated data can be stabilized by employing the elliptic regularization. Under suitable conditions, a sequence of bounded regularized solutions converge strongly to a solution of the original quasi variational inequality. Moreover, the conditions that ensure the boundedness of regularized solutions, become sufficient solvability conditions. It turns out that the regularization theory is quite strong for quasi variational inequalities with set-valued monotone maps but restrictive for generalized monotone maps. The results are quite general and are applicable to ill-posed variational inequalities, inverse problems, and split-feasibility problem, among others.

Keywords: Quasi variational inequalities; parameter identification, regularization, ill-posed, monotone maps, pseudo-monotone maps.

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1 Introduction

Let $\mathscr{B}$ be a real reflexive Banach space with $\mathscr{B}^*$ as its dual, and let J be the associated normalized duality mapping. We assume that $\mathscr{B}$ has been renormed so that $\mathscr{B}$ and $\mathscr{B}^*$ are locally uniformly convex. We specify the duality pairing between $\mathscr{B}$ and $\mathscr{B}^*$ by $\langle \cdot,\cdot \rangle$, whereas $\|\cdot\|$ stands for the norm in $\mathscr{B}$ as well as in $\mathscr{B}^*$. Let $\mathscr{C} \subset \mathscr{B}$ be nonempty, closed, and convex, and let $\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ be a set-valued map such that for every $v \in \mathscr{C}$, the set $\mathscr{K}(v)$ is a nonempty, closed, and convex subset of $\mathscr{C}$. Let $\mathscr{F} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ be a given set-valued map, let $\varphi : \mathscr{B} \to \mathbb{R}$ be a given functional, and let $f \in \mathscr{B}^*$ be arbitrary. The domain and the graph of $\mathscr{F}$ are given by $\mathscr{D}(\mathscr{F}) := \{x \in \mathscr{B} : \ \mathscr{F}(x) \neq \emptyset\}$ and $\mathscr{G}(\mathscr{F}) := \{(x,y) \in \mathscr{B} \times \mathscr{B}^* : \ x \in \mathscr{D}(\mathscr{F}), \ y \in \mathscr{F}(x)\}$, respectively. The strong convergence and the weak convergence in $\mathscr{B}$ as well as in $\mathscr{B}^*$ are specified by $\to$ and $\rightharpoonup$, respectively.

In this work we study the following quasi variational inequality: find $x \in \mathscr{C} \cap \mathscr{K}(x)$ such that for some $w \in \mathscr{F}(x)$, we have

$$\langle w - f, z - x \rangle \geq \varphi(x) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x). \tag{1}$$

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The above quasi variational inequality includes many important problems of interest as particular cases. For instance, if the map $\mathscr{F}$ is single-valued and $\varphi \equiv 0$, then (1) recovers the following quasi variational inequality: find $x \in \mathscr{C} \cap \mathscr{K}(x)$ such that

$$\langle \mathscr{F}(x) - f, z - x \rangle \geq 0, \quad \text{for every } z \in \mathscr{K}(x). \tag{2}$$

The above problem was introduced by Bensoussan and Lions [7] in connection with an impulse control problem. A general treatment of (2) was given by Mosco [41]. If additionally $\mathscr{K}(x) = \mathscr{C}$ for every $x \in \mathscr{C}$, then (1) recovers the following variational inequality: find $x \in \mathscr{C}$ such that

$$\langle \mathscr{F}(x) - f, z - x \rangle \geq 0, \quad \text{for every } z \in \mathscr{C}. \tag{3}$$

Variational inequality (3) appears as a necessary optimality condition in output least-squares (OLS) formulation of the inverse problem of identifying coefficients in partial differential equations (cf. [15]). Moreover, (3) also emerges as a necessary and sufficient optimality condition for the same inverse problem through the modified OLS (cf. [20]) and the equation error approach (cf. [21]).

Notice that, if for every $x \in \mathscr{C}$, $\mathscr{K}(x)$ is a proper, closed, and convex cone, $f = 0$, and $\varphi \equiv 0$, then (1) collapses to the generalized complementarity problem: find $x \in \mathscr{C}$ such that

$$x \in \mathscr{K}(x), \quad w \in \mathscr{F}(x) \cap \mathscr{K}^*(x), \quad \langle w, x \rangle = 0, \tag{4}$$

where $\mathscr{K}^*(x)$ is the positive polar of $\mathscr{K}(x)$. If additionally $\mathscr{K}(x) \equiv \mathscr{C}$, then (4) recovers the complementarity problem (see [22]). The equivalence between (1) and (4) is given by Giannessi [18].

In recent years the theory of variational and quasi variational inequalities emerged as one of the most promising branches of pure, applied, and industrial mathematics. This theory provides us with a convenient mathematical apparatus for studying a wide range of problems arising in diverse fields such a structural mechanics, elasticity, economics, optimization, optimal control, inverse problems, financial mathematics, and others (see [8],[31],[34]). The existence and the approximation theories for quasi variational inequalities are challenging and require that a variational inequality and a fixed point problem should be solved simultaneously. Consequently, many solution techniques which are available for variational inequalities have not been extended for quasi variational inequalities. For instance, regularization and penalization methods for monotone variational inequalities have almost reached a saturation point. In contrast, for quasi variational inequalities, these approaches have not been fully explored, and there are many questions to be answered.

In this study, our objective is to develop a regularization theory for quasi variational inequalities. It is known that, in general, quasi variational inequalities are ill-posed. That is, a quasi variational inequality may either have no solution or multiple solutions, and, most importantly, small errors in data could lead to uncontrollable errors in its solution(s). Not surprising, the ill-posedness of a quasi variational inequality is also evident from the fact that it recovers many severely ill-posed inverse problems and variational inequalities as particular cases. A few of these examples will be discussed below. The use of regularization will ensure the stable behavior of an ill-posed quasi variational inequality. To be more specific, we will study (an analogue of) the following regularized quasi variational inequality: find $x_\varepsilon \in \mathscr{C} \cap \mathscr{K}(x_\varepsilon)$ such that for some $w_\varepsilon \in \mathscr{F}(x_\varepsilon)$, we have

$$\langle w_\varepsilon + \varepsilon J(x_\varepsilon), z - x_\varepsilon \rangle \geq \varphi(x_\varepsilon) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x_\varepsilon),$$

where $\varepsilon > 0$. Due to the nice features of duality map J, the map $\mathscr{F}(\cdot) + \varepsilon J(\cdot)$ has significantly better properties than $\mathscr{F}$. The objective is to study the convergence of the regularized solutions $\{x_\varepsilon\}$ as $\varepsilon \to 0$. The importance of regularization methods for variational inequalities has long been recognized. In fact, Lions and Stampacchia [35] investigated the behavior of regularized solutions in the context of a linear variational inequality. They established an equivalence between the boundedness of regularized solutions $\{x_\varepsilon\}$ and the solvability of the corresponding variational inequality. Their work was significantly improved by Browder [11], who gave various conditions to ensure the boundedness of regularized solutions of a (nonlinear) variational inequality and as a consequence obtained new existence results. In another important contribution, Mosco [40] regularized a nonlinear variational inequality and established its stability for the case when the data is subjected to a perturbation. Bakushinskii and Polak [6] combined the regularization approach to a numerical scheme and proposed the principle of iterative regularization. In the same vein, A. Kaplan and R. Tichatschke, in their seminal contributions, investigated various aspects of iterative regularization and its connection with proximal point algorithms (see [29] and the cited reference therein). All of these works primarily focused on variational inequalities with monotone operators. In order to tackle more general boundary value problems, Liskovets [36] studied a regularization scheme for variational inequalities with pseudo-monotone operators. Recently, Gwinner [25] gave new convergence results for equilibrium problems with pseudo-monotone forms. Ya. I. Alber, along with his several coworkers, explored various aspects of regularization for general variational inequalities under milder conditions. A nice exposition of their work can be found in the recent monograph [4]. Liu and Nashed [39] focused on variational inequalities with co-coercive maps and gave new rates of convergence for regularized solutions. Some of the related ideas are available in Badriev et al. [5], Djafari Rouhani and Khan [16], Konnov et al. [33], Liu [37], and the cited references therein.

In this work, we present an extension of some of the above-mentioned contributions to quasi variational inequalities. More specifically, we will show that, under suitable conditions, regularized solutions converge to a solution of the original quasi variational inequality, provided that the original problem is solvable. This result is valid for quasi variational inequalities with non-coercive operators. To the best of our knowledge, a regularization theory for a quasi variational inequality with set-valued maps was initiated by Giannessi and Khan [19]. However, the results in [19] were proved under quite stringent conditions, and no results concerning the strong convergence of regularized solutions were given. In this work, besides studying the stable approximation of a quasi variational inequality with pseudo-monotone and generalized monotone operators, we give results concerning the strong convergence of regularized solutions. For the case of monotone operators, we heavily rely on some of the techniques developed by Alber et al. [2], to strengthen the results given in [19] and to give new results for quasi variational inequalities.

The rest of the paper is divided into seven sections. Section 2 recalls the basic definitions and results for their later use in this work. Section 3 deals with an existence theory for quasi variational inequalities. Section 4 discusses the solvability of a regularized quasi variational inequality with monotone operators. In Section 5, we work under the assumption that regularized solutions are bounded and study their weak and strong convergence. We focus on quasi variational inequalities with monotone and generalized monotone maps. As our results show, the convergence criteria for generalized monotone maps is slightly more restrictive. In Section 6, we explore the conditions that ensure boundedness of regularized solutions. Due to the results of Section 5, all these conditions, in fact, turn out to be sufficient conditions for the solvability of quasi variational inequalities. Section 7 discusses applications to quasi hemi-variational inequalities and inverse problems. Some remarks and open questions in Section 8 conclude the paper.

2 Preliminaries

One of the most commonly used techniques for the solvability of quasi variational inequalities is finding fixed points of the associated variational selection (see [1],[17],[40],[44]). To shed some light on this idea, we fix an element $v \in \mathscr{C}$ and consider the following parametric variational inequality with element v as the parameter: find $x \in \mathscr{K}(v)$ such that for some $w \in \mathscr{F}(x)$, we have

$$\langle w - f, z - x\rangle \geq \varphi(x) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(v). \tag{5}$$

This allows us to define a set-valued map, the so-called variational selection, $\mathscr{S} : \mathscr{C} \rightrightarrows \mathscr{C}$ such that for any $v \in \mathscr{C}$, $\mathscr{S}(v)$ is the set of all solutions of (5). It is evident that, if $x \in \mathscr{C}$ is a fixed point of $\mathscr{S}$, that is, if $x \in \mathscr{S}(x)$, then x is a solution of quasi variational inequality (1). The following fixed point theorem of Kluge [32], therefore, turns out to be an important tool for solving (1).

Theorem 2.1. [32] *Let Z be a reflexive Banach space and let $C \subset Z$ be nonempty, convex, and weakly closed. Assume that $\Psi : C \rightrightarrows C$ is a set-valued map such that for every $u \in C$, the set $\Psi(u)$ is nonempty, closed, and convex, and the graph of Ψ is weakly closed. Assume that either the set C is bounded or the set $\Psi(C)$ is bounded. Then the map Ψ has at least one fixed point in C.*

To employ the above theorem for the solvability of (1), we need to verify that the conditions imposed on Ψ are satisfied by $\mathscr{S}$. Since the properties of $\mathscr{S}$ are determined by the data $\{\mathscr{F}, \mathscr{K}, \mathscr{C}, f, \varphi\}$, we first recall the basic classes of functions that will be dealt with in this work.

Definition 2.1. *Let $\mathscr{A} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ be a set-valued map, and let $(x,u), (y,v) \in \mathscr{G}(\mathscr{A})$ be arbitrary. The map $\mathscr{A}$ is said to be:*

- *monotone, if $\langle u - v, x - y\rangle \geq 0$;*
- *strictly monotone, if $\langle u - v, x - y\rangle > 0, \quad$ only if $\; x \neq y$;*
- *m-relaxed monotone, if $\langle u - v, x - y\rangle \geq -m\|x - y\| > 0, \quad m > 0$;*
- *maximal monotone, if the graph of the monotone map $\mathscr{A}$ is not included in the graph of any other monotone map with the same domain;*
- *coercive, if $\langle u, x\rangle \geq m(\|x\|)\|x\|$, where $m : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{r\to\infty} m(r) = +\infty$.*

Definition 2.2. *Let $\mathscr{A} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ be a set-valued map.*

- *The map $\mathscr{A}$ is called pseudo-monotone, if for each $x \in \mathscr{D}(\mathscr{A})$, the set $\mathscr{A}(x)$ is bounded, closed, and convex, the map $\mathscr{A}$ is finitely continuous, and for any $\{(x_n, w_n)\} \subset \mathscr{G}(\mathscr{A})$ such that $x_n \rightharpoonup x$ and $\limsup_{n\to\infty}\langle w_n, x_n - x\rangle \leq 0$, it holds that for each $y \in \mathscr{B}$, there exists $w(y) \in \mathscr{A}(x)$ satisfying $\liminf_{n\to\infty}\langle w_n, x_n - y\rangle \geq \langle w(y), x - y\rangle$.*

- *The map $\mathscr{A}$ is called generalized monotone, if for any $\{(x_n, w_n)\} \subset \mathscr{G}(\mathscr{A})$ with $x_n \rightharpoonup x$ and $w_n \rightharpoonup w$ such that $\limsup_{n\to\infty}\langle w_n, x_n - x\rangle \leq 0$, we have $w \in \mathscr{F}(x)$ and $\langle w_n, x_n\rangle \to \langle w, x\rangle$.*
- *The map $\mathscr{A}$ is said to possess S_+ property if for any sequence $\{(x_n, w_n)\} \subset \mathscr{G}(\mathscr{A})$ with $x_n \rightharpoonup x \in \mathscr{D}(\mathscr{A})$ and $\limsup_{n\to\infty}\langle w_n, x_n - x\rangle \leq 0$, we have $x_n \to x$.*

For details on the maps of monotone type, the reader is referred to monographs [32],[43],[45].

Definition 2.3. *The map $\mathscr{K}$ is called M-continuous relative to φ, if the following conditions hold:*

(M1) *For any sequence $\{x_n\} \subset \mathscr{C}$ with $x_n \rightharpoonup x$, and for each $y \in \mathscr{K}(x)$, there exists $\{y_n\}$ such that $y_n \in \mathscr{K}(x_n)$, $y_n \to y$ and $\varphi(y_n) \to \varphi(y)$.*

(M2) *For $y_n \in \mathscr{K}(x_n)$ with $x_n \rightharpoonup x$ and $y_n \rightharpoonup y$, we have $y \in \mathscr{K}(x)$.*

Moreover, $\mathscr{K}$ is called fast M-continuous relative to φ and $\{\varepsilon_n\} \subset \mathbb{R}_+$ with $\varepsilon_n \downarrow 0$, if the following conditions hold:

(FM1) *For any sequence $\{x_n\} \subset \mathscr{C}$ with $x_n \rightharpoonup x$, and for each $y \in \mathscr{K}(x)$, there exists $\{y_n\}$ such that $y_n \in \mathscr{K}(x_n)$, $\varepsilon_n^{-1}\|y_n - y\| \to 0$ and $\varepsilon_n^{-1}|\varphi(y_n) - \varphi(y)| \to 0$.*

(FM2) *For $y_n \in \mathscr{K}(x_n)$ with $x_n \rightharpoonup x$, there exists $z_n \in \mathscr{K}(x)$ such that $\varepsilon_n^{-1}(z_n - y_n) \rightharpoonup 0$ and $\varepsilon_n^{-1}|\varphi(z_n) - \varphi(y_n)| \to 0$.*

Remark 2.1. In his celebrated paper, Mosco [40] introduced the above notion of set-convergence and used it for stable approximation of variational inequalities when the underlying convex set K is approximated by a sequence of convex sets $\{K_n\}$. Recently, Alber et al. [2] employed this convergence for regularization of set-valued variational inequalities. The notion of M-continuity when $\mathscr{K}$ is a set-valued map was used by Mosco [41] for the study of quasi variational inequalities.

The following estimate by Alber and Notik [3] turns out to be an indispensable technical tool:

Lemma 2.1. *Let Z be a reflexive Banach space with Z^* as its dual. Let $\mathscr{A} : Z \rightrightarrows Z^*$ be a monotone map with $\bar{x} \in int(\mathscr{D}(\mathscr{A}))$. Then there exists a constant $r = r(\bar{x}) > 0$ such that for every $(x, w) \in \mathscr{G}(\mathscr{A})$ and corresponding $c := \sup\{\|w\| : \ \|x - \bar{x}\| \leq r, \ \text{and} \ w \in \mathscr{A}(x)\} < \infty$, we have*

$$\langle w, x - \bar{x}\rangle \geq r\|w\| - (\|x - \bar{x}\| + r)c.$$

The above estimate is derived from the fact that $\mathscr{A}$ is locally bounded at $\bar{x}$ which belongs to the interior of its domain. Since Borwein and Fitzpatrick [9] showed that a monotone map is locally bounded at every absorbing point, the conclusions of Lemma 2.1 will still hold if we replace the condition $\bar{x} \in int(\mathscr{D}(\mathscr{A}))$ by a milder condition that $\bar{x}$ is an absorbing point of its domain.

We conclude this section by the following lemma, a proof of which is based on Banach-Steinhaus theorem, and can be found in Browder [12, Lemma 1].

Lemma 2.2. *Let Z be a Banach space with Z^* as its dual, let $\{x_n\} \subset Z$, and let $\{s_n\} \subset \mathbb{R}_+$ be such that $s_n \downarrow 0$. Fix $r > 0$, and assume that for every $h \in Z^*$ with $\|h\| \leq r$, there exists a constant C_h such that $\langle h, x_n\rangle \leq s_n\|x_n\| + C_h$, for every n. Then the sequence $\{x_n\}$ is bounded.*

3 Existence Theorems for Quasi Variational Inequalities

We begin by a new existence result for (1). In the following, by $\partial\varphi(\cdot)$ we denote the subdifferential of φ, and by Γ we represent the set of all $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\sigma(r) \to 0$ as $r \to \infty$.

Theorem 3.1. *Assume that the following conditions hold:*

($A_{\mathscr{F}}$) *$\mathscr{F} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ is a maximal monotone map.*

(A_φ) *$\varphi : \mathscr{B} \to \mathbb{R}$ is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{C}}$) *$\mathscr{C} \subset int(\mathscr{D}(\mathscr{F})) \cap int(\mathscr{D}(\partial\varphi))$.*

($A_{\mathscr{K}}$) *$\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ is M-continuous relative to φ.*

(A_{coer}) *For every $s \in \mathscr{B}^*$, there are $x_s \in \cap_{v\in\mathscr{C}}\mathscr{K}(v)$ with $\varphi(x_s) < \infty$ and $\sigma \in \Gamma$ such that for every $y \in \mathscr{D}(\mathscr{F})$ with $\|y\|$ sufficiently large and every $w \in \mathscr{F}(y)$, we have*

$$\langle w - s, y - x_s\rangle + \varphi(y) \geq -\sigma(\|y\|)\|y\|. \tag{6}$$

Then the quasi variational inequality (1) has at least one solution.

Proof. We will divide the proof into several parts. Our objective is to show that the solution map $\mathscr{S} : \mathscr{C} \rightrightarrows \mathscr{C}$ satisfies the assumptions imposed on the map Ψ in Theorem 2.1.
(I). *For every $v \in \mathscr{C}$, the set $\mathscr{S}(v)$ is nonempty.* Let $v \in \mathscr{C}$ be arbitrary. We will show that there exists $x \in \mathscr{K}(v)$ such that for some $w \in \mathscr{F}(x)$, we have

$$\langle w - f, z - x\rangle \geq \varphi(x) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(v). \tag{7}$$

Define a set-valued map $\mathscr{T} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ by

$$\mathscr{T}(x) = \mathscr{F}(x) + N_{\mathscr{K}(v)}(x) + \partial\varphi(x),$$

where $N_{\mathscr{K}(v)}$ is the normal map of $\mathscr{K}(v)$. It is known that $N_{\mathscr{K}(v)}$ is maximal monotone. Since $\mathscr{D}(N_{\mathscr{K}(v)}) \cap int(\mathscr{D}(\mathscr{F}) \cap int(\mathscr{D}(\partial\varphi)) \neq \emptyset$, we notice that $\mathscr{T}$ is a maximal monotone map with $\mathscr{D}(\mathscr{T}) = \mathscr{K}(v)$ (see [45]). Hence, by the classical surjectivity results, for every $n \in \mathbb{N}$, there exists $x_n \in \mathscr{D}(\mathscr{T})$ such that $f \in \mathscr{T}(x_n) + \varepsilon_n J(x_n)$, where $\{\varepsilon_n\} \subset \mathbb{R}_+$ is such that $\varepsilon_n \downarrow 0$. Therefore, for every $y \in \mathscr{K}(v)$ and some $w_n \in \mathscr{F}(x_n)$, $v_n \in N_{\mathscr{K}(v)}(x_n)$, $u_n \in \partial\varphi(x_n)$, we have $\langle w_n + v_n + u_n + \varepsilon_n J(x_n) - f, y - x_n\rangle = 0$, which, due to the definitions of $N_{\mathscr{K}(v)}(\cdot)$ and $\partial\varphi(\cdot)$, implies that

$$\langle w_n + \varepsilon_n J(x_n) - f, y - x_n\rangle \geq \varphi(x_n) - \varphi(y), \quad \text{for every } y \in \mathscr{K}(v). \tag{8}$$

We will show that $\{x_n\}$ is bounded. In view of the above inequality, for every $y \in \mathscr{K}(v)$, we have

$$\langle w_n - f, x_n - y\rangle + \varphi(x_n) \leq \langle \varepsilon_n J(x_n), y - x_n\rangle + \varphi(y) \leq -\varepsilon_n\|x_n\|\,[\|x_n\| - \|y\|] + \varphi(y),$$

where the second inequality follows from the properties of the duality map. By substituting $y = x_s$ in the above inequality, and using (A_{coer}), we obtain

$$\begin{aligned} -\sigma(\|x_n\|)\|x_n\| &\leq \langle w_n - s, x_n - x_s\rangle + \varphi(x_n) \\ &\leq -\langle s - f, x_n - x_s\rangle - \varepsilon_n\|x_n\|\,[\|x_n\| - \|x_s\|] + \varphi(x_s) \\ &\leq -\langle s - f, x_n - x_s\rangle + \varphi(x_s), \end{aligned}$$

because $\varepsilon_n\|x_n\|\,[\|x_n\|-\|x_s\|]$ is positive for $\|x_n\|$ sufficiently large. Therefore,

$$\langle s-f, x_n\rangle \leq \sigma(\|x_n\|)\|x_n\| + \langle s-f, x_s\rangle + \varphi(x_s).$$

The above inequality, in view of Lemma 2.2, ensures that $\{x_n\}$ is bounded. Due to the reflexivity of $\mathscr{B}$, we extract a subsequence $\{x_n\}$ converging weakly to some x. The Minty formulation (see (9) below) of (8) reads

$$\langle w_z + \varepsilon_n J(z) - f, z - x_n\rangle \geq \varphi(x_n) - \varphi(z), \quad \text{for every } w_z \in \mathscr{F}(z), \text{ for every } z \in \mathscr{K}(v),$$

which, due to (A_φ), under the limit $n\to\infty$, yields

$$\langle w_z - f, z - x\rangle \geq \varphi(x) - \varphi(z), \quad \text{for every } w_z \in \mathscr{F}(z), \text{ for every } z \in \mathscr{K}(v).$$

By invoking the Minty formulation once again, we obtain (7).
(II) *Minty formulation holds.* If $x \in \mathscr{K}(v)$ satisfies (7), then it is a solution of the following Minty variational inequality and vice versa: for every $z \in \mathscr{K}(v)$ and for every $u \in \mathscr{F}(z)$, we have

$$\langle u - f, z - x\rangle \geq \varphi(x) - \varphi(z). \tag{9}$$

The proof of the statement can be found in [2] or [19]. Notice that due to the assumption $(A_{\mathscr{C}})$, the arguments given in [2] for the case $\varphi \equiv 0$ are at once applicable to the present case. However, a direct proof can also be found in [19].
(III) *For every $v \in \mathscr{C}$, $\mathscr{S}(v)$ is closed and convex.* This is a direct consequence of (9) (see [19].)
(IV) *The graph of the variational selection $\mathscr{S}$ is weakly closed.* Let $\{(y_n, v_n)\} \subset \mathscr{G}(\mathscr{S})$ be such that $y_n \rightharpoonup y$ and $v_n \rightharpoonup v$. We will show that $(y, v) \in \mathscr{G}(\mathscr{S})$. The set $\mathscr{C}$ being convex and closed is also weakly closed, and consequently $v \in \mathscr{C}$. From the containment $(y_n, v_n) \in \mathscr{G}(\mathscr{S})$, we infer that $y_n \in \mathscr{K}(v_n)$ and that there exists $w_n \in \mathscr{F}(y_n)$ such that

$$\langle w_n - f, z - y_n\rangle \geq \varphi(y_n) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(v_n). \tag{10}$$

Let us first prove that $\{w_n\}$ is bounded. Notice that $y_n \in \mathscr{K}(v_n)$, in view of M-continuity of $\mathscr{K}$, implies that $y \in \mathscr{K}(v)$. Moreover, there exists $\{z_n\}$ converging strongly to y and such that $z_n \in \mathscr{K}(v_n)$ and $\varphi(z_n) \to \varphi(y)$. By substituting $z = z_n$ in (10), we obtain

$$\langle w_n - f, z_n - y_n\rangle \geq \varphi(y_n) - \varphi(z_n).$$

The view of Lemma 2.1 implies that there are constants $c > 0$ and $r > 0$ such that

$$\begin{aligned} r\|w_n - f\| &\leq \langle w_n - f, y_n - y\rangle + c(r + \|y_n - y\|) \\ &= \langle w_n - f, y_n - z_n\rangle + \langle w_n - f, z_n - y\rangle + c(r + \|y_n - y\|) \\ &\leq \|w_n - f\|\|z_n - y\| + c(r + \|y_n - y\|) + \varphi(z_n) - \varphi(y_n), \end{aligned}$$

and consequently,

$$[r - \|z_n - y\|]\|w_n - f\| \leq c(r + \|y_n - \tilde{x}\|) + \varphi(z_n) - \varphi(y_n).$$

Since the term $\|z_n - y\|$ can be made arbitrarily small, and since the right-hand side of the above inequality remains bounded, we infer the boundedness of $\{\|w_n - f\|\}$.

Let $z \in \mathscr{K}(x)$ be arbitrary. Then for some $\{z_n\}$ with $z_n \in \mathscr{K}(x_n)$, $z_n \to z$ and $\varphi(z_n) \to \varphi(z)$, and for any $w_z \in \mathscr{F}(z)$, we have

$$\begin{aligned} \langle w_z, y_n - z\rangle &\leq \langle w_z, y_n - z\rangle + \langle w_n - f, z_n - y_n\rangle + \varphi(z_n) - \varphi(y_n) \\ &\leq \langle w_n, z_n - z\rangle + \langle w_n - w_z, z - y_n\rangle + \langle f, y_n - z_n\rangle + \varphi(z_n) - \varphi(y_n), \end{aligned}$$

where we used the monotonicity of $\mathscr{F}$. By passing the above inequality to limit $n \to \infty$, we obtain

$$\langle w_z, y - z\rangle \leq \varphi(z) - \varphi(y) + \langle f, y - z\rangle.$$

In other words, we have shown that for every $z \in \mathscr{K}(v)$ and for every $w_z \in \mathscr{F}(z)$, we have

$$\langle w_z - f, z - y\rangle \geq \varphi(y) - \varphi(z).$$

By using the Minty formulation, we deduce that $(y, v) \in \mathscr{G}(\mathscr{S})$.

(V) *The set $\mathscr{S}(\mathscr{C})$ is bounded.* This follows from the condition (24) and its use in part (I).

Therefore, we have shown that $\mathscr{S} : \mathscr{C} \rightrightarrows \mathscr{C}$ is a set-valued map such that for every $v \in \mathscr{C}$, the set $\mathscr{S}(v)$ is nonempty, closed, and convex and the graph of $\mathscr{S}$ is weakly closed. Moreover, the set $\mathscr{S}(\mathscr{C})$ is bounded. This means that all assumptions of Theorem 2.1 are fulfilled for the set-valued map $\Psi = \mathscr{S} : \mathscr{C} \rightrightarrows \mathscr{C}$. From Theorem 2.1, we get that $\mathscr{S}$ has at least one fixed point in $\mathscr{C}$, and this yields that the quasi variational inequality (1) has at least one solution. □

Remark 3.1. Theorem 3.1 uses a milder coercivity condition (6). This condition was introduced by Gyuan and Kartsatos [23] (see also [24]), who used it extensively in connection with solvability of set-valued operator equations. We also remark that the condition $\mathscr{C} \subset int(\mathscr{D}(\partial\varphi))$ can be weakened by using the Moreau-Yosida regularization of $\partial\varphi$ (see Kenmochi [30]).

The following result holds by adopting the techniques of the above proof:

Corollary 3.1. *Assume that the following conditions hold:*

($A_{\mathscr{F}}$) *$\mathscr{F} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ is a maximal monotone map.*

(A_φ) *$\varphi : \mathscr{B} \to \mathbb{R}$ is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{C}}$) *$\mathscr{C} \subset int(\mathscr{D}(\mathscr{F})) \cap int(\mathscr{D}(\partial\varphi))$.*

($A_{\mathscr{K}}$) *$\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ is M-continuous relative to φ.*

(A'_{coer}) *There exists $x_0 \in \cap_{v\in\mathscr{C}}\mathscr{K}(v)$ with $\varphi(x_0) < \infty$ such that for every $w \in \mathscr{F}(x)$, we have*

$$\frac{\langle w, x - x_0\rangle + \varphi(x)}{\|x\|} \to 0, \quad as \quad \|x\| \to 0. \tag{11}$$

Then the quasi variational inequality (1) has at least one solution.

We also recover the following new existence theorem for a set-valued variational inequality.

Corollary 3.2. *Assume that the following conditions hold:*

($A_{\mathscr{F}}$) $\mathscr{F}:\mathscr{B}\rightrightarrows\mathscr{B}^*$ *is a maximal monotone map.*

(A_{φ}) $\varphi:\mathscr{B}\to\mathbb{R}$ *is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{C}}$) $\mathscr{C}\subset int(\mathscr{D}(\mathscr{F}))\cap int(\mathscr{D}(\partial\varphi))$.

(A''_{coer}) *For every* $s\in\mathscr{B}^*$ *there are* $x_s\in\mathscr{C}$ *with* $\varphi(x_s)<\infty$ *and* $\sigma\in\Gamma$ *such that for every* $y\in\mathscr{D}(\mathscr{F})$ *with* $\|y\|$ *sufficiently large and for every* $w\in\mathscr{F}(y)$, *we have*

$$\langle w-s,y-x_s\rangle\geq-\sigma(\|y\|)\|y\|.$$

Then there exists an element $x\in\mathscr{C}$ *such that for some* $w\in\mathscr{F}(x)$, *we have*

$$\langle w-f,z-x\rangle\geq\varphi(x)-\varphi(z),\quad \text{for every } z\in\mathscr{C}. \tag{12}$$

Moreover, the set of all solutions of (12) is closed and convex. The solution is unique if either the map $\mathscr{F}$ *is strictly monotone or the functional* φ *is strictly convex.*

Proof. The existence of a solution follows from Theorem 3.1 by taking $\mathscr{C}=\mathscr{K}(v)$ for every $v\in\mathscr{C}$. The second part can be deduced from [2] and also proved in [19]. □

4 Regularization

It is our objective to study the solvability of regularized quasi variational inequalities by employing the results given in the previous section. Since we are interested in the applications of our results to inverse and ill-posed problems, we emphasize on the case when instead of the data $(\mathscr{F},f,\varphi)$, only the contaminated data $(\mathscr{F}_{\alpha_n},f_{\beta_n},\varphi_{\gamma_n})$ are available. Here $\{\alpha_n\}$, $\{\beta_n\}$ $\{\gamma_n\}$ are sequences of positive reals, signifying the level of error in a certain sense which will be made clear shortly. To simplify the notation, instead of $(\mathscr{F}_{\alpha_n},f_{\beta_n},\varphi_{\gamma_n})$, we will use $(\mathscr{F}_n,f_n,\varphi_n)$. Let $\{\varepsilon_n\}$ be another sequence of positive reals, and let $\mathscr{R}:\mathscr{B}\to\mathscr{B}^*$ be a single-valued, hemi-continuous, monotone, and coercive map with $\mathscr{D}(\mathscr{R})=\mathscr{B}$ and $\|\mathscr{R}(x)\|\leq a\|x\|+b$, where a and b are constants.

We consider the following regularized quasi variational inequality: find $x_n\in\mathscr{C}\cap\mathscr{K}(x_n)$ such that for some $w_n\in\mathscr{F}_n(x_n)$, we have

$$\langle w_n+\varepsilon_n\mathscr{R}(x_n)-f_n,z-x_n\rangle\geq\varphi_n(x_n)-\varphi_n(z),\quad \text{for every } z\in\mathscr{K}(x_n). \tag{13}$$

In the following, by $S_n(RQVI)$, we will represent the set of all solutions of (13). Here, the map $\mathscr{R}$ is termed as the regularization map, and $\varepsilon_n>0$ is the regularization parameter. Moreover, any solution $x_n\in\mathscr{C}$ of (13) is referred to as a regularized solution of quasi variational inequality (1).

The conditions that ensure that $S_n(RQVI)$ is non-empty are similar to those which are needed for the solvability (1), except that the coercivity is now achieved through the regularization operator. The following result explains the procedure:

Theorem 4.1. *Assume for a fixed $n \in \mathbb{N}$, the following conditions hold:*

($A_{\mathscr{F}_n}$) *$\mathscr{F}_n : \mathscr{B} \rightrightarrows \mathscr{B}^*$ is a maximal monotone map.*

(A_{φ_n}) *$\varphi_n : \mathscr{B} \to \mathbb{R}$ is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{C}}$) *$\mathscr{C} \subset int(\mathscr{D}(\mathscr{F}_n)) \cap int(\mathscr{D}(\partial \varphi_n))$.*

($A_{\mathscr{K}}$) *$\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ is M-continuous relative to φ_n.*

($A_{\mathscr{R}}$) *$\mathscr{R} : \mathscr{B} \to \mathscr{B}^*$ is hemi-continuous and monotone.*

(A'''_{coer}) *$\mathscr{R} : \mathscr{B} \to \mathscr{B}^*$ is a coercive map with $\mathscr{D}(\mathscr{R}) = \mathscr{B}$ and $\|\mathscr{R}(x)\| \leq a\|x\| + b$, where a and b are constants. Furthermore, there exists $x_0 \in \cap_{v \in \mathscr{C}} \mathscr{K}(v)$.*

Then the regularized quasi variational inequality (13) admits a solution.

Proof. We claim that for any $x \in \mathscr{C}$ and any $w \in \mathscr{F}(x)$, the following coercivity condition holds:

$$\frac{\langle w + \varepsilon_n \mathscr{R}(x), x - x_0 \rangle + \varphi_n(x)}{\|x\|} \to \infty \quad \text{as} \quad \|x\| \to \infty.$$

In fact, from the definition of $\partial \varphi(\cdot)$, for any $z \in \mathscr{D}(\partial \varphi)$, we have

$$\varphi_n(x) \geq \varphi_n(x_0) + \langle x_0^*, x - x_0 \rangle, \quad \text{for every } x_0^* \in \partial \varphi_n(x_0), \quad \text{for every } x \in \mathscr{C}.$$

which implies that

$$\varphi_n(x) \geq \varphi_n(x_0) - \|x_0^*\| \|x - x_0\|.$$

Further, by the properties of the map $\mathscr{R}$, we have

$$\langle \mathscr{R}(x), x - x_0 \rangle \geq c(\|x\|)\|x\| - \|\mathscr{R}(x)\| \|x_0\| - \|\mathscr{R}(x_0)\| \|x - x_0\|,$$

where $c(r) \to \infty$ as $r \to \infty$. Therefore, for any $w \in \mathscr{F}_n(x)$ and for any $w_0 \in \mathscr{F}_n(x_0)$, we have

$$\begin{aligned} \langle w + \varepsilon_n \mathscr{R}(x), x - x_0 \rangle + \varphi_n(x) \quad \geq \quad & [\varepsilon_n \|\mathscr{R}(x_0)\| - \|w_0\| - \|x_0^*\|] \, \|x - x_0\| - \\ & \varepsilon_n \|\mathscr{R}(x)\| \|x_0\| + \varphi_n(x_0) + c(\|x\|)\|x\|, \end{aligned}$$

from which the coercivity readily follows. The existence of a regularized solution then is ensured by Corollary 3.1. This completes the proof. □

Remark 4.1. The coercivity of the regularized map $\mathscr{F} + \varepsilon_n \mathscr{R}$ is a direct consequence of the coercivity of $\mathscr{R}$ as long as $\mathscr{F}$ is monotone. However, for pseudo-monotone or generalized pseudo-monotone map $\mathscr{F}$, the coercivity of $\mathscr{F} + \varepsilon_n \mathscr{R}$ would demand additional conditions on $\mathscr{F}$.

5 Convergence of bounded regularized solutions

In this section we will study the convergence of bounded regularized solutions of quasi variational inequality (1) and postpone the discussion of boundedness of such solutions until the next section. Let Φ be the collection of all bounded functions $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$. We are now ready to introduce the conditions that characterize the perturbed data and connect it to the exact data.

Assumption (A1): There exists $\kappa \in \Phi$ such that for any $x \in \mathscr{C}$, and for every $w \in \mathscr{F}(x)$ and for every $\tilde{w} \in \mathscr{F}_n(x)$, we have

$$\|w - \tilde{w}\| \leq \alpha_n \kappa(\|x\|), \quad \alpha_n > 0. \tag{14}$$

Assumption (A2): There are constants $c_1 > 0$ and $c_2 > 0$ and $\ell \in \Phi$ with $\ell(r) \leq c_1 r + c_2$, such that for any $x \in \mathscr{C}$, we have

$$|\varphi_n(x) - \varphi(x)| \leq \gamma_n \ell(\|x\|), \quad \gamma_n > 0. \tag{15}$$

Assumption (A3): For $n \in \mathbb{N}$, $f_n \in \mathscr{B}^*$ and satisfies $\|f_n - f\| \leq \beta_n, \quad \beta_n > 0$.

Assumption (A4): For $\{\alpha_n, \beta_n, \gamma_n, \varepsilon_n\}$ defined above, we have

$$\left\{\varepsilon_n, \alpha_n, \beta_n, \gamma_n, \varepsilon_n^{-1}\alpha_n, \varepsilon_n^{-1}\beta_n, \varepsilon_n^{-1}\gamma_n\right\} \to 0 \quad \text{as } n \to \infty.$$

Theorem 5.1. *Assume that (A1)-(A4) hold, $S_n(RQVI) \neq \emptyset$ for every $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}} S_n(RQVI)$ is bounded. Furthermore, suppose*

($A'_{\mathscr{F}}$) *$\mathscr{F} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ is bounded and generalized monotone.*

(A_{φ}) *$\varphi : \mathscr{B} \to \mathbb{R}$ is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{K}}$) *$\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ is M-continuous relative to φ.*

Then, any sequence $\{x_n\}$, where $x_n \in S_n(RQVI)$, has a subsequence that converges weakly to a solution of (1). The convergence is strong if $\mathscr{F}$ possesses S_+ property.

Proof. Since the sequence $\{x_n\}$, where $x_n \in S_n(RQVI)$, is assumed to be bounded, and since $\mathscr{F}$ is bounded, the reflexivity of $\mathscr{B}$ ensures that we can extract a subsequence $\{(w_{i_n}, x_{i_n})\} \subset \mathscr{G}(\mathscr{F})$ such that w_{i_n} converges weakly to some $w \in \mathscr{B}^*$, and $\{x_{i_n}\}$ converges weakly to some $x \in \mathscr{B}$. For simplicity, we set $x_{i_n} = x_n$ and $w_{i_n} = w_n$. We will show that x is a solution of (1).

From the definition of x_n, we deduce that $x_n \in \mathscr{C} \cap \mathscr{K}(x_n)$, and for some $u_n \in \mathscr{F}_n(x_n)$, we have

$$\langle u_n + \varepsilon_n \mathscr{R}(x_n) - f_n, z - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z), \quad \text{for every } z \in \mathscr{K}(x_n). \tag{16}$$

The set $\mathscr{C}$ being closed is weakly closed and hence $x \in \mathscr{C}$. Moreover, $x_n \in \mathscr{K}(x_n)$, due to the M-continuity of $\mathscr{K}$, confirms that $x \in \mathscr{K}(x)$. Let $\{\tilde{x}_n\}$ be such that $\tilde{x}_n \in \mathscr{K}(x_n)$, $\tilde{x}_n \to x$ and $\varphi(\tilde{x}_n) \to \varphi(x)$. Substituting $z = \tilde{x}_n$ in (16) yields

$$\langle u_n + \varepsilon_n \mathscr{R}(x_n) - f_n, \tilde{x}_n - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(\tilde{x}_n). \tag{17}$$

Due to Assumption (A1), there exists a function $\kappa \in \Phi$ such that (14) holds for $w_n \in \mathscr{F}(x_n)$ which is defined above. This, by rearrangements of terms in (17), yields

$$\begin{aligned}\langle w_n, x_n - \tilde{x}_n\rangle &\leq \langle w_n - u_n, x_n - \tilde{x}_n\rangle + \varphi_n(\tilde{x}_n) - \varphi_n(x_n) + \langle \varepsilon_n \mathscr{R}(x_n) - f_n, \tilde{x}_n - x_n\rangle \\ &\leq \alpha_n \kappa(\|x_n\|)\|\tilde{x}_n - x_n\| + \varphi_n(\tilde{x}_n) - \varphi_n(x_n) + \langle \varepsilon_n \mathscr{R}(x_n) - f_n, \tilde{x}_n - x_n\rangle. \\ &\leq [\alpha_n \kappa(\|x_n\|)\| + \varepsilon_n \|\mathscr{R}(x_n)\| + \beta_n]\,\|\tilde{x}_n - x_n\| - \langle f, \tilde{x}_n - x\rangle + \langle f, x_n - x\rangle + \\ &\quad \varphi(\tilde{x}_n) - \varphi(x_n) + \gamma_n[\ell(\|x_n\|) + \ell(\|\tilde{x}_n\|)],\end{aligned}$$

where we used Assumptions (A1) and (A2). Therefore, by using the fact that $\{\varepsilon_n, \alpha_n, \beta_n, \gamma_n\} \to 0$, we get

$$\limsup_{n\to\infty} \langle w_n, x_n - \tilde{x}_n\rangle \leq \limsup_{n\to\infty} \{\varphi(\tilde{x}_n) - \varphi(x_n)\} \leq 0,$$

and finally

$$\limsup_{n\to\infty} \langle w_n, x_n - x\rangle \leq \limsup_{n\to\infty} \langle w_n, \tilde{x}_n - x\rangle \leq 0. \tag{18}$$

The above inequality, in view of the generalized monotonicity of $\mathscr{F}$, ensures that $w \in \mathscr{F}(x)$ and $\lim_{n\to\infty} \langle w_n, x_n\rangle = \langle w, x\rangle$. We will show that

$$\langle w - f, z - x\rangle \geq \varphi(x) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x).$$

Let $z \in \mathscr{K}(x)$ be arbitrary, and let $\{z_n\}$ be such that $z_n \in \mathscr{K}(x_n)$, $z_n \to z$ and $\varphi(z_n) \to \varphi(z)$. Moreover, (16) holds with $z = z_n$. Using all this information, we have

$$\begin{aligned}\langle w, x - z\rangle &= \liminf_{n\to\infty} \langle w_n, x_n - z\rangle \\ &\leq \limsup_{n\to\infty} \langle w_n - u_n, x_n - z\rangle + \limsup_{n\to\infty} \langle u_n, x_n - z_n\rangle + \limsup_{n\to\infty} \langle u_n, z_n - z\rangle \\ &\leq \limsup_{n\to\infty} \langle \alpha_n \kappa(\|x_n\|)\|x_n - z\| + \limsup_{n\to\infty} [\langle \varepsilon_n \mathscr{R}(x_n) - f_n, z_n - x_n\rangle + \varphi_n(z_n) - \varphi_n(x_n)] \\ &\leq \limsup_{n\to\infty} [\varphi(z_n) - \varphi(x_n)] + \langle f, x - z\rangle \\ &\leq \varphi(z) - \varphi(x) + \langle f, x - z\rangle,\end{aligned}$$

implying that

$$\langle w - f, z - x\rangle \geq \varphi(x) - \varphi(z).$$

Since $z \in \mathscr{K}(x)$ is arbitrary, and as we have already shown that $x \in \mathscr{C} \cap \mathscr{K}(x)$, we conclude that x is a solution of quasi variational inequality (1). The strong convergence of $\{x_n\}$ to x is a direct consequence of S_+ property and (18). This completes the proof. □

If the map $\mathscr{F}$ is monotone, then Assumption A1 can be considerably relaxed to the following:

Assumption (A5): There exists $\kappa \in \Phi$ such that for any $x \in \mathscr{C}$, and for every $w \in \mathscr{F}(x)$ (respectively $w_n \in \mathscr{F}(x)$), there exists $\tilde{w} \in \mathscr{F}_n(x)$ (respectively $w \in \mathscr{F}(x)$) satisfying (14).

The following result gives the convergence analysis for monotone quasi variational inequalities.

Theorem 5.2. *Assume that assumptions (A2)-(A5) hold, $S_n(RQVI) \neq \emptyset$ for every $n \in \mathbb{N}$, and $\cup_{n\in\mathbb{N}} S_n(RVI)$ is bounded. Furthermore, suppose that the following conditions hold:*

($A''_{\mathscr{F}}$) *$\mathscr{F} : \mathscr{B} \rightrightarrows \mathscr{B}^*$ is monotone.*

($A'_{\mathscr{C}}$) *$\mathscr{C} \subset int(\mathscr{D}(\mathscr{F}))$.*

(A_φ) *$\varphi : \mathscr{B} \to \mathbb{R}$ is a proper, convex, and lower-semicontinuous function.*

($A_{\mathscr{K}}$) *$\mathscr{K} : \mathscr{C} \rightrightarrows \mathscr{C}$ is M-continuous relative to φ.*

Then, any sequence $\{x_n\}$, where $x_n \in S_n(RQVI)$, has a subsequence that converges weakly to a solution x of (1).

If

($A'_{\mathscr{K}}$) *$\mathscr{K}$ is fast M-continuous relative to φ and $\{\varepsilon_n\}$,*

then $\lim_{n\to\infty} \langle \mathscr{R}(x_n) - \mathscr{R}(x), x_n - x\rangle \to 0$ *and we have*

$$\langle \mathscr{R}(x), \bar{x} - x\rangle \geq 0 \quad \text{for every } \bar{x} \in \mathscr{S}(x). \tag{19}$$

In particular, if $\mathscr{R} = J$, the normalized duality map, then the element x is a minimal norm element of $\mathscr{S}(x)$. If additionally the space $\mathscr{B}$ is uniformly convex, then the convergence to x is strong.

Proof. Let $\{x_n\}$, where $x_n \in S_n(RQVI)$, be the sequence of regularized solutions of (1). By the same line of arguments as in the proof of Theorem 5.1, there exists a subsequence $\{x_n\}$ converging weakly to some $x \in \mathscr{C} \cap \mathscr{K}(x)$. Moreover, for an arbitrary $z \in \mathscr{K}(x)$ and some $\{z_n\}$ with $z_n \in \mathscr{K}(x_n)$, $z_n \to z$ and $\varphi(z_n) \to \varphi(z)$, we have

$$\langle u_n + \varepsilon_n \mathscr{R}(x_n) - f_n, z_n - x_n\rangle \geq \varphi_n(x_n) - \varphi_n(z_n), \tag{20}$$

where $u_n \in \mathscr{F}_n(x_n)$. For the moment assume that $\{u_n\}$ is bounded. Due to Assumption (A5), there exists $\{w_n\}$ with $w_n \in \mathscr{F}(x_n)$ satisfying (14). For any $w_z \in \mathscr{F}(z)$, we have

$$\begin{aligned}
\langle w_z, x_n - z\rangle &\leq \langle w_z, x_n - z\rangle + \langle u_n + \varepsilon_n \mathscr{R}_n(x_n) - f_n, z_n - x_n\rangle + \varphi_n(z_n) - \varphi_n(x_n)\\
&\leq \langle u_n - w_n, z_n - x_n\rangle + \langle \varepsilon_n \mathscr{R}(x_n) - f_n, z_n - x_n\rangle +\\
&\quad \langle w_n, z_n - z\rangle + \langle w_n - w_z, z - x_n\rangle + \varphi_n(z_n) - \varphi_n(x_n)\\
&\leq \langle u_n - w_n, z_n - x_n\rangle + \langle \varepsilon_n \mathscr{R}(x_n) - f_n, z_n - x_n\rangle + \langle w_n, z_n - z\rangle + \varphi_n(z_n) - \varphi_n(x_n),
\end{aligned}$$

where we used the monotonicity of $\mathscr{F}$. Therefore,

$$\begin{aligned}
\langle w_z, x - z\rangle &= \liminf_{n\to\infty} \langle w_z, x_n - z\rangle\\
&\leq \liminf_{n\to\infty} [\alpha_n \kappa_n(\|x_n\|)\|z_n - x_n\| + (\varepsilon_n \|\mathscr{R}(x_n)\| + \beta_n)\|z_n - x_n\| +\\
&\quad \langle f, x_n - z_n\rangle + \|w_n\|\|z_n - z\| + \gamma_n(\ell(\|x_n\|) + \ell(\|z_n\|)) + \varphi(z_n) - \varphi(x_n)]\\
&\leq \varphi(z) - \varphi(x) + \langle f, x - z\rangle.
\end{aligned}$$

In the above, we used the fact that due to the boundedness of $\{u_n\}$, the sequence $\{w_n\}$ is bounded as well. In conclusion, we have shown that for any $z \in \mathscr{K}(x)$ and for any $w_z \in F(z)$, we have

$$\langle w_z - f, z - x \geq \varphi(x) - \varphi(z).$$

Employing the equivalent Minty formulation, for some $w \in \mathscr{F}(x)$, we have

$$\langle w - f, z - x \geq \varphi(x) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x).$$

This, in view of the fact that $x \in \mathscr{C} \cap \mathscr{K}(x)$, confirms that x is a solution of (1).

To finish the proof, we have to show that $\{u_n\}$ is bounded. Let $\tilde{x} \in \mathscr{K}(x)$ be arbitrary, and let $\{z_n\}$ be such that $z_n \in \mathscr{K}(x_n)$, $z_n \to \tilde{x}$ and $\varphi(z_n) \to \varphi(\tilde{x})$. In view of Lemma 2.1, for $w_n \in \mathscr{F}(x_n)$ satisfying (14), there are constants $c > 0$ and $r > 0$, such that

$$\begin{aligned}
r\|w_n - f\| &\leq \langle w_n - f, x_n - \tilde{x}\rangle + c(r + \|x_n - \tilde{x}\|) \\
&= \langle w_n - f, x_n - z_n\rangle + \langle w_n - f, z_n - \tilde{x}\rangle + c(r + \|x_n - \tilde{x}\|) \\
&\leq [\alpha_n \kappa(\|x_n\|) + \beta_n + \varepsilon_n \|\mathscr{R}x_n\|]\|x_n - z_n\| + \|w_n - f\|\|z_n - \tilde{x}\| \\
&\quad + c(r + \|x_n - \tilde{x}\|) + \varphi(z_n) - \varphi(x_n),
\end{aligned}$$

where we used (20). The above inequality, taking into account that $\lim_{n\to\infty} \|z_n - \tilde{x}\| = 0$, ensures the boundedness of $\{\|w_n - f\|\}$. This further confirms the boundedness of $\{u_n\}$.

We now proceed to establish the remaining claims. By the above arguments, we have shown that $x \in \mathscr{S}(x)$. If $\mathscr{S}(x)$ is a singleton, then (19) follows trivially. Otherwise, let $\bar{x} \in \mathscr{S}(x)$ be chosen arbitrarily. Therefore, for some $\bar{w} \in \mathscr{F}(\bar{x})$, we have

$$\langle \bar{w} - f, z - \bar{x}\rangle \geq \varphi(\bar{x}) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x). \tag{21}$$

For this $\bar{x}$, let $\{z_n\}$ be such that $z_n \in \mathscr{K}(x_n)$, $\varepsilon_n^{-1}\|z_n - \bar{x}\|$, and $\varepsilon_n^{-1}|\varphi(z_n) \to \varphi(\bar{x})|$. Furthermore, let $\{\tilde{x}_n\} \subset \mathscr{K}(\bar{x})$ be such that $\varepsilon_n^{-1}(z_n - y_n) \rightharpoonup 0$ and $\varepsilon_n^{-1}|\varphi(z_n) - \varphi(y_n)| \to 0$. These sequences exist due to the fast M-continuity of $\mathscr{K}$. By manipulating terms in (20) and by using (21), we have

$$\begin{aligned}
\varepsilon_n\langle \mathscr{R}(x_n), x_n - \bar{x}\rangle &\leq \varepsilon_n\langle \mathscr{R}(x_n), x_n - z_n\rangle + \varepsilon_n\langle \mathscr{R}(x_n), z_n - \bar{x}\rangle \\
&\leq \langle u_n - f_n, z_n - x_n\rangle + \varphi_n(z_n) - \varphi_n(x_n) + \varepsilon_n\langle \mathscr{R}(x_n), z_n - \bar{x}\rangle \\
&\leq \langle u_n - f_n, z_n - \bar{x}\rangle + \langle u_n - f_n, \bar{x} - x_n\rangle + \varphi_n(z_n) - \varphi_n(x_n) + \varepsilon_n\langle \mathscr{R}(x_n), z_n - \bar{x}\rangle \\
&\leq [\|u_n - f_n\| + \varepsilon_n\|\mathscr{R}(x_n)\|]\|z_n - \bar{x}\| + \langle u_n - \bar{w} - f_n + f, \bar{x} - x_n\rangle + \\
&\quad \varphi(z_n) - \varphi(x_n) + \gamma_n[\ell(\|z_n\|) + \ell(\|x_n\|)] + \langle \bar{w} - f, \bar{x} - x_n\rangle \\
&\leq [\|u_n - f_n\| + \varepsilon_n\|\mathscr{R}(x_n)\|]\|z_n - \bar{x}\| + \beta_n\|x_n - \bar{x}\| + \gamma_n[\ell(\|z_n\|) + \ell(\|x_n\|)] + \\
&\quad \langle \bar{w} - f, \tilde{x}_n - x_n\rangle + \varphi(z_n) - \varphi(x_n) + \varphi(\tilde{x}_n) - \varphi(\bar{x}),
\end{aligned}$$

where we used monotonicity of $\mathscr{F}$. This confirms that

$$\begin{aligned}
\langle \mathscr{R}(x_n), x_n - \bar{x}\rangle &\leq \varepsilon_n^{-1}\|z_n - \bar{x}\|[\|u_n - f_n\| + \varepsilon_n\|\mathscr{R}(x_n)\|] + \varepsilon_n^{-1}\gamma_n[\ell(\|z_n\|) + \ell(\|x_n\|)] + \\
&\quad + \varepsilon_n^{-1}\beta_n\|x_n - \bar{x}\| + \langle \bar{w} - f, \varepsilon_n^{-1}(\tilde{x}_n - x_n)\rangle + \varepsilon_n^{-1}[\varphi(z_n) - \varphi(\bar{x}) + \varphi(\tilde{x}_n) - \varphi(x_n)],
\end{aligned}$$

implying $\limsup_{n\to\infty}\langle \mathscr{R}(x_n), x_n - \bar{x}\rangle \leq 0$. This, in view of the monotonicity of $\mathscr{R}$, yields

$$\langle \mathscr{R}(\bar{x}), x - \bar{x}\rangle = \liminf_{n\to\infty}\langle \mathscr{R}(\bar{x}), x_n - \bar{x}\rangle \leq \limsup_{n\to\infty}\langle \mathscr{R}(x_n), x_n - \bar{x}\rangle.$$

Since $\bar{x} \in \mathscr{S}(x)$ was chosen arbitrarily, the Minty formulation ensures that (19) holds. Furthermore, by repeating the above arguments for the choice $\bar{x} = x$, we obtain $\langle \mathscr{R}(x_n) - \mathscr{R}(x), x_n - x\rangle \to 0$. In particular, if $\mathscr{R}$ is the normalized duality map J, then the above considerations at once imply that $\|x_n\| \to \|x\|$ and x is a minimal norm element of $\mathscr{S}(x)$. If additionally, $\mathscr{B}$ is uniformly convex, then we deduce that $\{x_n\}$ converges strongly to x. □

6 Boundedness of regularized solutions

In this section we give conditions that ensure that regularized solutions of (1) are bounded. We begin by showing that the solvability of (1), along with a boundedness assumption on the associated variational selection, confirms the boundedness of the regularized solutions.

Theorem 6.1. *Assume that assumptions (A2)-(A5) hold. Assume that $\mathscr{F}_n$ is δ_n-relaxed monotone with $0 < \delta_n < \delta$. Assume that for $v \in \mathscr{C}$, and for $n \in \mathbb{N}$, $\mathscr{S}_n(v) \neq \emptyset$, $\mathscr{S}(v) \neq \emptyset$, and $\mathscr{S}(\mathscr{C})$ is bounded. Then the sequence $\{x_n\}$, where $x_n \in S_n(RQVI)$, is bounded.*

Proof. Let $\{x_n\}$, where $x_n \in S_n(RQVI)$, be a sequence of regularized solutions. For a fixed but arbitrary $v \in \mathscr{C}$, let $x_n^v \in \mathscr{S}_n(v)$. Consequently, $x_n^v \in \mathscr{K}(v)$ and for some $w_n^v \in \mathscr{F}_n(x_n^v)$, we have

$$\langle w_n^v + \varepsilon_n \mathscr{R}(x_n^v) - f_n, z - x_n \rangle \geq \varphi_n(x_n^v) - \varphi_n(z), \quad \text{for every } z \in \mathscr{K}(v). \tag{22}$$

Now let $x^v \in \mathscr{S}(v)$ be arbitrary. Then $x^v \in \mathscr{K}(v)$ and for some $w^v \in \mathscr{F}(x^v)$, we have

$$\langle w^v - f, z - x^v \rangle \geq \varphi(x^v) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(v). \tag{23}$$

Due to Assumption (A5), there exists $\bar{w}_n^v \in \mathscr{F}_n(x^v)$ such that $\|\bar{w}_n^v - w^v\| \leq \alpha_n \kappa(\|x^v\|)$. Therefore,

$$\langle w^v - w_n^v, x_n^v - x^v \rangle = \langle w^v - \bar{w}_n^v, x_n^v - x^v \rangle - \langle \bar{w}_n^v - w_n^v, x^v - x_n^v \rangle \leq [\alpha_n \kappa(\|x^v\|) + \delta_n] \|x_n^v - x^v\|,$$

where we used the assumption that $\mathscr{F}_n$ is δ_n-relaxed monotone.

By setting $z = x^v$ in (22), $z = x_n^v$ in (23), and rearranging the resulting inequalities, we get

$$\begin{aligned} \varepsilon_n \langle \mathscr{R}(x_n^v), x_n^v - x^v \rangle &\leq \langle w^v - w_n^v, x_n^v - x^v \rangle + \langle f - f_n, x^v - x_n^v \rangle + \varphi_n(x_n^v) - \varphi_n(x^v) + \varphi(x^v) - \varphi(x_n^v) \\ &\leq [\alpha_n \kappa(\|x^v\|) + \beta_n + \gamma_n] \|x_n^v - x^v\| + \gamma_n [\ell(\|x^v\|) + \ell(\|x_n^v\|)]. \end{aligned}$$

Due to the boundedness of $\mathscr{S}(\mathscr{C})$, $\|x^v\|$ remains bounded, and consequently, the above estimate ensures that $\|x_n^v\| < C$, where C is a constant independent of $v \in \mathscr{C}$. This ensures that $\{x_n\}$ remains bounded. The proof is complete. $\square$

In the following result we prove the boundedness of regularized solution under a coercivity condition imposed on the perturbed operator and the exact operator, respectively.

Theorem 6.2. *Assume that $S_n(RQVI) \neq \emptyset$ for $n \in \mathbb{N}$, and assumptions (A3)-(A4) hold. Assume that for a given sequence $\{x_n\} \subset \mathscr{C}$, there is a bounded sequence $\{z_n\}$ with $z_n \in \mathscr{K}(x_n)$ and $\varphi_n(z_n) < \infty$ such that $\limsup \varphi_n(z_n) < \infty$. Assume that one of the following two conditions holds:*

(A). *For any sequence $\{(w_n, x_n)\} \subset \mathscr{G}(\mathscr{F}_n)$ with $\|x_n\| \to \infty$, we have*

$$\frac{\langle w_n, x_n - z_n \rangle + \varphi_n(x_n)}{\|x_n\|} \to \infty. \tag{24}$$

(B). *Assume that assumptions (A2)-(A5) hold, $\mathscr{F}_n$ is monotone, and $\mathscr{F}$ and φ are as in Corollary 3.2. Assume that for any sequence $\{(w_n, x_n)\} \subset \mathscr{G}(\mathscr{F})$ with $\|x_n\| \to \infty$, (24) holds.*

Then the sequence $\{x_n\}$, where $x_n \in S_n(RQVI)$ is chosen arbitrarily, is bounded.

Proof. (A). Let $\{x_n\}$ be unbounded and let $\{x_n\}$ be a subsequence such that $\|x_n\| \to \infty$. The definition of $\{x_n\}$ implies that $x_n \in \mathscr{C} \cap \mathscr{K}(x_n)$ and for some $w_n \in \mathscr{F}_n(x_n)$, we have

$$\langle w_n + \varepsilon_n \mathscr{R}(x_n) - f_n, z - x_n \rangle \geq \varphi_n(x_n) - \varphi_n(z), \quad \text{for every } z \in \mathscr{K}(x_n). \tag{25}$$

By substituting $z = z_n$ in the above inequality and rearranging the terms, we obtain

$$\begin{aligned} \langle w_n, x_n - z_n \rangle + \varphi_n(x_n) &\leq \varepsilon_n \langle \mathscr{R}(x_n) - f_n, z_n - x_n \rangle + \varphi_n(z_n) \\ &\leq [\varepsilon_n \|\mathscr{R}(z_n)\| + \|f\| + \beta_n] \|x_n - z_n\| + \varphi_n(z_n), \end{aligned}$$

where we used the monotonicity of $\mathscr{R}$. The above inequality implies that

$$\frac{\langle w_n, x_n - z_n \rangle + \varphi_n(x_n)}{\|x_n\|} \leq [\varepsilon_n \|\mathscr{R}(z_n)\| + \|f\| + \beta_n] \left[1 + \frac{\|z_n\|}{\|x_n\|}\right] + \frac{\varphi_n(z_n)}{\|x_n\|}.$$

Since the right-hand side term of above inequality remains bounded as $\|x_n\| \to \infty$, the coercivity condition (24) is violated. Therefore, $\{x_n\}$ must be bounded.

(B). Given $\{x_n\}$, where $x_n \in S_n(RQVI)$, we generate another sequence $\{y_n\}$ such that for $n \in \mathbb{N}$, y_n solves the following variational inequality on $\mathscr{K}(x_n)$: find $y \in \mathscr{K}(x_n)$ such that for some $u \in \mathscr{F}(y)$, we have

$$\langle u + \varepsilon_n \mathscr{R}(y) - f, z - y \rangle \geq \varphi(y) - \varphi(z), \quad \text{for every } z \in \mathscr{K}(x_n). \tag{26}$$

A solution $y_n \in \mathscr{K}(x_n)$ exists due to the imposed condition on the data. By following the same line of arguments as in the proof of part (A), we obtain the boundedness of the sequence $\{y_n\}$. We will now employ the boundedness of $\{y_n\}$ to show that $\{x_n\}$ is bounded as well. We substitute $z = x_n$ in (26) (with $y = y_n$ and $u = u_n$) and $z = y_n$ in (25), and add the resulting inequalities to obtain

$$\begin{aligned} \langle u_n - v_n, x_n - y_n \rangle + \langle v_n - w_n + f_n - f, x_n - y_n \rangle \quad &\geq \quad \varepsilon_n \langle \mathscr{R}(y_n) - \mathscr{R}(x_n), y_n - x_n \rangle + \\ &\quad\quad \varphi(y_n) - \varphi(x_n) + \varphi_n(x_n) - \varphi_n(y_n), \end{aligned}$$

where $v_n \in \mathscr{F}_n(y_n)$ is such that $\|v_n - u_n\| \leq \alpha_n \kappa(\|y_n\|)$. The above inequality implies

$$\begin{aligned} &[\alpha_n \kappa(\|y_n\|) + \beta_n + \varepsilon_n \|\mathscr{R}(y_n)\|] \|y_n - x_n\| + \varepsilon_n \|\mathscr{R}(x_n)\| \|y_n\| \\ &\qquad\qquad + \gamma_n [\ell(\|x_n\|) + \ell(\|y_n\|)] \quad \geq \quad \varepsilon_n \langle \mathscr{R}(x_n), x_n \rangle. \end{aligned}$$

where we used the monotonicity of $\mathscr{F}_n$. This further implies that

$$[\frac{\alpha_n}{\varepsilon_n} \kappa(\|y_n\|) + \frac{\beta_n}{\varepsilon_n} + \|\mathscr{R}(y_n)\|][\frac{\|y_n\|}{\|x_n\|} + 1] + \frac{\|\mathscr{R}(x_n)\|}{\|x_n\|} \|y_n\| + \frac{\gamma_n}{\varepsilon_n} [\frac{\ell(\|x_n\|)}{\|x_n\|} + \frac{\ell(\|y_n\|)}{\|x_n\|}] \geq m(\|x_n\|).$$

If there is a subsequence $\{x_n\}$ such that $\|x_n\| \to \infty$ as $n \to \infty$, then $m(\|x_n\|) \to \infty$; however, since the left side of the above inequality remains bounded, this is impossible. Therefore, $\{x_n\}$ must be bounded. This completes the proof. □

7 Applications

7.1 Quasi hemi-variational inequalities

Let $\mathscr{B}$ be a uniformly convex Banach space with a strictly convex topological dual $\mathscr{B}^*$. Let $\mathscr{C}$ be a nonempty, closed, and convex subset of $\mathscr{B}$, and let $\mathscr{K}: \mathscr{C} \rightrightarrows \mathscr{C}$ be a set-valued map such that for every $v \in \mathscr{C}$, the set $\mathscr{K}(v)$ is nonempty, closed and convex. Let $\mathscr{F}: \mathscr{B} \rightrightarrows \mathscr{B}^*$ be a given set-valued map, let $h: \mathscr{B} \to \mathbb{R}$ be a locally Lipschitz functional, and let $f \in \mathscr{B}^*$.

A genuine class of set-valued variational and quasi variational inequalities consists of subdifferential maps. Of particular relevance to this discussion is the Clarke's subgradient (see [14]). Given $h: \mathscr{B} \to \mathbb{R}$, locally Lipschitz near some $x \in \mathscr{B}$, the generalized derivative of h at x in direction $y \in \mathscr{B}$, denoted by $h^0(x,y)$, is defined by

$$h^0(x,y) = \limsup_{z \to x, \lambda \to 0} \lambda^{-1}[h(z+\lambda y) - h(z)],$$

where $z \in \mathscr{B}$, and $\lambda > 0$. Then the Clarke's subgradient of h at x, denoted by $\partial h(x)$, is given by

$$\partial h(x) = \{w \in \mathscr{B}^* | \quad h^0(x,y) \geq \langle w, y \rangle, \ \forall\, y \in \mathscr{B}\}.$$

Let us now consider the following quasi hemi-variational inequality: find $x \in \mathscr{C} \cap \mathscr{K}(x)$, such that for some $w \in \mathscr{F}(x)$ and some $u \in \partial h(x)$, we have

$$\langle w + u - f, z - x \rangle \geq 0 \quad \text{for every } z \in \mathscr{K}(x). \tag{27}$$

If $\mathscr{F}$ is m-strongly monotone, and ∂h is m-relaxed monotone, then the map $\mathscr{F} + \partial h$ is monotone, and our general theory can be applied to (27). An analogue of (27) for a single-valued $\mathscr{F}$ is studied in [37] (see also [38]).

7.2 Inverse problems

Assume that V is Hilbert space, $\mathscr{B}$ is a reflexive Banach space, and $A \subset \mathscr{B}$ is convex, and closed. We assume that $T: \mathscr{B} \times V \times V \to \mathbb{R}$ is a continuous and coercive trilinear form $T(a,u,v)$. Assume that $T(a,u,v)$ is symmetric in u, v. Finally, we assume that m is a bounded linear functional on V. Then, for any $a \in A$, it follows from the Riesz representation theorem that the following variational equation has a unique solution $u \in V$:

$$T(a,u,v) = m(v) \quad \text{for all} \quad v \in V. \tag{28}$$

We focus on the inverse problem associated with the direct problem (28) which is the following: Given a measurement of u, say z, estimate the coefficient a which together with u makes (28) true.

By the Riesz representation theorem, there is an isomorphism $E: V \to V^*$ defined by

$$(Eu)(v) = \langle u, v \rangle_V \quad \text{for all } v \in V.$$

For each $(a,u) \in A \times V$, $T(a,u,\cdot) - m(\cdot) \in V^*$. We define $e(a,u)$ to be the pre-image under E of this element:

$$\langle e(a,u), v \rangle_V = T(a,u,v) - m(v) \quad \text{for all } v \in V.$$

For a fixed $z \in V$, we consider the following minimization problem. Find $a^* \in A$ by solving

$$\min_{a \in A} J(a) = \|e(a,z)\|_V^2. \tag{29}$$

The functional J being convex, a necessary and sufficient optimality condition for (29) is a variational inequality involving the Fréchet derivative of $J(\cdot)$, defined by $\langle J'(a), b\rangle = 2\langle e(a,z), e_1(a,z)\rangle_V$, where e_1 is given by $\langle e_1(a,z), v\rangle = T(a,z,v)$ for all $v \in V$.

Since J is convex, the map J' is monotone, and hence our general theory can be applied to a perturbed analogue of the equation error approach (see [21]).

8 Concluding remarks

In this work, we studied weak and strong convergence of regularized solutions of quasi variational inequalities with set-valued maps. It would be of interest to extend our results to parabolic variational inequalities with set-valued maps. One of the main advantages of the regularization approach is that it can be successfully coupled with some iterative schemes, for instance in the so-called principle of iterative regularization (see [6]). It would be of advantage to extend this approach for quasi variational inequalities. Another useful topic that is well worth investigating, is the notion of generalized solution of quasi variational inequalities, investigated by Bruckner [10]. It seems that the generalized solutions could become a useful tool for quasi variational inequalities with pseudo-monotone or generalized monotone maps. In recent years there have been many developments in vector variational and quasi variational inequalities. However, the studies related to regularization and penalization of these problems are lacking. The results of this paper should be extended to vector quasi variational inequalities. We strongly believe that the scalarization techniques developed in the monograph [22] can be combined with the results of this paper to give new existence and perturbation results for vector variational and quasi variational inequalities.

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References

[1] R.P. Agarwal and D. O'Regan, Nonlinear generalized quasi-variational inequalities: a fixed point approach, *Math. Inequal. Appl.*, **6** (2003), 133–143.

[2] Y. I Alber, D. Butnariu, and I. Ryazantseva, Regularization of monotone variational inequalities with Mosco approximations of the constraint sets, *Set-Valued Anal.,* **13** (2005), 265–290.

[3] Y.I. Alber and A.I. Notik, Perturbed unstable variational inequalities with unbounded operators on approximately given sets, *Set-Valued Analysis*, **1**, (1993), 393-402.

[4] Y.I. Alber and I. Ryazantseva, *Nonlinear ill-posed problems of monotone type,* Springer, Dordrecht, 2006.

[5] I.B. Badriev, O.A. Zadvornov, and L.N. Ismagilov, On iterative regularization methods for variational inequalities of the second kind with pseudomonotone operators, *Comput. Methods Appl. Math.*, **3** (2003), 223–234.

[6] A.B. Bakushinskii and B.T. Polyak, The solution of variational inequalities, *Soviet Math. Dokl.*, **15** (1975), 1705-1710.

[7] A. Bensoussan and J.L. Lions, Nouvelles mthodes en contrle impulsionnel, *Appl. Math. Optim.*, **1**, (1974/75), 289–312.

[8] C. Baiocchi and A. Capelo, *Variational and quasivariational inequalities. Applications to free boundary problems,* John Wiley & Sons, Inc., New York, 1984.

[9] J.M. Borwein and S. Fitzpatrick, Local boundedness of monotone operators under minimal hypotheses, *Bull. Australian Math. Soc.,* **39** (1988), 439–441.

[10] G. Bruckner, On the existence of the solution of an abstract optimization problem related to a quasivariational inequality, *Z. Anal. Anwendungen*, **3** (1984), 81–86.

[11] F. E. Browder, Existence and approximation of solutions of nonlinear variational inequalities, *Proc. Nat. Acad. Sci. U.S.A.,* **56** (1966), 1080–1086.

[12] F. E. Browder, On a principle of H. Brézis and its applications, *J. Functional Analysis*, **25** (1977), 356–365.

[13] F. E. Browder and P. Hess, Nonlinear mappings of monotone type in Banach spaces, *J. Functional Analysis*, **11** (1972), 251–294.

[14] F. H. Clarke, *Optimization and nonsmooth analysis*, SIAM, Philadelphia, PA, 1990.

[15] F. Colonius and K. Kunisch, Output least squares stability in elliptic systems, *Appl. Math. Optim.*, **19** (1989), 33–63.

[16] B. Djafari Rouhani and A.A. Khan, On the embedding of variational inequalities, *Proc. Amer. Math. Soc.,* **131** (2003), 3861-3871.

[17] A. El Arni, Generalized quasi-variational inequalities on non-compact sets with pseudo-monotone operators, *J. Math. Anal. Appl.,* **249** (2000), 515–526.

[18] F. Giannessi, Embedding variational inequalities and their generalizations into a separation scheme, *J. Inequal. Appl.*, **1** (1997), no. 2, 139–147.

[19] F. Giannessi and A.A. Khan, Regularization of non-coercive quasi variational inequalities, *Control Cybernet*, **29** (2000), 91–110.

[20] M.S. Gockenbach and A.A. Khan, An abstract framework for elliptic inverse problems. Part 1: an output least-squares approach, *Math. Mech. Solids*, **12** (2007), 259-276.

[21] M.S. Gockenbach, B. Jadamba, and A.A. Khan, Equation error approach for elliptic inverse problems with an application to the identification of Lamé parameters, *Inverse Problems in Science and Engineering*, **16**, (2008), 349-367.

[22] A. Göpfert, Chr. Tammer, and H. Riahi, Existence and proximal point algorithms for nonlinear monotone complementarity problems, *Optimization*, **45** (1999), 57–68.

[23] Z. Gyuan and A.G. Kartsatos, Ranges of generalized pseudo-monotone perturbations of maximal monotone operators in reflexive Banach spaces, *Recent developments in optimization theory and nonlinear analysis*, 107–123, Amer. Math. Soc., Providence, RI, 1997.

[24] Z. Guan, A.G. Kartsatos and I.V. Skrypnik, Ranges of densely defined generalized pseudomonotone perturbations of maximal monotone operators, *J. Differential Equations*, **188** (2003), 332–351.

[25] J. Gwinner, Note on pseudomonotone functions, regularization, and relaxed coerciveness, *Nonlinear Anal.*, **30** (1997), 4217–4227.

[26] A. Hasanov and A. Erdem, Determination of unknown coefficient in a non-linear elliptic problem related to the elastoplastic torsion of a bar, *IMA J. Appl. Math.* **73** (2008), 579–591.

[27] G. Isac, Tikhonov regularization and the complementarity problem in Hilbert spaces, *J. Math. Anal. Appl.*, **174** (1993) 53-66.

[28] P. Jebelean, D. Motreanu, and V.V. Motreanu, A unified approach for a class of problems involving a pseudo-monotone operator, *Math. Nachr.*, **281** (2008), 1283–1293.

[29] A. Kaplan and R. Tichatschke, General view on proximal point methods to variational inequalities in Hilbert spaces—iterative regularization and approximation, *J. Nonlinear Convex Anal.*, **2** (2001), 305–332.

[30] N. Kenmochi, Nonlinear operators of monotone type in reflexive Banach spaces and nonlinear perturbations, *Hiroshima Math. J.*, **4** (1974), 229–263.

[31] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational inequalities and their applications*, SIAM, 1987.

[32] R. Kluge, *Nichtlineare variationsungleichungen und extremalaufgaben, theorie und nherungsverfahren,* VEB Deutscher Verlag der Wissenschaften, Berlin, 1979.

[33] I.V. Konnov, M.S.S. Ali and E.O. Mazurkevich, Regularization of nonmonotone variational inequalities, *Appl. Math. Optim.,* **53** (2006), 311-330.

[34] A.S. Kravchuk and P.J. Neittaanmaki, *Variational and quasi-Variational inequalities in Mechanics*, Springer, 2007.

[35] J.L. Lions and G. Stampacchia, Variational inequalities, *Comm. Pure Appl. Math.,* **20** (1967), 493–519.

[36] O.L. Liskovets, Regularization of mixed incorrect variational inequalities of monotonic type, *Soviet J. Numer. Anal. Math. Model.,* **6** (1991), 107-119.

[37] Z. Liu, Generalized quasi variational hemi-variational inequalities, *Appl. Math. Lett.*, **17** (2004), 741–745.

[38] Z. Liu, Browder-Tikhonov regularization of non-coercive evolution hemivariational inequalities, *Inverse Problems*, **21** (2005), 13–20.

[39] F. Liu and M.Z. Nashed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, *Set-Valued Anal.*, **6** (1998), 313–344.

[40] U. Mosco, Convergence of convex sets and of solutions of variational inequalities, *Adv. Math.*, **3** (1969), 512-585.

[41] U. Mosco, Implicit variational problems and quasi variational inequalities. In: *Nonlinear operators and the calculus of variations,* pp. 83–156. *Lecture Notes in Math.*, Vol. 543, Springer, Berlin, 1976.

[42] M.A. Noor and T.M. Rassias, Auxiliary principle technique for multi-valued mixed quasi-variational inequalities, *Math. Inequal. Appl.,* **7** (2004), 267–276.

[43] D. Pascali and S. Sburlan, *Nonlinear mappings of monotone type*, Sijthoff & Noordhoff International Publishers, 1978.

[44] B. Ricceri, Basic existence theorems for generalization variational and quasi-variational inequalities. In: Variational inequalities and network equilibrium problems, 251–255, Plenum, New York, 1995.

[45] E. Zeidler, *Nonlinear functional analysis and its applications.* II/B. Nonlinear monotone operators, Springer-Verlag, New York, 1990.

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