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Maxima for the expectation of the lifetime of a Brownian motion in the ball

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Abstract

Let $G_B(x,y)$ be the Green's function of the unit ball in R^3 and $\Gamma_B(x,y) = \int_B G_B(x,z)G_B(z,y)dz$ the iterated Green's function.

$$E_x^y = \frac{\Gamma_B(x,y)}{G_B(x,y)}$$

is the expectation of the lifetime of a Brownian motion in B. The aim of the paper is to prove

$$\sup_{x\in\partial B, y\in B} E_y^x = \sup_{x,y\in\partial B} E_y^x = E_{-x_0}^{x_o} = 2\pi - 6, x_o \in \partial B$$

and the maximum occurs if and only if x_o, y_o are diametrically opposite points on the sphere.

1 Introduction

Let $G_B(x,y)$ be the Green's function of the Laplacian of the domain B with zero Dirichlet boundary condition and

$$\Gamma_B(x,y) = \int_B G_B(x,z)G_B(z,y)dz \tag{1}$$

the iterated Green's function, which is one of the Green's functions of the biharmonic equation. The expectation of the lifetime of a Brownian motion in B, starting in x, conditioned to converge to and to be stopped at y and to be killed on exiting B is known to be equal [1, 3]

$$E_x^y = \frac{\Gamma_B(x,y)}{G_B(x,y)}. \tag{2}$$

For simply connected plane domains B it was proven by Griffin, McConnell and Verchota [7] there is no maximum of E_ζ^z if $z \in B$ and $\zeta \in \partial B$ and in the case of the unit disk the maximum occurs if and only if z and ζ are diametrically opposite on ∂B. For the unit disk D it is proven by Dall'acqua, Grunau and Sweers [4] and later by the author [6] there is no maximum for $z, \zeta \in D$. For more details of this problem and closely related problems compare [7, 8]. The purpose of this paper is to deal with the case of the ball B in R^3.

Using a maximum principle Dall'acqua [5] proved that the maximum has been attained for opposite boundary points. We will prove without any maximum principle

$$\sup_{x\in\partial B, y\in B} E_y^x = \sup_{x,y\in\partial B} E_y^x = E_{-x_0}^{x_o} = 2\pi - 6 = 0.2831..., x_o \in \partial B$$

and the maximum has been attained if and only if x and y are diametrically opposite points on the sphere.

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2 Notations and preliminary material

As usual R^3 denotes Euclidean space, a typical point in this being $x = (x_1, x_2, x_3)$ with $|x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$. The standard basis of R^3 is e_1, e_2, e_3 where, for example $e_1 = (1, 0, 0)$. For two points x, y we use the notation $x \circ y = \sum_1^3 x_j y_j$. Let $B_r = \{x \in R^3 : |x| < r\}$ and for shortness we write B for the unit ball B_1. Furthermore we have

$$G_B(x,y) = \frac{1}{4\pi}\left(\frac{1}{|x-y|} - \frac{1}{|y|}\frac{1}{|x-y^*|}\right) \tag{3}$$

the Green's function of B, where $y^* = y/|y|^2$.

We need the following versions of the well-known Poisson formula [9]

Lemma 1 *For a harmonic function u inside of the ball B_r with the boundary values $u(|z| = r)$ the following Poisson formula holds*

$$u(x) = \frac{1}{4\pi}\int_{|z|=r} \frac{r^2 - |x|^2}{r|x-z|^3} u(z) dO_z,$$

where $|x| < r$.

An analogous Poisson formula holds for the exterior A_r of the ball B_r with $A_r = R^3 \backslash \overline{B_r}$

Lemma 2 *If u is a harmonic function in A_r with the boundary values $u(|z| = r)$ then*

$$u(x) = \frac{1}{4\pi}\int_{|z|=r} \frac{|x|^2 - r^2}{r|x-z|^3} u(z) dO_z,$$

where $|x| > r$.

In the case of the disc in the plane we have the following Poisson formula

Lemma 3 *Let u be a harmonic function in the plane domain $\{z : |z| > r\}$, then*

$$\frac{1}{2\pi}\int_0^{2\pi} u(|z| = r)\frac{1 - r^2}{|re^{i\theta} - \zeta_1|^2} d\theta = u(\zeta_1).$$

The following lemma will be helpful in some cases.

Lemma 4 *If $x_o, y \in B_1$ with $|x_o| = 1, 0 < |y| \leq 1,$ and $0 \leq r < 1$ then*

$$\frac{(1-|y|)^2}{|y|^2(1-r^2|y|)^2} \leq \frac{|x_o - y|^2}{|x_o r^2|y|^2 - y|^2} \leq \frac{(1+|y|)^2}{|y|^2(1+r^2|y|)^2}. \tag{4}$$

Equality occurs if and only if in the right $y = -x_o|y|$ and in the left $y = x_o|y|$.

Proof. It follows with $y \circ x_o = |y|\cos\alpha$, where $\alpha = \angle(x_o, y)$

$$\frac{|x_o - y|^2}{|x_o r^2|y|^2 - y|^2} = \frac{1 + |y|^2 - 2|y|\cos\alpha}{|y|^2 + r^4|y|^4 - 2|y|^3 r^2 \cos\alpha} = f(r, |y|, \alpha)$$

and after a simple calculation

$$\frac{df}{d\alpha} = \frac{2|y|^3 \sin\alpha(1-r^2)(1-|y|^2r^2)}{(|y|^2 + r^4|y|^4 - 2|y|^3r^2\cos\alpha)^2} = \frac{2|y|^3 \sin\alpha(1-r^2)(1-|y|^2r^2)}{|x_o r^2|y|^2 - y|^4}. \tag{5}$$

It follows $\frac{df}{d\alpha} = 0$ if and only if $\sin\alpha = 0$ from which the desired result follows because

$$\frac{1-|y|}{|y|(1-r^2|y|)} < \frac{1+|y|}{|y|(1+r^2|y|)}.$$

3 The expectation for the lifetime on the boundary

First we want to calculate the expectation for the lifetime for the case that one point is on the boundary. Let $x_o, y, |y| < 1 = |x_o|$ be given. The first step is to calculate

$$\frac{\partial G_B(x_o, y)}{\partial r},$$

on the boundary, where $r = |x|$.

$$\frac{\partial}{\partial r}\frac{1}{|x-y|} = \frac{|y|\cos\alpha - 1}{|x-y|^3}, \qquad \frac{\partial}{\partial r}\frac{1}{|x-y^*|} = \frac{|y|^2(\cos\alpha - |y|)}{|x-y|^3}, \tag{6}$$

where we have used

$$|x - y^*|^3 = \frac{|x-y|^3}{|y|^3},$$

for $|x| = 1$. Putting all these together, we obtain

$$\frac{\partial G_B(x_o, y)}{\partial r} = \frac{1}{4\pi}\frac{-1+|y|^2}{|x_o - y|^3}.$$

We evaluate

$$\lim_{x \to x_o} E_x^y = \lim_{x \to x_o} \frac{\int_B G_B(x,z)G_B(z,y)dV_z}{G_B(x,y)}.$$

For that reason we use

$$\lim_{x \to x_o} \frac{G_B(x,z)}{G_B(x,y)} = \frac{-\frac{\partial}{\partial r}G_B(x_o,z)}{-\frac{\partial}{\partial r}G_B(x_o,y)} = \frac{|z|^2-1}{|y|^2-1}\frac{|x_o-y|^3}{|x_o-z|^3}$$

and it follows

$$E_{x_o}^y = \frac{|x_o - y|^3}{1-|y|^2} \int_B \frac{1-|z|^2}{|x_o - z|^3} G_B(z,y) dV_z. \tag{7}$$

Now we want to deal with the case that also y tends to the boundary. We see easily

$$\lim_{y \to y_o} \frac{G_B(z,y)}{1-|y|^2} = \frac{1}{2} \lim_{y \to y_o} \frac{G_B(z,y)}{1-|y|} = \frac{1}{8\pi} \frac{1-|z|^2}{|z-y_o|^3}.$$

It follows immediately for $|y_o| = 1$

$$\lim_{y \to y_o} E_{x_o}^y = \frac{|x_o - y_o|^3}{8\pi} \int_B \frac{(1-|z|^2)^2}{|x_o - z|^3 |y_o - z|^3} dV_z. \tag{8}$$

Now we are able to prove the first theorem.

Theorem 1 *If $x_o, y_o \in \partial B$, then*

$$E_{x_o}^{y_o} \leq E_{x_1}^{y_1} = 4 \int_o^1 r^2 \frac{1-r^2}{(1+r^2)^2} dr = 2\pi - 6 = 0.2831...,$$

where $x_1 = e_1$ and $y_1 = -e_1$. Equality occurs if and only if x_o, y_o is a rotation of $e_1, -e_1$.

Proof We have with (8)

$$\begin{aligned} E_{x_o}^{y_o} = \frac{|x_o - y_o|^3}{8\pi} \int_B \frac{(1-|z|^2)^2}{|x_o - z|^3 |y_o - z|^3} dV_z = \\ \frac{|x_o - y_o|^3}{8\pi} \int_0^1 (1-r^2)^2 \left(\int_{|z|=r} \frac{dO_z}{|x_o - z|^3 |y_o - z|^3} \right) dr = \\ = \frac{|x_o - y_o|^3}{2} \int_0^1 \frac{r^2(1-r^4) dr}{|y_o - x_o r^2|^3}, \end{aligned} \tag{9}$$

where we have used

$$\frac{1}{4\pi} \int_{|z|=r} \frac{dO_z}{|x_o - z|^3 |y_o - z|^3} = r^2 \frac{(1+r^2)}{(1-r^2)} \frac{1}{|y_o - x_o r^2|^3},$$

which follows from Lemma 2 with $u(z) = \frac{|z|^2 - r^4}{|z - x_o r^2|^3}, |z| > r, r < 1$. It is easy to check that u is harmonic for $|z| > r$ and it holds $u(|z| = r) = \frac{1-r^2}{r|z-x_o|^3}$. We have with (9)

$$E_{x_o}^{y_o} = \frac{|x_o - y_o|^3}{2} \int_0^1 \frac{r^2(1-r^4)dr}{|y_o - x_o r^2|^3}.$$

The proof is a straightforward consequence of the inequality

$$\frac{|x_o - y_o|^3}{|y_o - x_o r^2|^3} \leq \frac{8}{(1+r^2)^3}, \tag{10}$$

which follows from Lemma 4 and equality holds if and only if $x_o \circ y_o = -1$. These yields

$$E_{x_o}^{y_o} \leq 4 \int_0^1 r^2 \frac{1-r^2}{(1+r^2)^2} dr = E_{x_1}^{y_1}.$$

4 One point lying on the boundary

What we want to prove is the following formula for the expectation of the lifetime if one point lies on the boundary.

Lemma 5 *Let* $|y| < 1 = |x_o|$ *then*

$$E^y_{x_o} = \frac{|x_o - y|^2}{2} - |x_o - y|^3|y| \int_0^1 \frac{r^2 dr}{|y - x_o r^2 |y|^2|}. \tag{11}$$

Proof First we invoke formula (7)

$$E^y_{x_o} = \frac{|x_o - y|^3}{1 - |y|^2} \int_B \frac{1 - |z|^2}{|x_o - z|^3} G_B(z, y) dV_z.$$

A glance at (3) shows that we have to evaluate

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y|}, \tag{12}$$

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y^*|}, \tag{13}$$

for $0 < r < 1$. For the second integral we obtain using Lemma 2 with the harmonic function $|z - y^*|^{-1}$

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y^*|} = \int_{|z|=r} r^3 \frac{(1 - |z|^2)}{|r^2 x_o - z|^3} \frac{dO_z}{|z - y^*|} = \frac{4\pi r^2}{|r^2 x_o - y^*|},$$

where we have used

$$|x_o - z|^3 = |x_o r^2 - z|^3 r^{-3}, \; |z| = r. \tag{14}$$

Consider the first integral

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y|}.$$

If $|z| > |y|$ we obtain using Lemma 2 with the harmonic function $|z - y|^{-1}$

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y|} = r \frac{4\pi}{|x_o - y|}.$$

For $|z| < |y|$ it follows with Lemma 1 using the harmonic function $|z - y|^{-1}$ and (14)

$$\int_{|z|=r} \frac{(1 - |z|^2)}{|x_o - z|^3} \frac{dO_z}{|z - y|} = \int_{|z|=r} \frac{r^3(1 - r^2)}{|z - x_o r^2|} \frac{dO_z}{|z - y|} = r^2 \frac{4\pi}{|y - x_o r^2|}.$$

Putting all these together we get

$$E^y_{x_o} = \frac{|x_o - y|^3}{1 - |y|^2} \int_B \frac{1 - |z|^2}{|x_o - z|^3} G_B(z, y) dV_z =$$

$$
\begin{aligned}
&\frac{1}{4\pi}\frac{|x_o-y|^3}{1-|y|^2}\int_0^{|y|}\left(\int_{|z|=r}\frac{(1-|z|^2)}{|x_o-z|^3}\frac{dO_z}{|z-y|}\right)dr+\\
&\frac{1}{4\pi}\frac{|x_o-y|^3}{1-|y|^2}\int_{|y|}^{1}\left(\int_{|z|=r}\frac{(1-|z|^2)}{|x_o-z|^3}\frac{dO_z}{|z-y|}\right)dr\\
&-\frac{1}{4\pi}\frac{|x_o-y|^3}{1-|y|^2}\int_0^{1}\frac{1}{|y|}\left(\int_{|z|=r}\frac{(1-|z|^2)}{|x_o-z|^3}\frac{dO_z}{|z-y^*|}\right)dr=\\
&\frac{|x_o-y|^3}{1-|y|^2}\left(\frac{1}{2}\frac{(1-|y|^2)}{|x_o-y|}+\int_0^{|y|}\frac{r^2}{|x_or^2-y|}dr-\frac{1}{|y|}\int_0^1\frac{r^2}{|x_or^2-y^*|}dr\right).
\end{aligned}
\tag{15}
$$

The desired result follows with

$$
\int_0^{|y|}\frac{r^2}{|x_or^2-y|}dr-\frac{1}{|y|}\int_0^1\frac{r^2}{|x_or^2-y^*|}dr=|y|(|y|^2-1)\int_0^1\frac{r^2}{|y-x_or^2|y|^2|}dr
$$

which is due to

$$
\int_0^{|y|}\frac{r^2}{|x_or^2-y|}dr=|y|^3\int_0^1\frac{r^2}{|y-x_or^2|y|^2|}dr
$$

and

$$
-\frac{1}{|y|}\int_0^1\frac{r^2dr}{|x_or^2-y^*|}=-|y|\int_0^1\frac{r^2dr}{|x_or^2|y|^2-y|}.
$$

As a first conclusion we obtain the following lemma, which we need later.

Lemma 6 *If $x_o, y \in B_1$ with $|x_o| = 1, |y| < 1$, then*

$$
E^{x_o}_{x_o|y|} < E^{x_o}_{-x_o|y|}
$$

Proof. For fixed $|y|$ we have only to show that

$$
l(|y|)=-2|y|-(1-|y|)^2\int_0^1\frac{1-|y|}{1-r^2|y|}r^2dr+(1+|y|)^2\int_0^1\frac{1+|y|}{1+r^2|y|}r^2dr<0.
$$

We obtain for the left

$$
-2|y|+(1+|y|^2)\int_0^1\frac{2|y|(1-r^2)}{1-r^4|y|^2}r^2dr+2|y|\int_0^1\frac{2-2r^2|y|^2}{1-r^4|y|^2}r^2dr
$$

With $l(|y|)=|y|l_1(|y|)$,$t=|y|^2$, and

$$
l_1(t)=-2+2(1+t)\int_0^1\frac{(1-r^2)}{1-r^4t}r^2dr+4\int_0^1\frac{1-tr^2}{1-tr^4}r^2dr \tag{16}
$$

we get

$$
\frac{dl_1(t)}{dt}=\int_0^1\frac{2r^2(1-r^2)(1-r^2)^2}{(1-r^4t)^2}dr>0.
$$

It follows

$$l_1(t) \le l_1(1) = -2 + 8\int_0^1 \frac{r^2}{1+r^2}dr = 6 - 2\pi < 0,$$

which establishes the assertion of the lemma.

Lemma 7 *Let* $g(t) = 1/2 - (1+t)\int_0^1 \frac{r^2 dr}{1+r^2 t}$ *be then* $g'(t) + g(t) > 0$ *for* $0 < t < 1$*.*

Proof. We first compute

$$\begin{aligned} g(t) &= \frac{1}{2} - \frac{1+t}{t}(1 - \frac{\arctan\sqrt{t}}{\sqrt{t}}), \\ g'(t) &= \frac{1}{2t^2}\left(3 - (3+t)\frac{\arctan\sqrt{t}}{\sqrt{t}}\right), \\ g'(t) + g(t) &= \frac{(1-t)}{2t^2}\left((t+3) - (2t+3)\frac{\arctan\sqrt{t}}{\sqrt{t}}\right). \end{aligned} \tag{17}$$

Let

$$H(t) = \arctan\sqrt{t} - \sqrt{t}\frac{t+3}{2t+3}$$

be then $H'(t) < 0$ and $H(0) = 0$ and it follows $H(t) < 0$ for $0 < t < 1$ which proves the desired assertion.

Using this lemma we deduce the following corollary.

Corollary 1 $E^{x_o}_{-x_o|y|}$ *is a strictly monotone increasing function of* $|y|$ *for* $|y| < 1 = |x_o|$*, especially* $E^{x_o}_{-x_o|y|} < E^{e_1}_{-e_1} = 2\pi - 6$ *.*

Proof. We have

$$E^{x_o}_{-x_o|y|} = (1+|y|)^2\left(\frac{1}{2} - (1+|y|)\int_0^1 \frac{r^2 dr}{1+r^2|y|}\right) = (1+|y|)^2 g(|y|)$$

and it follows with Lemma 7

$$\frac{dE^{x_o}_{-x_o|y|}}{d|y|} = f'g + fg' \ge g(f' - f) = g(1 - |y|^2) > 0$$

with $f = (1+|y|)^2$.

Before we prove the main result of this chapter we need the following

Lemma 8 *For* $0 < t < 1$ *the following holds*

$$\frac{1}{2}\int_0^1 \frac{(1-t^2r^2)(1+t)^3}{(1+r^2t)^3}(r^2 - r^4)dr - (1+t)^2 g(t) < 0.$$

Proof. We first prove the inequality for $0 < t \le 0.25$. For the left of the inequality we get

$$\frac{1}{2}\int_0^1 \frac{(1-t^2r^2)}{(1+r^2t)^3}(r^2-r^4)dr - \frac{g(t)}{1+t} <$$

$$\frac{1}{2}\int_0^1 (1-t^2r^2)(r^2-r^4)dr - \frac{1}{2(1+t)} + \int_0^1 \frac{r^2}{1+r^2t}dr <$$

$$\frac{1}{2}\int_0^1 (1-t^2r^2)(r^2-r^4)dr - \frac{1}{2(1+t)} + \frac{1}{3} = \frac{6}{15} - \frac{t^2}{35} - \frac{1}{2(1+t)}.$$

For $0 < t \le 0.25$ we have

$$\frac{6}{15} - \frac{1}{2(1+t)} \le 0$$

which yields the assertion. Using $(1+t)/(1+r^2t) < 2/(1+r^2)$ for $0 < t < 1$ we get

$$\frac{1}{2}\int_0^1 \frac{(1+t)^3(1-t^2r^2)}{(1+r^2t)^3}(r^2-r^4)dr - (1+t)^2 g(t) \le$$

$$4\int_0^1 \frac{(r^2-r^4)}{(1+r^2)^3}(1-t^2r^2)dr - (1+t)^2 g(t) = K(t)$$

$K(t)$ is a decreasing function because $(1+t)^2 g(t)$ is increasing which was proven in Corollary 1. An elementary calculation yields $K(0.24) < 0$, which completes the proof of the lemma.

Now we are in a position to prove the main result of this chapter

Theorem 2 *If $x_o, y \in B_1$ with $|x_o| = 1, |y| < 1$ then for fixed $|y|$ and fixed x_o*

$$\max_{|y|=s} E_y^{x_o} = E_{-x_o s}^{x_o}$$

and equality occurs if and only if $y = -x_o s$.

Proof. We have with Lemma 5, $|y| = s$,

$$E_{x_o}^{y} = |x_o - y|^2\left(\frac{1}{2} - |y|\int_0^1 \frac{|x_o - y|r^2}{|y - x_o r^2|y|^2|}dr\right)$$

and obtain with $\alpha = \angle(x_o, y)$ using (5) the necessary condition

$$\frac{dE_y^{x_o}}{d\alpha} = 2|y|\sin\alpha\left(\frac{1}{2} - |y|\int_0^1 \frac{|x_o - y|r^2}{|y - x_o r^2|y|^2|}dr\right)$$

$$-\sin\alpha|x_o - y||y|\int_0^1 \frac{|y|^3(1-r^2)(1-|y|^2r^2)}{|y - x_o r^2|y|^2|^3}r^2 dr = 0. \tag{18}$$

For $\sin\alpha \ne 0$ it follows

$$E_y^{x_o} = \frac{|y|^3}{2}\int_0^1 \frac{|x_o - y|^3}{|y - x_o r^2|y|^2|^3}(1-r^2)(1-|y|^2r^2)r^2 dr \le$$

$$\frac{(1+|y|)^3}{2}\int_0^1 \frac{(1-|y|^2r^2)}{(1+r^2|y|)^3}(r^2-r^4)dr, \tag{19}$$

where the last inequality follows with Lemma 4. We see

$$\frac{(1+|y|)^3}{2}\int_0^1 \frac{(1-|y|^2r^2)}{1+r^2|y|)^3}(r^2-r^4)dr <$$
$$E_{x_o}^{-x_o|y|} = (1+|y|)^2\Big(\frac{1}{2}-\int_0^1 \frac{(1+|y|)r^2}{1+r^2|y|}dr\Big).$$

This is a consequence of

$$\frac{(1+|y|)^3}{2}\int_0^1 \frac{1-|y|^2r^2}{(1+|y|r^2)^3}(r^2-r^4)dr-(1+|y|)^2 g(|y|)<0,$$

which follows with Lemma 8. The necessary condition is satisfied for $\sin\alpha = 0$ and if are there other points ζ_1, ζ_2 then in these points is $E_{\zeta_1}^{\zeta_2} < E_{-x_o|y|}^{x_o}$. Our claim follows now with Lemma 6.

5 The case of the unit disk

The aim of this chapter is to give a new proof for a lemma which was used in [6] for the case of the unit disk. This proof follows the context of this paper. The lemma ([6, Lemma 3.1]) is the following.

Lemma 9 *Let $\zeta_1 = e^{i\varphi_1}, z_1 = e^{i\varphi_2},\ i^2=-1,$ and $0 \le r < 1$, then holds*

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{|re^{i\vartheta}-z_1|^2|re^{i\vartheta}-\zeta_1|^2} = \sum_0^\infty |A_n|^2 r^{2n},$$

with $A_n = \frac{\eta^{n+1}-1}{\eta^{n+1}-\eta^{n+2}}$, $\eta = z_1\zeta_1^{-1} = e^{i\varphi}$ and

$$|1-\eta|^2\sum_0^\infty |A_n|^2 r^{2n} = \sum_0^\infty (2-2\cos(n+1)\varphi)r^{2n} \le \frac{4}{1-r^4}. \tag{20}$$

Equality holds if and only if $\eta = -1$ for all r.

Proof. First it is easy to see

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{|re^{i\vartheta}-z_1|^2|re^{i\vartheta}-\zeta_1|^2} = \frac{(1+r^2)}{(1-r^2)}\frac{1}{|\zeta_1-r^2z_1|^2}. \tag{21}$$

The proof of the lemma given in [6] contains the following formula

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{|re^{i\vartheta}-z_1|^2|re^{i\vartheta}-\zeta_1|^2} = \frac{2}{|1-\eta|^2}\Big(\frac{1}{1-r^2}-\Re\frac{\eta}{1-\eta r^2}\Big) \tag{22}$$

and a straightforward calculation yields

$$\frac{(1+r^2)}{(1-r^2)}\frac{1}{|\zeta_1-r^2z_1|^2} = \frac{2}{|1-\eta|^2}\Big(\frac{1}{1-r^2}-\Re\frac{\eta}{1-\eta r^2}\Big).$$

Now we see that (22) follows from Lemma 3.

$$u(z) = 2\Re\frac{z}{z - z_1 r^2} - 1 = \Re\frac{z + z_1 r^2}{z - z_1 r^2}$$

is a harmonic function for $|z| > r^2$ and we obtain $u(|z| = r) = (1 - r^2)/(|z - z_1|^2)$. Using these we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{|re^{i\vartheta} - z_1|^2|re^{i\vartheta} - \zeta_1|^2} = \frac{1}{(1 - r^2)}\frac{1}{2\pi}\int_0^{2\pi} u(|z| = r)\frac{d\vartheta}{|re^{i\vartheta} - \zeta_1|^2} =$$
$$\frac{1}{(1 - r^2)^2}\left(2\Re\frac{\zeta_1}{\zeta_1 - r^2 z_1} - 1\right) = \frac{1 + r^2}{1 - r^2}\frac{1}{|\zeta_1 - z_1 r^2|^2},$$

which is the desired result. The inequality mentioned in the lemma is an easy matter and is analogous (10) .

6 A lower bound for E_ζ^z

In this chapter we will show how we can obtain lower bounds for the expectation of the lifetime using only the monotonicity of the Green's function for plane domains.

Theorem 3 *Let D be a simply connected domain in the plane with $D_1 = \{z : |z| < 1\} \subseteq D \subseteq D_R = \{z : |z| < R\}$ then*

$$\sup_{z,\zeta\in D} E_\zeta^z \geq \sup_r E_{-r}^r \frac{G_{D_1}(r, -r)}{G_{D_R}(r, -r)} \geq \sup_r(-r^2 + 2\ln(1 + r^2))\frac{\ln\frac{1+r^2}{2r}}{\ln\frac{R^2+r^2}{2rR}},$$

where E_{-r}^r denotes the expectation for the life time of the unit disk for $z = r$ and $\zeta = -r$. Equality occurs if and only if $D = D_1$.

Proof. We deduce

$$\sup_{z,\zeta\in D} E_\zeta^z \geq \sup_{z,\zeta\in D_1} E_\zeta^z \geq \sup_{z,\zeta\in D_1}\frac{\Gamma_{D_1}}{G_{D_R}} \geq$$

where the last inequality follows from the monotonicity of the Green's function and furthermore

$$\geq \sup_r \frac{\Gamma_{D_1}(r, -r)}{G_{D_1}(r, -r)}\frac{G_{D_1}(r, -r)}{G_{D_R}(r, -r)} \geq \sup_r E_{-r}^r \frac{G_{D_1}(r, -r)}{G_{D_R}(r, -r)},$$

which is the desired conclusion because

$$E_{-r}^r = -r^2 + 2\ln(1 + r^2) +$$
$$\frac{(1 - r^2)^2}{4r^2}\left(\ln^{-1}(1 + \frac{(1 - r^2)^2}{4r^2} - \frac{4r^2}{(1 - r^2)^2}\right)2\ln(1 + r^2) \geq -r^2 + 2\ln(1 + r^2),$$

which was proven in Lemma 5.2 in [6].

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