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Report No. 08 (2010)

Editors:
Professors of the Institute for Mathematics, Martin-Luther-University Halle-Wittenberg.

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Christiane Tammer
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät II
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany
Email: christiane.tammer@mathematik.uni-halle.de

Constantin Zălinescu
University Al.I.Cuza Iaşi
Faculty of Mathematics
700506 Iaşi, Romania
Email: zalinesc@uaic.ro

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Christiane Tammer and Constantin Zălinescu

Abstract Ekeland's variational principle (EVP) has many equivalent formulations and generalizations. In this chapter we present new minimal point theorems in product spaces and the corresponding vector variational principles for set-valued functions. As special cases we derive many of the existing variational principles of Ekeland's type. Moreover, we use our new approach to get extensions of EVPs of Isac–Tammer's and Ha's types, as well as extensions of EVPs for bi-functions. An important tool for deriving variational principles is a general nonlinear scalarization technique. We study useful properties of scalarizing functionals. Finally, we present several applications, especially necessary conditions for solutions of vector optimization problems.

1 Introduction

Deriving existence results and necessary conditions for approximate solutions of nonlinear optimization problems under week assumptions is an interesting and modern field in optimization theory. It is of interest to show corresponding results for optimization problems without any convexity and compactness assumptions. Ekeland's variational principle is a very deep assertion about the existence of an exact solution of a slightly perturbed optimization problem in a neighborhood of an approximate solution of the original problem. The importance of Ekeland's variational principle in nonlinear analysis is well known. Especially, this assertion is very useful for deriving necessary conditions under certain differentiability assumptions. In

Christiane Tammer
University Halle–Wittenberg, Institute of Mathematics, D-06126 Halle, e-mail: christiane.tammer@mathematik.uni-halle.de

Constantin Zălinescu
University Al.I.Cuza Iaşi, Faculty of Mathematics, 700506 Iaşi, Romania, e-mail: zalinesc@uaic.ro

optimal control Ekeland's principle can be used in order to prove an ε-maximum principle in the sense of Pontryagin and in approximation theory for deriving ε-Kolmogorov conditions.

Below we recall a versatile variant.

Proposition 1 (Ekeland's Variational Principle). *[21] Let (X,d) be a complete metric space and $f : X \to \mathbb{R} \cup \{+\infty\}$ a proper, lower semicontinuous function bounded below. Consider $\varepsilon > 0$ and $x_0 \in X$ such that $f(x_0) \leq \inf f + \varepsilon$. Then for every $\lambda > 0$ there exists $\overline{x} \in \operatorname{dom} f$ such that*

$$f(\overline{x}) + \lambda^{-1}\varepsilon d(\overline{x}, x_0) \leq f(x_0), \qquad d(\overline{x}, x_0) \leq \lambda, \tag{1}$$

and

$$f(\overline{x}) < f(x) + \lambda^{-1}\varepsilon d(\overline{x}, x) \quad \forall x \in X \setminus \{\overline{x}\}. \tag{2}$$

This means that for $\lambda, \varepsilon > 0$ and x_0 an ε-approximate solution of the minimization problem

$$f(x) \to \min \text{ s.t. } x \in X, \tag{3}$$

there exists a new point $\overline{x}$ that is not worse than x_0 and belongs to a λ-neighborhood of x_0, and especially, $\overline{x}$ satisfies the variational inequality (2). Relation (2) says, in fact, that $\overline{x}$ minimizes globally $f + \lambda^{-1}\varepsilon d(\overline{x}, \cdot)$, which is nothing else than a Lipschitz perturbation of f (for "smooth" principles, see [11]). Note that $\lambda = \sqrt{\varepsilon}$ gives a useful compromise in Proposition 1. For applications see Section 5 and, e.g., [24, 25, 58, 62, 61].

There are several statements that are equivalent to Ekeland's variational principle (EVP); see, e.g., [1, 2, 5, 16, 27, 52, 53, 54, 12, 15, 38, 29, 31, 30, 34, 33, 13, 14].

Phelps [54] introduced for $\varepsilon > 0$ the following closed convex cone K_ε in $X \times \mathbb{R}$, where X is a Banach space:

$$K_\varepsilon := \{(x,r) \in X \times \mathbb{R} \mid \varepsilon ||x|| \leq -r\} \tag{4}$$

(see Figure 1). Sometimes the cone K_ε is called a *Phelps cone*. Phelps has shown the existence of minimal points of a set $\mathscr{A} \subset X \times \mathbb{R}$ with respect to K_ε under a closedness assumption (H) and a boundedness assumption (B) concerning $\mathscr{A}$.

Proposition 2 (Phelps Minimal-Point Theorem). *[53, 54] Let X be a Banach space and $\mathscr{A} (\neq \emptyset) \subset X \times \mathbb{R}$. Assume*

(H) *$\mathscr{A}$ is closed,*
(B) $\inf\{r \in \mathbb{R} \mid (x,r) \in \mathscr{A}\} = 0$.

Suppose $\varepsilon > 0$. Then, for any point $(x_0, r_0) \in \mathscr{A}$ there exists a point $(\overline{x}, \overline{r}) \in \mathscr{A}$ such that

(a) $(\overline{x}, \overline{r}) \in \mathscr{A} \cap ((x_0, r_0) + K_\varepsilon)$,
(b) $\{(\overline{x}, \overline{r})\} = \mathscr{A} \cap ((\overline{x}, \overline{r}) + K_\varepsilon)$.

Remark 1. The assertion (a) in Proposition 2 can be considered as a *domination property* and assertion (b) describes a *minimal point* $(\bar{x}, \bar{r})$ of $\mathscr{A}$ with respect to K_ε.

In Phelps [53] and [54] it is shown that Ekeland's variational principle (Proposition 1) is a conclusion of a minimal-point theorem (Proposition 2) setting $\mathscr{A} = \operatorname{epi} f$ in Proposition 2. We will present extensions of Phelps minimal-point theorem to general product spaces and corresponding variational principles. The aim of this chapter is to give an overview on existing minimal-point theorems and variational principles of Ekeland's type for set-valued and vector-valued objective functions. In order to show such assertions a main tool is the application of a certain scalarization technique. In the following section we will discuss scalarizing functionals and their properties.

2 Preliminaries

Let us recall some notions and notation for sets and functions defined on locally convex spaces. So let (X, τ) be a locally convex space and $A \subset X$. By $\operatorname{cl} A$ (or $\operatorname{cl}_\tau A$ or $\overline{A}$ or $\overline{A}^\tau$), $\operatorname{int} A$ and $\operatorname{bd} A$ we denote the closure (with respect to τ when we want to emphasize the topology), the interior and the boundary of A; moreover $\operatorname{conv} A$ is the convex hull of A and $\overline{\operatorname{conv}} A := \operatorname{cl}(\operatorname{conv} A)$. As usual, for $A, B \subset X$, $a \in X$, $\Gamma \subset \mathbb{R}$ and $\alpha \in \mathbb{R}$ we set

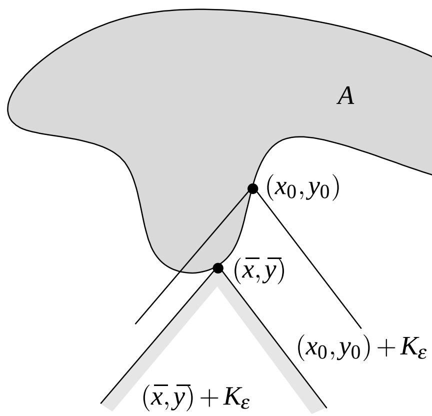

Fig. 1 Minimal point $(\bar{x}, \bar{y})$ of a set A with respect to K_ε

$$A+B := \{a+b \mid a \in A,\ b \in B\}, \quad a+B := \{a\}+B,$$
$$\Gamma A := \{\gamma a \mid \gamma \in \Gamma,\ a \in A\}, \quad \Gamma a := \Gamma\{a\}, \quad \alpha A := \{\alpha\}A, \quad -A := (-1)A.$$

The recession cone of the nonempty set $A \subset X$ is the set

$$A_\infty := \{u \in X \mid x+tu \in A\ \forall x \in A,\ \forall t \in \mathbb{R}_+\}.$$

It follows easily that A_∞ is a convex cone; A_∞ is also closed when A is closed. If A is a closed convex set then $A_\infty = \cap_{t \in \mathbb{P}} t(A-a)$, where $\mathbb{P} := \,]0,+\infty[$ and $a \in A$ (A_∞ does not depend on $a \in A$). Moreover, the indicator function associated to the set $A \subset X$ is the function $\iota_A : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, where $\infty := +\infty$. A cone $K \subset X$ is called pointed if $K \cap (-K) = \{0\}$.

Let $f : X \to \overline{\mathbb{R}}$; the domain and the epigraph of f are defined by

$$\operatorname{dom} f := \{x \in X \mid f(x) < +\infty\}, \quad \operatorname{epi} f := \{(x,t) \in X \times \mathbb{R} \mid f(x) \le t\}.$$

The function f is said to be *convex* if epi f is a convex set, and f is said to be *proper* if dom $f \neq \emptyset$ and f does not take the value $-\infty$. Of course, f is *lower semicontinuous* if epi f is closed. The class of lower semi-continuous (lsc for short) proper convex functions on X will be denoted by $\Gamma(X)$. Let $B \subset X$; $f : X \to \overline{\mathbb{R}}$ is called *B-monotone* if $x_2 - x_1 \in B \Rightarrow \varphi(x_1) \le \varphi(x_2)$. Furthermore, f is called *strictly B-monotone* if $x_2 - x_1 \in B \setminus \{0\} \Rightarrow \varphi(x_1) < \varphi(x_2)$.

We consider a proper closed convex cone $K \subset Y$ and $k^0 \in K \setminus (-K)$. As usual, we denote

$$K^+ := \{y^* \in Y^* \mid y^*(k) \ge 0\ \forall k \in K\},$$
$$K^\# := \{y^* \in Y^* \mid y^*(k) > 0\ \forall k \in K \setminus \{0\}\}$$

the positive dual cone of the convex cone $K \subset Y$ and the quasi interior of K^+, respectively.

In Section 3 we show several properties of scalarizing functionals. Motivated by papers on the field of economics, especially production theory (cf. Luenberger [50]) we assume that the sets A and K verify the *free-disposal condition* $A - K = A$ included in assumption $(A1)$ introduced in Section 3.2; for Lipschitz properties of φ_{A,k^0} (see (5) for its definition) we need the *strong free-disposal condition* $A - (K \setminus \{0\}) = \operatorname{int} A$, which is a part of assumption $(A2)$. The main results concerning Lipschitz properties are given in Section 3.4 under assumption $(A1)$: First, without convexity assumptions for the closed set $A \subset Y$ we prove that φ_{A,k^0} is Lipschitz on Y under the (stronger) assumption $k^0 \in \operatorname{int} K$ (Theorem 4); then, assuming that A is a convex set with nonempty interior and $k^0 \notin A_\infty$ we show that φ_{A,k^0} is locally Lipschitz on $\operatorname{int}(\operatorname{dom} \varphi_A) = \mathbb{R}k^0 + \operatorname{int} A$ (Proposition 5). Moreover, without assuming the convexity of A and without the assumption $k^0 \in \operatorname{int} K$ we give a characterization of Lipschitz continuity of φ_{A,k^0} on a neighbourhood of $y_0 \in Y$ using the notion of epi-Lipschitz set introduced by Rockafellar [55] (Theorem 5). In Section

3.5 we provide formulas for the conjugate and the subdifferential of φ_{A,k^0} when A is convex. Using the properties of the scalarizing functionals we present in Section 4 minimal-point theorems and corresponding variational principles. As an application of the Lipschitz properties of φ_{A,k^0}, we establish necessary conditions for properly efficient solutions of a vector optimization problem in terms of the Mordukhovich subdifferential in Section 5.2. Taking into account the fact that the conditions in the definition of properly efficient elements are related to the strong free disposal condition in $(A2)$ we get in Theorem 15 useful properties for the scalarizing functional φ_{A,k^0} as well as for the Mordukhovich subdifferential of the scalarized objective function.

3 Nonlinear Scalarization Functions

In order to show minimal-point theorems and corresponding variational principles in Section 4 we use a scalarization method by means of certain nonlinear functionals. In this section we discuss useful properties of these functionals (cf. Göpfert, Riahi, Tammer, Zălinescu [32] and Tammer, Zălinescu [63]).

3.1 Construction of scalarizing functionals

Having a nonempty subset A of a real linear space Y and an element $k^0 \neq 0$ of Y, Gerstewitz (Tammer) and Iwanow [28] introduced the function (see Figure 2)

$$\varphi_A := \varphi_{A,k^0} : Y \to \overline{\mathbb{R}}, \quad \varphi_{A,k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + A\}, \tag{5}$$

where, as usual, $\inf \emptyset := \infty$ (and $\sup \emptyset := -\infty$); we use also the convention $(+\infty) + (-\infty) := +\infty$.

This function was used by Chr. Tammer and her collaborators, as well as by D.T. Luc etc., mainly for scalarization of vector optimization problems. Luenberger [50, Def. 4.1] considered

$$\sigma(g; y) := \inf\{\xi \in \mathbb{R} \mid y - \xi g \in \mathscr{Y}\},$$

the corresponding function being called the shortage function associated to the production possibility set $\mathscr{Y} \subset \mathbb{R}^m$ and $g \in \mathbb{R}^m_+ \setminus \{0\}$. The case when $g = (1, \dots, 1)$ was introduced earlier by Bonnisseau and Cornet [10]. A similar function is introduced in [50, Def. 2.1] under the name of benefit function.

More recently such a function was considered in the context of mathematical finance beginning with Artzner et. al. [3]; see Heyde [42] and Hamel [39] for more historical facts. Under the name of topical function such functions were studied by Singer and his collaborators (see [59]). We discuss many important properties of φ_{A,k^0} in Section 3.2. Moreover, we study local continuity properties in Section

3.4. Very recently Bonnisseau and Crettez [4] obtained local Lipschitz properties for φ_{A,k^0} (called Luenberger shortage function in [4]) in a very special case, more general results are given by Tammer and Zălinescu [63]. Of course, φ_{A,k^0} is a continuous sublinear functional if A is a proper closed convex cone and $k^0 \in \operatorname{int} A$ (cf. Corollary 2) and so φ_{A,k^0} is Lipschitz continuous. Such Lipschitz properties of φ_{A,k^0} are of interest also in the case when $A \subset Y$ is an arbitrary (convex) set and the interior of the usual ordering cone in Y is empty like in mathematical finance where the acceptance sets are in function spaces as L_p and the corresponding risk measures are formulated by means of φ_{A,k^0} (see e.g. Föllmer and Schied [26]).

3.2 Properties of Scalarization Functions

Throughout this section Y is a separated locally convex space and Y^* is its topological dual, $K \subset Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subset Y$ is a nonempty set. The cone K determines the order $\leq_K$ on Y defined by $y_1 \leq_K y_2$ if $y_2 - y_1 \in K$.

Furthermore, we assume that A satisfies the following condition (see also [4]):

(A1) A is closed, satisfies the free-disposal assumption $A - K = A$, and $A \neq Y$.

We shall use also the (stronger) condition:

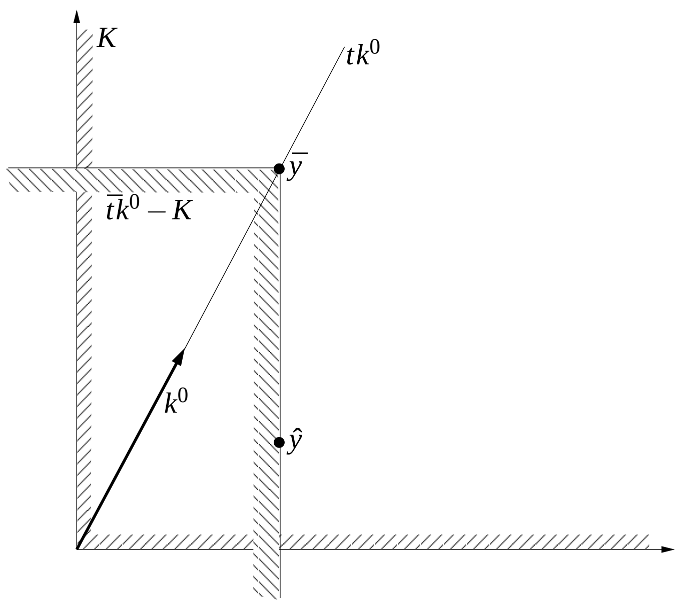

Fig. 2 Level sets of the function φ_{A,k^0} from (5), where $A = -K = -\mathbb{R}^2_+$ and $k^0 \in \operatorname{int} K$ holds.

(A2) A is closed, satisfies the strong free-disposal assumption $A-(K\setminus\{0\}) = \operatorname{int}A$, and $A \neq Y$.

Because $A-K = A\cup(A-(K\setminus\{0\}))$, we have that $(A2) \Rightarrow (A1)$. Moreover, the condition $A-(K\setminus\{0\}) = \operatorname{int}A$ is equivalent to $A-(K\setminus\{0\}) \subset \operatorname{int}A$.

Remark 2. Assume that the nonempty set A satisfies assumption $(A2)$. Then K is pointed, that is, $K\cap(-K) = \{0\}$, and $A-\mathbb{P}k^0 \subset \operatorname{int}A$ for $k^0 \in K\setminus\{0\}$.

The last assertion is obvious. For the first one, assume that $k \in K\cap(-K)\setminus\{0\}$. Take $a \in \operatorname{bd}A$ $(\subset A)$; such an a exists because $A \neq Y$. Then $a' := a-k \in \operatorname{int}A \subset A$, and so $a = a'-(-k) \in \operatorname{int}A$, a contradiction.

Remark 3. When A satisfies condition (A1) or (A2) with respect to K and $k^0 \in K\setminus(-K)$ then A satisfies condition (A1) or (A2), respectively, with respect to $\mathbb{R}_+k^0$. In fact in many situations it is sufficient to take $K = \mathbb{R}_+k^0$ for some $k^0 \in Y\setminus\{0\}$. In such a situation (A1) [resp. (A2)] means that A is a closed proper subset of Y and $A-\mathbb{R}_+k^0 = A$ [resp. $A-\mathbb{P}k^0 \subset \operatorname{int}A$].

The free-disposal condition $A = A-K$ shows that $K \subset -A_\infty$. As observed above A_∞ is also closed because A is closed. Hence $-A_\infty$ is the largest closed convex cone K verifying the free-disposal assumption $A = A-K$.

The aim of this section is to find a suitable functional $\varphi : Y \to \mathbb{R}$ and conditions such that two given nonempty subsets A and H of Y can be separated by φ.

To $A \subset Y$ satisfying (A1) and $k^0 \in K\setminus(-K)$ we associate the function φ_{A,k^0} defined in (5). We consider the set

$$A' := \{(y,t) \in Y\times\mathbb{R} \mid y \in tk^0 + A\}.$$

The assumption on A shows that A' is of **epigraph type**, i.e. if $(y,t) \in A'$ and $t' \geq t$, then $(y,t') \in A'$. Indeed, if $y \in tk^0 + A$ and $t' \geq t$, since

$$tk^0 + A = t'k^0 + A - (t'-t)k^0 \subset t'k^0 + A,$$

(because of (A1)) we obtain that $(y,t') \in A'$. Also observe that $A' = T^{-1}(A)$, where $T : Y\times\mathbb{R} \to Y$ is the continuous linear operator defined by $T(y,t) := tk^0 + y$. So, if A is closed (convex, cone), then A' is closed (convex, cone). Obviously, the domain of φ_A is the set $\mathbb{R}k^0 + A$ and $A' \subset \operatorname{epi}\varphi_A \subset \operatorname{cl}A'$ (because A' is of epigraph type), from which it follows that $A' = \operatorname{epi}\varphi_A$ if A is closed, and so φ_A is a lower semicontinuous (l.s.c.) function.

In the next results we collect several useful properties of φ_A (compare Göpfert, Riahi, Tammer, Zălinescu [32]).

Theorem 1. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in K\setminus(-K)$ and $A \subset Y$ is a nonempty set. Furthermore, suppose*

(A1) A is closed, satisfies the free-disposal assumption $A-K = A$, and $A \neq Y$.

Then φ_A (defined in (5)) is l.s.c., $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$*,*

$$\{y \in Y \mid \varphi_A(y) \le \lambda\} = \lambda k^0 + A \quad \forall \lambda \in \mathbb{R}, \tag{6}$$

and

$$\varphi_A(y + \lambda k^0) = \varphi_A(y) + \lambda \quad \forall y \in Y,\ \forall \lambda \in \mathbb{R}. \tag{7}$$

Moreover,

(a) φ_A is convex if and only if A is convex; $\varphi_A(\lambda y) = \lambda \varphi_A(y)$ for all $\lambda > 0$ and $y \in Y$ if and only if A is a cone.
(b) φ_A is proper if and only if A does not contain lines parallel to k^0, i.e.,

$$\forall y \in Y,\ \exists t \in \mathbb{R}\ :\ y + tk^0 \notin A. \tag{8}$$

(c) φ_A is finite-valued if and only if A does not contain lines parallel to k^0 and

$$\mathbb{R}k^0 + A = Y. \tag{9}$$

(d) Let $B \subset Y$; φ_A is B-monotone if and only if $A - B \subset A$.
(e) φ_A is subadditive if and only if $A + A \subset A$.

Proof. We have already observed that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$ and φ_A is l.s.c. when A is closed. From the definition of φ_A the inclusion $\supset$ in (6) is obvious, while the converse inclusion is immediate, taking into account the closedness of A. Formula (7) follows easily from (6).

(a) Since the operator T defined above is onto and $\operatorname{epi}\varphi_A = T^{-1}(A)$, we have that $\operatorname{epi}\varphi_A$ is convex (cone) if and only if $A = T(\operatorname{epi}\varphi_A)$ is so. The conclusion follows.

(b) We have

$$\varphi_A(y) = -\infty \Leftrightarrow y \in tk^0 + A\ \forall t \in \mathbb{R} \Leftrightarrow \{y + tk^0 \mid t \in \mathbb{R}\} \subset A.$$

The conclusion follows.

(c) The conclusion follows from (b) and the fact that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$.

(d) Suppose first that $A - B \subset A$ and take $y_1, y_2 \in Y$ with $y_2 - y_1 \in B$. Let $t \in \mathbb{R}$ be such that $y_2 \in tk^0 + A$. Then $y_1 \in y_2 - B \subset tk^0 + (A - B) \subset tk^0 + A$, and so $\varphi_A(y_1) \le t$. Hence $\varphi_A(y_1) \le \varphi_A(y_2)$. Assume now that φ_A is B-monotone and take $y \in A$ and $b \in B$. From (6) we have that $\varphi_A(y) \le 0$. Since $y - (y - b) \in B$, we obtain that $\varphi_A(y - b) \le \varphi_A(y) \le 0$, and so, using again (6), we obtain that $y - b \in A$.

(e) Suppose first that $A + A \subset A$ and take $y_1, y_2 \in Y$. Let $t_i \in \mathbb{R}$ be such that $y_i \in t_ik^0 + A$ for $i \in \{1, 2\}$. Then $y_1 + y_2 \in (t_1 + t_2)k^0 + (A + A) \subset (t_1 + t_2)k^0 + A$, and so $\varphi_A(y_1 + y_2) \le t_1 + t_2$. It follows that $\varphi_A(y_1 + y_2) \le \varphi_A(y_1) + \varphi_A(y_2)$. Assume now that φ_A is subadditive and take $y_1, y_2 \in A$. From (6) we have that $\varphi_A(y_1), \varphi_A(y_2) \le 0$. Since φ_A is subadditive, we obtain that $\varphi_A(y_1 + y_2) \le \varphi_A(y_1) + \varphi_A(y_2) \le 0$, and so, using again (6), we obtain that $y_1 + y_2 \in A$. $\square$

Remark 4. From Theorem 1 we get under assumption (A1) that φ_A is lower semicontinuous,

$$A = \{y \in Y \mid \varphi_A(y) \leq 0\}, \quad \operatorname{int} A \subset \{y \in Y \mid \varphi_A(y) < 0\}, \tag{10}$$

and so

$$\operatorname{bd} A = A \setminus \operatorname{int} A \supset \{y \in Y \mid \varphi_A(y) = 0\}. \tag{11}$$

In general the inclusion in (11) is strict.

Example 1. Consider $K := \mathbb{R}^2_+$, $k^0 := (1,0)$ and

$$A := (]-\infty, 0] \times]-\infty, 0]) \cup ([0,\infty[\times]-\infty, -1]).$$

Then $\varphi_A(u,v) = -\infty$ for $v \leq -1$, $\varphi_A(u,v) = u$ for $v \in (-1,0]$ and $\varphi_A(u,v) = \infty$ for $v > 0$. In particular, $\varphi_A(0,-1) = -\infty$ and $(0,-1) \in \operatorname{bd} A$ (see Figure 3).

Theorem 2. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subset Y$ is a nonempty set. Furthermore, suppose*

(A2) A is closed, satisfies the strong free-disposal assumption $A - (K \setminus \{0\}) = \operatorname{int} A$, and $A \neq Y$.

Then (a), (b), (c) from Theorem 1 holds, and moreover

(f) φ_A is continuous and

$$\{y \in Y \mid \varphi_A(y) < \lambda\} = \lambda k^0 + \operatorname{int} A, \quad \forall \lambda \in \mathbb{R}, \tag{12}$$

$$\{y \in Y \mid \varphi_A(y) = \lambda\} = \lambda k^0 + \operatorname{bd} A, \quad \forall \lambda \in \mathbb{R}. \tag{13}$$

(g) If φ_A is proper, then

$$\varphi_A \text{ is } B\text{-monotone} \Leftrightarrow A - B \subset A \Leftrightarrow \operatorname{bd} A - B \subset A.$$

Moreover, if φ_A is finite-valued, then

$$\varphi_A \text{ strictly } B\text{-monotone} \Leftrightarrow A - (B \setminus \{0\}) \subset \operatorname{int} A \Leftrightarrow \operatorname{bd} A - (B \setminus \{0\}) \subset \operatorname{int} A.$$

(h) Assume that φ_A is proper; then

$$\varphi_A \text{ is subadditive} \Leftrightarrow A + A \subset A \Leftrightarrow \operatorname{bd} A + \operatorname{bd} A \subset A.$$

Proof. Suppose now that (A2) holds.

(f) Let $\lambda \in \mathbb{R}$ and take $y \in \lambda k^0 + \operatorname{int} A$. Since $y - \lambda k^0 \in \operatorname{int} A$, there exists $\varepsilon > 0$ such that $y - \lambda k^0 + \varepsilon k^0 \in A$. Therefore $\varphi_A(y) \leq \lambda - \varepsilon < \lambda$, which shows that the inclusion $\supset$ always holds in (12. Let $\lambda \in \mathbb{R}$ and $y \in Y$ be such that $\varphi_A(y) < \lambda$. There exists $t \in \mathbb{R}, t < \lambda$, such that $y \in tk^0 + A$. It follows with (A2) that $y \in \lambda k^0 + A - (\lambda - t)k^0 \subset \lambda k^0 + \operatorname{int} A$. Therefore (12) holds, and so φ_A is upper semicontinuous. Because φ_A

is also lower semicontinuous, we have that φ_A is continuous. From (6) and (12) we obtain immediately that (13) holds.

(g) Let us prove the second part, the first one being similar to that of (and partially proved in) (d). So, let φ_A be finite-valued.

Assume that φ_A is strictly B-monotone and take $y \in A$ and $b \in -B \setminus \{0\}$. From (6) we have that $\varphi_A(y) \leq 0$, and so, by hypothesis, $\varphi_A(y-b) < 0$. Using (12) we obtain that $y - b \in \operatorname{int} A$. Assume now that $\operatorname{bd} A - (B \setminus \{0\}) \subset \operatorname{int} A$. Consider $y_1, y_2 \in Y$ with $y_2 - y_1 \in B \setminus \{0\}$. From (13) we have that $y_2 \in \varphi_A(y_2)k^0 + \operatorname{bd} A$, and so $y_1 \in \varphi_A(y_2)k^0 - (\operatorname{bd} A + (B \setminus \{0\})) \subset \varphi_A(y_2)k^0 + \operatorname{int} A$. From (12) we obtain that $\varphi_A(y_1) < \varphi_A(y_2)$. The remaining implication is obvious.

(h) Let φ_A be proper. One has to prove $\operatorname{bd} A + \operatorname{bd} A \subset A \Rightarrow \varphi_A$ is subadditive. Consider $y_1, y_2 \in Y$. If $\{y_1, y_2\} \not\subset \operatorname{dom} \varphi_A$, there is nothing to prove; hence let $y_1, y_2 \in \operatorname{dom} \varphi_A$. Then, by (13), $y_i \in \varphi_A(y_i)k^0 + \operatorname{bd} A$ for $i \in \{1,2\}$, and so $y_1 + y_2 \in (\varphi_A(y_1) + \varphi_A(y_2))k^0 + (\operatorname{bd} A + \operatorname{bd} A) \subset (\varphi_A(y_1) + \varphi_A(y_2))k^0 + A$. Therefore $\varphi_A(y_1 + y_2) \leq \varphi_A(y_1) + \varphi_A(y_2)$. $\square$

When $k^0 \in \operatorname{int} K$ we get an additional important property of φ_A (see also Theorem 4).

Corollary 1. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in \operatorname{int} K$ and $A \subset Y$ satisfies condition (A1). Then φ_A is finite-valued and continuous.*

Proof. Because $k^0 \in \operatorname{int} K$ we have that $\mathbb{R}k^0 + K = Y$. From Theorem 1 (c) it follows that

$$\operatorname{dom} \varphi_A = A + \mathbb{R}k^0 = A - K + \mathbb{R}k^0 = A + Y = Y.$$

Assuming that φ_A is not proper, from Theorem 1 (c) we get $y + \mathbb{R}k^0 \subset A$ for some $y \in Y$. Then $Y = y + \mathbb{R}k^0 - K \subset A - K = A$, a contradiction. Hence φ_A is finite-valued.

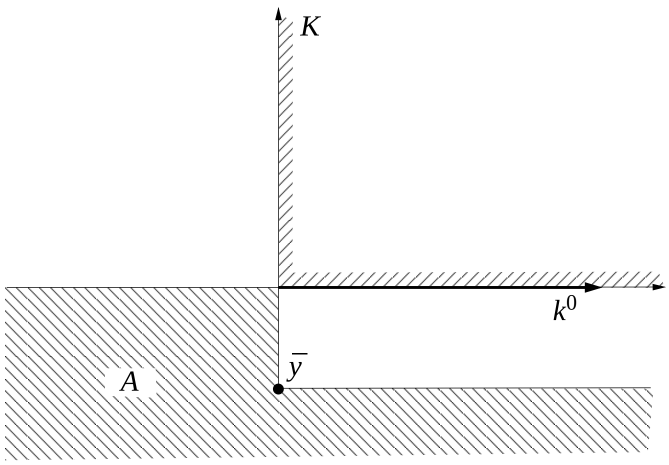

Fig. 3 $\bar{y} \in \operatorname{bd} A$ with $\varphi_A(\bar{y}) = -\infty$ in Example 1.

Moreover, we have that $A - \mathbb{P}k^0 \subset A - \operatorname{int} K \subset \operatorname{int}(A-K) = \operatorname{int} A$. Applying Theorem 2 (f) for K replaced by $\mathbb{R}_+ k^0$ we obtain that φ_A is continuous. $\square$

From the preceding results we get the following particular case.

Corollary 2. *Let $K \subset Y$ be a proper closed convex cone and $k^0 \in -\operatorname{int} K$. Then*

$$\varphi_K : Y \to \mathbb{R}, \qquad \varphi_K(y) := \inf\{t \in \mathbb{R} \mid y \in tk^0 + K\}$$

is a well-defined continuous sublinear function such that for every $\lambda \in \mathbb{R}$,

$$\{y \in Y \mid \varphi_K(y) \le \lambda\} = \lambda k^0 + K, \quad \{y \in Y \mid \varphi_K(y) < \lambda\} = \lambda k^0 + \operatorname{int} K.$$

Moreover, φ_K is strictly $(-\operatorname{int} K)$-monotone.

Proof. The assertions follow using Theorem 2 and Corollary 1 applied for $A := K$ and K replaced by $-K$. For the last part note that $K + \operatorname{int} K = \operatorname{int} K$. $\square$

Now all preliminaries are done, and we can prove the following nonconvex separation theorem.

Theorem 3 (Non-convex Separation Theorem). *Let $A \subset Y$ be a closed proper set with nonempty interior, $H \subset Y$ a nonempty set such that $H \cap \operatorname{int} A = \emptyset$. Let $K \subset Y$ be a proper closed convex cone and $k^0 \in \operatorname{int} K$. Furthermore, assume*

(A2) A is closed, satisfies the strong free-disposal assumption $A - (K \setminus \{0\}) = \operatorname{int} A$, and $A \ne Y$.

Then φ_A defined by (5) is a finite-valued continuous function such that

$$\varphi_A(x) \ge 0 > \varphi_A(y) \quad \forall x \in H,\ \forall y \in \operatorname{int} A; \tag{14}$$

moreover, $\varphi_A(x) > 0$ for every $x \in \operatorname{int} H$.

Proof. By Corollary 1 φ_A is a finite-valued continuous function. By Theorem 2 (f) we have that $\operatorname{int} A = \{y \in Y \mid \varphi_A(y) < 0\}$, and so (14) obviously holds.

Take $y \in \operatorname{int} H$; then there exists $t > 0$ such that $y - tk^0 \in H$. From (7) and (12) we obtain that $0 \le \varphi_A(y - tk^0) = \varphi_A(y) - t$, whence $\varphi_A(y) > 0$. $\square$

Of course, if we impose additional conditions on A, we have additional properties of the separating functional φ_A (see Theorems 1 and 2).

3.3 Continuity Properties

If A is a proper closed subset of Y (hence $\emptyset \ne A \ne Y$) and $A - \mathbb{P}k^0 \subset \operatorname{int} A$, applying Theorem 2 for $K := \mathbb{R}_+ k^0$ we obtain that φ_A is continuous (on Y) and (13) holds. In the next result we characterize the continuity of φ_A at a point $y_0 \in Y$ (compare Tammer and Zălinescu [63]).

Proposition 3. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subset Y$ is a nonempty set satisfying condition (A1). Then the function φ_A is (upper semi-) continuous at $y_0 \in Y$ if and only if $y_0 -]\varphi_A(y_0), \infty[\cdot k^0 \subset \operatorname{int} A$.*

Proof. If $\varphi_A(y_0) = \infty$ it is clear that φ_A is upper semicontinuous at y_0 and the inclusion holds. So let $\varphi_A(y_0) < \infty$.

Assume first that φ_A is upper semicontinuous at y_0. Let $\lambda \in]\varphi_A(y_0), \infty[$. Then there exists a neighbourhood V of y_0 such that $\varphi_A(y) < \lambda$ for every $y \in V$. It follows that for $y \in V$ we have $y \in \lambda k^0 + A$, that is, $V \subset \lambda k^0 + A$. Hence $y_0 \in \lambda k^0 + \operatorname{int} A$, whence $y_0 - \lambda k^0 \in \operatorname{int} A$.

Assume now that $y_0 -]\varphi_A(y_0), \infty[\cdot k^0 \subset \operatorname{int} A$ and take $\varphi_A(y) < \lambda < \infty$. Then, by our hypothesis, $V := \lambda k^0 + A$ is a neighbourhood of y_0 and from the definition of φ_A we have that $\varphi_A(y) \leq \lambda$ for every $y \in V$. Hence φ_A is upper semicontinuous at y_0. □

Corollary 3. *Under the hypotheses of Proposition 3 assume that φ_A is continuous at $y_0 \in \operatorname{bd} A$. Then $\varphi_A(y_0) = 0$.*

Proof. Of course, $\varphi_A(y_0) \leq 0$. If $\varphi_A(y_0) < 0$, from the preceding proposition we obtain the contradiction $y_0 = y_0 - 0k^0 \in \operatorname{int} A$. □

3.4 Lipschitz Properties

The primary goal of this section is to study local Lipschitz properties of the functional φ_{A,k^0} under as weak as possible assumptions concerning the subset $A \subset Y$ and $k^0 \in Y$ (compare Tammer and Zălinescu [63]).

When A is a convex set, as noticed above, φ_A is convex. In such a situation from the continuity of φ_A at a point in the interior of its domain one obtains the local Lipschitz continuity of φ_A on the interior of its domain (if the function is proper). Moreover, when $A = -K$ and $k^0 \in \operatorname{int} K$ then (it is well known that) φ_A is a continuous sublinear function, and so φ_A is Lipschitz continuous.

Recently in the case $Y = \mathbb{R}^m$ and for $K = \mathbb{R}^m_+$ Bonnisseau–Crettez [4] obtained the Lipschitz continuity of φ_A around a point $y \in \operatorname{bd} A$ when $-k^0$ is in the interior of the Clarke tangent cone of A at y. The (global) Lipschitz continuity of φ_A can be related to a result of Gorokhovik–Gorokhovik [35] established in normed vector spaces as we shall see in the sequel.

Theorem 4. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subset Y$ is a nonempty set satisfying condition (A1).*

(a) One has

$$\varphi_A(y) \leq \varphi_A(y') + \varphi_{-K}(y - y') \quad \forall y, y' \in Y. \tag{15}$$

(b) If $k^0 \in \operatorname{int} K$ then φ_A is finite-valued and Lipschitz on Y.

Proof. (a) By Theorem 1 (applied for A and $A := -K$, respectively) we have that φ_A and φ_{-K} are lower semicontinuous functions, φ_{-K} being sublinear and proper.

Let $y, y' \in Y$. If $\varphi_A(y') = +\infty$ or $\varphi_{-K}(y-y') = +\infty$ it is nothing to prove. In the contrary case let $t, s \in \mathbb{R}$ be such that $y - y' \in tk^0 - K$ and $y' \in sk^0 + A$. Then, taking into account assumption $(A1)$

$$y \in tk^0 - K + sk^0 + A = (t+s)k^0 + (A-K) = (t+s)k^0 + A.$$

It follows that $\varphi_A(y) \leq t+s$. Passing to infimum with respect to t and s satisfying the preceding relations we get (15).

(b) Assume that $k^0 \in \operatorname{int} K$. Let $V \subset Y$ be a symmetric closed and convex neighbourhood of 0 such that $k^0 + V \subset K$ and let $p_V : Y \to \mathbb{R}$ be the Minkowski functional associated to V; then p_V is a continuous seminorm and $V = \{y \in Y \mid p_V(y) \leq 1\}$. Let $y \in Y$ and $t > 0$ such that $y \in tV$. Then $t^{-1}y \in V \subset k^0 - K$, whence $y \in tk^0 - K$. Hence $\varphi_{-K}(y) \leq t$. Therefore, $\varphi_{-K}(y) \leq p_V(y)$. This inequality confirms that $(\mathbb{R}k^0 - K =) \operatorname{dom}\varphi_{-K} = Y$. Moreover, since φ_{-K} is sublinear we get $\varphi_{-K}(y) \leq \varphi_{-K}(y') + p_V(y-y')$ and so

$$\left|\varphi_{-K}(y) - \varphi_{-K}(y')\right| \leq p_V(y-y') \quad \forall y, y' \in Y, \tag{16}$$

that is, φ_{-K} is Lipschitz.

By Corollary 1 we have that φ_A is finite-valued (and continuous). From (15) we have that $\varphi_A(y) - \varphi_A(y') \leq \varphi_{-K}(y-y') \leq p_V(y-y')$, whence (interchanging y and y')

$$\left|\varphi_A(y) - \varphi_A(y')\right| \leq p_V(y-y') \quad \forall y, y' \in Y. \tag{17}$$

Hence φ_A is Lipschitz continuous (on Y). $\square$

Note that the condition $A - (K \setminus \{0\}) \subset \operatorname{int} A$ does not imply that φ_A is proper.

Example 2. Take $A := \{(x,y) \in \mathbb{R}^2 \mid y \geq -|x|^{-1}\}$, with the convention $0^{-1} := \infty$, and $K := \mathbb{R}_+ k^0$ with $k^0 := (0,-1)$. Then $A - (K \setminus \{0\}) = \operatorname{int} A$ and $\varphi_A(0,1) = -\infty$.

Note that, with our notation, [4, Proposition 7] asserts that φ_{A,k^0} is finite and locally Lipschitz provided $Y = \mathbb{R}^n$, $K = \mathbb{R}^n_+$ and $k^0 \in \operatorname{int} K$, which is much less than the conclusion of Theorem 4(ii).

Of course, in the conditions of Theorem 4(ii) we have that $-k^0 \in \operatorname{int} A_\infty$ because $K \subset -A_\infty$. In fact we have also a converse of Theorem 4(ii).

Proposition 4. *Assume that $K \subset Y$ is a proper closed convex cone, $k^0 \in K \setminus (-K)$ and $A \subset Y$ is a nonempty set satisfying condition (A1). If φ_A is finite-valued and Lipschitz then $-k^0 \in \operatorname{int} A_\infty$.*

Proof. By hypothesis there exists a closed convex and symmetric neighbourhood V of 0 such that (17) holds. We have that $A = \{y \in Y \mid \varphi_A(y) \leq 0\}$. Let $y \in A$, $v \in V$ and $\alpha \geq 0$. Then

$$\varphi_A(y + \alpha(v - k^0)) \leq \varphi_A(y + \alpha v) - \alpha \leq \varphi_A(y) + \alpha p_V(v) - \alpha \leq 0$$

because $V = \{y \in Y \mid p_V(y) \le 1\}$. Hence $V - k^0 \subset A_\infty$, which shows that $-k^0 \in \operatorname{int} A_\infty$. $\square$

Corollary 4. *Under the assumptions of Proposition 4, the function φ_A is finite-valued and Lipschitz if and only if $-k^0 \in \operatorname{int} A_\infty$.*

Proof. The necessity is given by Proposition 4. Assume that $-k^0 \in \operatorname{int} A_\infty$. Taking $K := -A_\infty$, using Theorem 4(b) we obtain that φ_A is finite-valued and Lipschitz. $\square$

If $\operatorname{int} K \neq \emptyset$ and $k^0 \notin \operatorname{int} K$, φ_{-K} is not finite-valued, and so it is not Lipschitz. One may ask if the restriction of φ_{-K} at its domain is Lipschitz. The next examples show that both situations are possible.

Example 3. Take $K = \mathbb{R}^2_+$ and $k^0 = (1,0)$. We have that $\varphi_{-K}(y_1,y_2) = y_1$ for $y_2 \le 0$, $\varphi_{-K}(y_1,y_2) = \infty$ for $y_2 > 0$, and so $\varphi_{-K}|_{\operatorname{dom}\varphi_{-K}}$ is Lipschitz.

Example 4. Take $K := \left\{(u,v,w) \in \mathbb{R}^3 \mid v,w \ge 0,\ u^2 \le vw\right\}$ and $k^0 := (0,0,1)$; then

$$\varphi_{-K}(x,y,z) = \begin{cases} \infty & \text{if } y > 0 \text{ or } [y = 0 \text{ and } x \neq 0], \\ z & \text{if } x = y = 0, \\ z - x^2/y & \text{if } y < 0. \end{cases}$$

It is clear that the restriction of φ_{-K} at its domain is not continuous at $(0,0,0) \in \operatorname{dom}\varphi_{-K}$ and the restriction of φ_{-K} at the interior of its domain is not Lipschitz. However, φ_{-K} is locally Lipschitz on the interior of its domain.

The last property mentioned in the previous example is a general one for φ_A when A is convex.

Proposition 5. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{0\}$ be such that $A - \mathbb{R}_+ k^0 = A$. If A is convex, has nonempty interior, and does not contain any line parallel with k^0 (or equivalently $k^0 \notin A_\infty$), then φ_A is locally Lipschitz on $\operatorname{int}(\operatorname{dom}\varphi_A) = \mathbb{R}k^0 + \operatorname{int} A$.*

Proof. Because A does not contain any line parallel with k^0, φ_A is proper (see Theorem 1 taking into account assumption $(A1)$). We know that $\operatorname{dom}\varphi_A = \mathbb{R}k^0 + A$, and so $\operatorname{int}(\operatorname{dom}\varphi_A) = \operatorname{int}(\mathbb{R}k^0 + A) = \mathbb{R}k^0 + \operatorname{int} A$ (see, e.g., [67, Exer. 1.4]). On the other hand it is clear that $A \subset \{y \in Y \mid \varphi_A(y) \le 0\}$. Since $\operatorname{int} A \neq \emptyset$, we have that φ_A is bounded above on a neighbourhood of a point, and so φ_A is locally Lipschitz on $\operatorname{int}(\operatorname{dom}\varphi_A) = \mathbb{R}k^0 + \operatorname{int} A$ (see e.g. [67, Cor. 2.2.13]). $\square$

We have seen in Theorem 4 that φ_A is Lipschitz even if A is not convex when $k^0 \in \operatorname{int} K$. So, in the sequel we are interested by the case in which A is not convex, $k^0 \notin \operatorname{int} K$ and A does not contain any line parallel with k^0.

Note that for A not convex and $y \in \operatorname{int}(\operatorname{dom}\varphi_A)$ we can have situations in which φ_A is not continuous at y or φ_A is continuous but not Lipschitz around y.

Example 5. Take $K := \mathbb{R}^2_+$, $k^0 := (1,0)$ and

$$A_1 := (]-\infty,0]\times]-\infty,1]) \cup ([0,1]\times]-\infty,0])$$
$$A_2 := \{(a,b) \mid a \in]0,\infty[,\ b \leq -a^2\} \cup (]-\infty,0]\times]-\infty,1]).$$

Then

$$\varphi_{A_1,k^0}(u,v) = \begin{cases} \infty & \text{if } v>1, \\ u & \text{if } 0<v\leq 1, \\ u-1 & \text{if } v\leq 0, \end{cases} \qquad \varphi_{A_2,k^0}(u,v) = \begin{cases} \infty & \text{if } v>1, \\ u & \text{if } 0<v\leq 1, \\ u-\sqrt{-v} & \text{if } v\leq 0. \end{cases}$$

It is clear that $(0,0) \in \operatorname{int}(\operatorname{dom}\varphi_{A_1})$ but φ_{A_1} is not continuous at $(0,0)$, and $(0,0) \in \operatorname{int}(\operatorname{dom}\varphi_{A_2})$, φ_{A_2} is continuous at $(0,0)$ but φ_{A_2} is not Lipschitz at $(0,0)$.

In what concerns the Lipschitz continuity of φ_A around a point $y \in \operatorname{dom}\varphi_A$ in finite dimensional spaces this can be obtained using the notion of epi-lipschitzianity of a set as introduced by Rockafellar [55] (see also [56]). We extend this notion in our context. We say that the set $A \subset Y$ is epi-Lipschitz at $\overline{y} \in A$ in the direction $v \in Y\setminus\{0\}$ if there exist $\varepsilon > 0$ and a (closed convex symmetric) neighbourhood V_0 of 0 in Y such that

$$\forall y \in (\overline{y}+V_0)\cap A,\ \forall w \in v+V_0,\ \forall \lambda \in [0,\varepsilon]:\ y+\lambda w \in A. \tag{18}$$

Note that (18) holds for $v=0$ if and only if $\overline{y} \in \operatorname{int}A$. Moreover, if $\overline{y} \in \operatorname{int}A$ then A is epi-Lipschitz at $\overline{y} \in A$ in any direction.

Theorem 5. *Let A be a proper closed subset of Y and $k^0 \in Y\setminus\{0\}$ be such that $A-\mathbb{R}_+k^0 = A$. Assume that $y_0 \in Y$ is such that $\varphi_A(y_0) \in \mathbb{R}$. Then φ_A is finite and Lipschitz on a neighbourhood of y_0 if and only if A is epi-Lipschitz at $\overline{y} := y_0 - \varphi_A(y_0)k^0$ in the direction $-k^0$.*

Proof. Using (7) we get $\varphi_A(\overline{y}) = 0$. Recall also that $A = \{y \in Y \mid \varphi_A(y) \leq 0\}$ and the finite values of φ_A are attained (because A is closed).

Assume that there exist a closed convex symmetric neighbourhood V of 0 in Y and $p : Y \to \mathbb{R}$ a continuous seminorm such that φ_A is finite on y_0+V and $|\varphi_A(y)-\varphi_A(y')| \leq p(y-y')$ for all $y,y' \in y_0+V$. Taking into account (7), we have that φ_A is finite on $\overline{y}+V$ and

$$\left|\varphi_A(y)-\varphi_A(y')\right| \leq p(y-y') \quad \forall y,y' \in \overline{y}+V.$$

Take $V_0 := \{y \in \frac{1}{3}V \mid p(y) \leq 1\}$ and $\varepsilon \in]0,1]$ such that $\varepsilon k^0 \in V_0$. Let us show that (18) holds with v replaced by $-k^0$. For this take $y \in (\overline{y}+V_0)\cap A$, $w \in -k^0+V_0$ and $\lambda \in [0,\varepsilon]$. Then $y-\lambda k^0-\overline{y} \in V_0+V_0 \subset V$ and $y+\lambda w-\overline{y} = y-\lambda k^0-\overline{y}+\lambda(w+k^0) \in V_0+V_0+V_0 \subset V$, and so

$$\begin{aligned}\varphi_A(y+\lambda w) &\leq \varphi_A(y-\lambda k^0)+p(\lambda(w+k^0)) = \varphi_A(y)-\lambda+\lambda p(w+k^0) \\ &\leq \lambda(p(w+k^0)-1) \leq 0.\end{aligned}$$

Hence $y+\lambda w \in A$.

Assume now that (18) holds with v replaced by $-k^0$. Let $r \in \,]0,\varepsilon]$ be such that $2r(1+p(k^0)) < 1$, where $p := p_{V_0}$. Of course, $\{y \mid p(y) \le \lambda\} = \lambda V_0$ for every $\lambda > 0$ and if $p(y) = 0$ then $y \in \lambda V_0$ for every $\lambda > 0$. Set

$$M := \{y \in \overline{y} + rV_0 \mid |\varphi_A(y)| \le p(y-\overline{y})\};$$

of course, $\overline{y} \in M$. We claim that $M = \overline{y}+rV_0$. Consider $y \in M$, $w \in V_0$ and $\lambda \in [0,r]$. Setting $y' := y - \varphi_A(y)k^0 \in A$, we have that $\varphi_A(y') = 0$ and

$$p(y'-\overline{y}) \le p(y-\overline{y}) + |\varphi_A(y)| \cdot p(k^0) \le r\left(1+p(k^0)\right) < \tfrac{1}{2} \le 1, \tag{19}$$

and so, by (18), $y' + \lambda(w-k^0) \in A$; hence $\varphi_A(y'+\lambda w) \le \lambda$.

Take $v \in rV_0$. On one hand one has

$$\varphi_A(y'+v) = \varphi_A\left(y' + p(v)\cdot \frac{1}{p(v)}v\right) \le p(v)$$

if $p(v) > 0$, and $\varphi_A(y'+v) = \varphi_A(y'+\lambda(\lambda^{-1}v)) \le \lambda$ for every $\lambda \in \,]0,r]$, whence $\varphi_A(y'+v) \le 0 = p(v)$. Therefore, $\varphi_A(y'+v) \le p(v)$.

On the other hand, assume that $\varphi_A(y'+v) < -p(v)$. Because $2r(1+p(k^0)) < 1$, there exists $t > 0$ such that $r + (t+r)p(k^0) \le 1/2$ and $\varphi_A(y'+v) < -p(v) - t =: t' < 0$. It follows that $y'+v-t'k^0 \in A$. Moreover, taking into account (19),

$$p(y'+v-t'k^0-\overline{y}) \le p(y'-\overline{y}) + p(v) + (t+p(v))p(k^0) \le 1/2 + r + (t+r)p(k^0) \le 1,$$

and so $y'+v-t'k^0 \in (\overline{y}+V_0) \cap A$. Using (18), if $p(v) > 0$ then

$$y'+tk^0 = y' - \left(t'+p(v)\right)k^0 = y'+v-t'k^0 + p(v)\left(-k^0 - \frac{1}{p(v)}v\right) \in A,$$

while if $p(v) = 0$ then

$$y' + (1-\gamma)tk^0 = y'+v-t'k^0 + \gamma t\left(-k^0 - (\gamma t)^{-1}v\right) \in A$$

for $\gamma := \min\{\frac{1}{2}, \varepsilon t^{-1}\}$. We get the contradiction $0 = \varphi_A(y') \le -t < 0$ in the first case and $0 = \varphi_A(y') \le -t(1-\gamma) < 0$ in the second case. Hence $\varphi_A(y'+v) \in \mathbb{R}$ and $|\varphi_A(y'+v) - \varphi_A(y')| \le p(v)$ for every $v \in rV_0$, or equivalently,

$$\varphi_A(y+v) \in \mathbb{R}, \quad |\varphi_A(y+v) - \varphi_A(y)| \le p(v) \quad \forall v \in rV_0. \tag{20}$$

When $y := \overline{y} \in M$, from (20) we get $\overline{y}+rV_0 \subset M$, and so $M = \overline{y}+rV_0$ as claimed. Moreover, if $y, y' \in \overline{y} + \frac{1}{2}rV_0$, then $y \in M$ and $y' = y+v$ for some $v \in rV_0$; using again (20) we have that $|\varphi_A(y') - \varphi_A(y)| \le p(y'-y)$. The conclusion follows. $\square$

The next result is similar to Corollary 3.

Corollary 5. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{0\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Consider $\overline{y} \in \operatorname{bd} A$. If A is epi-Lipschitz at $\overline{y}$ in the direction $-k^0$ then $\varphi_A(\overline{y}) = 0$.*

Proof. Consider $\varepsilon \in \,]0,1[$ and V_0 provided by (18) with $v := -k^0$. Assume that $\varphi_A(\overline{y}) \neq 0$. Then there exists $t > 0$ such that $t p_{V_0}(k^0) \leq \varepsilon$ and $y := \overline{y} + tk^0 \in A$. Taking $\lambda := t$ in (18) we obtain that $y + t(-k^0 + V_0) = \overline{y} + tV_0 \subset A$, contradicting the fact that $\overline{y} \in \operatorname{bd} A$. $\square$

Corollary 6. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{0\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Assume that $\dim Y < \infty$ and $\overline{y} \in \operatorname{bd} A$. Then φ_A is finite and Lipschitz on a neighbourhood of $\overline{y}$ if and only if $-k^0 \in \operatorname{int} T_{Cl}(A,\overline{y})$, where $T_{Cl}(A,\overline{y})$ is the Clarke tangent cone of A at $\overline{y}$.*

Proof. By [56, Th. 2I], $-k^0 \in \operatorname{int} T_{Cl}(A,\overline{y})$ if and only if A is epi-Lipschitz at $\overline{y}$ in the direction $-k^0$. The conclusion follows from Corollary 3, Theorem 5 and Corollary 5. $\square$

The fact that φ_A is Lipschitz on a neighbourhood of $\overline{y}$ under the condition $-k^0 \in \operatorname{int} T_{Cl}(A,\overline{y})$ is obtained in [4, Prop. 6] in the case $Y = \mathbb{R}^m$ (and $K = \mathbb{R}^m_+$).

Consider $y^* \in Y^*$ such that $\langle k^0, y^* \rangle \neq 0$, $H := \ker y^*$ and take

$$\varphi_0 : H \to \overline{\mathbb{R}}, \quad \varphi_0(z) := \varphi_A(z),$$

that is, $\varphi_0 = \varphi_A|_H$. Since φ_A is lsc, so is φ_0. Then any $y \in Y$ can be written uniquely as $z - tk^0$ with $z \in H$ and $t \in \mathbb{R}$. So, by (7), $\varphi_A(y) = \varphi_A(z - tk^0) = \varphi_0(z) - t$. Using (10) we obtain that $A = \{z - tk^0 \mid (z,t) \in \operatorname{epi} \varphi_0\}$. Conversely, if $g : H \to \overline{\mathbb{R}}$ is a lsc function and $A := \{z - tk^0 \mid (z,t) \in \operatorname{epi} g\}$, then A is a closed set with $A - \mathbb{R}_+ k^0 = A$ and $\varphi_0 = g$. Therefore, the closed set A with the property $A - \mathbb{R}_+ k^0 = A$ is uniquely determined by a lsc function $\varphi_0 : H \to \overline{\mathbb{R}}$. Moreover, for $\overline{y} = \overline{z} - \overline{t} k^0$ we have that φ_A is finite (resp. continuous) at $\overline{y}$ if and only if φ_0 is finite (resp. continuous) at $\overline{z}$. Moreover, because $Y = H + \mathbb{R}k^0$ and the sum is topological (that is, the projection onto H parallel to $\mathbb{R}k^0$ is continuous), we have that φ_A is finite and Lipschitz continuous on a neighbourhood of $\overline{y}$ if and only if φ_0 is finite and Lipschitz continuous on a neighbourhood of $\overline{z}$. Similarly, φ_A is finite and Lipschitz continuous if and only if φ_0 is finite and Lipschitz continuous.

Note that for Y a normed vector space in [35] one says that A is (globally) epi-Lipschitz in the direction $e \in Y \setminus \{0\}$ if there exist a closed linear subspace H of codimension 1 with $e \notin H$ and a Lipschitz function $g : H \to \mathbb{R}$ such that $A = \{y + \alpha e \mid y \in H,\ \alpha \in \mathbb{R},\ g(y) \leq \alpha\}$; A is epi-Lipschitz if there exists $e \in Y \setminus \{0\}$ such that A is epi-Lipschitz in the direction e. The main result of [35] asserts that the proper closed set $A \subset Y$ is epi-Lipschitz in the direction e if and only if $e \in \operatorname{int} A_\infty$, and so $A \subset Y$ is epi-Lipschitz if and only if $\operatorname{int} A_\infty \neq \emptyset$.

The discussion above shows that not only the main theorem of [35] can be obtained from Corollary 4, but this one extends the main theorem of [35] to locally convex spaces.

3.5 The Formula for the Conjugate and Subdifferential of φ_A for A Convex

The results of this section (less the second part of Corollary 7) were established in several papers; we give the proofs for reader's convenience. The formula for the conjugate of φ_A is derived by Hamel [40, Th. 3] and can be related also to [57, Th. 3] and [60, Th. 2.2]. Results concerning the subdifferential of φ_A are given in [17, Th. 2.2, Lem. 2.1]. Another proof of these assertions using the formula for the conjugates is presented in Hamel [40, Cor. 12]. In the statements below we use some usual notation from convex analysis. So, having X a separated locally convex space with topological dual X^* and $f : X \to \overline{\mathbb{R}}$, the conjugate of f is the function $f^* : X^* \to \overline{\mathbb{R}}$ defined by $f^*(x^*) := \sup\{x^*(x) - f(x) \mid x \in X\}$ and its subdifferential at $x \in X$ with $f(x) \in \mathbb{R}$ is the set $\partial f(x) := \{x^* \in X^* \mid x^*(x' - x) \leq f(x') - f(x) \ \forall x' \in X\}$; $\partial f(x) := \emptyset$ if $f(x) \notin \mathbb{R}$. Having a set $A \subset X$, the indicator of A is the function $\iota_A : X \to \overline{\mathbb{R}}$ defined by $\iota_A(x) := 0$ for $x \in A$ and $\iota_A(x) := \infty$ for $x \in X \setminus A$, while the support of A is the function $\sigma_A := (\iota_A)^*$. When A is nonempty the domain of σ_A is a convex cone which is called the barrier cone of A and is denoted by $\operatorname{bar} A$. Moreover, the normal cone of A at $a \in A$ is the set $N(A, a) := \partial \iota_A(a)$.

Proposition 6. *Let A be a proper closed subset of Y and $k^0 \in Y \setminus \{0\}$ be such that $A - \mathbb{R}_+ k^0 = A$. Assume that A is convex and $k^0 \notin A_\infty$. Then $\varphi_A \in \Gamma(Y)$, that is, φ_A is a proper lsc convex function,*

$$\varphi_A^*(y^*) = \begin{cases} \sigma_A(y^*) & \text{if } y^* \in \operatorname{bar} A,\ y^*(k^0) = 1, \\ \infty & \text{otherwise,} \end{cases} \tag{21}$$

and $\partial \varphi_A(y) \subset \{y^* \in \operatorname{bar} A \mid y^*(k^0) = 1\} \subset \{y^* \in K^+ \mid y^*(k^0) = 1\}$ for every $y \in Y$.

Proof. From [32, Th. 2.3.1]) we have that $\varphi_A \in \Gamma(Y)$. Consider $y^* \in Y^*$. Then

$$\begin{aligned} \varphi_A^*(y^*) &= \sup\{y^*(y) - \varphi_A(y) \mid y \in Y\} \\ &= \sup\left\{y^*(y) - t \mid y \in Y,\ t \in \mathbb{R},\ y \in tk^0 + A\right\} \\ &= \sup\left\{y^*(tk^0 + a) - t \mid y \in Y,\ t \in \mathbb{R},\ a \in A\right\} \\ &= \sup\{y^*(a) \mid a \in A\} + \sup\{t(y^*(k^0) - 1) \mid t \in \mathbb{R}\}. \end{aligned}$$

Hence (21) holds.

Since $\partial f(y) \subset \operatorname{dom} f^*$ for every proper function $f : Y \to \overline{\mathbb{R}}$ and every $y \in Y$, the first estimate for $\partial \varphi_A(y)$ follows. Moreover, because $A = A - K$ we have $\sigma_A = \sigma_{A-K} = \sigma_A + \sigma_{-K} = \sigma_A + \iota_{K^+}$, and so $\operatorname{bar} A \subset K^+$. □

The estimate $\operatorname{bar} A \subset K^+$ becomes more precise when $K = -A_\infty$; in fact one has $(A_\infty)^+ = -\operatorname{cl}_{w^*}(\operatorname{bar} A)$. Indeed, from [67, Exer. 2.23] we have that $\iota_{A_\infty} = (\iota_A)_\infty = \sigma_{\operatorname{dom} \iota_A^*} = \sigma_{\operatorname{dom} \sigma_A}$, whence $\iota_{-(A_\infty)^+} = (\iota_{A_\infty})^* = \iota_{\operatorname{cl}_{w^*}(\operatorname{dom} \sigma_A)}$, and so $\operatorname{cl}_{w^*}(\operatorname{dom} \sigma_A) = -(A_\infty)^+$.

Using Proposition 6 one deduces the expression of $\partial \varphi_A$ (see also [17, Th. 2.2] for Y a normed vector space).

Corollary 7. *Assume that A is convex and $k^0 \notin A_\infty$. Then for all $\bar{y} \in Y$ one has*

$$\partial\varphi_A(\bar{y}) = \{y^* \in \operatorname{bar} A \mid y^*(k^0) = 1,\ y^*(\bar{y}) - \varphi_A(\bar{y}) \geq y^*(y)\ \forall y \in A\}. \tag{22}$$

Moreover, if (A2) holds then $\partial\varphi_A(y) \subset K^\#$ for every $y \in Y$.

Proof. Fix $\bar{y} \in Y$. If $\bar{y} \notin \operatorname{dom}\varphi_A$ then both sets in (22) are empty. Let $\bar{y} \in \operatorname{dom}\varphi_A$. Then, of course, $\bar{y} - \varphi_A(\bar{y})k^0 \in A$. If $y^* \in \partial\varphi_A(\bar{y})$ then $\varphi_A(\bar{y}) + \varphi_A^*(y^*) = y^*(\bar{y})$. Taking into account (21) we obtain that

$$y^* \in \operatorname{bar} A,\ y^*(k^0) = 1 \text{ and } y^*(\bar{y}) - \varphi_A(\bar{y}) \geq y^*(y)\ \forall y \in A, \tag{23}$$

that is, the inclusion $\subset$ holds in (22). Conversely, if $y^* \in Y^*$ is such that (23) holds, since $\bar{y} - \varphi_A(\bar{y})k^0 \in A$ and $y^*(k^0) = 1$, we obtain that $y^*\big(\bar{y} - \varphi_A(\bar{y})k^0\big) = \sigma_A(y^*)$, which shows that $\varphi_A(\bar{y}) + \varphi_A^*(y^*) = y^*(\bar{y})$. Hence $y^* \in \partial\varphi_A(\bar{y})$. Therefore, (22) holds.

Assume now that $(A2)$ holds, that is, $A - (K \setminus \{0\}) \subset \operatorname{int} A$, and take $y^* \in \partial\varphi_A(\bar{y})$. Hence $\bar{y} \in \operatorname{dom}\varphi_A$. Consider $k \in K \setminus \{0\}$. Since $(\bar{y} - k) - \bar{y} = -k \in -(K \setminus \{0\})$, by Theorem 4 (iv), we have that $y^*(-k) \leq \varphi_A(\bar{y} - k) - \varphi_A(\bar{y}) < 0$, that is, $y^*(k) > 0$. Therefore, $y^* \in K^\#$. $\square$

4 Minimal-Point Theorems and Corresponding Variational Principles

4.1 Introduction

The celebrated Ekeland variational principle [21] (see Proposition 1) has many equivalent formulations and generalizations.

The aim of this section is to show general minimal-point theorems and corresponding variational principles. In Proposition 2 an existence result for minimal points of a set $\mathscr{A}$ with respect to the cone K_ε defined by (4) is presented. Taking into account (4) we get

$$(x_1 - x_2, r_1 - r_2) \in K_\varepsilon \quad \Longleftrightarrow \quad \varepsilon||x_1 - x_2|| \leq -(r_1 - r_2).$$

This means

$$r_2 \geq r_1 + \varepsilon||x_1 - x_2||. \tag{24}$$

Quite rapidly after the publication of the Ekeland variational principle (EVP) in 1974 there were formulated extensions to functions $f : (X,d) \to Y$, where Y is a real (topological) vector space. A systematization of such results was done in [34] (see also [32]), where instead of a function f it was considered a subset of $X \times Y$; said differently, it was considered a multifunction from X to Y. In [32] we have shown minimal-point theorems in product spaces $X \times Y$ with respect to a relation

$$(x_1, y_1) \preceq_{k^0} (x_2, y_2) \iff y_2 \in y_1 + d(x_1, x_2)k^0 + K, \tag{25}$$

where K is the convex ordering cone in Y and $k^0 \in K \setminus \{0\}$. This is an extension of the binary relation defined by (24) to product spaces $X \times Y$. Very recently the term $d(x_1,x_2)k^0$ in (25) was replaced by $d(x,x')H$ with H a bounded convex subset of K (see [8]) or by $F(x,x') \subset K$, F being a so called K-metric (see [36]); in both papers one deals with functions $f : X \to Y$.

In order to formulate general minimal-point theorems in this section we replace $d(x_1,x_2)k^0$ in (25) by a set-valued map F with certain properties (compare Tammer and Zălinescu [64]).

It is worth mentioning that a weaker result than a full (= *authentic*) *minimal-point theorem* gives an EVP, as shown in this section. Such a weaker result is called *not authentic minimal-point theorem*.

In this section we present new results with proofs very similar to the corresponding ones in [34], which have as particular cases most part of the existing EVPs, or they are very close to them. Moreover, we use the same approach to get extensions of EVPs of Isac–Tammer and Ha types, as well as extensions of EVPs for bi-functions.

In the sequel (X,d) is a complete metric space, Y is a real topological vector space, Y^* is its topological dual, and $K \subset Y$ is a proper convex cone.

If Y is just a real linear space we endow it with the finest locally convex topology, that is, the topology defined by all the seminorms on Y.

As in [6] and [7], we say that $E \subset Y$ is *quasi bounded* (from below) if there exists a bounded set $B \subset Y$ such that $E \subset B + K$; as in [36], we say that E is K-*bounded* (by scalarization) if $y^*(E)$ is bounded from below for every $y^* \in K^+$. It is clear that any quasi bounded set is K-bounded.

Let $F : X \times X \rightrightarrows K$ satisfy the conditions:

(F1) $0 \in F(x,x)$ for all $x \in X$,
(F2) $F(x_1,x_2) + F(x_2,x_3) \subset F(x_1,x_3) + K$ for all $x_1,x_2,x_3 \in X$.

Using F we introduce a preorder on $X \times Y$, denoted by $\preceq_F$, in the following manner:

$$(x_1,y_1) \preceq_F (x_2,y_2) \iff y_2 \in y_1 + F(x_1,x_2) + K. \tag{26}$$

Indeed, $\preceq_F$ is reflexive by (F1). If $(x_1,y_1) \preceq_F (x_2,y_2)$ and $(x_2,y_2) \preceq_F (x_3,y_3)$, then

$$y_2 = y_1 + v_1 + k_1, \quad y_3 = y_2 + v_2 + k_2 \tag{27}$$

with $v_1 \in F(x_1,x_2)$, $v_2 \in F(x_2,x_3)$ and $k_1,k_2 \in K$. By (F2) we have that $v_1 + v_2 = v_3 + k_3$ for some $v_3 \in F(x_1,x_3)$ and $k_3 \in K$, and so

$$y_3 = y_1 + v_1 + k_1 + v_2 + k_2 = y_1 + v_3 + k_1 + k_2 + k_3 \in y_1 + F(x_1,x_3) + K;$$

hence $(x_1,y_1) \preceq_F (x_3,y_3)$, and so $\preceq_F$ is transitive. Of course,

$$(x_1,y_1) \preceq_F (x_2,y_2) \Rightarrow y_1 \leq_K y_2; \tag{28}$$

moreover, by (F1), we have that

$$(x,y_1) \preceq_F (x,y_2) \iff y_2 \in y_1 + K \iff y_1 \leq_K y_2. \tag{29}$$

Besides conditions (F1) and (F2) we shall assume to be true the condition

(F3) there exists $z^* \in K^+$ such that

$$\eta(\delta) := \inf\left\{z^*(v) \mid v \in \cup_{d(x,x') \geq \delta} F(x,x')\right\} > 0 \quad \forall \delta > 0. \tag{30}$$

Clearly, by (F3) we have that $0 \notin \mathrm{cl}\ \mathrm{conv} F(x,x')$ for $x \neq x'$.
A sufficient condition for (30) is

$$\inf_{z \in F(x,x')} z^*(z) \geq d(x,x') \quad \forall x,x' \in X. \tag{31}$$

If (31) holds then

$$(x_1,y_1) \preceq_F (x_2,y_2) \Rightarrow d(x_1,x_2) \leq z^*(y_2) - z^*(y_1). \tag{32}$$

Indeed, since $F(x_1,x_2) \subset K$, from (28) we get first that $y_1 \leq_K y_2$; then from (27)

$$z^*(y_2) = z^*(y_1) + z^*(v_1) + z^*(k_1) \geq z^*(y_1) + \inf_{v \in F(x_1,x_2)} z^*(v) \geq z^*(y_1) + d(x_1,x_2),$$

and so (32) holds.
Using (32) we obtain that

$$[(x_1,y_1) \preceq_F (x_2,y_2),\ (x_2,y_2) \preceq_F (x_1,y_1)] \Rightarrow [x_1 = x_2,\ z^*(y_1) = z^*(y_2)]; \tag{33}$$

moreover, if $z^* \in K^\#$ then $\preceq_F$ is anti-symmetric, and so $\preceq_F$ is a partial order.
For F satisfying conditions (F1)–(F3), z^* being that from (F3), we introduce the order relation $\preceq_{F,z^*}$ on $X \times Y$ by

$$(x_1,y_1) \preceq_{F,z^*} (x_2,y_2) \iff \begin{cases} (x_1,y_1) = (x_2,y_2) \text{ or} \\ (x_1,y_1) \preceq_F (x_2,y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases} \tag{34}$$

It is easy to verify that $\preceq_{F,z^*}$ is reflexive, transitive, and antisymmetric.

4.2 Minimal Points in Product Spaces

We take X,Y,K,F as above, that is, F satisfies conditions (F1)–(F3), z^* being that from (F3).

Consider a nonempty set $\mathscr{A} \subset X \times Y$. In the sequel we shall use the condition (H1) on $\mathscr{A}$, where $\mathbb{N}$ is the set of nonnegative integers; moreover, we set $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$.

The next theorem is the main result of this section.

Theorem 6 (Minimal-Point Theorem with respect to $\preceq_{F,z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3) and $\mathscr{A} \subset X \times K$ satisfy the condition*

(H1) for every $\preceq_F$–decreasing sequence $((x_n,y_n)) \subset \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preceq_F (x_n,y_n)$ for every $n \in \mathbb{N}$.

Furthermore, suppose that

(B1) z^* (from (F3)) is bounded from below on $\mathrm{Pr}_Y(\mathscr{A})$.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists an element $(\bar{x},\bar{y})$ of $\mathscr{A}$ such that

(a) $(\bar{x},\bar{y}) \preceq_{F,z^*} (x_0,y_0)$,
(b) $(\bar{x},\bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{F,z^*}$.

Proof. Let

$$\alpha := \inf\{z^*(y) \mid \exists x \in X : (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,z^*} (x_0,y_0)\} \in \mathbb{R}.$$

Let us denote by $\mathscr{A}(x,y)$ the set of those $(x',y') \in \mathscr{A}$ with $(x',y') \preceq_{F,z^*} (x,y)$. We construct a sequence $((x_n,y_n))_{n\geq 0} \subset \mathscr{A}$ as follows: Having $(x_n,y_n) \in \mathscr{A}$, we take $(x_{n+1},y_{n+1}) \in \mathscr{A}(x_n,y_n)$ such that

$$z^*(y_{n+1}) \leq \inf\{z^*(y) \mid (x,y) \in \mathscr{A}(x_n,y_n)\} + 1/(n+1).$$

Of course, $((x_n,y_n))$ is $\preceq_{F,z^*}$–decreasing. It follows that $(y_n)_{n\geq 0}$ is $\leq_K$–decreasing, and so the sequence $(z^*(y_n))_{n\geq 0}$ is nonincreasing and bounded from below; hence $\gamma := \lim z^*(y_n) \in \mathbb{R}$.

If $\mathscr{A}(x_{n_0},y_{n_0})$ is a singleton (that is, $\{(x_{n_0},y_{n_0})\}$) for some $n_0 \in \mathbb{N}$, then clearly $(\bar{x},\bar{y}) := (x_{n_0},y_{n_0})$ is the desired element. In the contrary case the sequence $(z^*(y_n))$ is (strictly) decreasing; moreover, $\gamma < z^*(y_n)$ for every $n \in \mathbb{N}$.

Assume that (x_n) is not a Cauchy sequence. Then there exist $\delta > 0$ and the sequences (n_k), (p_k) from $\mathbb{N}^*$ such that $n_k \to \infty$ and $d(x_{n_k},x_{n_k+p_k}) \geq \delta$ for every k. Since $(x_{n_k+p_k},y_{n_k+p_k}) \preceq_{F,z^*} (x_{n_k},y_{n_k})$ we obtain that

$$z^*(y_{n_k}) - z^*(y_{n_k+p_k}) \geq \inf\left\{z^*(v) \mid v \in F(x_{n_k+p_k},x_{n_k})\right\} \geq \eta(\delta) \quad \forall k \in \mathbb{N}.$$

Since $\eta(\delta) > 0$ and $(z^*(y_n))$ is convergent, this is a contradiction. Therefore, (x_n) is a Cauchy sequence in the complete metric space (X,d), and so (x_n) converges to some $\bar{x} \in X$. Since $((x_n,y_n))$ is $\preceq_F$–decreasing, by (H1) there exists some $\bar{y} \in Y$ such that $(\bar{x},\bar{y}) \in \mathscr{A}$ and $(\bar{x},\bar{y}) \preceq_F (x_n,y_n)$ for every $n \in \mathbb{N}$. It follows that $z^*(\bar{y}) \leq \lim z^*(y_n)$, and so $z^*(\bar{y}) < z^*(y_n)$ for every $n \in \mathbb{N}$. Therefore $(\bar{x},\bar{y}) \preceq_{F,z^*} (x_n,y_n)$ for every $n \in \mathbb{N}$. Let $(x',y') \in \mathscr{A}$ be such that $(x',y') \preceq_{F,z^*} (\bar{x},\bar{y})$. Since $(\bar{x},\bar{y}) \preceq_{F,z^*} (x_n,y_n)$, we have that $(x',y') \preceq_{F,z^*} (x_n,y_n)$ for every $n \in \mathbb{N}$. It follows that

$$0 \leq z^*(\bar{y}) - z^*(y') \leq z^*(y_n) - z^*(y') \leq 1/n \quad \forall n \geq 1,$$

whence $z^*(y') = z^*(\bar{y})$. By the definition of $\preceq_{F,z^*}$ we obtain that $(x',y') = (\bar{x},\bar{y})$. □

As seen from the proof, for a fixed $(x_0,y_0) \in \mathscr{A}$ it is sufficient that z^* be bounded from below on the set $\{y \in Y \mid \exists x \in X : (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,z^*} (x_0,y_0)\}$ instead of being bounded from below on $\mathrm{Pr}_Y(\mathscr{A})$.

Remark 5. When $k^0 \in K$ and $F(x,x') := \{d(x,x')k^0\}$ we have that F satisfies conditions (F1) and (F2); moreover, if Y is a separated locally convex space and $-k^0 \notin \mathrm{cl}\,K$, then there exists $z^* \in K^+$ with $z^*(k^0) = 1$, and so (F3) is also satisfied (even (31) is satisfied). In this case condition (H1) becomes condition (H1) in [32, p. 199]. So Theorem 6 extends [32, Th. 3.10.7] to this framework, using practically the same proof.

In [36] one considers for a proper pointed convex cone $D \subset Y$ a so called set-valued D-metric, that is, a multifunction $F : X \times X \rightrightarrows D$ satisfying the following conditions:

(i) $F(x,y) \neq \emptyset$ and $F(x,x) = \{0\}$ $\forall x,y \in X$, and $0 \notin F(x,y)$ $\forall x \neq y$,
(ii) $F(x,y) = F(y,x)$ $\forall x,y \in X$,
(iii) $F(x,y) + F(y,z) \subset F(x,z) + D$ $\forall x,y,z \in X$.

The basic supplementary assumptions on D and F are:

(S1) D is w-normal and D_F is based,
(S2) $0 \notin \mathrm{cl}_w \left(\cup_{d(x,y)\geq\delta} F(x,y)\right)$ $\forall \delta > 0$.

Here $K_F := \mathrm{cone}(\mathrm{conv}\,(\cup\{F(x,y) \mid x,y \in X\}))$ and $D_F := (K_F \setminus \{0\} + D) \cup \{0\}$.

As observed in [36], D is w-normal iff $D^+ - D^+ = Y^*$, and D_F is based iff $D^+ \cap K_F^\# \neq \emptyset$.

Comparing with our assumptions on F, we see that (F1) is weaker than (i) because we ask just $0 \in F(x,x)$ for every $x \in X$, and we don't ask the symmetry condition (ii). From (F3) we obtain that (S2) is verified and that $z^* \in K_F^\#$ and so $(K_F \setminus \{0\} + K) \cup \{0\}$ is based, but we don't need either K be w-normal or even K be pointed.

Another possible choice for F, considered also in [36], is $F(x,x') := d(x,x')H$ with $H \subset K \setminus \{0\}$ a nonempty set such that $H + K$ is convex. Clearly (F1), (i), and (ii) are satisfied (for $D = K$). From the convexity of $H + K$ we obtain easily that (F2) (and (iii)) holds. When Y is a separated locally convex space condition (F3) is equivalent to $0 \notin \mathrm{cl}(H+K)$. In order to have that (S1) holds one needs $K^+ - K^+ = Y^*$ and the existence of $z^* \in K^+$ with $z^*(v) > 0$ for every $v \in H$, while for (S2) one needs $0 \notin \mathrm{cl}_w H$ (see [36, Lem. 5.9 (d)]); of course, if $H = H + K$, the last condition is equivalent to $0 \notin \mathrm{cl}(H+K)$. So it seems that our condition (F3) is more convenient than (S1) and (S2).

For H as above, that is, $H \subset K$ is a nonempty set such that $H + K$ is convex and $0 \notin \mathrm{cl}(H+K)$, we consider $F_H(x,x') := d(x,x')H$ for $x,x' \in X$, and we set

$$\preceq_H := \preceq_{F_H};$$

moreover, if $z^* \in K^+$ is such that $\inf z^*(H) > 0$ we set

$$\preceq_{H,z^*} := \preceq_{F_H,z^*} .$$

An immediate consequence of the preceding theorem is the next result.

Corollary 8 (Minimal-Point Theorem with respect to $\preceq_{H,z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space, $K \subset Y$ is a proper convex cone and $\mathscr{A} \subset X \times Y$ satisfies*

(H1) for every $\preceq_H$–decreasing sequence $((x_n,y_n)) \subset \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$.

Suppose that there exists $z^* \in K^+$ such that $\inf z^*(H) > 0$ and

(B1) $\inf z^* (\Pr_Y(\mathscr{A})) > -\infty$.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists $(\bar{x},\bar{y}) \in \mathscr{A}$ such that

(a) $(\bar{x},\bar{y}) \preceq_{H,z^*} (x_0,y_0)$,
(b) $(\bar{x},\bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{H,z^*}$.

A condition related to (H1) is the next one.

(H2) for every sequence $\big((x_n,y_n)\big) \subset \mathscr{A}$ with $x_n \to x \in X$ and (y_n) $\leq_K$–decreasing there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Remark 6. Note that (H2) holds if $\mathscr{A}$ is closed with $\Pr_Y(\mathscr{A}) \subset y_0 + K$ for some $y_0 \in Y$ and every $\leq_K$–decreasing sequence in K is convergent (i.e., K is a sequentially Daniell cone). In fact, instead of asking that $\mathscr{A}$ is closed we may assume that

$$\forall \left((x_n,y_n)\right)_{n\geq 1} \subset \mathscr{A} : [x_n \to x,\ y_n \to y,\ (y_n) \text{ is } \leq_K \text{–decreasing } \Rightarrow (x,y) \in \mathscr{A}] .$$

Remark 7. Note that (H1) is verified whenever $\mathscr{A}$ satisfies (H2) and

$$\forall u \in X,\ \forall X \supset (x_n) \to x \in X \ : \ \cap_{n\in\mathbb{N}} (F(x_n,u) + K) \subset F(x,u) + K.$$

Indeed, let $\big((x_n,y_n)\big) \subset \mathscr{A}$ be $\preceq_F$–decreasing with $x_n \to x$. It is obvious that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$. It follows that

$$y_n \in y_{n+p} + F(x_{n+p},x_n) + K \subset y + F(x_{n+p},x_n) + K \quad \forall n,p \in \mathbb{N}.$$

Fix n; then $y_n - y \in F(x_{n+p},x_n) + K$ for every $p \in \mathbb{N}$, and so, by our hypothesis, $y_n - y \in F(x,x_n) + K$ because $\lim_{p\to\infty} x_{n+p} = x$. Therefore, $(x,y) \preceq_F (x_n,y_n)$.

Remark 8. In the case $F = F_H$, (H1) is verified whenever $\mathscr{A}$ satisfies (H2) and $H+K$ is closed.

Indeed, let $\big((x_n,y_n)\big) \subset \mathscr{A}$ be a $\preceq_H$–decreasing sequence with $x_n \to x$. It is obvious that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Fix n. If $x_n = x$ then clearly $(x,y) = (x_n,y) \preceq_H (x_n,y_n)$. Else, because $d(x_{n+p},x_n) \to d(x,x_n) > 0$ for $p \to \infty$, we get $d(x_{n+p},x_n) > 0$ for sufficiently large p, and so

$$y_n \in y_{n+p} + d(x_{n+p}, x_n)H + K \subset y + d(x_{n+p}, x_n)H + K = y + d(x_{n+p}, x_n)(H + K)$$

for sufficiently large p. Since $H+K$ is closed we obtain that

$$y_n \in y + d(x_n, x)(H+K) = y + d(x_n, x)H + K,$$

that is, $(x,y) \preceq_H (x_n, y_n)$.

Another condition to be added to (H2) in order to have (H1) is suggested by the hypotheses of [8, Theorem 4.1]. Recall that a set $C \subset Y$ is cs-complete (see [67, p. 9]) if for all sequences $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ and $(y_n)_{n\geq 1} \subset C$ such that $\sum_{n\geq 1} \lambda_n = 1$ and the sequence $(\sum_{m=1}^{n} \lambda_m y_m)_{n\geq 1}$ is Cauchy, the series $\sum_{n\geq 1} \lambda_n y_n$ is convergent and its sum belongs to C. One says that $C \subset Y$ is cs-closed if the sum of the series $\sum_{n\geq 1} \lambda_n y_n$ belongs to C whenever $\sum_{n\geq 1} \lambda_n y_n$ is convergent and $(y_n) \subset C$, $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ and $\sum_{n\geq 1} \lambda_n = 1$. Of course, any cs-complete set is cs-closed; if Y is complete then the converse is true. Moreover, notice that any cs-closed set is convex.

Note that the sequence $(\sum_{m=1}^{n} \lambda_m y_m)_{n\geq 1}$ is Cauchy whenever $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ is such that the series $\sum_{n\geq 1} \lambda_n$ is convergent and $(y_n)_{n\geq 1} \subset Y$ is such that $\operatorname{conv}\{y_n \mid n \geq 1\}$ is bounded; of course, if Y is a locally convex space then $B \subset Y$ is bounded iff $\operatorname{conv} B$ is bounded. Indeed, let $(\lambda_n)_{n\geq 1} \subset [0,\infty)$ with $\sum_{n\geq 1} \lambda_n$ convergent and $(y_n)_{n\geq 1} \subset Y$ with $B := \operatorname{conv}\{y_n \mid n \geq 1\}$ bounded. Fix $V \subset Y$ a balanced neighborhood of 0. Because B is bounded, there exists $\alpha > 0$ such that $B \subset \alpha V$. Since the series $\sum_{n\geq 1} \lambda_n$ is convergent there exists $n_0 \geq 1$ such that $\sum_{k=n}^{n+p} \lambda_k \leq \alpha^{-1}$ for all $n, p \in \mathbb{N}$ with $n \geq n_0$. Then for such n, p and some $b_{n,p} \in B$ we have

$$\sum_{k=n}^{n+p} \lambda_k y_k = \left(\sum_{k=n}^{n+p} \lambda_k \right) b_{n,p} \in [0, \alpha^{-1}]B \subset [0, \alpha^{-1}]\alpha V = V.$$

Proposition 7. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper closed convex cone. Furthermore, suppose that $H \subset K$ is a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H+K)$. If $\mathscr{A}$ satisfies (H2) then $\mathscr{A}$ satisfies (H1), too.*

Proof. Let $\big((x_n, y_n)\big)_{n\geq 1} \subset \mathscr{A}$ be a $\preceq_H$–decreasing sequence with $x_n \to x$. It follows that (y_n) is $\leq_K$–decreasing. By (H2), there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$.

Because $\big((x_n, y_n)\big)_{n\geq 1}$ is $\preceq_H$–decreasing we have that

$$y_n = y_{n+1} + d(x_n, x_{n+1})h_n + k_n \tag{35}$$

with $h_n \in H$ and $k_n \in K$ for $n \geq 1$. If $x_n = x_{\overline{n}}$ for $n \geq \overline{n} \geq 1$ we take $x := x_{\overline{n}}$; then $(x,y) \preceq_H (x_n, y_n)$ for every $n \in \mathbb{N}$. Indeed, for $n \leq \overline{n}$ we have that $(x_{\overline{n}}, y_{\overline{n}}) \preceq_H (x_n, y_n)$; because $y \leq_K y_{\overline{n}}$, by (29) we get $(x,y) \preceq_H (x, y_{\overline{n}}) = (x_{\overline{n}}, y_{\overline{n}})$, and so $(x,y) \preceq (x_n, y_n)$. If $n > \overline{n}$, using again (29), we have $(x,y) = (x_n, y) \preceq_H (x_n, y_n)$.

Assume that (x_n) is not constant for large n. Fix $n \geq 1$. From (35), for $p \geq 0$, we have

$$y_n = y_{n+p+1} + \sum_{l=n}^{n+p} d(x_l, x_{l+1}) h_l + \sum_{l=n}^{n+p} k_l = y_{n+p+1} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right) h_{n,p} + \sum_{l=n}^{n+p} k_l$$
$$= y + k'_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right) h_{n,p} \tag{36}$$

for some $h_{n,p} \in H$ and $k'_{n,p} \in K$. Assuming that $\sum_{l \geq n} d(x_l, x_{l+1}) = \infty$, from

$$\left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right)^{-1} (y_n - y) = h_{n,p} + \left(\sum_{l=n}^{n+p} d(x_l, x_{l+1})\right)^{-1} k'_{n,p} \in H + K,$$

we get the contradiction $0 \in \operatorname{cl}(H+K)$ taking the limit for $p \to \infty$. Hence $0 < \mu := \sum_{l \geq n} d(x_l, x_{l+1}) < \infty$. Set $\lambda_l := \mu^{-1} d(x_l, x_{l+1})$ for $l \geq n$. Since H is cs-complete and $\operatorname{conv}\{h_l \mid l \geq n\}$ $(\subset H)$ is bounded we obtain that the series $\sum_{l \geq n} \lambda_l h_l$ is convergent and its sum $\overline{h}_n$ belongs to H. It follows that $\sum_{l \geq n} d(x_l, x_{l+1}) h_l = \mu \overline{h}_n$, and so

$$\overline{k}_n := \lim_{p \to \infty} k'_p = y_n - y - \mu \overline{h}_n \in K$$

because K is closed. Since $d(x_n, x_{n+p}) \leq \sum_{l=n}^{n+p-1} d(x_l, x_{l+1})$, we obtain that $d(x_n, x) \leq \mu$, and so

$$y_n = y + d(x_n, x)\overline{h}_n + \overline{k}_n + (\mu - d(x_n, x))\,\overline{h}_n \in y + d(x_n, x)H + K.$$

Hence $(x,y) \preceq_H (x_n, y_n)$ for every $n \in \mathbb{N}$. □

The most part of vector EVP type results are established for Y a separated locally convex space. However, there are topological vector spaces Y whose topological dual reduce to $\{0\}$. In such a case it is not possible to find z^* satisfying the conditions of Corollary 8. In [6, Th. 1], in the case H is a singleton, the authors consider such a situation.

Theorem 7 (Not authentic minimal-point theorem with respect to $\preceq_H$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space. Let $K \subset Y$ be a proper closed convex cone and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H+K)$. Suppose that $\mathscr{A} \subset X \times Y$ satisfies*

(H2) for every sequence $\big((x_n, y_n)\big) \subset \mathscr{A}$ with $x_n \to x \in X$ and (y_n) $\leq_K$–decreasing there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $y \leq_K y_n$ for every $n \in \mathbb{N}$

and

(B2) $\operatorname{Pr}_Y(\mathscr{A})$ is quasi bounded.

Then for every $(x_0, y_0) \in \mathscr{A}$ there exists $(\overline{x}, \overline{y}) \in \mathscr{A}$ such that

(a) $(\overline{x}, \overline{y}) \preceq_H (x_0, y_0)$,
(b) $(x,y) \in \mathscr{A}$, $(x,y) \preceq_H (\overline{x}, \overline{y})$ imply $x = \overline{x}$.

Proof. First observe that $\mathscr{A}$ satisfies condition (H1) by Proposition 7. Moreover, because $\mathrm{Pr}_Y(\mathscr{A})$ is quasi bounded, there exists a bounded set $B \subset Y$ such that $\mathrm{Pr}_Y(\mathscr{A}) \subset B+K$.

Note that for $(x,y) \in \mathscr{A}$ the set $\mathrm{Pr}_X(\mathscr{A}(x,y))$ is bounded, where $\mathscr{A}(x,y) := \{(x',y') \in \mathscr{A} \mid (x',y') \preceq_H (x,y)\}$. In the contrary case there exists a sequence $((x_n,y_n))_{n\geq 1} \subset \mathscr{A}(x,y)$ with $d(x_n,x) \to \infty$. Hence $y = y_n + d(x_n,x)h_n + k_n = b_n + d(x_n,x)h_n + k'_n$ with $h_n \in H$, $b_n \in B$, $k_n,k'_n \in K$. It follows that $d(x_n,x)^{-1}(y-b_n) \in H+K$, whence the contradiction $0 \in \mathrm{cl}(H+K)$.

Let us construct a sequence $((x_n,y_n))_{n\geq 0} \subset \mathscr{A}$ in the following way: Having $(x_n,y_n) \in \mathscr{A}$, where $n \in \mathbb{N}$, because $D_n := \mathrm{Pr}_X(\mathscr{A}(x_n,y_n))$ is bounded, there exists $(x_{n+1},y_{n+1}) \in \mathscr{A}(x_n,y_n)$ such that

$$d(x_{n+1},x_n) \geq \tfrac{1}{2}\sup\{d(x,x_n) \mid x \in D_n\} \geq \tfrac{1}{4}\,\mathrm{diam}\,D_n.$$

We obtain in this way the sequence $((x_n,y_n))_{n\geq 0} \subset \mathscr{A}$, which is $\preceq_H$–decreasing. Since $\mathscr{A}(x_{n+1},y_{n+1}) \subset \mathscr{A}(x_n,y_n)$, we have that $D_{n+1} \subset D_n$ for every $n \in \mathbb{N}$. Of course, $x_n \in D_n$. Let us show that $\mathrm{diam}\,D_n \to 0$. In the contrary case there exists $\delta > 0$ such that $\mathrm{diam}\,D_n \geq 4\delta$, and so $d(x_{n+1},x_n) \geq \delta$ for every $n \in \mathbb{N}$. As in the proof of Proposition 7, for every $p \in \mathbb{N}$, we obtain that

$$\begin{aligned} y_0 &= y_{p+1} + \left(\sum_{l=0}^{p} d(x_l,x_{l+1})\right) h_p + \sum_{l=0}^{p} k_l = b_p + \left(\sum_{l=0}^{p} d(x_l,x_{l+1})\right) h_p + k'_p \\ &= b_p + (p+1)\delta h_p + k''_p, \end{aligned}$$

where $h_p \in H$, $b_p \in B$, $k_l,k'_p,k''_p \in K$. It follows that $[(p+1)\delta]^{-1}(y_0 - b_p) \in H+K$ for every $p \in \mathbb{N}$. Since (b_p) is bounded we obtain the contradiction $0 \in \mathrm{cl}(H+K)$. Thus we have that the sequence $(\mathrm{cl}\,D_n)$ is a decreasing sequence of nonempty closed subsets of the complete metric space (X,d), whose diameters tend to 0. By Cantor's theorem, $\bigcap_{n\in\mathbb{N}} \mathrm{cl}\,D_n = \{\overline{x}\}$ for some $\overline{x} \in X$. Of course, $x_n \to \overline{x}$. Since $((x_n,y_n)) \subset \mathscr{A}$ is a $\preceq_H$–decreasing sequence, from (H1) we get an $\overline{y} \in Y$ such that $(\overline{x},\overline{y}) \preceq_H (x_n,y_n)$ for every $n \in \mathbb{N}$; $(\overline{x},\overline{y})$ is the desired element. Indeed, $(\overline{x},\overline{y}) \preceq_H (x_0,y_0)$. Let $(x',y') \in \mathscr{A}(\overline{x},\overline{y})$. It follows that $(x',y') \in \mathscr{A}(x_n,y_n)$, and so $x' \in D_n \subset \mathrm{cl}\,D_n$ for every n. Thus $x' = \overline{x}$. $\square$

If Y is a separated locally convex space, the preceding result follows immediately from Corollary 8.

Of course, the set $\mathscr{A} \subset X \times Y$ can be viewed as the graph of a multifunction $\Gamma : X \rightrightarrows Y$; then $\mathrm{Pr}_X(\mathscr{A}) = \mathrm{dom}\,\Gamma$ and $\mathrm{Pr}_Y(\mathscr{A}) = \mathrm{Im}\,\Gamma$. In [6] one assumes that Γ is *level-closed*, that is,

$$\begin{aligned} L(b) :=& \{x \in X \mid \exists y \in \Gamma(x) : y \leq_K b\} = \{x \in X \mid b \in \Gamma(x) + K\} \\ =& \{x \in X \mid \Gamma(x) \cap (b-K) \neq \emptyset\} \end{aligned}$$

is closed for every $b \in Y$.

For the nonempty set $E \subset Y$ let us set

$$\mathrm{BMMin}\,E := \{\bar{y} \in E \mid E \cap (\bar{y} - K) = \{\bar{y}\}\}$$

(see [7, (1.2)]); note that this set is different of the usual set

$$\mathrm{Min}\,E := \{\bar{y} \in E \mid E \cap (\bar{y} - K) \subset \bar{y} + K\},$$

but they coincide if K is pointed. As in [7, Definition 3.2], we say that $\Gamma : X \rightrightarrows Y$ satisfies the *limiting monotonicity condition* at $\bar{x} \in \mathrm{dom}\,\Gamma$ if for every sequence $((x_n, y_n))_{n\geq 1} \subset \mathrm{gph}\,\Gamma$ with (x_n) converging to $\bar{x}$ and (y_n) being $\leq_K$–decreasing, there exists $\bar{y} \in \mathrm{BMMin}\,\Gamma(\bar{x})$ such that $\bar{y} \leq y_n$ for every $n \geq 1$. As observed in [7], if Γ satisfies the limiting monotonicity condition at $\bar{x} \in \mathrm{dom}\,\Gamma$ then $\Gamma(\bar{x}) \subset \mathrm{BMMin}\,\Gamma(\bar{x}) + K$, that is, $\Gamma(\bar{x})$ satisfies the domination property.

In [7, Proposition 3.3], in the case Y a Banach space, there are mentioned sufficient conditions in order that Γ satisfy the limiting monotonicity condition at $\bar{x} \in \mathrm{dom}\,\Gamma$.

When X and Y are Banach spaces and H is a singleton the next result is practically [7, Th. 3.5].

Corollary 9 (Not authentic minimal-point theorem with respect to $\preceq_H$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space. Let $K \subset Y$ be a proper closed convex cone and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \mathrm{cl}(H+K)$. Suppose that*

(H3) $\Gamma : X \rightrightarrows Y$ *is level-closed, satisfies the limiting monotonicity condition on* $\mathrm{dom}\,\Gamma$,

(B3) $\mathrm{Im}\,\Gamma$ *is quasi-bounded.*

Then for every $(x_0, y_0) \in \mathrm{gph}\,\Gamma$ there exist $\bar{x} \in \mathrm{dom}\,\Gamma$ and $\bar{y} \in \mathrm{BMMin}\,\Gamma(\bar{x})$ such that

(a) $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$,

(b) $(x,y) \in \mathrm{gph}\,\Gamma$, $(x,y) \preceq_H (\bar{x}, \bar{y})$ *imply* $x = \bar{x}$.

Proof. In order to apply Theorem 7 for $\mathscr{A} := \mathrm{gph}\,\Gamma$ we have only to show that $\mathscr{A}$ verifies condition (H2). For this consider the sequence $((x_n, y_n))_{n\geq 1} \subset \mathscr{A}$ such that (y_n) is $\leq_K$–decreasing and $x_n \to \bar{x}$. Clearly, $x_n \in L(y_1)$ for every n; since Γ is level-closed, we have that $\bar{x} \in L(y_1) \subset \mathrm{dom}\,\Gamma$. Since Γ satisfies the limiting monotonicity condition at $\bar{x}$, we find $\bar{y} \in \mathrm{BMMin}\,\Gamma(\bar{x}) \subset \Gamma(\bar{x})$ such that $\bar{y} \leq y_n$ for every n. Hence (H2) holds. By Theorem 7 there exists $(x,y) \in \mathscr{A}$ such that $(x,y) \preceq_H (x_0, y_0)$ and $(x', y') \in \mathrm{gph}\,\Gamma$, $(x', y') \preceq_H (x,y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \mathrm{BMMin}\,\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. By (29) we have that $(\bar{x}, \bar{y}) \preceq_H (x_0, y_0)$. Let now $(x', y') \in \mathrm{gph}\,\Gamma = \mathscr{A}$ with $(x', y') \preceq_H (\bar{x}, \bar{y})$. Since $(\bar{x}, \bar{y}) = (x, \bar{y}) \preceq_H (x,y)$, we have that $(x', y') \preceq_H (x,y)$, and so $x' = x = \bar{x}$. The proof is complete. □

In the case when H is a singleton the next result is practically [6, Theorem 1] under the supplementary hypothesis that $\mathrm{Min}\,\Gamma(x)$ is compact for every $x \in X$; it seems that this condition has to be added in order that [6, Theorem 1] be true.

Corollary 10 (Not Authentic Minimal-Point Theorem with respect to $\preceq_H$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space. Let $K \subset Y$ be a proper closed convex cone and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \mathrm{cl}(H+K)$. Suppose that*

(H4) $\Gamma : X \rightrightarrows Y$ *is level-closed,* $\mathrm{Min}\,\Gamma(x)$ *is compact and* $\Gamma(x) \subset K + \mathrm{Min}\,\Gamma(x)$ *for every* $x \in \mathrm{dom}\,\Gamma$,
(B3) $\mathrm{Im}\,\Gamma$ *is quasi-bounded.*

Then for every $(x_0,y_0) \in \mathrm{gph}\,\Gamma$ *there exist* $\bar{x} \in \mathrm{dom}\,\Gamma$ *and* $\bar{y} \in \mathrm{Min}\,\Gamma(\bar{x})$ *such that*

(a) $(\bar{x},\bar{y}) \preceq_H (x_0,y_0)$,
(b) $(x,y) \in \mathrm{gph}\,\Gamma$, $(x,y) \preceq_H (\bar{x},\bar{y})$ *imply* $x = \bar{x}$.

Proof. In order to apply Theorem 7 for $\mathscr{A} := \mathrm{gph}\,\Gamma$ we have only to show that $\mathscr{A}$ verifies condition (H2). For this consider the sequence $((x_n,y_n))_{n\geq 1} \subset \mathscr{A}$ such that (y_n) is $\leq_K$–decreasing and $x_n \to \bar{x}$. As in the proof of the preceding corollary, $\bar{x} \in L(y_n)$ for every $n \in \mathbb{N}$. Because $\Gamma(\bar{x}) \subset K + \mathrm{Min}\,\Gamma(\bar{x})$, for every $n \in \mathbb{N}$ there exists $y'_n \in \mathrm{Min}\,\Gamma(\bar{x})$ such that $y'_n \leq y_n$. Because $\mathrm{Min}\,\Gamma(\bar{x})$ is compact, (y'_n) has a subnet $(y'_{\psi(i)})_{i\in I}$ converging to some $\bar{y} \in \mathrm{Min}\,\Gamma(\bar{x})$; here $\psi : (I,\succeq) \to \mathbb{N}$ is such that for every n there exists $i_n \in I$ with $\psi(i) \geq n$ for $i \succeq i_n$. Hence $y'_{\psi(i)} \leq y_{\psi(i)} \leq y_n$ for $i \succeq i_n$, whence $\bar{y} \leq y_n$ because K is closed. Therefore, (H2) holds. By Theorem 7, for $(x_0,y_0) \in \mathrm{gph}\,\Gamma$, there exists $(x,y) \in \mathscr{A}$ such that $(x,y) \preceq_H (x_0,y_0)$ and $(x',y') \in \mathrm{gph}\,\Gamma$, $(x',y') \preceq_H (x,y)$ imply $x' = x$. Set $\bar{x} := x$ and take $\bar{y} \in \mathrm{Min}\,\Gamma(\bar{x})$ such that $\bar{y} \leq_K y$. As in the proof of Corollary 9 we find that $(\bar{x},\bar{y})$ is the desired element. The proof is complete. □

4.3 Minimal-Point Theorems of Isac–Tammer's Type

Besides $F : X \times X \rightrightarrows K$ considered in the preceding section we consider also $F' : Y \times Y \rightrightarrows K$ satisfying conditions (F1) and F(2), that is, $0 \in F'(y,y)$ for all $y \in Y$ and $F'(y_1,y_2) + F'(y_2,y_3) \subset F'(y_1,y_3) + K$ for all $y_1,y_2,y_3 \in Y$. Then $\Phi : Z \times Z \rightrightarrows K$ with $Z := X \times Y$, defined by $\Phi((x_1,y_1),(x_2,y_2)) := F(x_1,x_2) + F'(y_1,y_2)$, satisfies conditions (F1) and (F2), too. As in Section 4.1 we obtain that the relation $\preccurlyeq_{F,F'}$ defined by

$$(x_1,y_1) \preccurlyeq_{F,F'} (x_2,y_2) \Longleftrightarrow y_2 \in y_1 + F(x_1,x_2) + F'(y_1,y_2) + K$$

is reflexive and transitive. Moreover, for $x,x_1,x_2 \in X$ and $y_1,y_2 \in Y$ we have

$$(x_1,y_1) \preccurlyeq_{F,F'} (x_2,y_2) \Longrightarrow (x_1,y_1) \preccurlyeq_F (x_2,y_2) \Longrightarrow y_1 \leq_K y_2,$$
$$(x,y_1) \preccurlyeq_{F,F'} (x,y_2) \Longleftrightarrow y_1 \leq_K y_2.$$

As in the preceding section, for F satisfying (F1)–(F3), F' satisfying (F1), (F2) and z^* from (F3) we define the partial order $\preceq_{F,F',z^*}$ by

$$(x_1,y_1) \preceq_{F,F',z^*} (x_2,y_2) \iff \begin{cases} (x_1,y_1) = (x_2,y_2) \text{ or} \\ (x_1,y_1) \preceq_{F,F'} (x_2,y_2) \text{ and } z^*(y_1) < z^*(y_2). \end{cases}$$

Theorem 8 (Minimal-Point Theorem with respect to $\preceq_{F,F',z^*}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $\mathscr{A} \subset X \times Y$ satisfy the condition*

(H1b) for every $\preccurlyeq_{F,F'}$–decreasing sequence $((x_n,y_n)) \subset \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preccurlyeq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$.

Suppose that

(B1) z^* (from (F3)) is bounded from below on $\mathrm{Pr}_Y(\mathscr{A})$.

Then for every $(x_0,y_0) \in \mathscr{A}$ there exists an element $(\bar{x},\bar{y}) \in \mathscr{A}$ such that

(a) $(\bar{x},\bar{y}) \preceq_{F,F',z^*} (x_0,y_0)$,
(b) $(\bar{x},\bar{y})$ is a minimal element of $\mathscr{A}$ with respect to $\preceq_{F,F',z^*}$.

Proof. It is easy to verify that $\preceq_{F,F',z^*}$ is reflexive, transitive and antisymmetric. To get the conclusion one follows the lines of the proof of Theorem 6. □

Clearly, taking $F' = 0$ in Theorem 8 we get Theorem 6. As mentioned after the proof of Theorem 6, this extends significantly [32, Theorem 3.10.7], keeping practically the same proof. We ask ourselves if [32, Theorem 3.10.15] could be extended to this framework, taking into account that the boundedness condition on $\mathscr{A}$ in [32, Theorem 3.10.15] is much less restrictive. In [32, Theorem 3.10.15] we used a functional φ_A (defined by (5) and in (38) below) in order to prove the minimal-point theorem. Because an element k^0 does not impose itself naturally, and we need a stronger condition on the functional φ_A even if $k^0 \in K \setminus \{0\} \subset \mathrm{int}C$, we consider an abstract K–monotone functional φ to which we impose some conditions φ_A has already.

Theorem 9 (Not Authentic Minimal-Point Theorem with respect to $\preceq_{F,F'}$). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3), let $F' : Y \times Y \rightrightarrows K$ satisfy (F1) and (F2), and let $\mathscr{A} \subset X \times Y$ satisfy the condition*

(H1b) for every $\preccurlyeq_{F,F'}$–decreasing sequence $((x_n,y_n)) \subset \mathscr{A}$ with $x_n \to x \in X$ there exists $y \in Y$ such that $(x,y) \in \mathscr{A}$ and $(x,y) \preccurlyeq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$.

Assume that there exists a functional $\varphi : Y \to \overline{\mathbb{R}}$ such that

(F4) $(x_1,y_1) \preceq_{F,F'} (x_2,y_2) \Longrightarrow \varphi(y_1) + d(x_1,x_2) \le \varphi(y_2)$.

Furthermore, suppose

(B4) φ is bounded below on $\mathrm{Pr}_Y(\mathscr{A})$.

Then for every point $(x_0,y_0) \in \mathscr{A}$ with $\varphi(y_0) \in \mathbb{R}$, there exists $(\bar{x},\bar{y}) \in \mathscr{A}$ such that

(a) $(\bar{x},\bar{y}) \preceq_{F,F'} (x_0,y_0)$,

(b) $(x',y') \in \mathscr{A}$, $(x',y') \preceq_{F,F'} (\bar{x},\bar{y})$ *imply* $x' = \bar{x}$ *(not authentic minimal point with respect to* $\preceq_{F,F'}$*).*

Moreover, if φ *is strictly* K*–monotone on* $\mathrm{Pr}_Y(\mathscr{A})$*, that is,* $y_1,y_2 \in \mathrm{Pr}_Y(\mathscr{A})$, $y_2 - y_1 \in K\setminus\{0\} \Longrightarrow \varphi(y_1) < \varphi(y_2)$*, then*

(b') $(\bar{x},\bar{y})$ *is a minimal point of* $\mathscr{A}$ *with respect to* $\preceq_{F,F'}$ *(minimal point with respect to* $\preceq_{F,F'}$*).*

Proof. First note that from (F4) we have that φ is K–monotone. Let us construct a sequence $((x_n,y_n))_{n\geq 0} \subset \mathscr{A}$ as follows: Having $(x_n,y_n) \in \mathscr{A}$, we take $(x_{n+1},y_{n+1}) \in \mathscr{A}$, $(x_{n+1},y_{n+1}) \preceq_{F,F'} (x_n,y_n)$, such that

$$\varphi(y_{n+1}) \leq \inf\{\varphi(y) \mid (x,y) \in \mathscr{A},\ (x,y) \preceq_{F,F'} (x_n,y_n)\} + 1/(n+1). \tag{37}$$

Of course, the sequence $((x_n,y_n))$ is $\preceq_{F,F'}$–decreasing, and so (y_n) $(\subset \mathrm{Pr}_Y(\mathscr{A}))$ is K–decreasing. It follows that the sequence $(\varphi(y_n))$ is non-increasing and bounded from below, hence convergent in $\mathbb{R}$. Because

$$(x_{n+p},y_{n+p}) \preceq_{F,F'} (x_n,y_n) \preceq_{F,F'} (x_{n-1},y_{n-1}),$$

using (F4) and (37) we get

$$d(x_{n+p},x_n) \leq \varphi(y_n) - \varphi(y_{n+p}) \leq 1/n \quad \forall n,p \in \mathbb{N}^*.$$

It follows that (x_n) is a Cauchy sequence in the complete metric space (X,d), and so (x_n) is convergent to some $\bar{x} \in X$.

By (H1b) there exists $\bar{y} \in Y$ such that $(\bar{x},\bar{y}) \in \mathscr{A}$ and $(\bar{x},\bar{y}) \preceq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N}$. Let us show that $(\bar{x},\bar{y})$ is the desired element. Indeed, $(\bar{x},\bar{y}) \preceq_{F,F'} (x_0,y_0)$. Suppose that $(x',y') \in \mathscr{A}$ is such that $(x',y') \preceq_{F,F'} (\bar{x},\bar{y})$ $(\preceq_{F,F'} (x_n,y_n)$ for every $n \in \mathbb{N})$. Thus $\varphi(y') + d(x',\bar{x}) \leq \varphi(\bar{y})$ by (F4), whence

$$d(x',\bar{x}) \leq \varphi(\bar{y}) - \varphi(y') \leq \varphi(y_n) - \varphi(y') \leq 1/n \quad \forall n \geq 1.$$

It follows that $d(x',\bar{x}) = \varphi(\bar{y}) - \varphi(y') = 0$. Hence $x' = \bar{x}$.

Assuming that φ is strictly K–monotone, because $y' \leq_K \bar{y}$ and $\varphi(\bar{y}) - \varphi(y') = 0$, we have necessarily $y' = \bar{y}$. Hence $(\bar{x},\bar{y})$ is a minimal point with respect to $\preceq_{F,F'}$. □

Note that if $C \subset Y$ is a proper closed convex cone such that $C - (K\setminus\{0\}) = \operatorname{int} C$ and $k^0 \in K\setminus\{0\}$ (see assumption (A2)), the functional $\varphi_C : Y \to \mathbb{R}$ defined by (see (5))

$$\varphi_C(y) := \inf\left\{t \in \mathbb{R} \mid y \in tk^0 + C\right\} \tag{38}$$

is a strictly K–monotone continuous sublinear functional (see Theorem 2). Moreover, if the condition

(B') $\mathrm{Pr}_Y(\mathscr{A}) \cap (\widetilde{y} - \operatorname{int} C) = \emptyset$ for some $\widetilde{y} \in Y$

holds, then $\varphi := \varphi_C$ is bounded from below on $\mathrm{Pr}_Y(\mathscr{A})$, i.e., (B4) holds. Indeed, by Theorem 2, we have that $\varphi(y) + \varphi(-\widetilde{y}) \geq \varphi(y - \widetilde{y}) \geq 0$ for $y \in \mathrm{Pr}_Y(\mathscr{A})$, whence $\varphi(y) \geq -\varphi(-\widetilde{y})$ for $y \in \mathrm{Pr}_Y(\mathscr{A})$.

Another example for a function φ is that defined by

$$\varphi(y) := \varphi_{K,k^0}(y - \widehat{y}), \tag{39}$$

where K is a proper convex cone, $k^0 \in K \setminus \{0\}$, and $\widehat{y} \in Y$ is such that

(B”) $y_0 - \widehat{y} \in \mathbb{R}k^0 - K, \quad \mathrm{Pr}_Y(\mathscr{A}) \cap (\widehat{y} - K) = \emptyset.$

Then φ is K–monotone, $\varphi(y_0) < \infty$ and $\varphi(y) \geq 0$ for every $y \in \mathrm{Pr}_Y(\mathscr{A})$, i.e., (B4) holds.

For both of these functions in (38) and (39) we have to impose condition (F4) in order to be used in Theorem 9.

Remark 9. Using the function $\varphi = \varphi_{K,k^0}(\cdot - \widehat{y})$ (defined by (39)) in Theorem 9 we can derive [41, Theorem 4.2] taking $F(x_1,x_2) := \{d(x_1,x_2)k^0\}$ and $F'(y_1,y_2) := \{\varepsilon \|y_1 - y_2\| k^0\}$ when Y is a Banach space; note that, at its turn, [41, Theorem 4.2] extends [46, Theorem 8].

4.4 Ekeland’s Variational Principles of Ha’s Type

The previous EVP type results correspond to Pareto optimality. Ha [37] established an EVP type result which corresponds to Kuroiwa optimality. The next result is an extension of this type of result. For its proof we use [65, Theorem 3.1] or [41, Theorem 2.2].

Theorem 10 (Variational Principle). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3) and $\Gamma : X \rightrightarrows Y$ be such that*

(H5) $\{x \in X \mid \Gamma(u) \subset \Gamma(x) + F(x,u) + K\}$ *is closed for every* $u \in X$.

Moreover, if

(B5) z^* *(from (F3)) is bounded below on* $\Gamma(X)$,

then for every $x_0 \in \mathrm{dom}\,\Gamma$ *there exists* $\overline{x} \in X$ *such that*

(a) $\Gamma(x_0) \subset \Gamma(\overline{x}) + F(\overline{x},x_0) + K$,
(b) $\Gamma(\overline{x}) \subset \Gamma(x) + F(x,\overline{x}) + K$ *implies* $x = \overline{x}$.

Proof. Let us consider the relation $\preccurlyeq$ on X defined by $x' \preccurlyeq x$ if $\Gamma(x) \subset \Gamma(x') + F(x',x) + K$. By our hypotheses we have that $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$ is closed for every $x \in X$. Note that for $x \in X \setminus \mathrm{dom}\,\Gamma$ we have that $S(x) = X$, while for $x \in \mathrm{dom}\,\Gamma$ we have that $S(x) \subset \mathrm{dom}\,\Gamma$. The relation $\preccurlyeq$ is reflexive and transitive. The reflexivity of $\preccurlyeq$ is obvious. Let $x' \preccurlyeq x$ and $x'' \preccurlyeq x'$. Then $\Gamma(x) \subset \Gamma(x') + F(x',x) + K$ and $\Gamma(x') \subset \Gamma(x'') + F(x'',x') + K$. Using (F2) we get

$$\Gamma(x) \subset \Gamma(x'') + F(x'',x') + K + F(x',x) + K \subset \Gamma(x'') + F(x'',x) + K,$$

that is, $x'' \preccurlyeq x$. Consider

$$\varphi : X \to \overline{\mathbb{R}}, \quad \varphi(x) := \inf z^* \left(\Gamma(x)\right),$$

with the usual convention $\inf \emptyset := +\infty$. Clearly, $\varphi(x) \geq m := \inf z^*(\Gamma(X)) > -\infty$. Moreover, if $x' \preccurlyeq x \in \operatorname{dom}\Gamma$ then $z^*(\Gamma(x)) \subset z^*(\Gamma(x')) + z^*(F(x',x)) + z^*(K)$, whence $\varphi(x) \geq \varphi(x') + \inf z^*(F(x',x)) \geq \varphi(x')$.

Fix $x_0 \in \operatorname{dom}\Gamma$. The conclusion of the theorem asserts that there exists $\overline{x} \in X$ such that $\overline{x} \in S(x_0)$ and $S(\overline{x}) = \{\overline{x}\}$. To get this conclusion we apply [41, Th. 2.2] or [65, Th. 3.1]. Because (X,d) is complete and $S(x)$ is closed for every $x \in X$, we may (and we do) assume that $\operatorname{dom}\Gamma = X$ (otherwise we replace X by $S(x_0)$). In order to apply [41, Th. 2.2] we have to show that $d(x_n, x_{n+1}) \to 0$ provided $(x_n)_{n\geq 1} \subset X$ is $\preccurlyeq$–decreasing. In the contrary case there exist $\delta > 0$ and $(n_p)_{p\geq 1} \subset \mathbb{N}^*$ an increasing sequence such that $d(x_{n_p}, x_{n_p+1}) \geq \delta$ for every $p \geq 1$. Then, as seen above, $\varphi(x_n) \geq \varphi(x_{n+1}) + \inf z^* \left(F(x_{n+1}, x_n)\right)$, and so

$$\varphi(x_{n_1}) \geq \varphi\left(x_{n_p+1}\right) + \sum_{l=n_1}^{n_p} \inf z^* \left(F(x_{l+1}, x_l)\right) \geq m + p \cdot \eta(\delta)$$

with $\eta(\delta) > 0$ from (F3). Letting $p \to \infty$ we get a contradiction. Hence $d(x_n, x_{n+1}) \to 0$. The conclusion follows. □

Note that instead of assuming $S(u)$ to be closed for every $u \in X$ it is sufficient to have that $S(u)$ is $\preccurlyeq$–lower closed, that is, for every $\preccurlyeq$–decreasing sequence $(x_n) \subset S(u)$ with $x_n \to x$ we have that $x \in S(u)$. Moreover, instead of using [41, Theorem 2.2] it is possible to give a slightly longer direct proof similar to that of Theorem 6 (and using φ instead of z^* in the construction of (x_n)).

Remark 10. Taking Y to be a separated locally convex space, $K \subset Y$ a pointed closed convex cone and $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K \setminus \{0\}$, we can deduce [37, Theorem 3.1]. For this assume that $\Gamma(X)$ is quasi bounded, $\Gamma(x) + K$ is closed for every $x \in X$ and Γ is level-closed (or K-l.s.c. with the terminology from [37]). Since clearly z^* is bounded from below on $\operatorname{Im}\Gamma$, in order to apply the preceding theorem we need to have that $S(u)$ is closed for every $u \in X$; this is done in [37, Lemmma 3.2]. Below we provide another proof for the closedness of $S(u)$.

First, if $x \notin L(b)$ then there exists $\delta > 0$ such that $B(x,\delta) \cap L(b + \delta k^0) = \emptyset$. Indeed, because $x \notin L(b)$ we have that $b \notin \Gamma(x) + K$, and so $b + \delta' k^0 \notin \Gamma(x) + K$, that is, $x \notin L(b + \delta' k^0)$, for some $\delta' > 0$ (since $\Gamma(x) + K$ is closed). Because $L(b + \delta' k^0)$ is closed, there exists $\delta \in (0, \delta']$ such that $B(x,\delta) \cap L(b + \delta' k^0) = \emptyset$, and so $B(x,\delta) \cap L(b + \delta k^0) = \emptyset$.

Fix $u \in X$ and take $x \in X \setminus S(u)$, that is, $\Gamma(u) \not\subset \Gamma(x) + d(x,u)k^0 + K$. Then there exists $y \in \Gamma(u)$ with $b := y - d(x,u)k^0 \notin \Gamma(x) + K$. By the argument above there exists $\delta' > 0$ such that $B(x,\delta') \cap L(b+\delta' k^0) = \emptyset$, that is, $y - d(x,u)k^0 + \delta' k^0 \notin \Gamma(x') + K$ for every $x' \in B(x,\delta')$. Taking $\delta \in (0,\delta']$ sufficiently small we have that $d(x',u) \geq d(x,u) - \delta'$ for $x' \in B(x,\delta)$, and so $y \notin \Gamma(x') + d(x',u)k^0 + K$ for every $x' \in B(x,\delta)$, that is, $B(x,\delta) \cap S(u) = \emptyset$.

If we assume that $\Gamma(x_0) \not\subset \Gamma(x) + k^0 + K$ for every $x \in X$, then $\overline{x}$ provided by the preceding theorem satisfies $d(\overline{x},x_0) < 1$. Indeed, in the contrary case, because $\Gamma(x_0) \subset \Gamma(\overline{x}) + d(\overline{x},x_0)k^0 + K$ and $d(\overline{x},x_0)k^0 + K \subset k^0 + K$, we get the contradiction $\Gamma(x_0) \subset \Gamma(\overline{x}) + k^0 + K$. Replacing k^0 by εk^0 and d by $\lambda^{-1} d$ for some $\varepsilon, \lambda > 0$ we obtain exactly the statement of [37, Theorem 3.1].

In the case in which Y is just a topological vector space we have the following version of the preceding theorem under conditions similar to those in Theorem 7.

Theorem 11 (Variational Principle). *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper closed convex cone. Let $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H+K)$, and $\Gamma : X \rightrightarrows Y$. If*

(H6) $\{x \in X \mid \Gamma(u) \subset \Gamma(x) + d(x,u)H + K\}$ *is closed for every $u \in X$,*
(B6) $\Gamma(X)$ *is quasi bounded,*

then for every $x_0 \in \operatorname{dom}\Gamma$ there exists $\overline{x} \in X$ such that

(a) $\Gamma(x_0) \subset \Gamma(\overline{x}) + d(\overline{x},x_0)H + K$,
(b) $\Gamma(\overline{x}) \subset \Gamma(x) + d(x,\overline{x})H + K$ *implies $x = \overline{x}$.*

Proof. Let $B \subset Y$ be a bounded set such that $\Gamma(X) \subset B + K$.

Consider $F(x,x') := d(x,x')H$ for $x,x' \in X$. As seen before, F satisfies conditions (F1) and (F2), and so the relation $\preccurlyeq$ defined in the proof of Theorem 10 is reflexive and transitive; moreover, by our hypotheses, $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$ is closed for every $x \in X$. As in the proof of Theorem 10 we may (and do) assume that $X = \operatorname{dom}\Gamma$ and it is sufficient to show that $d(x_n,x_{n+1}) \to 0$ provided $(x_n)_{n\geq 1} \subset X$ is $\preccurlyeq$–decreasing. In the contrary case there exist $\delta > 0$ and $(n_p)_{p\geq 1} \subset \mathbb{N}^*$ an increasing sequence such that $d(x_{n_p},x_{n_p+1}) \geq \delta$ for every $p \geq 1$.

Fixing $y_1 \in \Gamma(x_1)$, inductively we find the sequences $(y_n)_{n\geq 0} \subset Y$, $(h_n)_{n\geq 0} \subset H$ and $(k_n)_{n\geq 0} \subset K$ such that $y_n = y_{n+1} + d(x_n,x_{n+1})h_n + k_n$ for every $n \geq 1$. Using the convexity of H, and the facts that $H \subset K$ and $\Gamma(X) \subset B + K$, for $p \in \mathbb{N}$ we get $h'_p \in H$, $b_p \in B$ and $k'_p, k''_p \in K$ such that

$$y_1 = y_{n_{p+1}} + \sum_{l=1}^{n_p} d(x_l,x_{l+1})h_l + \sum_{l=1}^{n_p} k_l = b_p + \delta(h_{n_1} + \ldots + h_{n_p}) + k'_p = b_p + p\delta h'_p + k''_p.$$

It follows that $(p\delta)^{-1}(y_1 - b_p) \in H + K$ for every $p \geq 1$. Since (b_p) is bounded we obtain the contradiction $0 \in \operatorname{cl}(H+K)$. The conclusion follows. $\square$

Again, instead of assuming that $S(u)$ is closed for every $u \in X$, it is sufficient to assume that $S(u)$ is $\preccurlyeq$-lower closed for $u \in X$. A slightly longer direct proof, similar to that of Theorem 7, is possible. Also Theorem 11 covers [37, Th. 3.1].

4.5 Ekeland's Variational Principle for Bi-Multifunctions

In [9] Bianchi, Kassay and Pini obtained an EVP type result for vector functions of two variables; previously such results were obtained by Isac [45] and Li et al. [49]. The next result extends [9, Theorem 1] in two directions: d is replaced by F satisfying (F1)–(F3) and instead of a single-valued function $f : X \times X \to Y$ we take a multi-valued one. For its proof we use again [65, Theorem 3.1] or [41, Theorem 2.2].

Theorem 12. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy conditions (F1)–(F3). Assume that $G : X \times X \rightrightarrows Y$ has the properties:*

(i) $0 \in G(x,x)$ *for every* $x \in X$,
(ii) $G(x_1,x_2) + G(x_2,x_3) \subset G(x_1,x_3) + K$ *for all* $x_1,x_2,x_3 \in X$.

If

(H7) $\{x' \in X \mid [G(x,x') + F(x,x')] \cap (-K) \neq \emptyset\}$ *is closed for every* $x \in X$,
(B7) z^* *(from (F3)) is bounded below on the set* $\operatorname{Im} G(x,\cdot)$ *for every* $x \in X$,

then for every $x_0 \in X$ there exists $\overline{x} \in X$ such that

(a) $[G(x_0,\overline{x}) + F(x_0,\overline{x})] \cap (-K) \neq \emptyset$,
(b) $[G(\overline{x},x) + F(\overline{x},x)] \cap (-K) \neq \emptyset$ *implies* $x = \overline{x}$.

Proof. Let us consider the relation $\preccurlyeq$ on X defined by

$$x \preccurlyeq x' \iff \left[G(x',x) + F(x',x)\right] \cap (-K) \neq \emptyset.$$

Then $\preccurlyeq$ is reflexive and transitive. The reflexivity is immediate from (i) and (F1). Assume that $x \preccurlyeq x'$ and $x' \preccurlyeq x''$. Then $-k \in G(x',x) + F(x',x)$ and $-k' \in G(x'',x') + F(x'',x')$ with $k,k' \in K$. Hence, by (ii) and (F2),

$$-k - k' \in G(x',x) + F(x',x) + G(x'',x') + F(x'',x') \subset G(x'',x) + K + F(x'',x) + K,$$

whence $[G(x'',x) + F(x'',x)] \cap (-K) \neq \emptyset$, that is, $x \preccurlyeq x''$.

Setting $S(x) := \{x' \in X \mid x' \preccurlyeq x\}$, by (H7) we have that $S(x)$ is closed for every $x \in X$. We have to show that for $(x_n)_{n\geq 1} \subset X$ a $\preccurlyeq$–decreasing sequence one has $d(x_n,x_{n+1}) \to 0$. In the contrary case there exist an increasing sequence $(n_l)_{l\geq 1} \subset \mathbb{N}$ and $\delta > 0$ such that $d(x_{n_l},x_{n_l+1}) \geq \delta$ for every $l \geq 1$. Because (x_n) is $\preccurlyeq$–decreasing, we have that $-k_n \in G(x_n,x_{n+1}) + F(x_n,x_{n+1})$ for some $k_n \in K$ and every $n \geq 1$. Then

$$-k_1 - \ldots - k_n \in G(x_1,x_{n+1}) + F(x_1,x_2) + \ldots + F(x_n,x_{n+1}) + K,$$

and so

$$\inf z^*(\operatorname{Im} G(x_1,\cdot)) + \inf z^*(F(x_1,x_2)) + \ldots + \inf z^*(F(x_n,x_{n+1})) \leq 0 \quad \forall n \geq 1.$$

Since $\inf z^*\left(F(x_n,x_{n+1})\right) \geq 0$ for every $n \geq 1$ and $\inf z^*\left(F(x_{n_l},x_{n_l+1})\right) \geq \eta(\delta) > 0$ for every $l \geq 1$, taking $n := n_p$ with $p \geq 1$, we obtain that

$$p\eta(\delta) \leq -\inf z^*\left(\operatorname{Im} G(x_1,\cdot)\right) \text{ for every } p \geq 1.$$

This yields the contradiction $\eta(\delta) \leq 0$. Hence $d(x_n,x_{n+1}) \to 0$. Applying [41, Theorem 2.2] we get some $\bar{x} \in S(x_0)$ with $S(\bar{x}) = \{\bar{x}\}$, that is, our conclusion holds. □

Remark 11. If we need the conclusion only for a fixed (given) point $x_0 \in X$, we may replace condition (B7) by the fact that z^* (from (F3)) is bounded below on the set $\operatorname{Im} G(x_0,\cdot)$.

Indeed, $X_0 := S(x_0)$ is closed by (H7), and so (X_0,d) is complete. If $x \in X_0$ then $-k \in G(x_0,x)+F(x_0,x) \subset G(x_0,x)+K$ for some $k \in K$, and so $-k' \in G(x_0,x)$ for some $k' \in K$. It follows that $-k'+G(x,u) \subset G(x_0,x)+G(x,u) \subset G(x_0,u)+K$, whence $G(x,u) \subset G(x_0,u)+K$ for every $u \in X$. Hence condition (B7) is verified on X_0, and so the conclusion of the theorem holds for x_0.

Remark 12. For $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K \setminus \{0\}$ and G single-valued, using Theorem 12 and the preceding remark one obtains [45, Theorem 8] and [49, Theorem 3]; in [45] K is normal and closed, while in [49] $k^0 \in \operatorname{int} K$.

Note that condition (H7) in the preceding theorem holds when G is compact-valued, $G(u,\cdot)$ is level-closed, K is closed and $F(x,x') := \{d(x,x')k^0\}$ for some $k^0 \in K$. Indeed, assume that $-k_n \in G(u,x_n)+d(x_n,u)k^0$ for every $n \geq 1$, where $k_n \in K$. Take $\varepsilon > 0$. Then there exists $n_\varepsilon \geq 1$ such that $d(x_n,u) \geq d(x,u)-\varepsilon =: \gamma_\varepsilon$ for every $n \geq n_\varepsilon$. Then for such n we have that $G(u,x_n) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$, whence $G(u,x) \cap \left(-\gamma_\varepsilon k^0 - K\right) \neq \emptyset$. Hence there exists $y_\varepsilon \in G(u,x)$ such that $y_\varepsilon + \gamma_\varepsilon k^0 \in -K$. Since $G(u,x)$ is compact, $(y_\varepsilon)_{\varepsilon>0}$ has a subnet converging to $y \in G(x,u)$. Since $\lim_{\varepsilon\to 0}\gamma_\varepsilon = d(x,u)$ and K is closed, we obtain that $y+d(x,u)k^0 \in -K$.

If Y is a separated locally convex space then we may assume that G is weakly compact-valued instead of being compact-valued.

When G is single-valued and $F(x,x') := \{d(x,x')k^0\}$ with $k^0 \in K$, where K is closed and $z^*(k^0) = 1$, the preceding theorem reduces to [9, Theorem 1].

4.6 EVP type Results

The framework is the same as in the previous sections. We want to apply the preceding results to obtain vectorial EVPs. To envisage functions defined on subsets of X we add to Y an element ∞ not belonging to the space Y, obtaining thus the space $Y^\bullet := Y \cup \{\infty\}$. We consider that $y \leq_K \infty$ for all $y \in Y$. Consider now the function $f : X \to Y^\bullet$. As usual, the domain of f is $\operatorname{dom} f = \{x \in X \mid f(x) \neq \infty\}$; the epigraph of f is $\operatorname{epi} f = \{(x,y) \in X \times Y \mid f(x) \leq_K y\}$; the graph of f is $\operatorname{gph} f = \{(x, f(x)) \mid x \in \operatorname{epi} f\}$. Of course, f is proper if $\operatorname{dom} f \neq \emptyset$. For $y^* \in K^+$ we set $(y^* \circ f)(x) := +\infty$ for $x \in X \setminus \operatorname{dom} f$.

Theorem 13. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper convex cone. Let $F : X \times X \rightrightarrows K$ satisfy the conditions (F1)–(F3) and let $f : X \to Y^\bullet$ be proper. Assume that*

(H8) for every sequence $(x_n) \subset \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1},x_n) + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + F(x,x_n) + K$ for every $n \in \mathbb{N}$,
(B8) $z^* \circ f$ (with z^* from (F3)) is bounded from below.

Then for every $x_0 \in \operatorname{dom} f$ there exists $\bar{x} \in \operatorname{dom} f$ such that

(a) $f(x_0) \in f(\bar{x}) + F(\bar{x},x_0) + K$,
(b) $\forall x \in \operatorname{dom} f \ : \ f(\bar{x}) \in f(x) + F(\bar{x},x) + K \Rightarrow x = \bar{x}$.

Proof. Consider $\mathscr{A} := \operatorname{gph} f := \{(x, f(x)) \mid x \in \operatorname{dom} f\}$. Condition (H8) says nothing than (H1) is verified. Applying Theorem 6 we get the conclusion. □

As for Theorem 6, in the above theorem we may assume that z^* is bounded from below on the set

$$B_0 := \{ f(x) \mid x \in \operatorname{dom} f,\ f(x_0) \in f(x) + F(x,x_0) + K\}.$$

The preceding theorem is very close to [36, Theorem 3.8] for $\gamma = 1$, which is stated for F and K satisfying conditions (i), (ii), (iii), (S1), (S2) and $f : S \to Y$ (with $S \subset X$ a nonempty closed set) satisfying the conditions

(S3) Let us denote $A_x^{\gamma F} := \{z \in X \mid (f(z) + \gamma F(z,x)) \cap (f(x) - K) \neq \emptyset\}$ for $x \in S$. For each $x \in S$ and $(z_n) \subset A_x^{\gamma F}, z_n \to z$ such that $f(z_n) \leq f(z_m)$ for $n > m$, it follows that $z \in A_x^{\gamma F}$.
(S4) The set $(f(S) - f(x_0)) \cap (-D_F)$ is K-bounded.

Because S is closed one may assume that $S = X$ and $\operatorname{dom} f = X$. Observe that (S4) implies that $y^*(B_0)$ is bounded from below for every $y^* \in K^+$, and so $z^*(B_0)$ is bounded from below. Let us prove that (S3) implies (H8) (for $\gamma = 1$). Consider $(x_n) \subset X = \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + F(x_{n+1},x_n) + K$ for every $n \in \mathbb{N}$. Clearly, for a fixed $\bar{n} \in \mathbb{N}$ we have that $(x_n)_{n \geq \bar{n}} \subset A_{x_{\bar{n}}}^{1F}$ and $f(x_n) \leq f(x_m)$ for $n \geq m \geq \bar{n}$. By (S3) we have that $x \in A_{x_{\bar{n}}}^{1F}$, that is, $f(x_{\bar{n}}) \in f(x) + F(x,x_{\bar{n}}) + K$. Hence (H8) holds.

In the case in which $F(x,x') = d(x,x')H$ for some $H \subset K$ the condition (H8) becomes

(H9) for every sequence $(x_n) \subset \operatorname{dom} f$ with $x_n \to x \in X$ and $f(x_n) \in f(x_{n+1}) + d(x_{n+1},x_n)H + K$ for every $n \in \mathbb{N}$ one has $f(x_n) \in f(x) + d(x,x_n)H + K$ for every $n \in \mathbb{N}$.

In the case $H := \{k^0\}$ condition (H9) is nothing else than condition (E1) in [41].

Using Theorem 13 and Proposition 7 we have the following variant of the preceding result.

Theorem 14. *Assume that (X,d) is a complete metric space, Y is a real topological vector space and $K \subset Y$ is a proper closed convex cone. Let $f : X \to Y^\bullet$ be a proper function and $H \subset K$ be a nonempty cs-complete bounded set with $0 \notin \operatorname{cl}(H+K)$. If*

(H10) *for every sequence $(x_n) \subset \operatorname{dom} f$ such that $x_n \to x \in X$ and $(f(x_n))$ is $\leq_K$–decreasing one has $f(x) \leq_K f(x_n)$ for every $n \in \mathbb{N}$,*
(B10) *$f(\operatorname{dom} f)$ is quasi bounded*

hold, then for every $x_0 \in \operatorname{dom} f$ there exists $\overline{x} \in \operatorname{dom} f$ such that

(a) $(f(x_0) - K) \cap (f(\overline{x}) + d(\overline{x}, x_0)H) \neq \emptyset$,
(b) $(f(\overline{x}) - K) \cap (f(x) + d(\overline{x}, x)H) = \emptyset \quad \forall x \in \operatorname{dom} f \setminus \{\overline{x}\}$.

Proof. Since condition (H10) is exactly condition (H1) for $\mathscr{A} := \operatorname{gph} f$ and $F = F_H$, in order to have the conclusion of the theorem it is sufficient to show that (H2) is verified for this situation; then just use Proposition 7 and Theorem 13.

Let $((x_n, y_n)) \subset \operatorname{gph} f$ be such that $x_n \to x \in X$ and (y_n) is $\leq_K$–decreasing. Hence $y_n = f(x_n)$ for every n. By (H10) we have that $y := f(x) \leq_K f(x_n) = y_n$ for every $n \in \mathbb{N}$ and, of course, $(x, f(x)) \in \operatorname{gph} f$. The proof is complete. □

Remark 13. Taking H to be complete, convex and bounded, then H is cs-complete. In this case we obtain the main result in [8], that is, [8, Theorem 4.1].

Note that the closed convex subsets as well as the open convex subsets of a separated locally convex space are cs-closed; moreover, all the convex subsets of finite dimensional normed spaces are cs-closed (hence cs-complete).

Remark 14. Taking $H := \{k^0\}$ in the preceding theorem one obtains practically [32, Corollary 3.10.6]; there K is assumed to be closed in the direction k^0, the present condition (H10) being condition (H4) in [32, Corollary 3.10.6].

Remark 15. Similar results can be stated using Theorems 8 and 9. When specializing to $F(x_1,x_2) = \left\{d(x_1,x_2)k^0\right\}$ and $F'(y_1,y_2) = \left\{\varepsilon \|y_1 - y_2\| k^0\right\}$ one recovers [41, Corollary 3.1] and [41, Theorem 4.2].

5 Applications in Vector Optimization

5.1 Solution Concepts

Consider the vector minimization problem (VP) given as

$$V - \min f(x), \quad \text{s.t.} \quad x \in S,$$

where X and Y are separated locally convex spaces, $\{0\} \neq K \subset Y$ is a closed convex cone (which induces the partial order $\leq_K$ on Y), $f : X \to Y$ and $S \subseteq X$. As in the preceding sections $k^0 \in K \setminus (-K)$ is fixed. The solution concepts for the vector optimization problem (VP) are described in the next definition.

Definition 1.

- The element $y_0 \in F \subset Y$ is said to be a *minimal point* of F with respect to K if $F \cap (y_0 - K) \subset y_0 + K$. The set of minimal points of F with respect to K is denoted by $\mathrm{Eff}(F,K)$. An element $x_0 \in S$ is called an *efficient solution* of (VP) if $f(x_0) \in \mathrm{Eff}(f(S),K)$.
- The element $y_0 \in F$ is said to be a *properly minimal point* of F w.r.t. K if there is a closed convex set $A \subset Y$ with $0 \in \mathrm{bd}A$ and $A - (K \setminus \{0\}) \subset \mathrm{int}A$ such that $F \cap (y_0 + \mathrm{int}A) = \emptyset$. An element $x_0 \in S$ is called a *properly efficient* solution for (VP) if $f(x_0)$ is a properly minimal point of $f(S)$.
- The element $y_0 \in F$ is said to be a *weakly minimal point* of F if $\mathrm{int}K \neq \emptyset$ and $F \cap (y_0 - \mathrm{int}K) = \emptyset$. The set of weakly minimal points of F is denoted by $w\mathrm{Eff}(F,K)$. An element $x_0 \in S$ is a *weakly efficient solution* of (VP) if $f(x_0) \in w\mathrm{Eff}(f(S),K)$.

Note first that from the very definition of weakly minimal points of F one has

$$w\mathrm{Eff}(F,K) = F \setminus (F + \mathrm{int}K); \tag{40}$$

then observe that the set A appearing in the definition of a properly minimal point verifies Assumption $(A2)$. Moreover, note that what is called here a properly minimal point of F w.r.t. K is said to be an E-minimal element of F in [66] and was introduced by Iwanow and Nehse in [47] for $K = \mathbb{R}^n_+$ and Gerstewitz and Iwanow [28] for the general case.

Lemma 1. *Let $x_0 \in S$.*

(a) If x_0 is a properly efficient solution of (VP) and $A \subset Y$ is the set provided by Definition 1 (ii) then x_0 is a solution of the scalar minimization problem

$$\min\ \varphi_{A,k^0}(f(x) - f(x_0)) \quad s.t. \quad x \in S, \tag{41}$$

where $k^0 \in K \setminus \{0\}$.

(b) If x_0 is a weakly efficient solution of (VP) and $k^0 \in \mathrm{int}K$, then x_0 is a solution of problem (41) with $A := -K$.

Proof. In both cases we have that $f(S) \cap (f(x_0) + \mathrm{int}A) = \emptyset$. Moreover, because $0 \in \mathrm{bd}A$, we have that $\varphi_A(0) = 0$. Assuming that $\varphi_{A,k^0}(f(x) - f(x_0)) < \varphi_{A,k^0}(f(x_0) - f(x_0)) = 0$, we get the contradiction $f(x) - f(x_0) \in \mathrm{int}A$. $\square$

5.2 Necessary Optimality Conditions in Vector Optimization

We consider vector optimization problems on Asplund spaces without convexity assumptions. Recall that a Banach space X is said to be an Asplund space (cf. Phelps [53, Definition 1.22]) if every continuous convex function defined on a nonempty open convex subset D of X is Fréchet differentiable at each point of some G_δ subset of D. It is known that the Banach spaces with separable dual and the reflexive Banach spaces are Asplund spaces. So c_0 and ℓ^p, $L^p[0,1]$ for $1 < p < \infty$ are Asplund spaces, but ℓ^1 is not an Asplund space.

Under the assumption that the objective function is locally Lipschitz we derive Lagrangian necessary conditions on the basis of Mordukhovich subdifferential using the Lipschitz continuity properties of φ_A discussed in Section 3.4. In the following we provide necessary conditions for properly efficient solutions of a vector optimization problem that are related to the strong free-disposal assumption in $(A2)$.

In order to present our results concerning the existence of Lagrange multipliers, we work with the Mordukhovich subdifferential ∂_M and normal cone N_M (denoted ∂ and N in [51]). One says ([51, Definition 3.25]) that a function $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ if f is Lipschitz on a neighbourhood of $\bar{x}$ and there exists a neighbourhood U of the origin in X such that the sequence $(t_k^{-1}(f(x_k + t_k u) - f(x_k)))_{k\in\mathbb{N}}$ contains a (norm) convergent subsequence whenever $u \in U$, $x_k \to \bar{x}$ and $t_k \downarrow 0$. It is clear that this notion reduces to local Lipschitz continuity if Y is finite dimensional.

Remark 16. The function f is strictly Lipschitz at $\bar{x}$ if and only if the sequence $\left(\|u_n\|^{-1}[f(x_n + u_n) - f(x_n)]\right)$ has a norm converging subsequence whenever $(x_n) \subset X$ converges to $\bar{x}$, $(u_n) \subset X \setminus \{0\}$ converges to 0 and the sequence $\left(\|u_n\|^{-1} u_n\right)$ converges in X.

For more details regarding the class of strictly Lipschitz mappings (with values in infinite dimensional spaces) see [51, Section 3.1.3].

We need the following calculus rules from [51] (see [51, Theorem 3.36] and [51, Corollary 3.43]) for proving one of our main results.

Lemma 2. *Assume that X and Y are Asplund spaces.*

(a) If $f_1, f_2 : X \to \overline{\mathbb{R}}$ are proper functions and there exists a neighbourhood U of $\bar{x} \in \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ such that f_1 is Lipschitz on U and f_2 is lsc on U, then

$$\partial_M(f_1 + f_2)(\bar{x}) \subset \partial_M f_1(\bar{x}) + \partial_M f_2(\bar{x}).$$

(b) If $f : X \to Y$ is strictly Lipschitz at $\bar{x}$ and $\varphi : Y \to \overline{\mathbb{R}}$ is finite and Lipschitz on a neighbourhood of $f(\bar{x})$, then

$$\partial_M(\varphi \circ f)(\bar{x}) \subset \bigcup \{\partial_M(y^* \circ f)(\bar{x}) \mid y^* \in \partial_M \varphi(f(\bar{x}))\}.$$

In the next result we provide necessary optimality conditions for properly efficient solutions of problem (VP).

Theorem 15. *Assume that X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitz, S is a closed subset of X and $x_0 \in S$. If x_0 is a properly efficient solution of (VP) then there exists $v^* \in K^\#$ such that*

$$0 \in \partial_M(v^* \circ f)(x_0) + N_M(S, x_0). \tag{42}$$

Moreover, if f is strictly differentiable at x_0 then

$$(f'(x_0))^* v^* \in -N_M(S, x_0). \tag{43}$$

Proof. Assume that x_0 is a properly efficient solution for (VP). By Lemma 1, x_0 is a solution of the problem (41), or equivalently x_0 is a minimum point of

$$h : X \to \overline{\mathbb{R}}, \quad h(x) := \varphi_A(f(x) - f(x_0)) + \iota_S(x),$$

where $A \subset Y$ is a closed convex set such that $0 \in \operatorname{bd} A$ and $A - (K \setminus \{0\}) \subset \operatorname{int} A$. As seen in Remark 2 (ii), $\operatorname{dom} \varphi_A$ is open because $k^0 \in K \setminus \{0\}$ and $\varphi_A(0) = 0$ because $0 \in \operatorname{bd} A$. Note that $k^0 \notin A_\infty$; otherwise we get the contradiction $0 = (0 + k^0) - k^0 \in A - \mathbb{P}k^0 \subset \operatorname{int} A$. Since $0 \in A \subset \operatorname{dom} \varphi_A$, by Proposition 5 we have that φ_A is convex and Lipschitz on a neighbourhood of 0. It follows that $0 \in \partial_M h(x_0)$ (see [51, Prop. 1.114]). Since f is strictly Lipschitz and φ_A is Lipschitz on a neighbourhood of 0, the function $x \mapsto \varphi_A(f(x) - f(x_0))$ is Lipschitz on a neighbourhood of x_0. Moreover, since S is a closed subset of X we have that ι_S is a proper lower-semicontinuous function. Using both parts of Lemma 2 we have that

$$0 \in \partial_M (v^* \circ f)(x_0) + N_M(S, x_0)$$

for some $v^* \in \partial_M \varphi_A(0) = \partial \varphi_A(0)$, φ_A being convex and finite and Lipschitz on a neighbourhood of 0. From Corollary 7 we have that $v^* \in K^\#$ and $\langle k^0, v^* \rangle = 1$, and so $v^* \neq 0$. If f is strictly differentiable at x_0 then $\partial_M f(x_0) = \{f'(x_0)\}$, and so the last conclusion follows. □

For weakly efficient solutions of (VP) we have the following result.

Theorem 16. *Assume that X and Y are Asplund spaces, $f : X \to Y$ is strictly Lipschitz, S is a closed subset of X and $x_0 \in S$. If x_0 is a weakly efficient solution of (VP) then there exists $v^* \in K^+ \setminus \{0\}$ such that (42) holds. Moreover, if f is strictly differentiable at x_0 then (43) holds.*

Proof. If $\operatorname{int} K \neq \emptyset$ and x_0 is a weakly efficient solution for (VP), by Lemma 1 we have that x_0 is a minimum point of h for $A := -K$ and $k^0 \in \operatorname{int} K$. This time φ_A is Lipschitz and sublinear. The rest of the proof is similar. □

Remark 17. If X is an Asplund space and $g : X \to \overline{\mathbb{R}}$ is finite and Lipschitz on a neighbourhood of $x_0 \in S \subset X$ with S closed, the following well-known relations

$$\partial_{Cl} g(x_0) = \overline{\operatorname{conv}}^{w^*} \partial_M g(x_0) \text{ and } N_{Cl}(S, x_0) = \overline{\operatorname{conv}}^{w^*} N_M(S, x_0)$$

hold (see [51, Theorem 3.57]), where $\partial_{Cl} g(x_0)$ and $N_{Cl}(S, x_0)$ represent the Clarke's subdifferential of g at x_0 and the Clarke's normal cone of S at x_0, respectively. In the hypotheses of Theorem 15 from (42) we get the necessary optimality condition

$$\exists v^* \in K^+ \setminus \{0\} \ : \ 0 \in \partial_{Cl}(v^* \circ f)(x_0) + N_{Cl}(S, x_0) \tag{44}$$

in terms of the Clarke's subdifferential and normal cone. However, the optimality condition given by (42) is sharper than the condition given by (44).

Remark 18. Note that Theorems 15 and 16 remain valid when the Mordukhovich subdifferential ∂_M is replaced by any subdifferential ∂ which verifies conditions (H1)–(H4) in [17]. In such a situation Theorem 16 corresponds to Lagrangian necessary condition for weakly efficient solutions in [17, Theorem 3.1] (compare also [20, Theorem 3.2] for the case $\dim Y < \infty$). In Theorem 15 we have established the result for properly efficient solutions without assuming $\operatorname{int} K \neq \emptyset$.

Another application envisage fuzzy necessary optimality conditions for approximate minimizers of a Lipschitz vectorial function (compare Durea and Tammer [17]). First, we need a definition.

Definition 2. If $\alpha > 0$ and $k^0 \in \operatorname{int} K$, a point $x_0 \in S$ is said to be (α, k^0)-*efficient solution* of (VP) if $(f(S) - f(x_0)) \cap (-\alpha k^0 - K) = \emptyset$.

Of course, every weakly efficient solution of (VP) is a (α, k^0)-efficient solution for every $\alpha > 0$ and $k^0 \in \operatorname{int} K$, but the converse is false, in general.

We introduce now the concept of abstract subdifferential (see, e.g. [43]; see also [19] for a theory of subdifferentials for vector-valued functions). Let $\mathscr{X}$ be a class of Banach spaces which contains the class of finite dimensional normed vector spaces. By an abstract subdifferential ∂ we mean a map which associates to every lsc function $h : X \in \mathscr{X} \to \overline{\mathbb{R}}$ and to every $x \in X$ a (possible empty) subset $\partial h(x) \subset X^*$; $\partial h(x) = \emptyset$ if $f(x) \notin \mathbb{R}$. Let $X, Y \in \mathscr{X}$ and denote by $\mathscr{F}(X,Y)$ a class of functions acting between X and Y having the property that by composition at left with a lsc function from Y to $\overline{\mathbb{R}}$ the resulting function is still lsc. In the sequel we shall work in every specific case with some of the next properties of the abstract subdifferential ∂.

(C1) If $h : X \to \overline{\mathbb{R}}$ is a proper lsc convex function then $\partial h(x)$ coincides with the Fenchel subdifferential.

(C2) If $x \in X$ is a local minimum point for the lsc function h and $h(x) \in \mathbb{R}$ then $0 \in \partial h(x)$.

Note that (C1) and (C2) are very natural requirements for any subdifferential.

The counterparts of "exact calculus rules" are the far more general "fuzzy calculus rules".

(C3) If $X \in \mathscr{X}$, $\varphi : X \to \mathbb{R}$ is a locally Lipschitz functions and $x \in \operatorname{dom} h$, then

$$\partial(h + \varphi)(x) \subset \|\cdot\|^* - \limsup_{y \to_h x, z \to x} (\partial h(y) + \partial \varphi(z)),$$

(C4) If $\varphi : Y \to \mathbb{R}$ is locally Lipschitz and $\psi \in \mathscr{F}(X,Y)$, then for every x,

$$\partial(\varphi \circ \psi)(x) \subset \|\cdot\|^* - \limsup_{u \to_\psi x, v \to \psi(x)} \bigcup\nolimits_{u^* \in \partial \varphi(v)} \partial(u^* \circ \psi)(u).$$

where the following notations are used:

1. $u \to_h x$ means that $u \to x$ and $h(u) \to h(x)$; note that if h is continuous, then $u \to_h x$ is equivalent with $u \to x$.

2. $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ means that for every $\varepsilon > 0$ there exist x_ε and x_ε^* such that $x_\varepsilon^* \in \partial h(x_\varepsilon)$ and $\|x_\varepsilon - x\| < \varepsilon$, $\|x_\varepsilon^* - x^*\| < \varepsilon$; the notation $x^* \in \|\cdot\|^* - \limsup_{u \to_h x} \partial h(u)$ has a similar interpretation and is equivalent with $x^* \in \|\cdot\|^* - \limsup_{u \to x} \partial h(u)$ provided that h is continuous.

The property (C3) is called fuzzy sum rule and a space X on which such a property holds is called trustworthiness space for the subdifferential ∂. For example, for the Fréchet subdifferential the trustworthiness spaces are the Asplund spaces (see [23]). This rule is also satisfied (see [48, pp. 41], [18], [44] and the references therein) by:

- the proximal subdifferential when $\mathscr{X}$ is the class of Hilbert spaces;
- the Fréchet subdifferential of viscosity when $\mathscr{X}$ is the class of Banach spaces which admit a C^1 Lipschitz bump function;
- the β-subdifferential of viscosity when $\mathscr{X}$ is the class of Banach spaces which admit a β-differentiable bump function.

The next result goes back to Durea and Tammer [17].

Theorem 17. *Let $X, Y \in \mathscr{X}$, let $f \in \mathscr{F}(X,Y)$ be locally Lipschitz and let S be a closed subset of X. Let $x_0 \in S$ be a weakly efficient solution of (VP). Then for every $k^0 \in \operatorname{int} K$ and $\varepsilon > 0$ there exist $u \in B(x_0, \varepsilon)$, $z \in B(x_0, \varepsilon/2) \cap S$ and $u^* \in K^+$ with $u^*(k^0) = 1$ such that*

$$0 \in \partial(u^* \circ f)(u) + N_\partial(S,z) + B(0,\varepsilon),$$

provided that ∂ satisfies conditions (C1), (C2), (C3), (C4). Moreover, for some $x \in B(x_0, \varepsilon/2)$ and $v \in B(f(x) - f(x_0), \varepsilon/2)$ we have that $u^*(v) = \varphi(v)$.

Proof. Let us consider $\varepsilon > 0$ and the functional φ given by (5) corresponding to a fixed $k^0 \in \operatorname{int} K$. We have that

$$f(x_0) \in w\operatorname{Eff}(f(S), K)$$

which means that

$$0 \in w\operatorname{Eff}(f(S) - f(x_0), K).$$

Thus, $\varphi(0) = 0$ and $\varphi(f(S) - f(x_0)) \geq 0$, whence x_0 is a minimum point for $(\varphi \circ g) + \iota_S$, where g is defined by $g(x) = f(x) - f(x_0)$. From (C2) we get

$$0 \in \partial(\varphi \circ g + \iota_S)(x_0)$$

and from (C3) (φ is Lipschitz, g is locally Lipschitz and ι_S is lsc because S is closed), there exist $x \in B(x_0, \varepsilon/2)$, $z \in B(x_0, \varepsilon/2) \cap S$, $p^* \in \partial(\varphi \circ g)(x)$, and $q^* \in N_\partial(S,z)$ such that

$$\|p^* + q^*\| < \varepsilon/2.$$

Since $p^* \in \partial(\varphi \circ g)(x)$, by (C4) there exist $u_1 \in B(x, \varepsilon/3) \subset B(x_0, 5\varepsilon/6)$, $v \in B(g(x), \varepsilon/2)$, $u^* \in \partial\varphi(v)$ and $v_1^* \in \partial(u^* \circ g)(u_1)$ such that

$$\|v_1^* - p^*\| < \varepsilon/2.$$

It follows that

$$\|v_1^* + q^*\| = \|v_1^* - p^* + p^* + q^*\| < 5\varepsilon/6.$$

This means that

$$0 \in \partial(u^* \circ g)(u_1) + N_\partial(S, z) + B(0, 5\varepsilon/6).$$

But

$$\partial(u^* \circ g)(u_1) = \partial(u^* \circ (f(\cdot) - f(x_0)))(u_1) = \partial(u^* \circ f)(u_1)$$

because the function $u \mapsto -u^*(f(x_0))$ is constant (in particular convex). Applying (C3) we find $u \in B(u_1, \varepsilon/6) \subset B(x_0, \varepsilon)$ and $v^* \in \partial(u^* \circ f)(u)$ such that

$$\|v_1^* - v^*\| < \varepsilon/6.$$

We deduce that

$$0 \in \partial(u^* \circ f)(u) + N_\partial(S, z) + B(0, \varepsilon).$$

The assertions concerning u^* follow from Corollary 7 and this completes the proof. □

Concerning (α, k^0)-efficient solutions of (VP) we have the following result (compare Durea and Tammer [17]).

Theorem 18. *Assume that S is a closed subset of X and f is a λ-Lipschitz function. Let $x_0 \in S$ be an (α, k^0)-efficient solution of (VP). Then for every $e \in \operatorname{int} K$ and $\varepsilon > 0$, there exist $u \in B(x_0, \sqrt{\alpha} + \varepsilon)$, $z \in B(x_0, \sqrt{\alpha} + \varepsilon/2) \cap S$, $u^* \in K^+$ with $u^*(e) = 1$ and $x^* \in X^*$ with $\|x^*\| \leq 1$ such that*

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\alpha}\, u^*(k^0) x^* + N_\partial(S, z) + B(0, \varepsilon),$$

provided that ∂ satisfies conditions (C1), (C2), (C3), (C4). Moreover, for some $x \in B(x_0, \sqrt{\alpha} + \varepsilon/2)$ and $v \in B(f(x) - f(x_0), \lambda\sqrt{\alpha} + \varepsilon)$ one has $u^*(v) = \varphi(v)$.

Proof. Since the function f is Lipschitz, it is continuous as well, and since S is a closed set in the Banach space X, S is a complete metric space with respect to the metric given by the norm. Thus, it is easy to see that we are in the conditions of the vectorial variant of Ekeland principle given in Theorem 13. Applying this result we get an element $\bar{x} \in S$ such that $\|\bar{x} - x_0\| < \sqrt{\alpha}$ and having the property that it is minimal element (whence weak minimal as well) over S for the function h defined by

$$h(x) := f(x) + \sqrt{\alpha}\, \|x - \bar{x}\|\, k^0.$$

Let $\varepsilon > 0$. One can apply now Theorem 17 for ε replaced $\delta \in]0, \varepsilon/2[$ with $\delta(1 + \sqrt{\alpha}\, \|k^0\|\, \delta) < 2\varepsilon$. Accordingly, we can find $\bar{u} \in B(\bar{x}, \delta) \subset B(x_0, \sqrt{\alpha} + \delta)$,

$x \in B(\overline{x}, \delta/2) \subset B(x_0, \sqrt{\alpha} + \delta/2)$, $v \in B(h(x) - h(\overline{x}), \delta/2)$, $z \in B(\overline{x}, \delta/2) \cap S \subset B(x_0, \sqrt{\alpha} + \delta/2) \cap S$ and $u^* \in \partial\varphi(v)$ such that

$$0 \in \partial(u^* \circ h)(\overline{u}) + N_\partial(S, z) + B(0, \delta). \tag{45}$$

Let us take the element $\overline{x}^* \in \partial(u^* \circ h)(\overline{u})$ involved in (45). Since

$$\partial(u^* \circ h)(\overline{u}) = \partial(u^* \circ (f(\cdot) + \sqrt{\alpha}\, \|\cdot - \overline{x}\|\, k^0))(\overline{u}),$$

by use of (C3) and (C1), there exist $u \in B(\overline{u}, \delta) \subset B(x_0, \sqrt{\alpha} + 2\delta)$ and $u' \in B(\overline{u}, \delta)$ such that

$$\overline{x}^* \in \partial(u^* \circ f)(u) + \sqrt{\alpha} u^*(k^0) \partial(\|\cdot - \overline{x}\|)(u') + B(0, \delta). \tag{46}$$

By the calculation rule for the subdifferential of the norm and combining relations (45) and (46) it follows that there exists $x^* \in X^*$ with $\|x^*\| \leq 1$ such that

$$0 \in \partial(u^* \circ f)(u) + \sqrt{\alpha} u^*(k^0) x^* + N_\partial(S, z) + B(0, 2\delta).$$

Since $2\delta < \varepsilon$, it remains only to prove the estimation about the ball containing v. We can write

$$\begin{aligned} \|v - (f(x) - f(x_0))\| &\leq \|v - (h(x) - h(\overline{x}))\| + \|(h(x) - h(\overline{x})) - (f(x) - f(x_0))\| \\ &\leq \delta/2 + \left\| \sqrt{\alpha} k^0\, \|x - \overline{x}\| - f(\overline{x}) + f(x_0)) \right\| \\ &\leq \delta/2 + \sqrt{\alpha}\, \left\| k^0 \right\| \delta/2 + \lambda\sqrt{\alpha} \\ &< \lambda\sqrt{\alpha} + \varepsilon, \end{aligned}$$

where for the last inequality we used the assumptions made on δ. The proof is complete. □

References

1. Al-Homidan, S., Ansari, Q.H., Yao, J.-C.: *Some generalizations of Ekeland-type Variational Principle with Applications to Equilibrium Problems and Fixed Point Theory*, Nonlinear Analysis, Theory, Methods and Applications, **69** (1), 126-139 (2008).
2. Ansari, Q.H.: *Vectorial Form of Ekeland-type Variational Principle with Applications to Vector Equilibrium Problems and Fixed Point Theory*, Journal of Mathematical Analysis and Applications, **334** (2), 561-575 (2007).
3. P. Artzner, F. Delbaen, J.-M. Eber, D. Heath: *Coherent measures of risk*, Mathematical Finance 9 (1999), 203–228.
4. J.-M. Bonnisseau, B. Crettez: *On the characterization of efficient production vectors*, Economic Theory 31 (2007), 213–223
5. H. Attouch, H. Riahi: *Stability results for Ekeland's ε-variational principle and cone extremal solutions*, Math. Oper. Res. 18 (1993), 173–201
6. T. Q. Bao, B. S. Mordukhovich: *Variational principles for set-valued mappings with applications to multiobjective optimization*, Control Cybernet. 36 (2007), 531–562
7. T. Q. Bao, B. S. Mordukhovich: *Relative Pareto minimizers for multiobjective problems: existence and optimality conditions*, Math. Program. 122 (2010), 301–347

8. E. M. Bednarczuk, D. Zagrodny: *Vector variational principle*, Arch. Math. (Basel) 93 (2009), 577–586
9. M. Bianchi, G. Kassay, R. Pini: *Ekeland's principle for vector equilibrium problems,* Nonlinear Anal. 66 (2007), 1454–1464.
10. J.-M. Bonnisseau B. Cornet: *Existence of equilibria when firms follow bounded losses pricing rules*, Journal of Mathematical Economics 17 (1988), 119–147
11. J.M. Borwein, D. Preiss: *A smooth variational principle with applications to subdifferentiability and to differentiability of convex functions*, Trans. Amer. Math. Soc. 303 (1987), 517–527
12. A. Brøndsted: *On a lemma of Bishop and Phelps*, Pacific J. Math. 55 (1974), 335–341
13. G.Y. Chen, X.X. Huang: *A unified approach to the existing three types of variational principles for vector valued functions*, Math. Meth. Oper. Res. 48 (1998), 349–357
14. G.Y. Chen, X.X. Huang: *Ekeland's ε-variational principle for set-valued mappings*, Math. Meth. Oper. Res. 48 (1998), 181–186
15. G.Y. Chen, X.X. Huang, G.M. Lee: *Equivalents of an approximate variational principle for vector-valued functions and applications*, Math. Meth. Oper. Res. 49 (1999), 125–136
16. J. Daneš: *A geometric theorem useful in nonlinear functional analysis*, Boll. Un. Mat. Ital. 6 (1972), 369–375
17. M. Durea, Chr. Tammer: *Fuzzy necessary optimality conditions for vector optimization problems*, Optimization 58 (2009), 449–467
18. R. Deville, E. El Haddad: *The viscosity subdifferential of the sum of two functions in Banach spaces I: First order case*, Journal of Convex Analysis 3 (1996), 259–308
19. S. Dolecki, C. Malivert: *General duality in vector optimization*, Optimization 27 (1993), 97–119
20. J. Dutta, Chr. Tammer: *Lagrangian conditions for vector optimization in Banach spaces*, Mathematical Methods of Operations Research 64 (2006), 521–541
21. I. Ekeland: *On the variational principle*, J. Math. Anal. Appl. 47 (1974), 324–353
22. I. Ekeland: *Nonconvex minimization problems*, Bull. Amer. Math. Soc. 1 (1979), 443–474
23. M. Fabian: *Subdifferentiability and trustworthiness in the light of a new variational principle of Borwein and Preiss*, Acta. Univ. Carolin. 30 (1989), 51–56
24. F: Ferro: *A new ABB theorem in Banach spaces*, Optimization 46 (1999), 353–362
25. D.G. de Figueiredo: *Lectures on the Ekeland's Variational Principle with Applications and Detours*, (Tata Res. Inst. Bombay), (Springer, Berlin, 1989)
26. H. Föllmer, A. Schied: *Convex measures of risk and trading constraints*, Finance and Stochastics 4 (2002), 429–447
27. P.G. Georgiev: *The strong Ekeland variational principle, the strong drop theorem and applications*, J. Math. Anal. Appl. 131 (1988), 1–21
28. Chr. Gerstewitz (Tammer), E. Iwanow: *Dualität für nichtkonvexe Vektoroptimierungsprobleme*, Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau, 31 (2) (1985), 61–81
29. A. Göpfert, Chr. Tammer: *ε-approximate solutions and conical support points. A new maximal point theorem*, ZAMM 75 (1995), 595–596
30. A. Göpfert, Chr. Tammer: *A new maximal point theorem*, Z. Anal. Anwendungen 14 (1995), 379–390
31. Göpfert, A., Tammer, Chr.: *Maximal point theorems in product spaces and applications for multicriteria approximation problems*, in Y.Y. Haimes, R.E. Steuer (eds.) Research and Practice in Multiple Criteria Decision Making. Lecture Notes in Econom. and Math. Systems, 487 (Springer, Berlin, 1999) pp. 93–104
32. A. Göpfert, H. Riahi, Chr. Tammer, C. Zălinescu: *Variational Methods in Partially Ordered Spaces*, CMS Books in Mathematics 17 (Springer, New York, 2003)
33. A. Göpfert, Chr. Tammer, C. Zălinescu: *A new minimal point theorem in product spaces*, Z. Anal. Anwendungen, 18 (1999), 767–770
34. A. Göpfert, Chr. Tammer, C. Zălinescu: *On the vectorial Ekeland's variational principle and minimal points in product spaces*, Nonlinear Anal. 39 (2000), 909–922

35. V.V. Gorokhovik, S.Ya. Gorokhovik: *A criterion of the global epi-Lipschitz property of sets* (Russian), Izvestiya Natsional'no Akademii Nauk Belarusi. Seriya Fiziko-Matematicheskikh Nauk, 1995, 118–120
36. C. Gutiérrez, B. Jiménez, V. Novo: *A set-valued Ekeland's variational principle in vector optimization*, SIAM J. Control Optim. 47 (2008), 883–903
37. T. X. D. Ha: *Some variants of the Ekeland variational principle for a set-valued map*, J. Optim. Theory Appl. 124 (2005), 187–206
38. A. Hamel: *Phelps' lemma, Danes' drop theorem and Ekeland's principle in locally convex topological vector spaces*, Proc. Amer. Math. Soc. 131 (2003), 3025–3038
39. A. Hamel: *From real to set-valued coherent risk measures*, Reports of the Institute of Optimization and Stochastics, Martin-Luther-University Halle-Wittenberg, No. 19 (2004), 10–22
40. A. Hamel: *Translative sets and functions and its applications to risk measure theory and nonlinear separation*, Preprint (Halle, 2007)
41. A. Hamel, Chr. Tammer: *Minimal elements for product orders*, Optimization 57 (2008), 263–275
42. F. Heyde: *Coherent risk measures and vector optimization*, in K.-H. Küfer, H. Rommelfanger, Chr. Tammer, K. Winkler (eds) Multicriteria Decision Making and Fuzzy Systems, (SHAKER Verlag, 2006), 3–12
43. A. D. Ioffe: *Metric regularity and subdifferential calculus*, Russian Mathematical Surveys 55 (2000), 501–558
44. A. D. Ioffe, J.-P. Penot: *Subdifferentials of performance functions and calculus of coderivatives of set-valued mappings*, Serdica Mathematical Journal, **22** (1996), 359–384
45. G. Isac: *The Ekeland principle and the Pareto-efficiency*, in Tamiz M. (ed.) Multiobjective Programming and Goal Programming, Theory and Applications. Lecture notes in economics and mathematical systems, vol. 432 (Springer, Berlin, 1996), pp. 148–163.
46. G. Isac, Chr. Tammer: *Nuclear and full nuclear cones in product spaces: Pareto efficiency and an Ekeland type variational principle*, Positivity 9 (2005), 511–539
47. E. Iwanow, R. Nehse: *On proper efficient solutions of multi-criterial problems* (Russian), Wissenschaftliche Zeitschrift der Technischen Hochschule Ilmenau 30 (5) (1984), 55–60
48. F. Jules: *Sur la somme de sous-différentiels de fonctions semi-continues inférieurement*, Dissertationes Math. 423 (2003)
49. S. J. Li, X. Q. Yang, G. Y. Chen: *Vector Ekeland variational principle*, in Giannessi F. (ed.) Vector varational inequalities and vector equilibria. Nonconvex optimization and its applications, vol. 38 (Kluwer, Dordrecht, 2000), pp. 321–333
50. D. G. Luenberger: *New optimality principles for economic efficiency and equilibrium*, Journal of Optimization Theory and Applications 75 (1992), 221–264
51. B. S. Mordukhovich: *Variational Analysis and Generalized Differentiation, Vol. I: Basic Theory, Vol. II: Applications* (Springer, Berlin, 2006)
52. J.-P. Penot: *The drop theorem, the petal theorem and Ekeland's variational principle*, Nonlinear Anal. 10 (1986), 813–822
53. R.R. Phelps: *Convex Functions, Monotone Operators and Differentiability*, Lecture Notes Math., 1364 (Springer, Berlin, 1989)
54. R.R. Phelps: *Convex Functions, Monotone Operators and Differentiability* (2nd ed.) Lect. Notes Math., 1364 (Springer, Berlin, 1993)
55. R.T. Rockafellar: *Clarke's tangent cones and the boundaries of closed sets in* $\mathbb{R}^n$, Nonlinear Analysis. Theory, Methods & Applications 3 (1979), 145–154
56. R.T. Rockafellar: *The Theory of Subgradients and its Applications to Problems of Optimization. Convex and Nonconvex Functions* (Heldermann, Berlin, 1981)
57. R. T. Rockafellar, S. Uryasev, M. Zabarankin: *Deviation measures in risk analysis and optimization*, *Finance Stochastics* 10 (2006), 51–74
58. S. Rolewicz: *On drop property*, Studia Math. 85 (1987), 27–35
59. A.M. Rubinov, I. Singer: *Topical and sub-topical functions. Downward sets and abstract convexity*, *Optimization* 50 (2001), 307–351
60. A. Ruszczynski, A. Shapiro: *Optimization of convex risk measures*, Mathematics of Operations Research 31 (2006), 433–452

61. Chr. Tammer: *A generalization of Ekeland's variational principle*, Optimization 25 (1992), 129–141
62. Chr. Tammer: *A variational principle and a fixed point problem*, In: Henry, J., Yvon, J.-P. (eds) System Modelling and Optimization. Lecture Notes in Control and Inform. Sci., 197 (Springer, London, 1994) pp. 248–257
63. Chr. Tammer, C. Zălinescu: *Lipschitz properties of the scalarization function and applications*, Optimization 59 (2) (2010), 305-319
64. Chr. Tammer, C. Zălinescu: *Vector variational principles for set-valued functions*, Report Institut fr Mathematik, Report No. 17, 2009. Accepted: Optimization.
65. M. Turinici: *Maximal elements in a class of order complete metric spaces*, Math. Japonica 25 (1980), 511–517
66. C. Zălinescu: *On two notions of proper efficiency*, In: Optimization in mathematical physics, Pap. 11th Conf. Methods Techniques Math. Phys., Oberwolfach/Ger. 1985, Methoden Verfahren Math. Phys., 34 (1987), 77–86
67. C. Zălinescu:, *Convex Analysis in General Vector Spaces* (World Scientific, Singapore, 2002)

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