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On conserved Penrose-Fife type models

Jan Prüss and Mathias Wilke

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. In this paper we investigate quasilinear parabolic systems of conserved Penrose-Fife type. We show maximal L_p - regularity for this problem with inhomogeneous boundary data. Furthermore we prove global existence of a solution, provided that the absolute temperature is bounded from below and above. Moreover, we apply the Lojasiewicz-Simon inequality to establish the convergence of solutions to a steady state as time tends to infinity.

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1. Introduction and the Model

We are interested in the conserved Penrose-Fife type equations

$$\begin{aligned} \partial_t\psi = \Delta\mu, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, \quad t \in J,\ x \in \Omega, \end{aligned} \tag{1.1}$$

where $\vartheta = 1/\theta$ and θ denotes the absolute temperature of the system, ψ is the order parameter and $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^4$. The function Φ' is the derivative of the physical potential, which characterizes the different phases of the system. A typical example is the *double well* potential $\Phi(s) = (s^2-1)^2$ with the two distinct minima $s = \pm 1$. Typically, the nonlinear function λ is a polynomial of second order.

For an explanation of (1.1) we will follow the lines of Alt & Pawlow [2] (see also Brokate & Sprekels [4, Section 4.4]). We start with the rescaled Landau-Ginzburg functional (total Helmholtz free energy)

$$\mathcal{F}(\psi,\theta) = \int_\Omega \left(\frac{\gamma(\theta)}{2\theta}|\nabla\psi|^2 + \frac{f(\psi,\theta)}{\theta}\right)\, dx,$$

where the free energy density $F(\psi,\theta) := \frac{\gamma(\theta)}{2}|\nabla\psi|^2 + f(\psi,\theta)$ is rescaled by $1/\theta$. The reduced chemical potential μ is given by the variational derivative of $\mathcal{F}$ with respect to ψ, i.e.

$$\mu = \frac{\delta \mathcal{F}}{\delta \psi}(\psi,\theta) = \frac{1}{\theta}\left(-\gamma(\theta)\Delta\psi + \frac{\partial f(\psi,\theta)}{\partial\psi}\right).$$

Assuming that ψ is a conserved quantity, we have the conservation law

$$\partial_t\psi + \mathrm{div} j = 0.$$

Here j is the flux of the order parameter ψ, for which we choose the well accepted constitutive law $j = -\nabla\mu$, i.e. the phase transition is driven by the chemical potential μ (see [4, (4.4)]). The kinetic equation for ψ thus reads

$$\partial_t\psi = \Delta\mu, \quad \mu = \frac{1}{\theta}\left(-\gamma(\theta)\Delta\psi + \frac{\partial f(\psi,\theta)}{\partial\psi}\right).$$

If the volume of the system is preserved, the internal energy e is given by the variational derivative

$$e = \frac{\delta\mathcal{F}(\psi,\theta)}{\delta(1/\theta)}.$$

This yields the expression

$$e(\psi,\theta) = f(\psi,\theta) - \theta\frac{\partial f(\psi,\theta)}{\partial\theta} + \frac{1}{2}\left(\gamma(\theta) - \theta\frac{\partial\gamma(\theta)}{\partial\theta}\right)|\nabla\psi|^2.$$

It can be readily checked that the GIBBS relation

$$e(\psi,\theta) = F(\psi,\theta) - \theta\frac{\partial F(\psi,\theta)}{\partial\theta}.$$

holds. If we assume that no mechanical stresses are active, the internal energy e satisfies the conservation law

$$\partial_t e + \mathrm{div} q = 0,$$

where q denotes the heat flux of the system. Following ALT & PAWLOW [2], we assume that $q = \nabla\left(\frac{1}{\theta}\right)$, so that the kinetic equation for e reads

$$\partial_t e + \Delta\left(\frac{1}{\theta}\right) = 0.$$

Let us now assume that $\gamma(\theta) = \theta$ and $f(\psi,\theta) = \theta\Phi(\psi) - \lambda(\psi) - \theta\log\theta$. In this case we obtain $e = \theta - \lambda(\psi)$ and

$$\mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\frac{1}{\theta},$$

hence system (1.1) for $\vartheta = 1/\theta$ and $b(s) = -1/s$, $s > 0$. Suppose $(j|\nu) = (q|\nu) = 0$ on $\partial\Omega$ with $\nu = \nu(x)$ being the outer unit normal in $x \in \partial\Omega$. This yields the boundary conditions $\partial_\nu\mu = 0$ and $\partial_\nu\vartheta = 0$ for the chemical potential μ and the function ϑ, respectively. Since (1.1) is of fourth order with respect to the function ψ we need an additional boundary condition. An appropriate and classical one from a variational point of view is $\partial_\nu \psi = 0$. Finally, this yields the initial-boundary value problem

$$\begin{aligned} \partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = f_2, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu\mu = g_1,\ \partial_\nu\psi = g_2,\ \partial_\nu\vartheta = g_3, &\quad t \in J,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{1.2}$$

The functions $f_j, g_j, \psi_0, \vartheta_0, \Phi, \lambda$ and b are given. Note that if θ has only a small deviation from a constant value $\theta_* > 0$, then the term $1/\theta$ can be linearized around θ_* and (1.2) turns into the nonisothermal Cahn-Hilliard equation for the order parameter ψ and the relative temperature $\theta - \theta_*$, provided $b(s) = -1/s$.

In the case of the Penrose-Fife equations, Brokate & Sprekels [4] and Zheng [18] proved global well-posedness in an L_2-setting if the spatial dimension is equal to 1. Sprekels & Zheng showed global well-posedness of the non-conserved equations (that is $\partial_t\psi = -\mu$) in higher space dimensions in [16], a similar result can be found in the article of Laurencot [10]. Concerning asymptotic behavior we refer to the articles of Kubo, Ito & Kenmochi [9], Shen & Zheng [15], Feireisl & Schimperna [8] and Rocca & Schimperna [13]. The last two authors studied well-posedness and qualitative behavior of solutions to the non-conserved Penrose-Fife equations. To be precise, they proved that each solution converges to a steady state, as time tends to infinity. Shen & Zheng [15] established the existence of attractors for the non-conserved equations, whereas Kubo, Ito & Kenmochi [9] studied the non-conserved as well as the conserved Penrose-Fife equations. Beside the proof of global well-posedness in the sense of weak solutions they also showed the existence of a global attractor. Finally, we want to mention that the physical potential Φ may also be of logarithmic type, such that $\Phi'(s)$ has singularities at $s = \pm 1$. This forces the order parameter to stay in the physically reasonable interval $(-1, 1)$, provided that the initial value $\psi(0) = \psi_0 \in (-1, 1)$. In general, such a result cannot be obtained in the case of the double well potential, since there is no maximum principle available for the fourth order equation $(1.2)_1$. For a result on global existence, uniqueness and asymptotic behaviour of solutions of the *Cahn-Hilliard equation* in case of a logarithmic potential, we refer the reader to Abels & Wilke [1]. However, in this paper we will only deal with smooth potentials.

In the following sections we will prove well-posedness of (1.2) for solutions in the maximal L_p-regularity classes

$$\psi \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)),$$

$$\vartheta \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)),$$

where $J = [0, T]$, $T > 0$. In Section 2 we investigate a linearized version of (1.2) and prove maximal L_p-regularity. Section 3 is devoted to local well-posedness of (1.2). To this end we apply the contraction mapping principle. In Section 4, we show that the solution exists globally in time, provided that the absolute temperature

ϑ is uniformly bounded from below and above. Finally, in Section 5, we study the asymptotic behavior of the solution to (1.2) as $t \to \infty$. The Lojasiewicz-Simon inequality will play an important role in the analysis.

2. The Linear Problem

In this section we deal with a linearized version of (1.2).

$$\begin{aligned} \partial_t u + \Delta^2 u + \Delta(\eta_1 v) &= f_1, \quad t \in J,\ x \in \Omega, \\ \partial_t v - a_0 \Delta v + \eta_2 \partial_t u &= f_2, \quad t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u + \partial_\nu(\eta_1 v) &= g_1, \quad t \in J,\ x \in \partial\Omega, \\ \partial_\nu u = g_2,\ \partial_\nu v &= g_3, \quad t \in J,\ x \in \partial\Omega, \\ u(0) = u_0,\ v(0) &= v_0, \quad t = 0,\ x \in \Omega. \end{aligned} \tag{2.1}$$

Here $\eta_1 = \eta_1(x), \eta_2 = \eta_2(x), a_0 = a_0(x)$ are given functions such that

$$\eta_1 \in B_{pp}^{4-4/p}(\Omega),\ \eta_2 \in B_{pp}^{2-2/p}(\Omega) \quad \text{and} \quad a_0 \in C(\overline{\Omega}). \tag{2.2}$$

We assume furthermore that $a_0(x) \geq \sigma > 0$ for all $x \in \overline{\Omega}$ and some constant $\sigma > 0$. Hence equation $(2.1)_2$ does not degenerate. We are interested in solutions

$$u \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^4(\Omega)) =: E_1(T)$$

and

$$v \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^2(\Omega)) =: E_2(T)$$

of (2.1). By the well-known trace theorems (cf. [3, Theorem 4.10.2])

$$E_1(T) \hookrightarrow C(J; B_{pp}^{4-4/p}(\Omega)) \quad \text{and} \quad E_2(T) \hookrightarrow C(J; B_{pp}^{2-2/p}(\Omega)), \tag{2.3}$$

we necessarily have $u_0 \in B_{pp}^{4-4/p}(\Omega) =: X_\gamma^1$, $v_0 \in B_{pp}^{2-2/p}(\Omega) =: X_\gamma^2$ and the compatibility conditions

$$\partial_\nu \Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0}, \quad \partial_\nu u_0 = g_2|_{t=0}, \quad \text{as well as} \quad \partial_\nu v_0 = g_3|_{t=0},$$

whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively (cf. [6, Theorem 2.1]). In the sequel we will assume that $p > (n+2)/2$ and $p \geq 2$. This yields the embeddings

$$B_{pp}^{4-4/p}(\Omega) \hookrightarrow H_p^2(\Omega) \cap C^1(\bar{\Omega}) \text{ and } B_{pp}^{2-2/p}(\Omega) \hookrightarrow H_p^1(\Omega) \cap C(\bar{\Omega}).$$

We are going to prove the following theorem.

Theorem 2.1. *Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ a bounded domain with boundary $\partial\Omega \in C^4$ and let $p > (n+2)/2$, $p \geq 2$, $p \neq 3, 5$. Assume in addition that $\eta_1 \in B_{pp}^{4-4/p}(\Omega)$, $\eta_2 \in B_{pp}^{2-2/p}(\Omega)$ and $a_0 \in C(\bar{\Omega})$, $a_0(x) \geq \sigma > 0$ for all $x \in \bar{\Omega}$. Then the linear problem* (2.1) *admits a unique solution*

$$(u, v) \in H_p^1(J_0; L_p(\Omega)^2) \cap L_p(J_0; (H_p^4(\Omega) \times H_p^2(\Omega))),$$

if and only if the data are subject to the following conditions.

1. $f_1, f_2 \in L_p(J_0; L_p(\Omega)) = X(J_0)$,

2. $g_1 \in W_p^{1/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega)) = Y_1(J_0)$,
3. $g_2 \in W_p^{3/4-1/4p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{3-1/p}(\partial\Omega)) = Y_2(J_0)$,
4. $g_3 \in W_p^{1/2-1/2p}(J_0; L_p(\partial\Omega)) \cap L_p(J_0; W_p^{1-1/p}(\partial\Omega)) = Y_3(J_0)$,
5. $u_0 \in B_{pp}^{4-4/p}(\Omega) = X_\gamma^1$, $v_0 \in B_{pp}^{2-2/p}(\Omega) = X_\gamma^2$,
6. $\partial_\nu \Delta u_0 + \partial_\nu(\eta_1 v_0) = g_1|_{t=0},\ p > 5$,
7. $\partial_\nu u_0 = g_2|_{t=0},\ \partial_\nu v_0 = g_3|_{t=0},\ p > 3$.

Proof. Suppose that the function $u \in E_1(T)$ in (2.1) is already known. Then in a first step we will solve the linear heat equation

$$\partial_t v - a_0 \Delta v = f_2 - \eta_2 \partial_t u, \tag{2.4}$$

subject to the boundary and initial conditions $\partial_\nu v = g_3$ and $v(0) = v_0$. By the properties of the function a_0 we may apply [6, Theorem 2.1] to obtain a unique solution $v \in E_2(T)$ of (2.4), provided that $f_2 \in L_p(J \times \Omega)$, $v_0 \in B_{pp}^{2-2/p}(\Omega)$,

$$g_3 \in W_p^{1/2-1/2p}(J; L_p(\partial\Omega)) \cap L_p(J; W_p^{1-1/p}(\partial\Omega)) =: Y_3(J),$$

and the compatibility condition $\partial_\nu v_0 = g_3|_{t=0}$ if $p > 3$ is valid. The solution may then be represented by the variation of parameters formula

$$v(t) = v_1(t) - \int_0^t e^{-A(t-s)} \eta_2 \partial_t u(s)\, ds, \tag{2.5}$$

where A denotes the L_p-realization of the differential operator $\mathcal{A}(x) = -a_0(x)\Delta_N$, Δ_N means the Neumann-Laplacian and e^{-At} stands for the bounded analytic semigroup, which is generated by $-A$ in $L_p(\Omega)$. Furthermore the function $v_1 \in E_2(T)$ solves the linear problem

$$\partial_t v_1 - a_0 \Delta v_1 = f_2, \quad \partial_\nu v_1 = g_3, \quad v_1(0) = v_0.$$

We fix a function $w^* \in E_1(T)$ such that $w^*|_{t=0} = u_0$ and make use of (2.5) and the fact that $(u - w^*)|_{t=0} = 0$ to obtain

$$v(t) = v_1(t) + v_2(t) - (\partial_t + A)^{-1} \eta_2 \partial_t (u - w^*)$$

with $v_2(t) := -\int_0^t e^{-A(t-s)} \eta_2 \partial_t w^*$. Set $v^* = v_1 + v_2 \in E_2(T)$ and

$$F(u) = -(\partial_t + A)^{-1} \eta_2 \partial_t (u - w^*).$$

Then we may reduce (2.1) to the problem

$$\begin{aligned} \partial_t u + \Delta^2 u = \Delta G(u) + f_1, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu \Delta u = \partial_\nu G(u) + g_1, &\quad t \in J,\ x \in \partial\Omega, \\ \partial_\nu u = g_2 &\quad t \in J,\ x \in \partial\Omega, \\ u(0) = u_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.6}$$

where $G(u) := -\eta_1(F(u) + v^*)$. For a given $T \in (0, T_0]$ we set

$${}_0E_1(T) = \{u \in E_1(T) : u|_{t=0} = 0\}$$

and

$$E_0(T) := X(T) \times Y_1(T) \times Y_2(T)$$
$${}_0E_0(T) := \{(f,g,h) \in E_0(T) : g|_{t=0} = h|_{t=0} = 0\},$$

where $X(T) := L_p((0,T)\times\Omega)$,

$$Y_1(T) := W_p^{1/4-1/4p}(0,T;L_p(\partial\Omega)) \cap L_p(0,T;W_p^{1-1/p}(\partial\Omega)),$$

and

$$Y_2(T) := W_p^{3/4-1/4p}(0,T;L_p(\partial\Omega)) \cap L_p(0,T;W_p^{3-1/p}(\partial\Omega)).$$

The spaces $E_1(T)$ and $E_0(T)$ are endowed with the canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. We introduce the new function $\tilde{u} := u - w^* \in {}_0E_1(T)$ and we set

$$\tilde{F}(\tilde{u}) := -(\partial_t + A)^{-1}\eta_2\partial_t\tilde{u}$$

as well as $\tilde{G}(\tilde{u}) := -\eta_1\tilde{F}(\tilde{u})$. If $u \in E_1(T)$ is a solution of (2.6), then the function $\tilde{u} \in {}_0E_1(T)$ solves the problem

$$\begin{aligned} \partial_t\tilde{u} + \Delta^2\tilde{u} &= \Delta\tilde{G}(\tilde{u}) + \tilde{f}_1, \quad t \in J,\ x \in \Omega,\\ \partial_\nu\Delta\tilde{u} &= \partial_\nu\tilde{G}(\tilde{u}) + \tilde{g}_1, \quad t \in J,\ x \in \partial\Omega,\\ \partial_\nu\tilde{u} &= \tilde{g}_2 \quad t \in J,\ x \in \partial\Omega,\\ \tilde{u}(0) &= 0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{2.7}$$

with the modified data

$$\begin{aligned} \tilde{f}_1 &:= f_1 - \Delta(\eta_1 v^*) - \partial_t w^* - \Delta^2 w^* \in X(T),\\ \tilde{g}_1 &:= g_1 - \partial_\nu(\eta v^*) - \partial_\nu\Delta w^* \in {}_0Y_1(T), \end{aligned}$$

and

$$\tilde{g}_2 := g_2 - \partial_\nu w^* \in {}_0Y_2(T).$$

Observe that by construction we have $\tilde{g}_1|_{t=0} = 0$ and $\tilde{g}_2|_{t=0} = 0$ if $p > 5$ and $p > 5/3$, respectively.

Let us estimate the term $\Delta\tilde{G}(u)$ in $L_p(J;L_p(\Omega))$, where $u \in {}_0E_1(T)$. We compute

$$\begin{aligned} |\Delta\tilde{G}(u)|_{L_p(J;L_p(\Omega))} \le{}& |\tilde{F}(u)\Delta\eta_1|_{L_p(J;L_p(\Omega))}\\ &+ 2|(\nabla\tilde{F}(u)|\nabla\eta_1)|_{L_p(J;L_p(\Omega))} + |\eta_1\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))}. \end{aligned}$$

Since $\eta_1 \in B_{pp}^{4-4/p}(\Omega)$ does not depend on the variable t, we obtain

$$\begin{aligned} |\tilde{F}(u)\Delta\eta_1|_{L_p(J;L_p(\Omega))} &\le |\Delta\eta_1|_{L_p(\Omega)}|\tilde{F}(u)|_{L_p(J;L_\infty(\Omega))},\\ |(\nabla\tilde{F}(u)|\nabla\eta_1)|_{L_p(J;L_p(\Omega))} &\le |\nabla\eta_1|_{L_\infty(\Omega)}|\nabla\tilde{F}(u)|_{L_p(J;L_p(\Omega))}, \end{aligned}$$

and

$$|\eta_1\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))} \le |\eta_1|_{L_\infty(\Omega)}|\Delta\tilde{F}(u)|_{L_p(J;L_p(\Omega))}.$$

Therefore we have to estimate $\tilde{F}(u)$ for each $u \in {}_0E_1(T)$ in the topology of the spaces $L_p(J;L_\infty(\Omega))$ and $L_p(J;H_p^2(\Omega))$. Let $u \in {}_0E_1$ and recall that $\tilde{F}(u)$ is defined by $\tilde{F}(u) = -(\partial_t + A)^{-1}\eta_2\partial_t u$. The operator $(\partial_t + A)^{-1}$ is a bounded linear operator from $L_p(J;L_p(\Omega))$ to ${}_0H_p^1(J;L_p(\Omega))\cap L_p(J;H_p^2(\Omega))={}_0E_2(T)$. Moreover, by the trace theorem and by Sobolev embedding, it holds that

$$ {}_0H_p^1(J;L_p(\Omega))\cap L_p(J;H_p^2(\Omega))\hookrightarrow C(J;B_{pp}^{2-2/p}(\Omega))\hookrightarrow C(J;C(\bar\Omega)). $$

Note that the bound of $(\partial_t+A)^{-1}$ as well as the embedding constant do not depend on the length of the interval $J=[0,T]\subset[0,T_0]=J_0$, since the time trace at $t=0$ vanishes. With these facts, we obtain

$$ \begin{aligned} |(\partial_t+A)^{-1}\eta_2\partial_t u|_{L_p(J;L_\infty(\Omega))} &\le T^{1/p}|(\partial_t+A)^{-1}\eta_2\partial_t u|_{L_\infty(J;L_\infty(\Omega))}\\ &\le T^{1/p}C|(\partial_t+A)^{-1}\eta_2\partial_t u|_{E_2(T)}\\ &\le T^{1/p}C|\eta_2\partial_t u|_{L_p(J;L_p(\Omega))}\\ &\le T^{1/p}C|\eta_2|_{L_\infty(\Omega)}|u|_{E_1(T)}. \end{aligned} $$

To estimate $\tilde F(u)$ in $L_p(J;H_p^2(\Omega))$ we need another representation of $\tilde F(u)$. To be precise, we rewrite $\tilde F(u)$ as follows

$$ \tilde F(u)=-(\partial_t+A)^{-1}\eta_2\partial_t u=-\partial_t^{1/2}(\partial_t+A)^{-1}\partial_t^{1/2}(\eta_2 u). $$

This is possible, since $u\in{}_0E_1(T)$. Now observe that for each $u\in{}_0E_1$ it holds that $\eta_2 u\in{}_0H_p^{3/4}(J;H_p^1(\Omega))$. This can be seen as follows. First of all, it suffices to show that $\eta_2 u\in L_p(J;H_p^1(\Omega))$, since η_2 does not depend on the variable t. But

$$ \begin{aligned} |\eta_2 u|_{L_p(J;H_p^1(\Omega))} &\le |\eta_2\nabla u|_{L_p(J;L_p(\Omega))}+|u\nabla\eta_2|_{L_p(J;L_p(\Omega))}\\ &\le C\left(|\eta_2|_{L_\infty(\Omega)}|u|_{E_1(T)}+|u|_{L_p(J;L_\infty(\Omega))}|\eta_2|_{H_p^1(\Omega)}\right)\\ &\le C|u|_{E_1(T)}|\eta_2|_{B_{pp}^{2-2/p}(\Omega)}, \end{aligned} $$

and this yields the claim, since

$$ u\in{}_0H_p^1(J;L_p(\Omega))\cap L_p(J;H_p^4(\Omega))\hookrightarrow{}_0H_p^{3/4}(J;H_p^1(\Omega)), $$

by the mixed derivative theorem. It follows readily that $\partial_t^{1/2}(\eta_2 u)\in{}_0H_p^{1/4}(J;H_p^1(\Omega))$ and

$$ (\partial_t+A)^{-1}(I+A)^{1/2}\partial_t^{1/2}(\eta_2 u)\in{}_0H_p^{5/4}(J;L_p(\Omega))\cap{}_0H_p^{1/4}(J;H_p^2(\Omega)). $$

Since the operator $(I+A)^{1/2}$ with domain $D((I+A)^{1/2})=H_p^1(\Omega)$ commutes with the operator $(\partial_t+A)^{-1}$, this yields

$$ (\partial_t+A)^{-1}\partial_t^{1/2}(\eta_2 u)\in{}_0H_p^{5/4}(J;H_p^1(\Omega))\cap{}_0H_p^{1/4}(J;H_p^3(\Omega)) $$

for each fixed $u\in{}_0E_1(T)$. By the mixed derivative theorem we obtain furthermore

$$ {}_0H_p^{5/4}(J;H_p^1(\Omega))\cap{}_0H_p^{1/4}(J;H_p^3(\Omega))\hookrightarrow{}_0H_p^{3/4}(J;H_p^2(\Omega)). $$

Therefore

$$ \tilde F(u)=-\partial_t^{1/2}(\partial_t+A)^{-1}\partial_t^{1/2}(\eta_2 u)\in{}_0H_p^{1/4}(J;H_p^2(\Omega)), $$

and there exists a constant $C > 0$ being independent of $T > 0$ and $u \in {}_0E_1(T)$ such that

$$|\tilde{F}(u)|_{H_p^{1/4}(J;H_p^2(\Omega))} \le C|u|_{E_1(T)},$$

for each $u \in {}_0E_1(T)$. In particular this yields the estimate

$$\begin{aligned}|\tilde{F}(u)|_{L_p(J;H_p^2(\Omega))} &\le T^{1/2p}|\tilde{F}(u)|_{L_{2p}(J;H_p^2(\Omega))}\\ &\le T^{1/2p}|\tilde{F}(u)|_{H_p^{1/4}(J;H_p^2(\Omega))} \le T^{1/2p}C|u|_{E_1(T)},\end{aligned}$$

by Hölders inequality and $C > 0$ does not depend on the length T of the interval J. We have thus shown that

$$|\Delta\tilde{G}(u)|_{L_p(J;L_p(\Omega))} \le \mu_1(T)C|u|_{E_1(T)},$$

where we have set $\mu_1(T) := T^{1/2p}(1+T^{1/2p})$. Observe that $\mu_1(T) \to 0_+$ as $T \to 0_+$. The next step consists of estimating the term $\partial_\nu\tilde{G}(u)$ in ${}_0W_p^{1/4-1/4p}(J;L_p(\partial\Omega)) \cap L_p(J;W_p^{1-1/p}(\partial\Omega))$. To this end, we recall the trace map

$${}_0H_p^{1/2}(J;L_p(\Omega)) \cap L_p(J;H_p^2(\Omega)) \hookrightarrow {}_0W_p^{1/4-1/4p}(J;L_p(\partial\Omega)) \cap L_p(J;W_p^{1-1/p}(\partial\Omega))$$

for the Neumann derivative on $\partial\Omega$. Therefore, by the results above, it remains to estimate $\tilde{G}(u)$ in ${}_0H_p^{1/2}(J;L_p(\Omega))$. By the complex interpolation method we have

$$|w|_{H_p^{1/2}(J;L_p(\Omega))} \le C|w|^{1/2}_{L_p(J;L_p(\Omega))}|w|^{1/2}_{H_p^1(J;L_p(\Omega))}$$

for each $w \in {}_0H_p^1(J;L_p(\Omega))$, and $C > 0$ does not depend on $T > 0$. Using Hölders inequality, this yields

$$\begin{aligned}|w|_{H_p^{1/2}(J;L_p(\Omega))} &\le T^{1/4p}C|w|^{1/2}_{L_{2p}(J;L_p(\Omega))}|w|^{1/2}_{H_p^1(J;L_p(\Omega))}\\ &\le T^{1/4p}C|w|_{H_p^1(J;L_p(\Omega))}.\end{aligned}$$

Finally we obtain the estimate

$$|\tilde{G}(u)|_{H_p^{1/2}(J;L_p(\Omega))} \le T^{1/2p}|\eta_1|_{L_\infty(\Omega)}C|u|_{\mathbb{E}_1(T)},$$

which in turn implies

$$|\partial_\nu\tilde{G}(u)|_{Y_1(J)} \le |\tilde{G}(u)|_{H_p^{1/2}(J;L_p(\Omega))} + |\tilde{G(u)}|_{L_p(J;H_p^2(\Omega))} \le \mu_2(T)C|u|_{E_1(T)},$$

where $\mu_2(T) := T^{1/4p}(1+T^{1/4p})$ and $\mu_2(T) \to 0_+$ as $T \to 0_+$. Define two operators $L, B : {}_0E_1(T) \to {}_0E_0(T)$ by means of

$$Lu := \begin{bmatrix}\partial_t u + \Delta^2 u\\ \partial_\nu\Delta u\\ \partial_\nu u\end{bmatrix} \text{ and } Bu := \begin{bmatrix}\Delta\tilde{G}(u)\\ \partial_\nu\tilde{G}(u)\\ 0\end{bmatrix}.$$

With these definitions, we may rewrite (2.7) in the abstract form

$$Lu = Bu + f, \quad f := (\tilde{f}_1, \tilde{g}_1, \tilde{g}_2) \in {}_0E_0(T).$$

By [6, Theorem 2.1], the operator L is bijective with bounded inverse L^{-1}, hence $u \in {}_0E_1(T)$ is a solution of (2.7) if and only if $(I - L^{-1}B)u = L^{-1}f$. Observe that $L^{-1}B$ is a bounded linear operator from ${}_0E_1(T)$ to ${}_0E_1(T)$ and

$$|L^{-1}Bu|_{E_1(T)} \leq |L^{-1}|_{\mathcal{B}(E_0(T),E_1(T))}|Bu|_{E_0(T)} \leq (\mu_1(T) + \mu_2(T))C|u|_{E_1(T)}.$$

Here the constant $C > 0$ as well as the bound of L^{-1} are independent of $T > 0$. This shows that choosing $T > 0$ sufficiently small, we may apply a Neumann series argument to conclude that (2.7) has a unique solution $u \in {}_0E_1(T)$ on a possibly small time interval $J = [0, T]$. Since the linear system (2.7) is invariant with respect to time shifts, we may set $J = J_0$. □

3. Local Well-Posedness

In this section we will use the following setting. For $T_0 > 0$, to be fixed later, and a given $T \in (0, T_0]$ we define

$$\mathbb{E}_1(T) := E_1(T) \times E_2(T), \qquad {}_0\mathbb{E}_1(T) := \{(u,v) \in \mathbb{E}_1(T) : (u,v)|_{t=0} = 0\}$$

and

$$\mathbb{E}_0(T) := X(T) \times X(T) \times Y_1(T) \times Y_2(T) \times Y_3(T),$$

as well as

$${}_0\mathbb{E}_0(T) := \{(f_1, f_2, g_1, g_2, g_3) \in \mathbb{E}_0(T) : g_1|_{t=0} = g_2|_{t=0} = g_3|_{t=0} = 0\},$$

with canonical norms $|\cdot|_1$ and $|\cdot|_0$, respectively. The aim of this section is to find a local solution $(\psi, \vartheta) \in \mathbb{E}_1(T)$ of the quasilinear system

$$\begin{aligned} \partial_t\psi - \Delta\mu = f_1, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t \in J,\ x \in \Omega, \\ \partial_t\left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = f_2, &\quad t \in J,\ x \in \Omega, \\ \partial_\nu\mu = g_1,\ \partial_\nu\psi = g_2,\ \partial_\nu\vartheta = g_3, &\quad t \in J,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega. \end{aligned} \tag{3.1}$$

To this end, we will apply Banach's fixed point theorem. For this purpose let $p > (n+2)/2$, $p \geq 2$, $f_1, f_2 \in X(T_0)$, $g_j \in Y_j(0, T_0)$, $j = 1, 2, 3$, $\psi_0 \in X_\gamma^1$ and $\vartheta_0 \in X_\gamma^2$ be given such that the compatibility conditions

$$\partial_\nu\Delta\psi_0 - \partial_\nu\Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0},\ \partial_\nu\psi_0 = g_2|_{t=0} \quad \text{and} \quad \partial_\nu\vartheta_0 = g_3|_{t=0}$$

are satisfied, whenever $p > 5$, $p > 5/3$ and $p > 3$, respectively. In the sequel we will assume that $\lambda, \phi \in C^{4-}(\mathbb{R})$, $b \in C^{3-}(0, \infty)$ and $b'(s) > 0$ for all $s > 0$. Note that by the Sobolev embedding theorem we have $\vartheta_0 \in C(\bar{\Omega})$ as well as $b'(\vartheta_0) \in C(\bar{\Omega})$. Since ϑ represents the inverse absolute temperature of the system, it is reasonable to assume $\vartheta_0(x) > 0$ for all $x \in \bar{\Omega}$. Therefore, there exists a constant $\sigma > 0$ such that $\vartheta_0(x), b'(\vartheta_0(x)) \geq \sigma > 0$ for all $x \in \bar{\Omega}$. We define $a_0(x) := 1/b'(\vartheta_0(x))$, $\eta_1(x) = \lambda'(\psi_0(x))$ and $\eta_2(x) = a_0(x)\eta_1(x)$. By assumption, it holds that $a_0 \in B_{pp}^{2-2/p}(\Omega)$, $\eta_1 \in B_{pp}^{4-4/p}(\Omega)$ and $\eta_2 \in B_{pp}^{2-2/p}(\Omega)$, cf. [14, Section 4.6 & Section 5.3.4].

Thanks to Theorem 2.1 we may define a pair of functions $(u^*, v^*) \in \mathbb{E}_1(T_0)$ as the solution of the problem

$$\begin{aligned} \partial_t u^* + \Delta^2 u^* + \Delta(\eta_1 v^*) &= f_1, \quad t \in [0,T_0],\ x \in \Omega, \\ \partial_t v^* - a_0 \Delta v^* + \eta_2 \partial_t u^* &= a_0 f_2, \quad t \in [0,T_0],\ x \in \Omega, \\ \partial_\nu \Delta u^* + \partial_\nu(\eta_1 v^*) &= -g_1 - e^{-B^2 t} g_0, \quad t \in [0,T_0],\ x \in \partial\Omega, \\ \partial_\nu u^* &= g_2, \quad t \in [0,T_0],\ x \in \partial\Omega, \\ \partial_\nu v^* &= g_3, \quad t \in [0,T_0],\ x \in \partial\Omega, \\ u^*(0) = \psi_0,\ v^*(0) &= \vartheta_0, \quad t = 0,\ x \in \Omega, \end{aligned} \tag{3.2}$$

where $B = -\Delta_{\partial\Omega}$ is the Laplace-Beltrami operator on $\partial\Omega$ and $e^{-B^2 t}$ is the analytic semigroup which is generated by $-B^2$. Furthermore $g_0 = 0$ if $p < 5$ and $g_0 = -g_1|_{t=0} - (\partial_\nu \Delta\psi_0 + \partial_\nu(\eta_1 \vartheta_0))$ if $p > 5$.

Define a linear operator $\mathbb{L} : {}_0\mathbb{E}_1(T_0) \to {}_0\mathbb{E}_0(T_0)$ by

$$\mathbb{L}(u,v) = \begin{bmatrix} \partial_t u + \Delta^2 u + \eta_1 \Delta v \\ \partial_t v - a_0 \Delta v + \eta_2 \partial_t u \\ \partial_\nu \Delta u + \partial_\nu(\eta_1 v) \\ \partial_\nu u \\ \partial_\nu v \end{bmatrix}.$$

Then, by Theorem 2.1, the operator $\mathbb{L} : {}_0\mathbb{E}_1(T_0) \to {}_0\mathbb{E}_0(T_0)$ is bounded and bijective, hence an isomorphism with bounded inverse $\mathbb{L}^{-1}$. For all $(u,v) \in {}_0\mathbb{E}_1(T)$ we set

$$G_1(u,v) = (\lambda'(\psi_0) - \lambda'(u))v + \Phi'(u),$$
$$G_2(u,v) = (a_0 \lambda'(\psi_0) - a(v)\lambda'(u))\partial_t u - (a_0 - a(v))\Delta v - (a_0 - a(v)) f_2,$$

where $a(v(t,x)) = 1/b'(v(t,x))$ and $a_0 = a(\vartheta_0)$. Lastly we define a nonlinear mapping $G : \mathbb{E}_1(T) \times {}_0\mathbb{E}_1(T) \to {}_0\mathbb{E}_0(T)$ by

$$G((u^*,v^*);(u,v)) = \begin{bmatrix} \Delta G_1(u+u^*, v+v^*) \\ G_2(u+u^*, v+v^*) \\ \partial_\nu G_1(u+u^*, v+v^*) - \tilde{g}_0 \\ 0 \\ 0 \end{bmatrix},$$

where $\tilde{g}_0 = 0$ if $p < 5$ and $\tilde{g}_0 = e^{-B^2 t} \partial_\nu G_1(\psi_0, \vartheta_0)$ if $p > 5$. Then it is easy to see that $\psi = u + u^* \in E_1(T)$ and $\vartheta = v + v^* \in E_2(T)$ is a solution of (1.2) if and only if

$$\mathbb{L}(u,v) = G((u^*,v^*);(u,v))$$

or equivalently

$$(u,v) = \mathbb{L}^{-1} G((u^*,v^*);(u,v)).$$

In order to apply the contraction mapping principle we consider a ball $\mathbb{B}_R = \mathbb{B}_R^1 \times \mathbb{B}_R^2 \subset {}_0\mathbb{E}_1(T)$, where $R \in (0,1]$. Furthermore we define a mapping $\mathcal{T} : \mathbb{B}_R \to {}_0\mathbb{E}_1(T)$ by $\mathcal{T}(u,v) = \mathbb{L}^{-1} G((u^*,v^*);(u,v))$. We shall prove that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ and that $\mathcal{T}$ defines a strict contraction on $\mathbb{B}_R$. To this end we define the shifted ball $\mathbb{B}_R(u^*, v^*) = \mathbb{B}_R^1(u^*) \times \mathbb{B}_R^2(v^*) \subset \mathbb{E}_1(T)$ by

$$\mathbb{B}_R(u^*, v^*) = \{(u, v) \in \mathbb{E}_1(T) : (u, v) = (\tilde{u}, \tilde{v}) + (u^*, v^*),\ (\tilde{u}, \tilde{v}) \in \mathbb{B}_R\}.$$

To ensure that the mapping G_2 is well defined, we choose $T_0 > 0$ and $R > 0$ sufficiently small. This yields that all functions $v \in \mathbb{B}_R^2(v^*)$ have only a small deviation from the initial value ϑ_0. To see this, write

$$|\vartheta_0(x) - v(t,x)| \le |\vartheta_0(x) - v^*(t,x)| + |v^*(t,x) - v(t,x)| \le \mu(T) + R,$$

for all functions $v \in \mathbb{B}_R^2(v^*)$, where $\mu = \mu(T)$ is defined by

$$\mu(T) = \max_{(t,x)\in[0,T]\times\Omega} |v^*(t,x) - \vartheta_0(x)|.$$

Observe that $\mu(T) \to 0$ as $T \to 0$, by the continuity of v^* and ϑ_0. This in turn implies that $v(t,x) \ge \sigma/2 > 0$ and $b'(v(t,x)) \ge \sigma/2 > 0$ for $(t,x) \in [0,T] \times \bar{\Omega}$ and all $v \in \mathbb{B}_R^2(v^*)$, with $T_0 > 0$, $R > 0$ being sufficiently small. Moreover, for all $v, \bar{v} \in \mathbb{B}_R^2(v^*)$ we obtain the estimates

$$|a(\vartheta_0(x)) - a(v(t,x))| \le C|\vartheta_0(x) - v(t,x)| \tag{3.3}$$

and

$$|a(\bar{v}(t,x)) - a(v(t,x))| \le C|\bar{v}(t,x) - v(t,x)|, \tag{3.4}$$

valid for all $(t,x) \in [0,T] \times \bar{\Omega}$, with some constant $C > 0$, since b' is locally Lipschitz continuous.

The next proposition provides all the facts to show the desired properties of the operator $\mathcal{T}$.

Proposition 3.1. *Let $n \in \mathbb{N}$ and $p > (n+2)/2$, $p \ge 2$, $b \in C^{2-}(0,\infty)$, $b'(s) > 0$ for all $s > 0$, $\lambda, \Phi \in C^{4-}(\mathbb{R})$ and $\vartheta_0(x) > 0$ for all $x \in \bar{\Omega}$. Then there exists a constant $C > 0$, independent of T, and functions $\mu_j = \mu_j(T)$ with $\mu_j(T) \to 0$ as $T \to 0$, such that for all $(u,v), (\bar{u}, \bar{v}) \in \mathbb{B}_R(u^*, v^*)$ the following statements hold.*

1. $|\Delta G_1(u,v) - \Delta G_1(\bar{u},\bar{v})|_{X(T)} \le (\mu_1(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$,
2. $|G_2(u,v) - G_2(\bar{u},\bar{v})|_{X(T)} \le C(\mu_2(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$,
3. $|\partial_\nu G_1(u,v) - \partial_\nu G_1(\bar{u},\bar{v})|_{Y_1(T)} \le C(\mu_3(T) + R)|(u,v) - (\bar{u},\bar{v})|_{\mathbb{E}_1(T)}$.

The proof is given in the Appendix.

It is now easy to verify the self-mapping property of $\mathcal{T}$. Let $(u,v) \in \mathbb{B}_R$. By Proposition 3.1 there exists a function $\mu = \mu(T)$ with $\mu(T) \to 0$ as $T \to 0$ such that

$$\begin{aligned}|\mathcal{T}(u,v)|_1 &= |\mathbb{L}^{-1}G((u^*,v^*),(u,v))|_1 \le |\mathbb{L}^{-1}||G((u^*,v^*),(u,v))|_0\\ &\le C(|G((u^*,v^*),(u,v)) - G((u^*,v^*),(0,0))|_0 + |G((u^*,v^*),(0,0))|_0)\\ &\le C(|\Delta G_1(u+u^*,v+v^*) - \Delta G_1(u^*,v^*)|_{X(T)}\\ &\qquad + |G_2(u+u^*,v+v^*) - G_2(u^*,v^*)|_{X(T)}\\ &\qquad + |\partial_\nu G_1(u+u^*,v+v^*) - \partial_\nu G_1(u^*,v^*)|_{Y_1(T)}\\ &\qquad + |G((u^*,v^*),(0,0))|_0)\\ &\le C(\mu(T)+R)|(u,v)|_1 + |G((u^*,v^*),(0,0))|_0\\ &\le C(\mu(T)+R)R + |G((u^*,v^*),(0,0))|_0.\end{aligned}$$

Hence we see that $\mathcal{T}\mathbb{B}_R \subset \mathbb{B}_R$ if T and R are sufficiently small, since $G((u^*,v^*),(0,0))$ is a fixed function. Furthermore for all $(u,v),(\bar{u},\bar{v}) \in \mathbb{B}_R$ we have

$$\begin{aligned}|\mathcal{T}(u,v) - \mathcal{T}(\bar{u},\bar{v})|_1 &= |\mathbb{L}^{-1}(G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v})))|_1\\ &\le |\mathbb{L}^{-1}||G((u^*,v^*),(u,v)) - G((u^*,v^*),(\bar{u},\bar{v}))|_0\\ &\le C(|\Delta G_1(u+u^*,v+v^*) - \Delta G_1(\bar{u}+u^*,\bar{v}+v^*)|_{X(T)}\\ &\qquad + |\partial_\nu G_1(u+u^*,v+v^*) - \partial_\nu G_1(\bar{u}+u^*,\bar{v}+v^*)|_{Y_1(T)}\\ &\qquad + |G_2(u+u^*,v+v^*) - G_2(\bar{u}+u^*,\bar{v}+v^*)|_{X(T)})\\ &\le C(\mu(T)+R)|(u,v)-(\bar{u},\bar{v})|_1.\end{aligned}$$

Thus $\mathcal{T}$ is a strict contraction on $\mathbb{B}_R$, if T and R are again small enough. Therefore we may apply the contraction mapping principle to obtain a unique fixed point $(\tilde{u},\tilde{v}) \in \mathbb{B}_R$ of $\mathcal{T}$. In other words the pair $(\psi,\vartheta) = (\tilde{u}+u^*,\tilde{v}+v^*) \in \mathbb{E}_1(T)$ is the unique local solution of (1.2). We summarize the preceding calculations in

Theorem 3.2. *Let $n \in \mathbb{N}$, $p > (n+2)/2$, $p \ge 2$, $p \ne 3,5$, $b \in C^{3-}(0,\infty)$, $b'(s) > 0$ for all $s > 0$ and let $\lambda, \Phi \in C^{4-}(\mathbb{R})$. Then there exists an interval $J = [0,T] \subset [0,T_0] = J_0$ and a unique solution (ψ,ϑ) of (1.2) on J, with*

$$\psi \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^4_p(\Omega))$$

and

$$\vartheta \in H^1_p(J;L_p(\Omega)) \cap L_p(J;H^2_p(\Omega)), \quad \vartheta(t,x) > 0 \text{ for all } (t,x) \in J \times \bar{\Omega},$$

provided the data are subject to the following conditions.

1. $f_1, f_2 \in L_p(J_0 \times \Omega)$,
2. $g_1 \in W_p^{1/4-1/4p}(J_0;L_p(\partial\Omega)) \cap L_p(J_0;W_p^{1-1/p}(\partial\Omega))$,
3. $g_2 \in W_p^{3/4-1/4p}(J_0;L_p(\partial\Omega)) \cap L_p(J_0;W_p^{3-1/p}(\partial\Omega))$,
4. $g_3 \in W_p^{1/2-1/2p}(J_0;L_p(\partial\Omega)) \cap L_p(J_0;W_p^{1-1/p}(\partial\Omega))$,
5. $\psi_0 \in B_{pp}^{4-4/p}(\Omega)$, $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega)$,
6. $\partial_\nu\Delta\psi_0 - \partial_\nu\Phi'(\psi_0) + \partial_\nu(\lambda'(\psi_0)\vartheta_0) = -g_1|_{t=0}$, *if* $p > 5$,
7. $\partial_\nu\psi_0 = g_2|_{t=0}$, $\partial_\nu\vartheta_0 = g_3|_{t=0}$, *if* $p > 3$,
8. $\vartheta_0(x) > 0$ *for all* $x \in \bar{\Omega}$.

The solution depends continuously on the given data and if the data are independent of t, the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ defines a local semiflow on the natural (nonlinear) phase manifold

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in B_{pp}^{4-4/p}(\Omega) \times B_{pp}^{2-2/p}(\Omega) : \psi_0 \text{ and } \vartheta_0 \text{ satisfy } 6. - 8.\}.$$

4. Global Well-Posedness

In this section we will investigate the global existence of the solution to the conserved Penrose-Fife type system

$$\begin{aligned} \partial_t \psi - \Delta\mu = 0, \quad \mu = -\Delta\psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, \quad & t > 0,\ x \in \Omega, \\ \partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta\vartheta = 0, \quad & t > 0,\ x \in \Omega, \\ \partial_\nu \mu = 0,\ \partial_\nu \psi = 0,\ \partial_\nu \vartheta = 0, \quad & t > 0,\ x \in \partial\Omega, \\ \psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, \quad & t = 0,\ x \in \Omega, \end{aligned} \tag{4.1}$$

with respect to time if the spatial dimension n is less or equal to 3. Note that the boundary conditions are equivalent to $\partial_\nu\vartheta = \partial_\nu\psi = \partial_\nu\Delta\psi = 0$. A successive application of Theorem 3.2 yields a maximal interval of existence $J_{\max} = [0, T_{\max})$ for the solution $(\psi, \vartheta) \in E_1(T) \times E_2(T)$ of (4.1), where $T \in (0, T_{\max})$. In the sequel we will make use of the following assumptions.

(H1) $\Phi \in C^{4-}(\mathbb{R})$ and there exist some constants $c_j > 0$, $\gamma \geq 0$ such that

$$\Phi(s) \geq -\frac{\eta}{2}s^2 - c_1, \ |\Phi'''(s)| \leq c_2(1 + |s|^\gamma),$$

for all $s \in \mathbb{R}$, where $\eta < \lambda_1$ with λ_1 being the smallest nontrivial eigenvalue of the negative Laplacian on Ω with Neumann boundary conditions and $\gamma < 3$ if $n = 3$.

(H2) $\lambda \in C^{4-}(\mathbb{R})$ and $\lambda'', \lambda''' \in L_\infty(\mathbb{R})$. In particular, there is a constant $c > 0$ such that $|\lambda'(s)| \leq c(1 + |s|)$ for all $s \in \mathbb{R}$.

(H3) $b \in C^{3-}((0, \infty))$, $b'(s) > 0$ on $(0, \infty)$ and there is a constant $\kappa > 1$ such that

$$\frac{1}{\kappa} \leq \vartheta(t, x) \leq \kappa$$

on $J_{\max} \times \Omega$. In particular, there exists $\sigma > 1$ such that

$$\frac{1}{\sigma} \leq b'(\vartheta(t, x)) \leq \sigma,$$

on $J_{\max} \times \Omega$.

Remark: Condition (H1) is certainly fulfilled, if Φ is a polynomial of degree $2m$, $m < 3$.

We prove global well-posedness with respect to time by contradiction. For this purpose, assume that $T_{\max} < \infty$. Multiply $\partial_t\psi = \Delta\mu$ by μ and integrate by parts to the result

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla\psi|_2^2+\int_\Omega \Phi(\psi)\,dx\right)+|\nabla\mu|_2^2-\int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\,dx=0. \tag{4.2}$$

Next we multiply $(4.1)_2$ by ϑ and integrate by parts. This yields

$$\int_\Omega \vartheta b'(\vartheta)\partial_t\vartheta\,dx+|\nabla\vartheta|_2^2+\int_\Omega \lambda'(\psi)\vartheta\partial_t\psi\,dx=0. \tag{4.3}$$

Set $\beta'(s)=sb'(s)$ and add (4.2) to (4.3) to obtain the equation

$$\frac{d}{dt}\Big(\frac{1}{2}|\nabla\psi|_2^2+\int_\Omega \Phi(\psi)\,dx+\int_\Omega \beta(\vartheta)\,dx\Big)+|\nabla\mu|_2^2+|\nabla\vartheta|_2^2=0. \tag{4.4}$$

Integrating (4.4) with respect to t, we obtain

$$E(\psi(t),\vartheta(t))+\int_0^t\left(|\nabla\mu(s)|_2^2+|\nabla\vartheta(s)|_2^2\right)\,dt=E(\psi_0,\vartheta_0), \tag{4.5}$$

for all $t\in J_{\max}$, where

$$E(u,v):=\frac{1}{2}|\nabla u|_2^2+\int_\Omega \Phi(u)\,dx+\int_\Omega \beta(v)\,dx.$$

It follows from (H1) and the Poincaré-Wirtinger inequality that

$$\begin{aligned}\frac{\varepsilon}{2}\int_\Omega &|\nabla\psi(t)|^2\,dx+\frac{1-\varepsilon}{2}\int_\Omega|\nabla\psi(t)|^2\,dx+\int_\Omega\Phi(\psi(t))\,dx\\ &\geq\frac{\varepsilon}{2}\int_\Omega|\nabla\psi(t)|^2\,dx+\frac{(1-\varepsilon)\lambda_1-\eta}{2}|\psi(t)|_2^2-c_1|\Omega|-\frac{\lambda_1}{2|\Omega|}\left(\int_\Omega\psi_0\,dx\right),\end{aligned}$$

since by equation $\partial_t\psi=\Delta\mu$ and the boundary condition $\partial_\nu\mu=0$, it holds that

$$\int_\Omega\psi(t,x)\,dx\equiv\int_\Omega\psi_0(x)\,dx,\quad t\in J_{\max}.$$

Hence for a sufficiently small $\varepsilon>0$ we obtain the a priori estimates

$$\psi\in L_\infty(J_{\max};H_2^1(\Omega))\quad\text{and}\quad|\nabla\mu|,|\nabla\vartheta|\in L_2(J_{\max};L_2(\Omega)), \tag{4.6}$$

since $\beta(\vartheta(t,x))$ is uniformly bounded on $J_{\max}\times\Omega$, by (H3). However, things are more involved for higher order estimates. Here we have the following result.

Proposition 4.1. *Let $n\leq 3$, $p>(n+2)/2$, $p\geq 2$ and let (ψ,ϑ) be the maximal solution of (4.1) with initial value $\psi_0\in B_{pp}^{4-4/p}(\Omega)$ and $\vartheta_0\in B_{pp}^{2-2/p}(\Omega)$. Suppose furthermore $b\in C^{3-}(0,\infty)$, $b'(s)>0$ for all $s>0$, $\lambda,\Phi\in C^{4-}(\mathbb{R})$ and let (H1)-(H3) hold.*

Then $\psi\in L_\infty(J_{\max}\times\Omega)$ and $\vartheta\in H_2^1(J_{\max};L_2(\Omega))\cap L_\infty(J_{\max};H_2^1(\Omega))$. Moreover, it holds that $\partial_t\psi\in L_r(J_{\max}\times\Omega)$, where $r:=\min\{p,2(n+4)/n\}$.

Proof. The proof is given in the Appendix.

□

Define the new function $u = b(\vartheta)$. Then u satisfies the nonautonomous linear differential equation in divergence form

$$\partial_t u - \operatorname{div}(a(t,x)\nabla u) = f, \tag{4.7}$$

subject to the boundary and initial conditions $\partial_\nu u = 0$ and $u(0) = b(\vartheta_0) =: u_0$, where $a(t,x) := 1/b'(\vartheta(t,x))$ and $f := -\lambda'(\psi)\partial_t\psi$. With (H3), the regularity of ϑ from Proposition 4.1 carries over to the function u; in particular $u_0 \in B_{pp}^{2-2/p}(\Omega)$. This yields, that u is a *weak solution* of (4.7) in the sense of Lieberman [11] & DiBenedetto [7], and u is bounded by (H3).

Furthermore, by (H3)

$$0 < \frac{1}{\sigma} \le a(t,x) \le \sigma < \infty,$$

for all $(t,x) \in J_{\max}\times\Omega$. Note that by Proposition 4.1 it holds that $f = -\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$, $r := \min\{p, 2(n+4)/n\}$. Consider the case $r = 2(n+4)/n$. Then it can be readily checked that

$$\frac{n+2}{2} < \frac{2(n+4)}{n} = r$$

provided $n \le 5$. It follows from Lieberman [11] & DiBenedetto [7] that there exists a real number $\alpha \in (0,1/2)$ such that $u \in C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$, provided $f \in L_p(J_{\max}\times\Omega)$ and $p > (n+2)/2$. Here $C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$ is defined as

$$C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}}) := \{v \in C(\overline{\Omega_{T_{\max}}}) : \sup_{(t,x),(s,y)\in\overline{\Omega_{T_{\max}}}} \frac{|v(t,x)-v(s,y)|}{|t-s|^\alpha + |x-y|^{2\alpha}} < \infty\}.$$

and we have set $\Omega_{T_{\max}} = (0,T_{\max})\times\Omega$. The properties of the function b then yield that $\vartheta = b^{-1}(u) \in C^{\alpha,2\alpha}(\overline{\Omega_{T_{\max}}})$. In a next step we solve the initial-boundary value problem

$$\begin{aligned} \partial_t\vartheta - a(t,x)\Delta\vartheta = g, &\quad t \in J_{\max},\ x \in \Omega, \\ \partial_\nu\vartheta = 0, &\quad t \in J_{\max},\ x \in \partial\Omega, \\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega, \end{aligned} \tag{4.8}$$

with $g := -a(t,x)\lambda'(\psi)\partial_t\psi \in L_r(J_{\max}\times\Omega)$ and $r = 2(n+4)/n > (n+2)/2$. By [6, Theorem 2.1] we obtain

$$\vartheta \in H_r^1(J_{\max}; L_r(\Omega)) \cap L_r(J_{\max}; H_r^2(\Omega)),$$

of (4.8), since

$$\vartheta_0 \in B_{pp}^{2-2/p}(\Omega) \hookrightarrow B_{rr}^{2-2/r}(\Omega), \quad p \ge r.$$

At this point we use equation (6.8) from the proof of Proposition 4.1 to conclude $\partial_t\psi \in L_s(J_{\max}\times\Omega)$, with $s = \min\{p,q\}$ where q is restricted by

$$\frac{1}{q} \ge \frac{1}{r} - \frac{2}{n+4}.$$

For the case $r = 2(n+4)/n$, this yields

$$\frac{1}{q} \geq \frac{n-4}{2(n+4)},$$

i.e. q may be arbitrarily large in case $n \leq 3$ and we may set $s = p$. Now we solve (4.8) again, this time with $g \in L_p(J_{\max} \times \Omega)$, to obtain

$$\vartheta \in H_p^1(J_{\max}; L_p(\Omega)) \cap L_p(J_{\max}; H_p^2(\Omega))$$

and therefore $\vartheta(T_{\max}) \in B_{pp}^{2-2/p}(\Omega)$ is well defined. Next, consider the equation

$$\partial_t \psi + \Delta^2 \psi = \Delta \Phi'(\psi) - \Delta(\lambda'(\psi)\vartheta),$$

subject to the initial and boundary conditions $\psi(0) = \psi_0$ and $\partial_\nu \psi = \partial_\nu \Delta \psi = 0$. By maximal L_p-regularity there exists a constant $M = M(J_{\max}) > 0$ such that

$$|\psi|_{E_1(T)} \leq M(1 + |\Delta \Phi'(\psi)|_{X(T)} + |\Delta(\lambda'(\psi)\vartheta)|_{X(T)}). \tag{4.9}$$

for each $T \in J_{\max}$. Since $\vartheta \in E_2(T_{\max})$ we may apply [12, Lemma 4.1] to the result

$$|\Delta \Phi'(\psi)|_{X(T)} + |\Delta(\lambda'(\psi)\vartheta)|_{X(T)} \leq C(1 + |\psi|_{E_1(T)}^\delta), \tag{4.10}$$

with some $\delta \in (0,1)$ and $C > 0$ being independent of $T \in J_{\max}$. Combining (4.9) with (4.10), we obtain the estimate

$$|\psi|_{E_1(T)} \leq C(1 + |\psi|_{E_1(T)}^\delta),$$

which in turn yields that $|\psi|_{E_1(T)}$ is bounded as $T \nearrow T_{\max}$, since $\delta \in (0,1)$. Therefore the value $\psi(T_{\max}) \in B_{pp}^{4-4/p}(\Omega)$ is well defined and we may continue the solution (ψ, ϑ) beyond the point $T_{\max}$, contradicting the assumption that $J_{\max} = [0, T_{\max})$ is the maximal interval of existence. We summarize these considerations in

Theorem 4.2. *Let $n \leq 3$, $p > (n+2)/2$, $p \geq 2$ and $p \neq 3, 5$. Assume that (H1)-(H3) hold. Then for each $T_0 > 0$ there exists a unique solution*

$$\psi \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^4(\Omega)) = E_1(T_0)$$

and

$$\vartheta \in H_p^1(J_0; L_p(\Omega)) \cap L_p(J_0; H_p^2(\Omega)) = E_2(T_0),$$

of (1.2)*, provided the data are subject to the following conditions.*

1. $\psi_0 \in B_{pp}^{4-4/p}(\Omega)$, $\vartheta_0 \in B_{pp}^{2-2/p}(\Omega)$;
2. $\partial_\nu \Delta \psi_0 = 0$, *if* $p > 5$, $\partial_\nu \psi_0 = 0$;
3. $\partial_\nu \vartheta_0 = 0$, *if* $p > 3$, $\vartheta_0(x) > 0$ *for all* $x \in \bar{\Omega}$.

The solution depends continuously on the given data and the map $(\psi_0, \vartheta_0) \mapsto (\psi, \vartheta)$ defines a semiflow on the natural phase manifold

$$\mathcal{M}_p := \{(\psi_0, \vartheta_0) \in B_{pp}^{4-4/p}(\Omega) \times B_{pp}^{2-2/p}(\Omega) : \psi_0 \text{ and } \vartheta_0 \text{ satisfy } 2. \text{ \& } 3.\}.$$

5. Asymptotic Behavior

Let $n \leq 3$. In the following we will investigate the asymptotic behavior of global solutions of the homogeneous system

$$
\begin{aligned}
\partial_t \psi - \Delta \mu = 0, \quad \mu = -\Delta \psi + \Phi'(\psi) - \lambda'(\psi)\vartheta, &\quad t > 0,\ x \in \Omega,\\
\partial_t \left(b(\vartheta) + \lambda(\psi)\right) - \Delta \vartheta = 0, &\quad t > 0,\ x \in \Omega,\\
\partial_\nu \mu = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\partial_\nu \psi = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\partial_\nu \vartheta = 0, &\quad t > 0,\ x \in \partial\Omega,\\
\psi(0) = \psi_0,\ \vartheta(0) = \vartheta_0, &\quad t = 0,\ x \in \Omega,
\end{aligned}
\tag{5.1}
$$

as $t \to \infty$. To this end let $(\psi_0, \vartheta_0) \in \mathcal{M}_p$, $p > (n+2)/2$, $p \geq 2$ and denote by $(\psi(t), \vartheta(t))$ the unique global solution of (5.1). In the sequel we will make use of the following assumptions.

(H4) $b \in C^{3-}((0,\infty))$, $b'(s) > 0$ on $(0,\infty)$ and there is a constant $\kappa > 1$ such that

$$\frac{1}{\kappa} \leq \vartheta(t,x) \leq \kappa$$

on $J_{\max} \times \Omega$. In particular, there exists $\sigma > 1$ such that

$$\frac{1}{\sigma} \leq b'(\vartheta(t,x)) \leq \sigma,$$

on $J_{\max} \times \Omega$.

(H5) The functions Φ, λ and b are real analytic on $\mathbb{R}$.

We remark that assumption (H4) is identical to (H3) for a global solution. We stated it here for the sake of readability.

Note that the boundary conditions $(5.1)_{3,5}$ yield

$$\int_\Omega \psi(t,x)\ dx \equiv \int_\Omega \psi_0(x)\ dx,$$

and

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\ dx \equiv \int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\ dx.$$

Replacing ψ by $\tilde{\psi} = \psi - c$, where $c := \frac{1}{|\Omega|}\int_\Omega \psi_0(x)\ dx$ we see that $\int_\Omega \tilde{\psi}\ dx \equiv 0$, if $\Phi(s)$ and $\lambda(s)$ are replaced by $\tilde{\Phi}(s) = \Phi(s+c)$ and $\tilde{\lambda}(s) = \lambda(s+c)$, respectively. Similarly we can achieve that

$$\int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\ dx \equiv 0,$$

by a shift of λ, to be precise $\bar{\lambda}(s) := \lambda(s) - d$, where

$$d := \frac{1}{|\Omega|}\int_\Omega (b(\vartheta_0(x)) + \lambda(\psi_0(x)))\ dx.$$

With these modifications of the data we obtain the constraints

$$\int_\Omega \psi(t,x)\;dx \equiv 0 \quad \text{and} \quad \int_\Omega (b(\vartheta(t,x)) + \lambda(\psi(t,x)))\;dx \equiv 0. \tag{5.2}$$

Recall from Section 4 the energy functional

$$E(u,v) = \frac{1}{2}|\nabla u|_2^2 + \int_\Omega \Phi(u)\;dx + \int_\Omega \beta(v)\;dx,$$

defined on the energy space $V = V_1 \times V_2$, where

$$V_1 := \left\{ u \in H_2^1(\Omega) : \int_\Omega u\;dx = 0 \right\}, \qquad V_2 := H_2^r(\Omega),\; r \in (n/4, 1).$$

and V is equipped with the canonical norm $|(u,v)|_V := |u|_{H_2^1(\Omega)} + |v|_{H_2^r(\Omega)}$. It is convenient to embed V into a Hilbert space $H = H_1 \times H_2$ where

$$H_1 := \left\{ u \in L_2(\Omega) : \int_\Omega u\;dx = 0 \right\} \quad \text{and} \quad H_2 := L_2(\Omega).$$

Proposition 5.1. *Let* $(\psi,\vartheta) \in E_1 \times E_2$ *be a global solution of* (5.1) *and assume (H1)-(H4). Then*

1. $\psi \in L_\infty(\mathbb{R}_+; H_p^{2s}(\Omega)),\; s \in [0,1),\; p \in (1,\infty),\; \partial_t\psi \in L_2(\mathbb{R}_+ \times \Omega)$;
2. $\vartheta \in L_\infty(\mathbb{R}_+; H_2^1(\Omega)),\; \partial_t\vartheta \in L_2(\mathbb{R}_+ \times \Omega)$.

In particular the orbits $\psi(\mathbb{R}_+)$ *and* $\vartheta(\mathbb{R}_+)$ *are relatively compact in* $H_2^1(\Omega)$ *and* $H_2^r(\Omega)$*, respectively, where* $r \in [0,1)$.

Proof. Assertions 1 & 2 follow directly from (H1)-(H4) and the proof of Proposition 4.1, which is given in the Appendix. Indeed, one may replace the interval $J_{\max}$ by $\mathbb{R}_+$, since the operator $-A^2 = -\Delta_N^2$ generates an exponentially stable, analytic semigroup e^{-A^2t} in the space

$$\mathbb{X}_p := \{u \in L_p(\Omega) : \int_\Omega u\;dx = 0\}$$

with domain

$$D(A^2) = \{u \in H_p^4(\Omega) \cap \mathbb{X}_p : \partial_\nu u = \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\}.$$

$\square$

By Assumption (H4), there exists some bounded interval $J_\vartheta \subset \mathbb{R}_+$ with $\vartheta(t,x) \in J_\vartheta$ for all $t \geq 0,\; x \in \Omega$. Therefore we may modify the nonlinearities b and β outside J_ϑ in such a way that $b, \beta \in C_b^{3-}(\mathbb{R})$.

Unfortunately the energy functional E is not yet the right one for our purpose, since we have to include the nonlinear constraint

$$\int_\Omega (\lambda(\psi) + b(\vartheta))\;dx = 0,$$

into our considerations. The linear constraint $\int_\Omega \psi\, dx = 0$ is part of the definition of the space H_1. For the nonlinear constraint we use a functional of Lagrangian type which is given by

$$L(u,v) = E(u,v) - \overline{v}F(u,v),$$

defined on V, where $F(u,v) := \int_\Omega (\lambda(u) + b(v))\, dx$ and $\bar{w} = \frac{1}{|\Omega|}\int_\Omega w\, dx$ for a function $w \in L_1(\Omega)$. Concerning the differentiability of L we have the following result.

Proposition 5.2. *Under the conditions (H1)-(H4), the functional L is twice continuously Fréchet differentiable on V and the derivatives are given by*

$$\langle L'(u,v),(h,k)\rangle_{V^*,V} = \langle E'(u,v),(h,k)\rangle_{V^*,V} - \overline{k}F(u,v) - \overline{v}\langle F'(u,v),(h,k)\rangle_{V^*,V} \quad (5.3)$$

and

$$\begin{aligned}\langle L''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} = \langle E''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} - \\ \overline{k_1}\langle F'(u,v),(h_2,k_2)\rangle_{V^*,V} - \overline{k_2}\langle F'(u,v),(h_1,k_1)\rangle_{V^*,V} - \\ \overline{v}\langle F''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V},\end{aligned} \quad (5.4)$$

where $(h,k),(h_j,k_j) \in V,\ j = 1,2,$ and

$$\langle E'(u,v),(h,k)\rangle_{V^*,V} = \int_\Omega \nabla u \nabla h\, dx + \int_\Omega \Phi'(u)h\, dx + \int_\Omega \beta'(v)k\, dx,$$

$$\langle E''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} = \int_\Omega \nabla h_1 \nabla h_2\, dx + \int_\Omega \Phi''(u)h_1h_2\, dx + \int_\Omega \beta''(v)k_1k_2\, dx,$$

$$\langle F'(u,v),(h,k)\rangle_{V^*,V} = \int_\Omega \lambda'(u)h\, dx + \int_\Omega b'(v)k\, dx$$

and

$$\langle F''(u,v)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V} = \int_\Omega \lambda''(u)h_1h_2\, dx + \int_\Omega b''(v)k_1k_2\, dx.$$

Proof. We only consider the first derivative, the second one is treated in a similar way. Since the bilinear form

$$a(u,v) := \int_\Omega \nabla u(x)\nabla v(x)\, dx \quad (5.5)$$

defined on $V_1 \times V_1$ is bounded and symmetric, the first term in E is twice continuously Fréchet differentiable. For the functional

$$G_1(u) := \int_\Omega \Phi(u)\, dx, \quad u \in V_1,$$

we argue as follows. With $u, h \in V_1$ it holds that

$$\begin{aligned}
&\Phi(u(x)+h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x)\\
&\qquad = \int_0^1 \frac{d}{dt}\,\Phi(u(x)+th(x))\;dt - \int_0^1 \Phi'(u(x))h(x)\;dt\\
&\qquad = \int_0^1 \Big(\Phi'(u(x)+th(x)) - \Phi'(u(x))\Big)h(x)\;dt\\
&\qquad = \int_0^1 \int_0^t \frac{d}{ds}\,\Phi'(u(x)+sh(x))h(x)\;ds\;dt\\
&\qquad = \int_0^1 \int_0^t \Phi''(u(x)+sh(x))h(x)^2\;ds\;dt\\
&\qquad = \int_0^1 \Phi''(u(x)+sh(x))h(x)^2(1-s)\;ds.
\end{aligned}$$

From the growth condition (H1), Hölder's inequality and the Sobolev embedding theorem it follows that

$$\begin{aligned}
&\Big|\int_\Omega \Big(\Phi(u(x)+h(x)) - \Phi(u(x)) - \Phi'(u(x))h(x)\Big)\;dx\Big|\\
&\qquad \le C\int_\Omega (1+|u(x)|^4+|h(x)|^4)|h(x)|^2\;dx\\
&\qquad \le C(1+|u|_6^4+|h|_6^4)|h|_6^2\\
&\qquad \le C(1+|u|_{V_1}^4+|h|_{V_1}^4)|h|_{V_1}^2.
\end{aligned}$$

This proves that G_1 is Fréchet differentiable and also $G_1'(u) = \Phi'(u) \in L_{6/5}(\Omega) \hookrightarrow V_1^*$. The next step is the proof of the continuity of $G_1' : V_1 \to V_1^*$. We make again use of (H1), the Hölder inequality and the Sobolev embedding theorem to obtain

$$\begin{aligned}
&|G_1'(u) - G_1'(\bar u)|_{V_1^*}\\
&\qquad \le C\left(\int_\Omega |\Phi'(u(x)) - \Phi'(\bar u(x))|^{\frac{6}{5}}\;dx\right)^{\frac{5}{6}}\\
&\qquad \le C\left(\int_\Omega\int_0^1 |\Phi''(tu(x)+(1-t)\bar u(x))|^{\frac{6}{5}}|u(x)-\bar u(x)|^{\frac{6}{5}}\;dt\;dx\right)^{\frac{5}{6}}\\
&\qquad \le C\left(\int_\Omega (1+|u(x)|^{\frac{24}{5}}+|\bar u(x)|^{\frac{24}{5}})|u(x)-\bar u(x)|^{\frac{6}{5}}\;dx\right)^{\frac{5}{6}}\\
&\qquad \le C\left(\int_\Omega (1+|u(x)|^6+|\bar u(x)|^6)\;dx\right)^{\frac{2}{3}}\left(\int_\Omega |u(x)-\bar u(x)|^6\right)^{\frac{1}{6}}\\
&\qquad \le C(1+|u|_{V_1}^4+|\bar u|_{V_1}^4)|u-\bar u|_{V_1}.
\end{aligned}$$

Actually this proves that G_1' is even locally Lipschitz continuous on V_1. The Fréchet differentiability of G_1' and the continuity of G_1'' can be proved in an analogue way. The fundamental theorem of differential calculus and the Sobolev embedding theorem yield the estimate

$$|\Phi'(u+h) - \Phi'(u) - \Phi''(u)h|_{V_1^*} \le C\left(\int_\Omega\int_0^1 |\Phi'''(u(x)+sh(x))|^{\frac{6}{5}}|h(x)|^{\frac{12}{5}}\ ds\ dx\right)^{\frac{5}{6}}.$$

We apply Assumption (H1) and Hölder's inequality to the result

$$\begin{aligned}|\Phi'(u+h) - \Phi'(u) - \Phi''(u)h|_{V_1^*} &\le C\left(\int_\Omega (1+|u(x)|^{\frac{18}{5}}+|h(x)|^{\frac{18}{5}})|h(x)|^{\frac{12}{5}}\ dx\right)^{\frac{5}{6}}\\ &\le C\left(\int_\Omega (1+|u(x)|^6+|h(x)|^6)\ dx\right)^{\frac{1}{2}}\left(\int_\Omega |h(x)|^6\ dx\right)^{\frac{1}{3}}\\ &= C(1+|u|_{V_1}^3+|h|_{V_1}^3)|h|_{V_1}^2.\end{aligned}$$

Hence the Fréchet derivative is given by the multiplication operator $G_1''(u)$ defined by $G_1''(u)v = \Phi''(u)v$ for all $v \in V_1$ and $\Phi''(u) \in L_{3/2}(\Omega)$. We will omit the proof of continuity of G_1''. The way to show the C^2-property of the functional

$$G_2(u) := \int_\Omega \lambda(u(x))\ dx, \quad u \in V_1,$$

is identical to the one above, by Assumption (H2). Concerning the C^2-differentiability of the functionals

$$G_3(v) := \int_\Omega \beta(v(x))\ dx \quad \text{and} \quad G_4(v) := \int_\Omega b(v(x))\ dx, \quad v \in V_2,$$

one may adopt the proof for G_1 and G_2. In fact, this time it is easier, since β and b are assumed to be elements of the space $C_b^{3-}(\mathbb{R})$, however one needs the assumption $r \in (n/4, 1)$. We will skip the details. Finally the product rule of differentiation yields that L is twice continuously Fréchet differentiable on $V_1 \times V_2$. □

The corresponding stationary system to (5.1) will be of importance for the forthcoming calculations. Setting all time-derivatives in (5.1) equal to 0 yields

$$\Delta\mu = 0 \quad \text{and} \quad \Delta\vartheta = 0,$$

subject to the boundary conditions $\partial_\nu\mu = \partial_\nu\vartheta = 0$. Thus we have $\mu \equiv \mu_\infty = const$, $\vartheta \equiv \vartheta_\infty = const$ and there remains the nonlinear elliptic problem of second order

$$\begin{cases}-\Delta\psi_\infty + \Phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty = \mu_\infty, & x \in \Omega,\\ \partial_\nu\psi_\infty = 0, & x \in \partial\Omega,\end{cases} \tag{5.6}$$

with the constraints (5.2) for the unknowns ψ_∞ and ϑ_∞. The following proposition collects some properties of the functional L and the ω-limit set

$$\omega(\psi,\vartheta) := \{(\varphi,\theta)\in V_1\times V_2 : \exists\ (t_n)\nearrow\infty \text{ s.t. } (\psi(t_n),\vartheta(t_n))\to(\varphi,\theta) \text{ in } V_1\times V_2\}.$$

Proposition 5.3. *Under Hypotheses (H1)-(H4) the following assertions are true.*

1. *The ω-limit set is nonempty, connected and compact.*
2. *Each point $(\psi_\infty,\vartheta_\infty)\in\omega(\psi,\vartheta)$ is a strong solution of the stationary problem* (5.6)*, where $\vartheta_\infty,\mu_\infty = const$ and $(\psi_\infty,\vartheta_\infty)$ satisfies the constraints* (5.2) *for the unknowns $\vartheta_\infty,\mu_\infty$.*
3. *The functional L is constant on $\omega(\psi,\vartheta)$ and each point $(\psi_\infty,\vartheta_\infty)\in\omega(\psi,\vartheta)$ is a critical point of L, i.e. $L'(\psi_\infty,\vartheta_\infty)=0$ in V^*.*

Proof. The fact that $\omega(\psi,\vartheta)$ is nonempty, connected and compact follows from Proposition 5.1 and some well-known facts in the theory of dynamical systems.

Now we turn to 2. Let $(\psi_\infty,\vartheta_\infty)\in\omega(\psi,\vartheta)$. Then there exists a sequence $(t_n)\nearrow+\infty$ such that $(\psi(t_n),\vartheta(t_n))\to(\psi_\infty,\vartheta_\infty)$ in V as $n\to\infty$. Since $\partial_t\psi,\partial_t\vartheta\in L_2(\mathbb{R}_+\times\Omega)$ it follows that $\psi(t_n+s)\to\psi_\infty$ and $\vartheta(t_n+s)\to\vartheta_\infty$ in $L_2(\Omega)$ for all $s\in[0,1]$ and by relative compactness also in V. This can be seen as follows.

$$\begin{aligned}|\psi(t_n+s)-\psi_\infty|_2 &\le |\psi(t_n+s)-\psi(t_n)|_2+|\psi(t_n)-\psi_\infty|_2\\ &\le \int_{t_n}^{t_n+s}|\partial_t\psi(t)|_2\,dt+|\psi(t_n)-\psi_\infty|_2\\ &\le s^{1/2}\left(\int_{t_n}^{t_n+s}|\partial_t\psi(t)|_2^2\,dt\right)^{1/2}+|\psi(t_n)-\psi_\infty|_2.\end{aligned}$$

Then, for $t_n\to\infty$ this yields $\psi(t_n+s)\to\psi_\infty$ for all $s\in[0,1]$. The proof for ϑ is the same. Integrating (4.4) with $f_1=f_2=0$ from t_n to t_n+1 we obtain

$$E(\psi(t_n+1),\vartheta(t_n+1))-E(\psi(t_n),\vartheta(t_n)) + \int_0^1\int_\Omega\left(|\nabla\mu(t_n+s,x)|^2+|\nabla\vartheta(t_n+s,x)|^2\right)\,dx\,ds=0.$$

Letting $t_n\to+\infty$ yields

$$|\nabla\mu(t_n+\cdot,\cdot)|,|\nabla\vartheta(t_n+\cdot,\cdot)|\to 0\quad\text{in } L_2([0,1]\times\Omega).$$

This in turn yields a subsequence (t_{n_k}) such that $\nabla\mu(t_{n_k}+s),\nabla\vartheta(t_{n_k}+s)\to 0$ in $L_2(\Omega;\mathbb{R}^n)$ for a.e. $s\in[0,1]$. Hence $\nabla\vartheta_\infty=0$, since the gradient is a closed operator in $L_2(\Omega;\mathbb{R}^n)$. This in turn yields that ϑ_∞ is a constant.

Furthermore the Poincaré-Wirtinger inequality implies that

$$
\begin{aligned}
&|\mu(t_{n_k}+s^*)-\mu(t_{n_l}+s^*)|_2 \\
&\le C_p\Big(|\nabla\mu(t_{n_k}+s^*)-\nabla\mu(t_{n_l}+s^*)|_2+\int_\Omega |\Phi'(\psi(t_{n_k}+s^*))-\Phi'(\psi(t_{n_l}+s^*))|\,dx \\
&\qquad +\int_\Omega |\lambda'(\psi(t_{n_k}+s^*))\vartheta(t_{n_k}+s^*)-\lambda'(\psi(t_{n_l}+s^*))\vartheta(t_{n_l}+s^*)|\,dx,
\end{aligned}
$$

for some $s^*\in[0,1]$. Taking the limit $k,l\to\infty$ we see that $\mu(t_{n_k}+s^*)$ is a Cauchy sequence in $L_2(\Omega)$, hence it admits a limit, which we denote by μ_∞. In the same manner as for ϑ_∞ we therefore obtain $\nabla\mu_\infty=0$, hence μ_∞ is a constant. Observe that the relation

$$
\mu_\infty=\frac{1}{|\Omega|}\left(\int_\Omega(\Phi'(\psi_\infty)-\lambda'(\psi_\infty)\vartheta_\infty)\,dx\right)
$$

is valid. Multiplying $(5.1)_1$ by a function $\varphi\in H_2^1(\Omega)$ and integrating by parts we obtain

$$
\begin{aligned}
&(\mu(t_{n_k}+s^*),\varphi)_2=(\nabla\psi(t_{n_k}+s^*),\nabla\varphi)_2+ \\
&\qquad (\Phi'(\psi(t_{n_k}+s^*)),\varphi)_2-(\lambda'(\psi(t_{n_k}+s^*))\vartheta(t_{n_k}+s^*),\varphi)_2.
\end{aligned}
$$

As $t_{n_k}\to\infty$ it follows that

$$
(\mu_\infty,\varphi)_2=(\nabla\psi_\infty,\nabla\varphi)_2+(\Phi'(\psi_\infty),\varphi)_2-\vartheta_\infty(\lambda'(\psi_\infty),\varphi)_2. \tag{5.7}
$$

By the Lax-Milgram theorem the bounded, symmetric and elliptic form

$$
a(u,v):=\int_\Omega \nabla u\nabla v\,dx,
$$

defined on the space $V_1\times V_1$ induces a bounded operator $A:V_1\to V_1^*$ with nonempty resolvent, such that

$$
a(u,v)=\langle Au,v\rangle_{V_1^*,V_1},
$$

for all $(u,v)\in V_1\times V_1$. It is well-known that the domain of the part A_p of the operator A in

$$
\mathbb{X}_p=\{u\in L_p(\Omega):\int_\Omega u\,dx=0\}
$$

is given by

$$
D(A_p)=\{u\in\mathbb{X}_p\cap H_p^2(\Omega),\ \partial_\nu u=0\}.
$$

Going back to (5.7) we obtain from (H1) and (H2) that $\psi_\infty\in D(A_q)$, where $q=6/(\beta+2)$. Since $q>6/5$ we may apply a bootstrap argument to conclude $\psi_\infty\in D(A_2)$. Integrating (5.7) by parts, assertion 2 follows.

In order to prove 3. , we make use of (5.3) to obtain

$$\begin{aligned}
&\langle L'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V}\\
&\qquad= \langle E'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V} - \vartheta_\infty\langle F'(\psi_\infty,\vartheta_\infty),(h,k)\rangle_{V^*,V}\\
&\qquad= \int_\Omega(-\Delta\psi_\infty+\Phi'(\psi_\infty))h\;dx + \int_\Omega\beta'(\vartheta_\infty)k\;dx\\
&\qquad\qquad\qquad - \vartheta_\infty\int_\Omega(\lambda'(\psi_\infty)h+b'(\vartheta_\infty)k)\;dx\\
&\qquad= \int_\Omega \mu_\infty h\;dx = 0,
\end{aligned}$$

for all $(h,k)\in V$, since μ_∞ and ϑ_∞ are constant. A continuity argument finally yields the last statement of the proposition.

□

The following result is crucial for the proof of convergence.

Proposition 5.4 (Lojasiewicz-Simon inequality). *Let $(\psi_\infty,\vartheta_\infty)\in\omega(\psi,\vartheta)$ and assume (H1)-(H5). Then there exist constants $s\in(0,\frac{1}{2}], C,\delta>0$ such that*

$$|L(u,v)-L(\psi_\infty,\vartheta_\infty)|^{1-s}\le C|L'(u,v)|_{V^*},$$

whenever $|(u,v)-(\psi_\infty,\vartheta_\infty)|_V\le\delta$.

Proof. We show first that $\dim N(L''(\psi_\infty,\vartheta_\infty))<\infty$. By (5.4) we obtain

$$\begin{aligned}
&\langle L''(\psi_\infty,\vartheta_\infty)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V}\\
&\qquad= \int_\Omega\nabla h_1\nabla h_2\;dx + \int_\Omega\Phi''(\psi_\infty)h_1h_2\;dx + \int_\Omega\beta''(\vartheta_\infty)k_1k_2\;dx\\
&\qquad- \overline{k_1}\int_\Omega(\lambda'(\psi_\infty)h_2+b'(\vartheta_\infty)k_2)\;dx\\
&\qquad- \overline{k_2}\int_\Omega(\lambda'(\psi_\infty)h_1+b'(\vartheta_\infty)k_1)\;dx\\
&\qquad- \overline{\vartheta_\infty}\int_\Omega(\lambda''(\psi_\infty)h_1h_2+b''(\vartheta_\infty)k_1k_2)\;dx.
\end{aligned}$$

Since $\beta''(s)=b'(s)+sb''(s)$ and $\vartheta_\infty\equiv const$ we have

$$\begin{aligned}
&\langle L''(\psi_\infty,\vartheta_\infty)(h_1,k_1),(h_2,k_2)\rangle_{V^*,V}\\
&\quad= \int_\Omega\nabla h_1\nabla h_2\;dx + \int_\Omega\left(\Phi''(\psi_\infty)h_1-\overline{k_1}\lambda'(\psi_\infty)-\vartheta_\infty\lambda''(\psi_\infty)h_1\right)h_2\;dx\\
&\quad+ \int_\Omega(b'(\vartheta_\infty)(k_1-2\overline{k_1})-\overline{\lambda'(\psi_\infty)h_1})k_2\;dx
\end{aligned}$$

for all $(h_j,k_j)\in V$. If $(h_1,k_1)\in N(L''(\psi_\infty,\vartheta_\infty))$, it follows that

$$b'(\vartheta_\infty)(k_1-2\overline{k_1})-\overline{\lambda'(\psi_\infty)h_1}=0.$$

It is obvious that a solution k_1 to this equation must be constant, hence it is given by

$$k_1 = -(b'(\vartheta_\infty))^{-1}\overline{\lambda'(\psi_\infty)h_1}, \tag{5.8}$$

where we also made use of (H4). Concerning h_1 we have

$$\langle Ah_1, h_2\rangle_{V_1^*,V_1} = \int_\Omega (k_1\lambda'(\psi_\infty) + \vartheta_\infty\lambda''(\psi_\infty)h_1 - \Phi''(\psi_\infty)h_1)h_2\, dx, \tag{5.9}$$

since k_1 is constant. By Proposition 5.3 it holds that $\psi_\infty \in D(A_2) \hookrightarrow L_\infty(\Omega)$, hence $Ah_1 \in H_1$, which means that $h_1 \in D(A_2)$ and from (5.9) we obtain

$$A_2h_1 + P(\Phi''(\psi_\infty)h_1 - \vartheta_\infty\lambda''(\psi_\infty)h_1 - k_1\lambda'(\psi_\infty)) = 0,$$

where P denotes the projection $P : H_2 \to H_1$, defined by $Pu = u - \overline{u}$. It is an easy consequence of the embedding $D(A_2) \hookrightarrow L_\infty(\Omega)$ that the linear operator $B : H_1 \to H_1$ given by

$$Bh_1 = P(\Phi''(\psi_\infty)h_1 - \vartheta_\infty\lambda''(\psi_\infty)h_1 - k_1\lambda'(\psi_\infty))$$

is bounded, where k_1 is given by (5.8). Furthermore the operator A_2 defined in the proof of Proposition 5.3 is invertible, hence $A_2^{-1}B : H_1 \to D(A_2)$ is a compact operator by compact embedding and this in turn yields that $(I + A_2^{-1}B)$ is a Fredholm operator. In particular it holds that $\dim N(I + A_2^{-1}B) < \infty$, whence $N(L''(\psi_\infty, \vartheta_\infty))$ is finite dimensional and moreover

$$N(L''(\psi_\infty, \vartheta_\infty)) \subset D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega)) \hookrightarrow L_\infty(\Omega) \times L_\infty(\Omega).$$

By Hypothesis (H5), the restriction of L' to the space $D(A_2) \times (H_2^r(\Omega) \cap L_\infty(\Omega))$ is analytic in a neighbourhood of $(\psi_\infty, \theta_\infty)$. For the definition of analyticity in Banach spaces we refer to [5, Section 3]. Now the claim follows from [5, Theorem 3.10 & Corollary 3.11].

□

Let us now state the main result of this section.

Theorem 5.5. *Assume (H1)-(H5) and let (ψ, ϑ) be a global solution of* (5.1). *Then the limits*

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty, \quad \textit{and} \quad \lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty = const$$

exist in $H_2^1(\Omega)$ and $H_2^r(\Omega)$, $r \in (0, 1)$, respectively, and $(\psi_\infty, \vartheta_\infty)$ is a strong solution of the stationary problem (5.6).

Proof. Since by Proposition 5.3 the ω-limit set is compact, we may cover it by a union of *finitely* many balls with center $(\varphi_i, \theta_i) \in \omega(\psi, \vartheta)$ and radius $\delta_i > 0$, $i = 1, \ldots, N$. Since $L(u, v) \equiv L_\infty$ on $\omega(\psi, \vartheta)$ and each (φ_i, θ_i) is a critical point of L, there are *uniform* constants $s \in (0, \frac{1}{2}]$, $C > 0$ and an open set $U \supset \omega(\psi, \vartheta)$, such that

$$|L(u, v) - L_\infty|^{1-s} \le C|L'(u, v)|_{V^*}, \tag{5.10}$$

for all $(u, v) \in U$. Define $H : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$H(t) := (L(\psi(t), \vartheta(t)) - L_\infty)^s.$$

The function H is nonincreasing and $\lim_{t\to\infty} H(t) = 0$, since $L(\psi(t),\vartheta(t)) = E(\psi(t),\vartheta(t))$ and since E is a strict Lyapunov functional for (5.1), which follows from (4.4). Furthermore we have $\lim_{t\to\infty} \mathrm{dist}((\psi(t),\vartheta(t)),\omega(\psi,\vartheta)) = 0$, i.e. there exists $t^* \geq 0$, such that $(\psi(t),\vartheta(t)) \in U$, for all $t \geq t^*$. Next, we compute and estimate the time derivative of H. By (4.4) and Proposition 5.4 we obtain

$$\begin{aligned} -\frac{d}{dt}\, H(t) &= s\left(-\frac{d}{dt}\, L(\psi(t),\vartheta(t))\right)|L(\psi(t),\vartheta(t)) - L_\infty|^{s-1} \\ &\geq C\frac{|\nabla\mu(t)|_2^2 + |\nabla\vartheta(t)|_2^2}{|L'(\psi(t),\vartheta(t))|_{V^*}} \end{aligned} \tag{5.11}$$

So have to estimate the term $|L'(\psi(t),\vartheta(t))|_{V^*}$. For convenience we will write $\psi = \psi(t)$ and $\vartheta = \vartheta(t)$. From (5.3) we obtain with $\bar{h} = 0$

$$\begin{aligned} &\langle L'(\psi,\vartheta),(h,k)\rangle_{V^*,V} \\ &\quad= \int_\Omega (-\Delta\psi + \Phi'(\psi))h\ dx + \int_\Omega \vartheta b'(\vartheta)k\ dx - \overline{\vartheta}\int_\Omega (\lambda'(\psi)h + b'(\vartheta)k)\ dx \\ &\quad= \int_\Omega (\mu - \overline{\mu})h\ dx + \int_\Omega (\vartheta - \overline{\vartheta})\lambda'(\psi)h\ dx + \int_\Omega (\vartheta - \overline{\vartheta})b'(\vartheta)k\ dx \end{aligned} \tag{5.12}$$

An application of the Hölder and Poincaré-Wirtinger inequality yields the estimates

$$|\int_\Omega (\vartheta - \overline{\vartheta})\lambda'(\psi)h\ dx| \leq |\lambda'(\psi)|_\infty |\vartheta - \overline{\vartheta}|_2 |h|_2 \leq c|\nabla\vartheta|_2|h|_2, \tag{5.13}$$

$$|\int_\Omega (\vartheta - \overline{\vartheta})b'(\vartheta)k\ dx| \leq |b'(\vartheta)|_\infty |\vartheta - \overline{\vartheta}|_2 |k|_2 \leq c|\nabla\vartheta|_2|k|_2 \tag{5.14}$$

and

$$|\int_\Omega (\mu - \overline{\mu})h\ dx| \leq c|\nabla\mu|_2|h|_2, \tag{5.15}$$

whence we obtain

$$|L'(\psi(t),\vartheta(t))|_{V^*} \leq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2),$$

by taking the supremum over all functions $(h,k) \in V$ with norm less than 1 in (5.12)-(5.15). This in connection with (5.11) yields

$$-\frac{d}{dt}H(t) \geq C(|\nabla\mu(t)|_2 + |\nabla\vartheta(t)|_2),$$

hence $|\nabla\mu|, |\nabla\vartheta| \in L_1([t^*,\infty), L_2(\Omega))$. Using the equation $\partial_t\psi = \Delta\mu$ we see that $\partial_t\psi \in L_1([t^*,\infty), H_2^1(\Omega)^*)$, hence the limit

$$\lim_{t\to\infty} \psi(t) =: \psi_\infty$$

exists in $H_2^1(\Omega)^*$ and even in $H_2^1(\Omega)$ thanks to Proposition 5.1. From equation $(5.1)_2$ it follows that $\partial_t e \in L_1([t^*,\infty); H_2^1(\Omega)^*)$, where $e := b(\vartheta) + \lambda(\psi)$, i.e. the limit $\lim_{t\to\infty} e(t)$ exists in $H_2^1(\Omega)^*$. This in turn yields that the limit

$$\lim_{t\to\infty} b(\vartheta(t)) =: b_\infty$$

exists in $L_2(\Omega)$, by relative compactness, cf. Proposition 5.1. By the monotonicity assumption (H3) we obtain $\vartheta(t) = b^{-1}(b(\vartheta(t)))$ and thus the limit of $\vartheta(t)$ as t tends to infinity exists in $L_2(\Omega)$. From the relative compactness of the orbit $\vartheta(\mathbb{R}_+)$ it follows that the limit

$$\lim_{t\to\infty} \vartheta(t) =: \vartheta_\infty$$

also exists in $H_2^r(\Omega)$, $r \in [0,1)$. Finally Proposition 5.3 yields the last statement of the theorem.

□

6. Appendix

Proof of Proposition 3.1

Let $(u,v),(\bar u,\bar v) \in \mathbb{B}_R(u^*,v^*)$. By Sobolev embedding it holds that $u,\bar u$ and $v,\bar v$ are uniformly bounded in $C^1(\overline{\Omega})$ and $C(\overline{\Omega})$, respectively. Furthermore, we will use the following inequality, which has been proven in [17, Lemma 6.2.3].

$$|f(w)-f(\bar w)|_{H_p^s(L_p)} \le \mu(T)(|w-\bar w|_{H_p^{s_0}(L_p)}+|w-\bar w|_{\infty,\infty}), \quad 0<s<s_0<1, \quad (6.1)$$

valid for every $f \in C^{2-}(\mathbb{R})$ and all $w,\bar w \in \mathbb{B}_R^1(u^*)\cup\mathbb{B}_R^2(v^*)$. Here $\mu=\mu(T)$ denotes a function, with the property $\mu(T)\to 0$ as $T\to 0$. The proof consists of several steps

(i) By Hölders inequality it holds that

$$\begin{aligned}
&|\Delta\Phi'(u)-\Delta\Phi'(\bar u)|_{X(T)}\\
&\quad\le |\Delta u\Phi''(u)-\Delta\bar u\Phi''(\bar u)|_{X(T)} + ||\nabla u|^2\Phi'''(u)-|\nabla\bar u|^2\Phi'''(\bar u)|_{X(T)}\\
&\quad\le |\Delta u|_{rp,rp}|\Phi''(u)-\Phi''(\bar u)|_{r'p,r'p} + |\Delta u-\Delta\bar u|_{rp,rp}|\Phi''(\bar u)|_{r'p,r'p}\\
&\qquad + T^{1/p}\left(|\nabla u|^2_{\infty,\infty}|\Phi'''(u)-\Phi'''(\bar u)|_{\infty,\infty} + |\nabla u-\nabla\bar u|_{\infty,\infty}|\Phi'''(\bar u)|_{\infty,\infty}\right)\\
&\quad\le T^{1/r'p}\left(|\Delta u|_{rp,rp}|\Phi''(u)-\Phi''(\bar u)|_{\infty,\infty} + |\Delta u-\Delta\bar u|_{rp,rp}|\Phi''(\bar u)|_{\infty,\infty}\right)\\
&\qquad + T^{1/p}\left(|\nabla u|^2_{\infty,\infty}|\Phi'''(u)-\Phi'''(\bar u)|_{\infty,\infty} + |\nabla u-\nabla\bar u|_{\infty,\infty}|\Phi'''(\bar u)|_{\infty,\infty}\right),
\end{aligned}$$

since $u,\bar u \in C(J;C^1(\overline{\Omega}))$. We have

$$\Delta w \in H_p^{\theta_2/2}(J;H_p^{2(1-\theta_2)}(\Omega)) \hookrightarrow L_{rp}(J\times\Omega), \quad \theta_2\in[0,1],$$

for every function $w \in E_1(T)$, since $r>1$ may be chosen close to 1. Therefore we obtain

$$|\Delta\Phi'(u)-\Delta\Phi'(\bar u)|_{X(T)} \le \mu(T)\,(R+|u^*|_1)\,|u-\bar u|_1,$$

due to the assumption $\Phi \in C^{4-}(\mathbb{R})$.

(ii) Consider the term $(\lambda'(\psi_0)-\lambda'(u))\Delta v-(\lambda'(\psi_0)-\lambda'(\bar u))\Delta\bar v$.

$$\begin{aligned}|(\lambda'(\psi_0)&-\lambda'(u))\Delta v-(\lambda'(\psi_0)-\lambda'(\bar u))\Delta\bar v|_{X(T)}\\ &\le|(\lambda'(\psi_0)-\lambda'(u))\Delta(v-\bar v)|_{X(T)}+|(\lambda'(u)-\lambda'(\bar u))\Delta\bar v|_{X(T)}\\ &\le|\psi_0-u|_{\infty,\infty}|v-\bar v|_{E_2(T)}+|u-\bar u|_{\infty,\infty}|\bar v|_{E_2(T)}\\ &\le(|\psi_0-u^*|_{\infty,\infty}+|u^*-u|_{\infty,\infty})|v-\bar v|_{E_2(T)}\\ &\quad+|u-\bar u|_{E_1(T)}(|\bar v-v^*|_{E_2(T)}+|v^*|_{E_2(T)})\\ &\le C(\mu(T)+R)|(u,v)-(\bar u,\bar v)|_1,\end{aligned}$$

since $\lambda\in C^{4-}(\mathbb{R})$. Next, we consider the term $\nabla(\lambda'(\psi_0)-\lambda'(u))\nabla v-\nabla(\lambda'(\psi_0)-\lambda'(\bar u))\nabla\bar v$. We obtain

$$\begin{aligned}|\nabla(\lambda'(\psi_0)&-\lambda'(u))\nabla v-\nabla(\lambda'(\psi_0)-\lambda'(\bar u))\nabla\bar v|_{X(T)}\\ &\le|\nabla(\lambda'(\psi_0)-\lambda'(u))|_\infty|\nabla(v-\bar v)|_{X(T)}+|\nabla(\lambda'(u)-\lambda'(\bar u))|_\infty|\nabla\bar v|_{X(T)}.\end{aligned}$$

Since

$$\nabla(\lambda'(\psi_0)-\lambda'(u))=\nabla\psi_0(\lambda''(\psi_0)-\lambda''(u))+\lambda''(u)(\nabla\psi_0-\nabla u),$$

and the same for $\nabla(\lambda'(u)-\lambda'(\bar u))$, we may argue as above, to conclude

$$\begin{aligned}|\nabla(\lambda'(\psi_0)-\lambda'(u))|_{\infty,\infty}|\nabla(v-\bar v)|_{X(T)}+|\nabla(\lambda'(u)-\lambda'(\bar u))|_{\infty,\infty}|\nabla\bar v|_{X(T)}\\ \le(\mu(T)+R)|(u,v)-(\bar u,\bar v)|_1.\end{aligned}$$

Finally, we estimate the remaining part with Hölder's inequality to the result

$$\begin{aligned}&|v\Delta(\lambda'(\psi_0)-\lambda'(u))-\bar v\Delta(\lambda'(\psi_0)-\lambda'(\bar u))|_{X(T)}\\ &\quad\le|v-\bar v|_{\infty,\infty}|\Delta(\lambda'(\psi_0)-\lambda'(u))|_{X(T)}+|\bar v|_{r'p,r'p}|\Delta(\lambda'(u)-\lambda'(\bar u))|_{rp,rp},\end{aligned}\tag{6.2}$$

where $1/r+1/r'=1$. For the first part, we obtain

$$\begin{aligned}|\Delta(\lambda'(\psi_0)&-\lambda'(u))|_{X(T)}\\ &\le|\Delta\psi_0|_p|\lambda''(\psi_0)-\lambda''(u)|_{\infty,\infty}+|\Delta\psi_0-\Delta u|_p|\lambda''(u)|_{\infty,\infty}\\ &\quad+|\nabla\psi_0|^2_{\infty,\infty}|\lambda'''(\psi_0)-\lambda'''(u)|_{\infty,\infty}+|\lambda'''(u)|_{\infty,\infty}|\nabla\psi_0-\nabla u|_{\infty,\infty}\\ &\le C(|\psi_0-u|_{\infty,\infty}+|\nabla\psi_0-\nabla u|_{\infty,\infty}+|\Delta\psi_0-\Delta u|_{p,p})\\ &\le C(\mu(T)+R),\end{aligned}$$

since $\psi_0\in H^2_p(\Omega)\cap C^1(\overline{\Omega})$ and $\lambda\in C^{4-}(\mathbb{R})$. For the second term in (6.2) we obtain

$$\begin{aligned}|\Delta(\lambda'(u)&-\lambda'(\bar u))|_{rp,rp}\\ &\le|\Delta u|_{rp,rp}|\lambda''(u)-\lambda''(\bar u)|_{\infty,\infty}+|\lambda''(\bar u)|_{\infty,\infty}|\Delta u-\Delta\bar u|_{rp,rp}\\ &\quad+|\nabla u|^2_{\infty,\infty}|\lambda'''(u)-\lambda'''(\bar u)|_{\infty,\infty}+|\lambda'''(\bar u)|_{\infty,\infty}|\nabla u-\nabla\bar u|_{\infty,\infty}\\ &\le C|u-\bar u|_{E_1(T)},\end{aligned}$$

since $u, \bar{u} \in C(J; C^1(\overline{\Omega}))$ and $r > 1$ can be chosen close enough to 1, due to the fact that $\bar{v} \in C(J; C(\overline{\Omega}))$. Finally, we observe

$$|\bar{v}|_{r'p,r'p} \leq |\bar{v} - v^*|_{r'p,r'p} + |v^*|_{r'p,r'p} \leq \mu(T) + R.$$

(iii) For simplicity we set $f(u, v) = a_0\lambda'(\psi_0) - a(v)\lambda'(u)$. Then we compute

$$\begin{aligned}
&|f(u, v)\partial_t u - f(\bar{u}, \bar{v})\partial_t \bar{u}|_{X(T)} \\
&\quad \leq |\partial_t u(f(u, v) - f(\bar{u}, \bar{v}))|_{X(T)} + |f(\bar{u}, \bar{v})(\partial_t u - \partial_t \bar{u})|_{X(T)} \qquad (6.3)\\
&\quad \leq (|\partial_t u - \partial_t u^*|_{X(T)} + |\partial_t u^*|_{X(T)})|f(u, v) - f(\bar{u}, \bar{v})|_{\infty,\infty} \\
&\qquad\qquad + |f(\bar{u}, \bar{v})|_{\infty,\infty}|\partial_t u - \partial_t \bar{u}|_{X(T)} \\
&\quad \leq C(\mu_3(T) + R)|f(u, v) - f(\bar{u}, \bar{v})|_{\infty,\infty} \\
&\qquad\qquad + |f(\bar{u}, \bar{v})|_{\infty,\infty}|\partial_t u - \partial_t \bar{u}|_{X(T)}.
\end{aligned}$$

Next we estimate

$$\begin{aligned}
&|f(u, v) - f(\bar{u}, \bar{v})|_{\infty,\infty} \\
&\quad \leq |a(v)(\lambda'(u) - \lambda'(\bar{u}))|_{\infty,\infty} + |\lambda'(\bar{u})(a(v) - a(\bar{v}))|_{\infty,\infty} \\
&\quad \leq |a(v)|_{\infty,\infty}|\lambda'(u) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a(v) - a(\bar{v})|_{\infty,\infty} \\
&\quad \leq C(|u - \bar{u}|_{\infty,\infty} + |v - \bar{v}|_{\infty,\infty}) \leq C|(u, v) - (\bar{u}, \bar{v})|_1.
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
|f(\bar{u}, \bar{v})|_{\infty,\infty} &\leq |a_0|_{\infty,\infty}|\lambda'(\psi_0) - \lambda'(\bar{u})|_{\infty,\infty} + |\lambda'(\bar{u})|_{\infty,\infty}|a_0 - a(\bar{v})|_{\infty,\infty} \\
&\leq C(|\psi_0 - \bar{u}|_{\infty,\infty} + |\vartheta_0 - \bar{v}|_{\infty,\infty}) \\
&\leq C(|\psi_0 - u^*|_{\infty,\infty} + |u^* - \bar{u}|_{\infty,\infty} + |\vartheta_0 - v^*|_{\infty,\infty} + |v^* - \bar{v}|_{\infty,\infty}) \\
&\leq C(\mu(T) + R).
\end{aligned}$$

The estimate of $(a_0 - a(v))\Delta v - (a_0 - a(\bar{v}))\Delta\bar{v}$ in $L_p(J; L_p(\Omega))$ can be carried out in a similar way.

(iv) We compute

$$\begin{aligned}
|(a(v) - a(\bar{v})f_2|_{X(T)} &\leq |a(v) - a(\bar{v})|_{\infty,\infty}|f_2|_{X(T)} \leq |v - \bar{v}|_{\infty,\infty}|f_2|_{X(T)} \\
&\leq \mu(T)|v - \bar{v}|_{E_2(T)} \leq \mu(T)|(u, v) - (\bar{u}, \bar{v})|_1,
\end{aligned}$$

since $f_2 \in X(T)$ is a fixed function, hence $|f_2|_{X(T)} \to 0$ as $T \to 0$.

(v) By trace theory, we obtain

$$\begin{aligned}
&|\partial_\nu(\Phi'(u) - \Phi'(\bar{u}))|_{Y_1(T)} \\
&\quad \leq C|\Phi'(u) - \Phi'(\bar{u})|_{H_p^{1/2}(J;L_p(\Omega))} + |\Phi'(u) - \Phi'(\bar{u})|_{L_p(J;H_p^2(\Omega))}.
\end{aligned}$$

The second norm has already been estimated in (i), so it remains to estimate $\Phi'(u) - \Phi'(\bar{u})$ in $H_p^{1/2}(J; L_p(\Omega))$. Here we will use (6.1), to obtain

$$\begin{aligned}
|\Phi'(u) - \Phi'(\bar{u})|_{H_p^{1/2}(L_p)} &\leq \mu(T)(|u - \bar{u}|_{H_p^{s_0}(L_p)} + |u - \bar{u}|_{\infty,\infty}) \\
&\leq \mu(T)C|u - \bar{u}|_{E_1(T)} \leq \mu(T)C|(u, v) - (\bar{u}, \bar{v})|_1,
\end{aligned}$$

since $s_0 < 1$.

(vi) We may apply (ii) and trace theory, to conclude that it suffices to estimate

$$(\lambda'(\psi_0) - \lambda'(u))v - (\lambda'(\psi_0) - \lambda'(\bar{u}))\bar{v} = (\lambda'(\psi_0) - \lambda'(u))(v - \bar{v}) - (\lambda'(u) - \lambda'(\bar{u}))\bar{v}$$

in $H_p^{1/2}(J; L_p(\Omega))$. This yields

$$\begin{aligned}
&|(\lambda'(\psi_0) - \lambda'(u))(v - \bar{v})|_{H_p^{1/2}(L_p)} \\
&\quad \le |\lambda'(\psi_0) - \lambda'(u)|_{H_p^{1/2}(L_p)} |v - \bar{v}|_{\infty,\infty} + |\lambda'(\psi_0) - \lambda'(u)|_{\infty,\infty} |v - \bar{v}|_{H_p^{1/2}(L_p)} \\
&\quad \le (|\lambda'(\psi_0) - \lambda'(u^*)|_{H_p^{1/2}(L_p)} + |\lambda'(u^*) - \lambda'(u)|_{H_p^{1/2}(L_p)}) |v - \bar{v}|_{E_2(T)} \\
&\qquad + (|\psi_0 - u^*|_{\infty,\infty} + |u^* - u|_{\infty,\infty}) |v - \bar{v}|_{E_2(T)} \\
&\quad \le \Big(|\lambda'(\psi_0) - \lambda'(u^*)|_{H_p^{1/2}(L_p)} + \mu(T)R + (\mu(T) + R) \Big) |v - \bar{v}|_{E_2(T)}.
\end{aligned}$$

Clearly $\lambda'(\psi_0) - \lambda'(u^*) \in {}_0H_p^{1/2}(J; L_p(\Omega))$, since ψ_0 does not depend on t and since $\lambda \in C^{4-}(\mathbb{R})$. Therefore it holds that

$$|\lambda'(\psi_0) - \lambda'(u^*)|_{H_p^{1/2}(L_p)} \to 0$$

as $T \to 0$. The second part $(\lambda'(u) - \lambda'(\bar{u}))\bar{v}$ can be treated as follows.

$$\begin{aligned}
&|(\lambda'(u) - \lambda'(\bar{u}))\bar{v}|_{H_p^{1/2}(L_p)} \\
&\quad \le |\lambda'(u) - \lambda'(\bar{u})|_{H_p^{1/2}(L_p)} |\bar{v}|_{\infty,\infty} + |\lambda'(u) - \lambda'(\bar{u})|_{\infty,\infty} |\bar{v}|_{H_p^{1/2}(L_p)} \\
&\quad \le C(\mu(T) + R + \mu(T)) |u - \bar{u}|_{E_1(T)},
\end{aligned}$$

where we applied again (6.1). This completes the proof of the proposition.

Proof of Proposition 4.1
Let $J_{\max}^\delta := [\delta, T_{\max}]$ for some small $\delta > 0$. Setting $A^2 = \Delta_N^2$ with domain

$$D(A^2) = \{u \in H_p^4(\Omega) : \partial_\nu u = \partial_\nu \Delta u = 0 \text{ on } \partial\Omega\},$$

the solution $\psi(t)$ of equation $(4.1)_1$ may be represented by the variation of parameters formula

$$\psi(t) = e^{-A^2 t}\psi_0 + \int_0^t A e^{-A^2(t-s)} \Big(\lambda'(\psi(s))\vartheta(s) - \Phi'(\psi(s)) \Big)\, ds, \quad t \in J_{\max}, \tag{6.4}$$

where $e^{-A^2 t}$ denotes the analytic semigroup, generated by $-A^2 = -\Delta_N^2$ in $L_p(\Omega)$. By (H1), (H2) and (4.6) it holds that

$$\Phi'(\psi) \in L_\infty(J_{\max}; L_{q_0}(\Omega)) \quad \text{and} \quad \lambda'(\psi) \in L_\infty(J_{\max}; L_6(\Omega)),$$

with $q_0 = 6/(\gamma + 2)$. We then apply A^r, $r \in (0, 1)$, to (6.4) and make use of semigroup theory to obtain

$$\psi \in L_\infty(J_{\max}^\delta; H_{q_0}^{2r}(\Omega)), \tag{6.5}$$

valid for all $r \in (0,1)$, since $q_0 < 6$. It follows from (6.5) that $\psi \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega))$ if $2r - 3/q_0 \geq -3/p_1$, and

$$\Phi'(\psi) \in L_\infty(J^\delta_{\max}; L_{q_1}(\Omega)) \quad \text{as well as} \quad \lambda'(\psi) \in L_\infty(J^\delta_{\max}; L_{p_1}(\Omega)),$$

with $q_1 = p_1/(\gamma+2)$. Hence we have this time

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_{q_1}(\Omega)), \quad r \in (0,1).$$

Iteratively we obtain a sequence $(p_n)_{n\in\mathbb{N}_0}$ such that

$$2r - \frac{3}{q_n} \geq -\frac{3}{p_{n+1}}, \quad n \in \mathbb{N}_0$$

with $q_n = p_n/(\gamma+2)$ and $p_0 = 6$. Thus the sequence $(p_n)_{n\in\mathbb{N}_0}$ may be recursively estimated by

$$\frac{1}{p_{n+1}} \geq \frac{\gamma+2}{p_n} - \frac{2r}{3},$$

for all $n \in \mathbb{N}_0$ and $r \in (0,1)$. From this definition it is not difficult to obtain the following estimate for $1/p_{n+1}$.

$$\begin{aligned}
\frac{1}{p_{n+1}} &\geq \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\sum_{k=0}^{n}(\gamma+2)^k \\
&= \frac{(\gamma+2)^{n+1}}{p_0} - \frac{2r}{3}\left(\frac{(\gamma+2)^{n+1}-1}{\gamma_1+1}\right) \\
&= (\gamma+2)^{n+1}\left(\frac{1}{p_0} - \frac{2r}{3\gamma+3}\right) + \frac{2r}{3\gamma+3}, \quad n \in \mathbb{N}_0.
\end{aligned} \tag{6.6}$$

By the assumption (H1) on γ we see that the term in brackets is negative if $r \in (0,1)$ is sufficiently close to 1 and therefore, after finitely many steps the entire right side of (6.6) is negative as well, whence we may choose p_n arbitrarily large or we may even set $p_n = \infty$ for $n \geq N$ and a certain $N \in \mathbb{N}_0$. In other words this means that for those $r \in (0,1)$ we have

$$\psi \in L_\infty(J^\delta_{\max}; H^{2r}_p(\Omega)), \tag{6.7}$$

for all $p \in [1,\infty]$. It is important, that we can achieve this result in *finitely* many steps!

Next we will derive an estimate for $\partial_t\psi$. For all forthcoming calculations we will use the abbreviation $\psi = \psi(t)$ and $\vartheta = \vartheta(t)$. Since we only have estimates on the interval $J^\delta_{\max}$, we will use the following solution formula.

$$\psi(t) = e^{-A^2(t-\delta)}\psi_\delta + \int_0^{t-\delta} Ae^{-A^2 s}\Big(\lambda'(\psi)\vartheta - \Phi'(\psi)\Big)(t-s)\, ds, \quad t \in J^\delta_{\max}$$

where $\psi_\delta := \psi(\delta)$. Differentiating with respect to t, we obtain

$$\begin{aligned}
\partial_t\psi(t) = A\int_0^{t-\delta} e^{-A^2 s}(\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi)(t-s)\, ds \\
+ F(t,\psi_\delta,\vartheta_\delta),
\end{aligned} \tag{6.8}$$

for all $t \geq \delta$ and with

$$F(t,\psi_\delta,\vartheta_\delta) := Ae^{-A^2(t-\delta)}(\lambda'(\psi_\delta)\vartheta_\delta - \Phi'(\psi_\delta)) - A^2e^{-A^2(t-\delta)}\psi_\delta.$$

Let us discuss the function F in detail. By the trace theorem we have $\psi_\delta \in B^{4-4/p}_{pp}(\Omega)$ and $\vartheta_\delta \in B^{2-2/p}_{pp}(\Omega)$. Since we assume $p > (n+2)/2$, it holds that $\psi_\delta, \vartheta_\delta \in L_\infty(\Omega)$. Furthermore, the semigroup e^{-A^2t} is analytic. Therefore there exist some constants $C > 0$ and $\omega \in \mathbb{R}$ such that

$$|F(t,\psi_\delta,\vartheta_\delta)|_{L_p(\Omega)} \leq C\left(\frac{1}{(t-\delta)^{1/2}} + \frac{1}{t-\delta}\right)e^{\omega t},$$

for all $t > \delta$. This in turn implies that

$$F(\cdot,\psi_\delta,\vartheta_\delta) \in L_p(J^{\delta'}_{\max} \times \Omega)$$

for all $p \in (1,\infty)$, where $0 < \delta < \delta' < T_{\max}$. We will now use equations $(5.1)_{1,2}$ to rewrite the integrand in (6.8) in the following way.

$$\begin{aligned}
&(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta \\
&\quad = (\lambda''(\psi)\vartheta - \Phi''(\psi))\Delta\mu + \frac{\lambda'(\psi)}{b'(\vartheta)}\Delta\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\Delta\mu \\
&\quad = \operatorname{div}\left[\left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi)\right)\nabla\mu\right] + \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)}\nabla\vartheta\right] \\
&\quad - \nabla\left(\lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi)\right)\cdot\nabla\mu - \nabla\frac{\lambda'(\psi)}{b'(\vartheta)}\cdot\nabla\vartheta.
\end{aligned} \tag{6.9}$$

Thus we obtain a decomposition of the following form

$$\begin{aligned}
&(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta \\
&\qquad = \operatorname{div}(f_\mu\nabla\mu + f_\vartheta\nabla\vartheta) + g_\mu\nabla\mu + g_\vartheta\nabla\vartheta + h_\mu\nabla\vartheta\nabla\mu + h_\vartheta|\nabla\vartheta|^2,
\end{aligned}$$

with

$$\begin{aligned}
f_\mu &:= \lambda''(\psi)\vartheta - \frac{\lambda'(\psi)^2}{b'(\vartheta)} - \Phi''(\psi), \quad f_\vartheta := \frac{\lambda'(\psi)}{b'(\vartheta)}, \\
g_\mu &:= -\left(\lambda'''(\psi)\vartheta - 2\frac{\lambda'(\psi)\lambda''(\psi)}{b'(\vartheta)} - \Phi''(\psi)\right)\nabla\psi, \quad g_\vartheta := -\frac{\lambda''(\psi)}{b'(\vartheta)}\nabla\psi, \\
h_\mu &:= \lambda''(\psi) - \frac{b''(\vartheta)\lambda'(\psi)^2}{b'(\vartheta)^2}, \quad h_\vartheta := \frac{b''(\vartheta)\lambda'(\psi)}{b'(\vartheta)^2}.
\end{aligned}$$

By Assumption (H3) and the first part of the proof it holds that $f_j, g_j, h_j \in L_\infty(J^\delta_{\max} \times \Omega)$ for each $j \in \{\mu,\vartheta\}$ and this in turn yields that

$$\begin{aligned}
\operatorname{div}(f_\mu\nabla\mu + f_\vartheta\nabla\vartheta) &\in L_2(J^\delta_{\max}; H^1_2(\Omega)^*), \\
g_\mu\cdot\nabla\mu + g_\vartheta\cdot\nabla\vartheta &\in L_2(J^\delta_{\max} \times \Omega), \\
h_\mu\nabla\vartheta\cdot\nabla\mu + h_\vartheta|\nabla\vartheta|^2 &\in L_1(J^\delta_{\max} \times \Omega),
\end{aligned}$$

where we also made use of (4.6). Setting

$$T_1 = Ae^{-A^2t} * \operatorname{div}(f_\mu \nabla\mu + f_\vartheta \nabla\vartheta), \quad T_2 = Ae^{-A^2t} * (g_\mu \cdot \nabla\mu + g_\vartheta \cdot \nabla\vartheta)$$

and

$$T_3 = Ae^{-A^2t} * (h_\mu \nabla\vartheta \cdot \nabla\mu + h_\vartheta |\nabla\vartheta|^2),$$

we may rewrite (6.8) as

$$\partial_t\psi = T_1 + T_2 + T_3 + F(t, \psi_0, \vartheta_0).$$

Going back to (6.8) we obtain

$$\begin{aligned}
&T_1 \in H_2^{1/4}(J^\delta_{\max}; L_2(\Omega)) \cap L_2(J^\delta_{\max}; H_2^1(\Omega)) \hookrightarrow L_2(J^\delta_{\max} \times \Omega),\\
&T_2 \in H_2^{1/2}(J^\delta_{\max}; L_2(\Omega)) \cap L_2(J^\delta_{\max}; H_2^2(\Omega)) \hookrightarrow L_2(J^\delta_{\max} \times \Omega), \quad \text{and}\\
&F(\cdot, \psi_\delta, \vartheta_\delta) \in L_2(J^{\delta'}_{\max} \times \Omega).
\end{aligned}$$

Observe that we do not have full regularity for T_3 since A has no maximal regularity in $L_1(\Omega)$, but nevertheless we obtain

$$T_3 \in H_1^{1/2-}(J^\delta_{\max}; L_1(\Omega)) \cap L_1(J^\delta_{\max}; H_1^{2-}(\Omega)).$$

Here we used the notation $H_p^{s-} := H_p^{s-\varepsilon}$ and $\varepsilon > 0$ is sufficiently small. An application of the mixed derivative theorem then yields

$$H_1^{1/2-}(J^\delta_{\max}; L_1(\Omega)) \cap L_1(J^\delta_{\max}; H_1^{2-}(\Omega)) \hookrightarrow L_p(J^\delta_{\max}; L_2(\Omega)),$$

if $p \in (1, 8/7)$, whence

$$\partial_t\psi \in L_2(J^{\delta'}_{\max} \times \Omega) + L_p(J^{\delta'}_{\max}; L_2(\Omega))$$

for some $1 < p < 8/7$. Now we go back to (6.9) where we replace this time only $\partial_t\vartheta$ by the differential equation $(5.1)_2$ to obtain

$$\begin{aligned}
&(\lambda''(\psi)\vartheta - \Phi''(\psi))\partial_t\psi + \lambda'(\psi)\partial_t\vartheta\\
&\quad= \left(\lambda''(\psi)\vartheta - \Phi''(\psi) - \frac{\lambda'(\psi)^2}{b'(\vartheta)}\right)\partial_t\psi\\
&\quad+ \operatorname{div}\left[\frac{\lambda'(\psi)}{b'(\vartheta)}\nabla\vartheta\right] - \frac{\lambda''(\psi)}{b'(\vartheta)}\nabla\psi \cdot \nabla\vartheta + \frac{\lambda'(\psi)b''(\vartheta)}{b'(\vartheta)^2}|\nabla\vartheta|^2\\
&\quad= f\partial_t\psi + \operatorname{div}[g\nabla\vartheta] + h \cdot \nabla\vartheta + k|\nabla\vartheta|^2.
\end{aligned}$$

Rewrite (6.8) in the following way

$$\partial_t\psi = S_1 + S_2 + S_3 + S_4 + F(t, \psi_0, \vartheta_0), \tag{6.10}$$

where the functions S_j are defined in the same manner as T_j. Since $f, g, h \in L_\infty(J^\delta_{\max} \times \Omega)$ it follows again from regularity theory that

$$\begin{aligned}
S_1 \in\, &H_2^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^2(\Omega))\\
&\qquad + H_p^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_p(J^{\delta'}_{\max}; H_2^2(\Omega)),\\
S_2 \in\, &H_2^{1/4}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^1(\Omega)),
\end{aligned}$$

$$S_3 \in H_2^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_2(J^{\delta'}_{\max}; H_2^2(\Omega)),$$

and it can be readily verified that

$$H_p^{1/2}(J^{\delta'}_{\max}; L_2(\Omega)) \cap L_p(J^{\delta'}_{\max}; H_2^2(\Omega)) \hookrightarrow L_2(J^{\delta'}_{\max} \times \Omega),$$

whenever $p \in [1,2]$. Now we turn our attention to the term $S_4 = Ae^{-A^2 t} * k|\nabla\vartheta|^2$. First we observe that by the mixed derivative theorem the embedding

$$Z_q := H_q^{1/2-}(J^{\delta'}_{\max}; L_1(\Omega)) \cap L_q(J^{\delta'}_{\max}; H_1^{2-}(\Omega)) \hookrightarrow L_2(J^{\delta'}_{\max} \times \Omega)$$

is valid, provided that $q \in (8/5, 2]$. Hence it holds that

$$|S_4|_{2,2} \leq C|S_4|_{Z_q} \leq C|k|\nabla\vartheta|^2|_{q,1} \leq C|\nabla\vartheta|^2_{2q,2},$$

with some constant $C > 0$. Taking the norm of $\partial_t\psi$ in $L_2(J^{\delta'}_{\max} \times \Omega)$ we obtain from (6.10)

$$|\partial_t\psi|_{2,2} \leq C\left(\sum_{j=1}^{3} |S_j|_{2,2} + |\nabla\vartheta|^2_{2q,2} + |F(\cdot,\psi_\delta,\vartheta_\delta)|_{2,2}\right).$$

The Gagliardo-Nirenberg inequality in connection with (4.6) yields the estimate

$$|\nabla\vartheta|^2_{2q,2} \leq c|\nabla\vartheta|^{2a}_{2,2}|\nabla\vartheta|^{2(1-a)}_{\infty,2} \leq c|\nabla\vartheta|^{2(1-a)}_{\infty,2},$$

provided that $a = 1/q$. Multiply $(4.1)_2$ by $\partial_t\vartheta$ and integrate by parts to the result

$$\int_\Omega b'(\vartheta(t,x))|\partial_t\vartheta(t,x)|^2\, dx + \frac{1}{2}\frac{d}{dt}|\nabla\vartheta(t)|_2^2 = -\int_\Omega \lambda'(\psi(t,x))\partial_t\psi(t,x)\partial_t\vartheta(t,x)\, dx.$$

Making use of (H3) and Young's inequality we obtain

$$C_1|\partial_t\vartheta|^2_{2,2} + \frac{1}{2}|\nabla\vartheta(t)|_2^2 \leq C_2(|\partial_t\psi|^2_{2,2} + |\nabla\vartheta_0|_2^2), \tag{6.11}$$

after integrating w.r.t. t. This in turn yields the estimate

$$|\nabla\vartheta|^2_{2q,2} \leq c|\nabla\vartheta|^{2(1-a)}_{\infty,2} \leq c(1 + |\partial_t\psi|^{2(1-a)}_{2,2}).$$

In order to gain something from this inequality we require that $2(1-a) < 1$, i.e. q is restricted by $1 < q < 2$. Finally, if we choose $q \in (8/5, 2)$ and use the uniform boundedness of the L_2 norms of S_j, $j \in \{1,2,3\}$ we obtain

$$|\partial_t\psi|_{2,2} \leq C(1 + |\partial_t\psi|^{2(1-a)}_{2,2}).$$

Since by construction $2(1-a) < 1$, it follows that the L_2-norm of $\partial_t\psi$ is bounded on $J^{\delta'}_{\max} \times \Omega$. In particular, this yields the statement for ϑ by equation (6.11).

Now we go back to (6.8) with δ replaced by δ'. By Assumption (H5), by the bounds $\partial_t\vartheta, \partial_t\psi \in L_2(J^{\delta'}_{\max}; L_2(\Omega))$ and by the first part of the proof we obtain

$$\lambda''(\psi)\vartheta\partial_t\psi + \lambda'(\psi)\partial_t\vartheta - \Phi''(\psi)\partial_t\psi \in L_2(J^{\delta'}_{\max}; L_2(\Omega)).$$

Since the operator $A^2 = \Delta^2$ with domain

$$D(A^2) = \{u \in H_p^4(\Omega) : \partial_\nu u = \partial_\nu\Delta u = 0\}$$

has the property of maximal L_p-regularity (cf. [6, Theorem 2.1]), we obtain from (6.8)

$$\partial_t\psi - F(\cdot,\psi_{\delta'},\vartheta_{\delta'}) \in H_2^{1/2}(J_{\max}^{\delta'};L_2(\Omega)) \cap L_2(J_{\max}^{\delta'};H_2^2(\Omega)) \hookrightarrow L_r(J_{\max}^{\delta'};L_r(\Omega)),$$

and the last embedding is valid for all $r \le 2(n+4)/n$. By the properties of the function F it follows

$$\partial_t\psi \in L_r(J_{\max}^{\delta''};L_r(\Omega)),$$

for all $r \le 2(n+4)/n$ and some $0 < \delta'' < T_{\max}$. To obtain an estimate for the whole interval $J_{\max}$, we use the fact that we already have a local strong solution, i.e. $\partial_t\psi \in L_p(0,\delta'';L_p(\Omega))$, $p > (n+2)/2$. The proof is complete.

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