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Marius Durea
Faculty of Mathematics
"Al. I. Cuza" University
Bd. Carol I, nr. 11
700506-Iaşi, Romania
Email: durea@uaic.ro

Joydeep Dutta
Department of Mathematics and Statistics
Indian Institute of Technology
Kanpur-208016, India
Email: jdutta@iitk.ac.in

Christiane Tammer
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät III
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany
Email: christiane.tammer@mathematik.uni-halle.de

Stability properties of KKT points in vector optimization

M. Durea* J. Dutta† Chr. Tammer‡

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Abstract: In this paper we introduce the notion of approximate KKT points for smooth, convex and nonsmooth, nonconvex vector optimization problems. We study a kind of stability of these points and KKT points of vector optimization problems. In the convex case we also introduce and study the notion of modified approximate KKT points motivated by Ekeland's variational principle. We prove stability properties of these points for several optimization problems.

Key words: set-valued optimization · Lagrange multipliers · (approximate) KKT points · coderivatives of set-valued mappings

Mathematics Subject Classification (2000): 90C29 · 90C26 · 49J52

1 Introduction

The aim of this paper is to introduce the notions of KKT points and approximate KKT points in nonlinear (smooth, convex and nonsmooth) vector optimization and to study a kind of stability of these points. Our work is motivated by the following ideas. In general, the exact solutions of concrete constrained optimization problems are difficult to find. The classical approach to this kind of problems is the Lagrange multiplier method whose roots can be traced back to Lagrange's work on calculus of variations in 18th century. In optimization theory, this method was considered for the first time in the paper by John [12] and, subsequently, in the paper by Kuhn and Tucker [13]. The method and conditions used by Kuhn and Tucker in their paper in 1951 lead to the seek of an important category of points as candidates for solutions. Later, it was discovered that similar results were presented by Karush back in 1939. Therefore, in the case of the scalar problems these points are called Karush-Kuhn-Tucker (KKT for short) points and it is well known that, under (generalized) convexity assumptions, these points turn to be solutions for the initial problem. Many algorithms in scalar optimization are looking for KKT points which are quite difficult to find

*Faculty of Mathematics, "Al. I. Cuza" University, Bd. Carol I, nr. 11, 700506-Iaşi, Romania, e-mail: durea@uaic.ro. The research of this author was supported by CNCSIS Grant Code 379/2007

†Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, Kanpur-208016, India, e-mail: jdutta@iitk.ac.in

‡Martin-Luther-Universität Halle-Wittenberg, Fachbereich Mathematik und Informatik, Institut für Optimierung und Stochastik, D-06099 Halle (Saale), Germany, e-mail: christiane.tammer@mathematik.uni-halle.de

exactly and therefore one uses as a stopping criterion an approximate fulfillment of optimality conditions. This gives rise to several so-called "sequential optimality conditions", which in particular deal with approximate KKT points (see [1]).

In [6] we have presented several results concerning the Lagrange multiplier method for vector single-valued or set-valued optimization problems. In particular, in the aforementioned paper, we have shown boundedness properties for the sets of Lagrange multipliers. In the present paper, making use of a similar technique, we develop several results concerning stability properties of KKT points for general vector optimization problems having the meaning explained above: Even if, numerically, it is difficult to find exact KKT points, then, if we find approximate KKT points they are not far from a genuine KKT point. We see this theoretical paper as a first step in developing numerical algorithms for the reference problems.

The paper is organized, in the sequel, into three main sections relatively independent: The first one deals with smooth vector optimization problems, the second one with cone-convex vector optimization problems, while the third one concerns the general situation of set-valued problems. Moreover, the settings of these sections are somehow different, because in the second and third sections we work on finite dimensional spaces, mainly in order to avoid technical complications and to emphasize the main ideas better. We show in all these three sections the stability of approximate KKT points in various situations.

2 The smooth case

In this section we consider the case of vectorial problems under inequalities and equalities constraints with smooth data. Let X, Y, Z be normed vector spaces and let $K \subset Y$ and $Q \subset Z$ be some proper, closed, convex and pointed cones which introduce partial order relations on Y and Z, respectively. The topological dual of Y is denoted by Y^*, while w^* denotes the weak-star topology on Y^*. We need to mention that $||\cdot||$ denotes the norm on the current working space and $||\cdot||_*$ the corresponding dual norm. The dual cone of K is the cone $K^* := \{y^* \in Y^* \mid y^*(k) \geq 0, \forall k \in Y\}$.

We recall now some basic facts concerning the set-valued maps. Let $H : X \rightrightarrows Y$ be a set-valued map. As usual, the graph of H is $\operatorname{Gr} H = \{(x, y) \in X \times Y \mid y \in H(x)\}$. If $A \subset X$, $H(A) := \bigcup_{x \in A} H(x)$ and the inverse set-valued map of H is $H^{-1} : Y \rightrightarrows X$ given by $(y, x) \in \operatorname{Gr} H^{-1}$ if and only if $(x, y) \in \operatorname{Gr} H$. The upper limit of H at a point $\overline{x} \in X$ is defined by:

$$\operatorname*{Limsup}_{x \to \overline{x}} H(x) = \{y \in Y \mid \exists x_n \to \overline{x}, \exists y_n \to y, y_n \in H(x_n) \text{ for all } n \in \mathbb{N}\}.$$

One says that H is upper semicontinuous (usc, for short) at $\overline{x}$ if $\operatorname{Limsup}_{x \to \overline{x}} H(x) \subset H(\overline{x})$.

Let $f : X \to Y$ and $g : X \to Z$ be two functions. In this section, we are mainly interested in the study of the following nonlinear optimization problem with functional constraints:

$$(\mathcal{P}_1) \qquad \text{minimize } f(x) \text{ s.t. } g(x) \in -Q.$$

We denote by M the set of feasible points of the problem $(\mathcal{P}_1)$, i.e., $M := \{x \in X \mid g(x) \in -Q\}$. Throughout this section, in various situations, we assume the following assumptions:

(A_K) $\operatorname{int} K \neq \emptyset$;

(A_Q) $\operatorname{int} Q \neq \emptyset$;

(A_f) the function f is of class C^1 on X;

(A_g) the function g is of class C^1 on X;

(A_Y) the closed unit ball of Y^* is w^* sequentially compact;

(A_Z) the closed unit ball of Z^* is w^* sequentially compact.

Note that (A_Y) holds if Y is a weakly compactly generated space, in particular, if Y is a reflexive Banach space or a separable Banach space (see [4, pp. 148]).

Accordingly, assuming (A_K), one can speak about weak solutions: A point $\overline{x} \in M$ is called weakly minimal solution of the problem $(\mathcal{P}_1)$, if for every $x \in M$

$$f(x) - f(\overline{x}) \notin -\operatorname{int} K.$$

Assuming (A_Q), one says that the Mangasarian-Fromovitz condition ((MF) for short) holds at $\overline{x}$ for $(\mathcal{P}_1)$ if there is an element $\overline{u} \in X$ with $\nabla g(\overline{x})(\overline{u}) \in -\operatorname{int} Q$. It is proved in [6] that (MF) is equivalent to

$$z^* \in Q^*, z^* \circ \nabla g(\overline{x}) = 0 \Rightarrow z^* = 0.$$

In the above conditions and notations one has the following result concerning the existence of the Lagrange multipliers.

Proposition 2.1 *[6, Proposition 3.4] Assume $(A_K), (A_Q)$, (A_f), (A_g). Let $\overline{x} \in M$ be a weakly minimal point for the problem $(\mathcal{P}_1)$ s.t. the condition (MF) holds at $\overline{x}$. Then there exist $y^* \in K^* \setminus \{0\}$ and $z^* \in Q^*$ s.t.*

$$y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}) = 0. \tag{1}$$

Take an element $y^* \in K^* \setminus \{0\}$ and consider the set $L_{y^*} := \{z^* \in Q^* \mid (1) \text{ holds}\}$. Of course, y^* can be chosen such that $||y^*||_* = 1$.

Theorem 2.2 *[6, Theorem 3.1] Assume (A_Z), $(A_K), (A_Q)$, (A_f), (A_g). Let $y^* \in K^* \setminus \{0\}$. If the Mangasarian-Fromovitz condition (MF) holds, then the set L_{y^*} is (norm) bounded. Conversely, if L_{y^*} is non-empty and bounded, then (MF) holds.*

In this paper we introduce the notion of $\varepsilon - KKT$ point as follows.

Definition 2.3 *Assume (A_f), (A_g). Let $\overline{x} \in M$ and $\varepsilon \geq 0$. One says that $\overline{x}$ is an $\varepsilon - KKT$ point for $(\mathcal{P}_1)$ if there exist $y^* \in K^*, \|y^*\|_* = 1$ and $z^* \in Q^*$ s.t.*

$$\|y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x})\|_* \leq \varepsilon.$$

It is natural to use, from now on, the term "KKT point" for a $0-KKT$ point. As one can see, in the assumptions of Proposition 2.1, a weakly minimal point of $(\mathcal{P}_1)$ is a KKT point provided that (MF) holds. Another observation is that under (MF) condition, if there exists $0 \neq z^* \in L_{y^*}$ then $0 \notin L_{y^*}$. Indeed, otherwise we would have $y^* \circ \nabla f(\overline{x}) = 0$, whence $z^* \circ \nabla g(\overline{x}) = 0$, in contradiction with (MF).

Let us consider the set valued map T which assigns to every $\varepsilon \in \mathbb{R}_+ := [0, +\infty)$ the subset of X of all $\varepsilon - KKT$ points of $(\mathcal{P}_1)$. For $\varepsilon < 0$ we put $T(\varepsilon) = \emptyset$. The main result of this section reads as follows.

Theorem 2.4 *Assume* (A_Y), (A_Z), $(A_K), (A_Q)$, (A_f), (A_g). *Suppose that* (MF) *holds at* $\overline{x} \in X$.

(i) If $\overline{x} \in \operatorname{Limsup}_{\mu \to 0} T(\mu)$, *then* $\overline{x} \in T(0)$.

(ii) Suppose that Y *is finite dimensional and* $\varepsilon > 0$. *If* $\overline{x} \in \operatorname{Limsup}_{\mu \to \varepsilon} T(\mu)$, *then* $\overline{x} \in T(\varepsilon)$.

Proof. Let $\varepsilon \in \mathbb{R}_+$ and take $\overline{x} \in \operatorname{Limsup}_{\mu \to \varepsilon} T(\mu)$. Therefore, there exist $(\mu_n) \subset \mathbb{R}_+$, $(\mu_n) \to \varepsilon$ and $(x_n) \to \overline{x}$ s.t. x_n is a $\mu_n - KKT$ point of $(\mathcal{P}_1)$ for every $n \in \mathbb{N}$. Following Definition 2.3, this means that there exist $y_n^* \in K^*$ with $||y_n^*||_* = 1$ and $z_n^* \in Q^*$ s.t.

$$\|y_n^* \circ \nabla f(x_n) + z_n^* \circ \nabla g(x_n)\|_* \leq \mu_n. \tag{2}$$

The proof consists of two steps. Firstly, we claim that every sequence (z_n^*) satisfying the above relations is bounded. Suppose, by way of contradiction, that one can find such a sequence (z_n^*) with $\|z_n^*\|_* \to +\infty$. We divide by $\|z_n^*\|_*$ and we get

$$\left\| \|z_n^*\|_*^{-1} y_n^* \circ \nabla f(x_n) + \|z_n^*\|_*^{-1} z_n^* \circ \nabla g(x_n) \right\|_* \leq \|z_n^*\|_*^{-1} \mu_n. \tag{3}$$

It is clear that $\nabla f(x_n) \to \nabla f(\overline{x})$, $\nabla g(x_n) \to \nabla g(\overline{x})$, $\|z_n^*\|_*^{-1} \mu_n \to 0$ and $\|z_n^*\|_*^{-1} y_n^* \to 0$. Moreover, the sequence $(\|z_n^*\|^{-1} z_n^*) \subset Q^*$ is bounded and contains, by (A_Z), a subsequence w^*-convergent towards some $z^* \in Q^*$. It is well known that in these conditions we have that

$$\|z_n^*\|_*^{-1} z_n^* \circ \nabla g(x_n) \to z^* \circ \nabla g(\overline{x}).$$

But, since Q has nonempty interior, one gets that $z^* \neq 0$ (see [16], [6]). Finally, passing to the limit in the relation (3) one has

$$z^* \circ \nabla g(\overline{x}) = 0,$$

in contradiction with (MF). The claim is proved and if we select a sequence (z_n^*) s.t. relation (2) holds, then it is bounded, whence it is w^*-convergent (on a sequence, if necessary) towards $z^* \in Q^*$. If we look now at the bounded sequence (y_n^*) it is (by (A_Y)) w^*-convergent (again on a sequence, if necessary) towards an element $y^* \in K^* \setminus \{0\}$ (employ again the same argument as above to get $y^* \neq 0$). One passes now to the limit in relation (2) and one gets

$$\|y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x})\|_* \leq \varepsilon.$$

Now, if $\varepsilon = 0$, this is exactly the fact that $\overline{x}$ is a KKT point, whence the first part of the conclusion. If Y is finite dimensional, then $\|y^*\|_* = 1$ and we get that $\overline{x}$ is a $\varepsilon - KKT$ point, whence the second conclusion holds. □

Of course, by imposing the (MF) condition at every $x \in \operatorname{Limsup}_{\mu \to \varepsilon} T(\mu)$ we get the upper semicontinuity of T at ε. The next consequence is straightforward.

Corollary 2.5 *Assume* $(A_Y), (A_Z), (A_K), (A_Q), (A_f), (A_g)$. *Consider* $(\varepsilon_n) \subset [0, +\infty), (\varepsilon_n) \to 0$. *For every* n, *let* x_n *be an* $\varepsilon_n - KKT$ *point for* $(\mathcal{P}_1)$ *for every* n. *Suppose that* $x_n \to \overline{x}$ *and* (MF) *holds at* $\overline{x}$. *Then* $\overline{x}$ *is a* KKT *point for* $(\mathcal{P}_1)$.

Remark 2.6 *In the above result, if all* x_n *are* KKT *points, then taking into account that* (MF) *holds at* $\overline{x}$ *and* $x_n \to \overline{x}$ *it is easy to see that* (MF) *holds at* x_n *as well for every* n *large enough, whence (from Theorem 2.2)* $L_{y_n^*}$ *is bounded for* $n \geq n_0 \in \mathbb{N}$. *So, it is easy to show that the collection of sets* $(L_{y_n^*})$ *is in fact uniformly bounded. We find a well known result from the scalar finite dimensional case (see [9, Lemma 2.1]).*

In the particular case $\varepsilon_n = 0$ for every n one has a stability property for KKT points.

Corollary 2.7 *Assume* (A_Y), (A_Z), $(A_K), (A_Q)$, (A_f), (A_g). *Let* (x_n) *a sequence of* KKT *points of* $(\mathcal{P}_1)$. *Suppose that* $x_n \to \overline{x}$ *and* (MF) *holds at* $\overline{x}$. *Then* $\overline{x}$ *is a* KKT *point as well.*

We would like now to interpret a KKT point as a solution of a set-valued optimization problem. This approach will give us the possibility to put the above results into relation with some well-known well-posedness criteria for optimization problems. To this aim, let us consider the set-valued map $F : X \rightrightarrows \mathbb{R}$ given by

$$F(x) := \{\|y^* \circ \nabla f(x) + z^* \circ \nabla g(x)\|_* \mid y^* \in K^*, \|y^*\|_* = 1, z^* \in Q^*\},$$

and the scalar set-valued problem

$$\min F(x) \text{ s.t. } x \in M = \{x \in X \mid g(x) \in -Q\}.$$

In certain conditions (cf. Proposition 2.1) the value of this problem is 0. Then, observe that the set of KKT points coincides with the set $\operatorname{argmin}(F, M)$ of points where the optimal value of the afore mentioned problem is realized. Also observe that, for $\varepsilon > 0$, the set of $\varepsilon - KKT$ points coincides with the set $\varepsilon-$ $\operatorname{argmin}(F, M)$ which consists of the points x where $F(x)$ contains values less or equal to ε. Therefore, under conditions of Theorem 2.4, the set-valued map

$$\varepsilon \mapsto \varepsilon - \operatorname{argmin}(F, M)$$

has some properties close to the upper semicontinuity at $\varepsilon = 0$. Note that this is similar to a well-posedness criterion for single-valued functions as it is done in [5, pp. 25]. Consequently, our result can be described as a kind of generalized well-posedness for an optimization problem derived from $(\mathcal{P}_1)$.

In order to complete this section, we consider now a problem with mixed constraints, i.e., problems having some additional constraints given as $h(x) = 0$, where h is a smooth function acting between X and another normed vector space W. Therefore, we shall use:

(A_h) the function h is of class C^1 on X;

and we shall consider the problem:

$$(\mathcal{P}_1') \qquad \text{minimize } f(x) \text{ s.t. } g(x) \in -Q, h(x) = 0.$$

Under the assumption (A_Q), the (MF) condition for this case (referred as $(MF)'$) is that there exists an $\overline{u} \in X$ with $\nabla g(\overline{x})(\overline{u}) \in -\operatorname{int} Q$ and $\nabla h(\overline{x})(\overline{u}) = 0$ (see [6]). The Lagrange multiplier condition is as following.

Proposition 2.8 *[6, Proposition 3.6] Assume* $(A_K),(A_Q)$, (A_f), $(A_g),(A_h)$. *Assume that* $\overline{x} \in M$ *is a weakly minimal point for the problem* $(\mathcal{P}_1')$ *such that the condition* $(MF)'$ *holds at* $\overline{x}$ *and* $\nabla h(\overline{x})$ *is surjective. Then there exist* $y^* \in K^* \setminus \{0\}$, $z^* \in Q^*$, $p^* \in W^*$ *s.t.*

$$y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}) + p^* \circ \nabla h(\overline{x}) = 0. \tag{4}$$

Fix $y^* \in K^* \setminus \{0\}$ and consider the set $L'_{y^*} := \{(z^*, p^*) \in Q^* \times W^* \mid (4) \text{ holds}\}$. It is proved in [6] that, under (A_Z), $(A_K),(A_Q)$, (A_f), (A_g), (A_h), if W is finite dimensional, $\nabla h(\overline{x})$ is surjective and $(MF)'$ holds at $\overline{x}$, then the set L'_{y^*} is (norm) bounded. On the other hand, it is easy to observe that, under the surjectivity of $\nabla h(\overline{x})$, condition $(MF)'$ is equivalent to the following regularity condition $(RC)'$:

$$z^* \in Q^*, p^* \in W^*, z^* \circ \nabla g(\overline{x}) + p^* \circ \nabla h(\overline{x}) = 0 \Longrightarrow z^* = 0, p^* = 0.$$

Definition 2.9 *Assume* (A_f), (A_g), (A_h). *Let* $\overline{x} \in M'$ *and* $\varepsilon \geq 0$. *One says that* $\overline{x}$ *is an* $\varepsilon - KKT$ *point for* $(\mathcal{P}_1')$ *if there exist* $y^* \in K^*, \|y^*\|_* = 1$ *and* $z^* \in Q^*, p^* \in W^*$ *s.t.*

$$\|y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}) + p^* \circ \nabla h(\overline{x})\|_* \leq \varepsilon.$$

As above, we denote by T' the set-valued map which assigns to every $\varepsilon \geq 0$ the subset of X of all $\varepsilon - KKT$ points of $(\mathcal{P}_1')$. We have the next stability result.

Theorem 2.10 *Assume* (A_Y), (A_Z), $(A_K),(A_Q)$, (A_f), (A_g), (A_h). *Suppose that* $(MF)'$ *holds at* $\overline{x}$ *and* $\nabla h(\overline{x})$ *is surjective.*

(i) If $\overline{x} \in \operatorname{Limsup}_{\mu \to 0} T'(\mu)$, *then* $\overline{x} \in T'(0)$;

(ii) Suppose that Y *is finite dimensional and* $\varepsilon > 0$. *If* $\overline{x} \in \operatorname{Limsup}_{\mu \to \varepsilon} T'(\mu)$, *then* $\overline{x} \in T'(\varepsilon)$.

Proof. Let $\varepsilon \geq 0$ and take $\overline{x} \in \operatorname{Limsup}_{\mu \to \varepsilon} T'(\mu)$. Therefore, there exist $(\mu_n) \to \varepsilon$ and $(x_n) \to \overline{x}$ s.t. x_n is a $\mu_n - KKT$ point of $(\mathcal{P}_1')$ for every $n \in \mathbb{N}$. Following the definition, this means that there exist $y_n^* \in K^*$ with $||y_n^*||_* = 1$ and $z_n^* \in Q^*, p_n^* \in W^*$ s.t.

$$\|y_n^* \circ \nabla f(x_n) + z_n^* \circ \nabla g(x_n) + p_n^*(x_n)\|_* \leq \mu_n.$$

First, we claim that both sequences (z_n^*) and (p_n^*) satisfying the above relations are bounded. Suppose, by way of contradiction, that $\|(z_n^*, p_n^*)\|_* \to +\infty$. We proceed as in the proof of Theorem 2.4 and we get a pair $(z^*, p^*) \in Q^* \times W^* \setminus \{(0,0)\}$ s.t.

$$z^* \circ \nabla g(\overline{x}) + p^* \circ \nabla h(\overline{x}) = 0,$$

in contradiction with $(MF)'$. The claim is proved and with the same arguments as above we get $y^* \in K^* \setminus \{0\}, z^* \in Q^*$ and $p^* \in W^*$ with

$$\|y^* \circ \nabla f(\overline{x}) + z^* \circ \nabla g(\overline{x}) + p^* \circ \nabla h(\overline{x})\|_* \leq \varepsilon.$$

The final arguments follow the lines of the final part of the proof of Theorem 2.4. $\square$

Corollary 2.11 *Assume* (A_Y), (A_Z), $(A_K),(A_Q)$, (A_f), (A_g), (A_h). *Consider* $(\varepsilon_n) \to 0$ *s.t.* $\varepsilon_n \geq 0$ *for every* n. *Let* x_n *be an* $\varepsilon_n - KKT$ *point for* $(\mathcal{P}_1')$ *for every* n. *Suppose that* $x_n \to \overline{x}$, $(MF)'$ *holds at* $\overline{x}$ *and* $\nabla h(\overline{x})$ *is surjective. Then* $\overline{x}$ *is a* KKT *point for* $(\mathcal{P}_1')$.

3 The convex case

In this section we will consider the case where the data of the problem $(\mathcal{P}_1)$ are convex. For simplicity of the presentation and also for some technical reasons we will consider the finite dimensional situation. Let us assume that

$(A_{X,Y,Z})$ X, Y, Z are finite dimensional normed vector spaces.

As before, K and Q are proper, closed, convex and pointed cones with non-empty interior which induce a partial order on Y and Z, respectively. It is important to note that in many practical vector optimization problems one needs to consider an ordering cone different from the natural ordering cone in the image space of the objective function (see, for example Eichfelder [8]). First of all, let us fix up some symbols used in this section. For a convex function $\varphi : X \to \mathbb{R}$ the symbol $\partial\varphi(x)$ denotes the classical subdifferential of Convex Analysis for the function φ at the point x. Further, if $\varphi : X \to \mathbb{R}$ denotes a locally Lipschitz function, then the symbol $\partial^{\circ}\varphi(x)$ denotes the Clarke subdifferential of φ at x. For a vector valued locally Lipschitz function $h : X \to Z$, let us denote by $\partial_C h(x)$ the Clarke Jacobian of h at x. In this section we assume:

$(A_{f,K})$ f is $K-$convex, i.e., for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\lambda f(y) + (1 - \lambda) f(y) - f(x + \lambda(y - x)) \in K$$

$(A_{g,Q})$ g is $Q-$convex, i.e., for any $x, y \in X$ and $\lambda \in [0, 1]$,

$$\lambda g(y) + (1 - \lambda) g(x) - g(x + \lambda(y - x)) \in Q.$$

The subdifferential of the $K-$convex function f at a the point $x \in X$ is denoted as $\partial_K f(x)$ and consists of all linear operators from X into Y which satisfy the following condition:

$$f(y) - f(x) - A(y - x) \in K \quad \forall y \in X.$$

Of course, a similar definition holds for a $Q-$convex function. In the paper [14], the authors give a very detailed study of the subdifferential of a cone-convex function and we make use of their results. It has been proved in [14] that every cone-convex function defined on the whole space is locally Lipschitz and the subdifferential of a cone-convex function at $x \in X$ is for all $x \in X$ a non-empty, closed convex and compact set. An important property of the K-convex and Q-convex functions f and g is that their Clarke Jacobians are contained in their subdifferentials. To be more precise, $\partial_C f(x) \subset \partial_K f(x)$ and $\partial_C g(x) \subset \partial_Q g(x)$ for each $x \in X$. This fact leads us to the following simple calculus rule whose proof we give for the sake of completion.

Lemma 3.1 *Assume that X and Y are finite dimensional and $f : X \to Y$ is a K-convex function. Then for any $y^* \in K^*$ the function $y^* \circ f : X \to \mathbb{R}$ is convex and we have for any $x \in X$,*

$$\partial(y^* \circ f)(x) = y^* \circ \partial_K f(x).$$

Proof. The first part is straightforward. In order to prove the calculus formula, first observe that $\partial(y^* \circ f)(x) = \partial^\circ(y^* \circ f)(x)$. Now using the calculus rules for the Clarke subdifferential (see, for example, [3, Theorem 4.1]) we have

$$\partial^\circ(y^* \circ f)(x) \subset y^* \circ \partial_C f(x).$$

Further, noting that $\partial_C f(x) \subset \partial_K f(x)$ we have

$$\partial(y^* \circ f)(x) \subset y^* \circ \partial_K f(x).$$

Now consider $\xi \in y^* \circ \partial_K f(x)$. Hence there exists $A \in \partial_K f(x)$ such that $\xi = A^T y^*$. Now this shows that

$$f(y) - f(x) - A(y - x) \in K \quad \forall y \in X.$$

Since $y^* \in K^*$ we have

$$\langle y^*, f(y)\rangle - \langle y^*, f(x)\rangle \geq \langle A^T y^*, y - x\rangle.$$

This shows that $\xi = A^T y^* \in \partial(y^* \circ f)(x)$, whence the result follows. $\square$

Remark 3.2 *The assertion in Lemma 3.1 is a special case of a corresponding result by Valadier [18].*

The above lemma allows us to quickly establish a necessary and sufficient condition for the convex problem $(\mathcal{P}_1)$ in terms of the subdifferential of the cone-convex functions f and g. As a constraint qualification condition we use a generalized Slater condition:

(S) there exists $\hat{x}$ such that $g(\hat{x}) \in -\operatorname{int} Q$.

Theorem 3.3 *Assume $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$, (S). Then $\bar{x}$ is a weakly minimal solution for the problem $(\mathcal{P}_1)$ if and only if there exist $y^* \in K^*$ with $||y^*||_* = 1$ and $z^* \in Q^*$ such that*
(i) $0 \in y^* \circ \partial_K f(\bar{x}) + z^* \circ \partial_Q g(\bar{x})$,
(ii) $\langle z^*, g(\bar{x})\rangle = 0$.

Proof. Let us begin by assuming that $\bar{x}$ is a weakly minimal solution of the convex problem $(\mathcal{P}_1)$. Then it is simple to observe that the following system

$$(f(x) - f(\bar{x}), g(x)) \in -\operatorname{int}(K \times Q)$$

has no solutions. Thus from simple separation arguments or generalized Gordon's theorem of the alternative (see, for example, [2]) we have that there exists $0 \neq (y^*, z^*) \in K^* \times Q^*$ such that

$$\langle y^*, f(x) - f(\bar{x})\rangle + \langle z^*, g(x)\rangle \geq 0 \quad \forall x \in X. \tag{5}$$

Now setting $x = \bar{x}$ shows that $\langle z^*, g(\bar{x}) \geq 0$. Further since $g(\bar{x}) \in -Q$ we have

$$\langle z^*, g(\bar{x})\rangle \leq 0. \tag{6}$$

This shows that

$$\langle z^*, g(\bar{x})\rangle = 0. \tag{7}$$

Now using the Slater condition (S), we can easily show that $y^* \neq 0$. Let us consider the scalar Lagrangian function associated with the problem $(\mathcal{P}_1)$ where $L : X \times K^* \times Q^* \to \mathbb{R}$ is given as

$$L(x, y^*, z^*) = (y^* \circ f)(x) + (z^* \circ g)(x), \tag{8}$$

where $(y^* \circ f)(x) = \langle y^*, f(x)\rangle$ and $(z^* \circ g)(x) = \langle z^*, g(x)\rangle$ for all $x \in X$.
Taking into account (5), (6) and (7), if $\bar{x}$ is a weakly minimal solution of the convex problem $(\mathcal{P}_1)$, then there exist $y^* \in K^*$ with $||y^*||_* = 1$ and $z^* \in Q^*$ such that

(a) $L(\bar{x}, y^*, z^*) = \min_{x \in X} L(x, y^*, z^*)$

(b) $\langle z^*, g(\bar{x})\rangle = 0$.

Now from (a) we have that

$$0 \in \partial(y^* \circ f)(\bar{x}) + \partial(z^* \circ g)(\bar{x}).$$

Furthermore, using Lemma 3.1 we have

$$0 \in y^* \circ \partial_K f(\bar{x}) + z^* \circ \partial_Q g(\bar{x}),$$

such that (i) and (ii) hold. The converse of the result can be proved using arguments which are now standard in nonsmooth optimization. Indeed, (i) implies that there exists $u^* \in \partial(y^* \circ f)(\bar{x}) \cap -\partial(z^* \circ g)(\bar{x}))$ s.t. for every $y \in X$,

$$\begin{aligned}\langle y^*, f(y) - f(\overline{x})\rangle - u^*(y - \overline{x}) \geq 0,\\ \langle z^*, g(y) - g(\overline{x})\rangle + u^*(y - \overline{x}) \geq 0,\end{aligned}$$

whence, using (ii),

$$\langle y^*, f(y) - f(\overline{x})\rangle + \langle z^*, g(y)\rangle \geq 0.$$

Now, for any feasible point $y \in M$, $\langle z^*, g(y)\rangle \leq 0$, therefore

$$\langle y^*, f(y) - f(\overline{x})\rangle \geq 0,$$

for every $y \in M$. This shows that $f(y) - f(\overline{x}) \notin -\operatorname{int} K$, for every $y \in M$, whence $\overline{x}$ is a weakly minimum point for $(\mathcal{P}_1)$. □

Let us now define the notion of an approximate *KKT* point or ε-*KKT* point for the convex problem $(\mathcal{P}_1)$.

Definition 3.4 *Assume* $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$. *Let* $\bar{x} \in M$ *and* $\varepsilon > 0$ *be given. Then* $\bar{x}$ *is said to be an* ε*-KKT point of the convex problem* $(\mathcal{P}_1)$ *if there exists* $A \in \partial_K f(\bar{x})$, $B \in \partial_Q g(\bar{x})$, $y^* \in K^*$, $||y^*||_* = 1$ *and* $z^* \in Q^*$ *such that*

$$||A^T y^* + B^T z^*||_* \leq \varepsilon.$$

For a given $\varepsilon > 0$ let us denote by $\hat{T}(\varepsilon)$ the set of all ε-*KKT* point of the convex problem $(\mathcal{P}_1)$.

Theorem 3.5 *Let us consider the problem* $(\mathcal{P}_1)$ *under* $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$, (S). *Assume that* $\bar{x} \in X$ *is a given point such that* $g(\bar{x}) = 0$ *and* $\bar{x} \in \limsup_{\varepsilon \to 0} \hat{T}(\varepsilon)$. *Then* $\bar{x}$ *is a weakly minimal solution of* $(\mathcal{P}_1)$.

Proof. From the hypothesis of the theorem there exist some sequences $\varepsilon_n \to 0$ and $x_n \to \bar{x}$ such that $x_n \in \hat{T}(\varepsilon_n)$. Taking into account Definition 3.4 we get the existence of $A_n \in \partial_K f(x_n)$, $B_n \in \partial_Q g(x_n)$, $y_n^* \in K^*$, $||y_n^*||_* = 1$ and $z_n^* \in Q^*$ such that

$$||A_n^T y_n^* + B_n^T z_n^*||_* \leq \varepsilon_n. \tag{9}$$

We claim that the sequence (z_n^*) is bounded. On the contrary, let us assume that (z_n^*) is unbounded that is $||z_n^*||_* \to \infty$. Now dividing both sides of (9) by $||z_n^*||_*$ we get for each n,

$$||A_n^T \frac{y_n^*}{||z_n^*||_*} + B_n^T \frac{z_n^*}{||z_n^*||_*}||_* \leq \frac{\varepsilon_n}{||z_n^*||_*}. \tag{10}$$

Since K^* and Q^* are cones we have $\dfrac{y_n^*}{||z_n^*||_*} \in K^*$ and $\dfrac{z_n^*}{||z_n^*||_*} \in Q^*$. Let us set for each n

$$\xi_n := A_n^T \frac{y_n^*}{||z_n^*||_*}$$

and

$$\psi_n := B_n^T \frac{z_n^*}{||z_n^*||_*}$$

Since Lemma 3.1 we have

$$\xi_n \in \partial(\frac{y_n^*}{||z_n^*||_*} \circ f)(\bar{x})$$

and

$$\psi_n \in \partial(\frac{z_n^*}{||z_n^*||_*} \circ g)(\bar{x}).$$

Now since the sequence (y_n^*) is bounded we have that

$$\frac{y_n^*}{||z_n^*||_*} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence using, for example, [7, Lemma 3.1] or [15, Lemma 3.27] we have $\xi_n \to 0$ as $n \to \infty$. Moreover as the sequence $(\frac{z_n^*}{||z_n^*||_*})$ is a bounded sequence without loss of generality we can assume that $\frac{z_n^*}{||z_n^*||_*} \to w^* \in Q^*$ and $||w^*||_* = 1$. Again using [15, Lemma 3.27] we can conclude without loss of generality that $\psi_n \to \psi$ for $n \to \infty$ and further $\psi \in \partial(w^* \circ g)(\bar{x})$. Now as (ε_n) converges to zero the sequence is bounded and thus as $n \to \infty$ we conclude from (10) that $||\psi||_* \leq 0$ and hence $\psi = 0$. So we get

$$0 \in \partial(w^* \circ g)(\bar{x}).$$

Now, using the definition of the subgradient of a convex function we have for any $x \in X$

$$\langle w^*, g(x) \rangle - \langle w^*, g(\bar{x}) \rangle \geq 0. \tag{11}$$

Since $g(\bar{x}) = 0$, we have

$$\langle w^*, g(x)\rangle \geq 0 \quad \forall x \in X.$$

Note that the last conclusion could be proven in a standard way on the base of (11) by means of Slater condition, without using $g(\bar{x}) = 0$. Hence using the generalized Gordan's theorem of the alternative (see, for example, [2]) we conclude that there exists no $x \in X$ such that $-g(x) \in \operatorname{int} Q$. This clearly contradicts the assumption that there exists $\hat{x} \in X$ such that $-g(\hat{x}) \in \operatorname{int} Q$. Hence we conclude that sequence (z_n^*) is bounded. Hence without loss of generality we conclude that $z_n^* \to z^* \in Q^*$. Further, since (y_n^*) is also bounded, we conclude again that $y_n^* \to y^* \in K^*$ without loss of generality. From Lemma 3.1 we get that $\alpha_n = A_n^T y_n^* \in \partial(y_n^* \circ f)(x_n)$ and $\beta_n = B_n^T z_n^* \in \partial(z_n^* \circ g)(x_n)$. By the same arguments, we conclude that $\alpha_n \to \alpha$ and $\beta_n \to \beta$ as $n \to \infty$ and further $\alpha \in \partial(y^* \circ f)(\bar{x})$ and $\beta \in \partial(z^* \circ g)(\bar{x})$. Now using Lemma 3.1 we see that there exist $A \in \partial_K f(\bar{x})$ and $B \in \partial_Q g(\bar{x})$ such that $\alpha = A^T y^*$ and $\beta = B^T z^*$. Thus as $n \to \infty$ we conclude from (9) that

$$A^T y^* + B^T z^* = 0.$$

This means $0 \in y^* \circ \partial_K f(\bar{x}) + z^* \circ \partial_Q g(\bar{x})$. Furthermore, noting that $g(\bar{x}) = 0$ we conclude using Theorem 3.3 that $\bar{x}$ is a weakly minimal solution of $(\mathcal{P}_1)$. □

Motivated by the well known Ekeland's variational principle we introduce the notion of a modified ε-*KKT* point as follow:

Definition 3.6 *Assume $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$. Let $\bar{x} \in M$ and $\varepsilon > 0$ be given. Then $\bar{x}$ is said to be a modified ε-KKT point of $(\mathcal{P}_1)$ if there exists $\hat{x} \in X$ such that $||\bar{x} - \hat{x}|| \leq \sqrt{\varepsilon}$ and there exists $y^* \in K^*$, $\|y^*\|_* = 1$, $z^* \in Q^*$, $\hat{A} \in \partial_K f(\hat{x})$ and $\hat{B} \in \partial_Q g(\hat{x})$ such that*

$$||\hat{A}^T y^* + \hat{B}^T z^*||_* \leq \sqrt{\varepsilon}$$

and

$$\langle z^*, g(\bar{x})\rangle \geq -\varepsilon.$$

We denote the set valued map which assigns to every $\varepsilon > 0$ the subset of X of all modified $\varepsilon - KKT$ points of the convex problem $(\mathcal{P}_1)$ by $\hat{T}_{mod}$.
Let us now present the well known notion of an ε-weakly minimal solution for the problem $(\mathcal{P}_1)$. For $\varepsilon > 0$ we say that $\bar{x} \in M$ is an ε-weak minimum for the problem $(\mathcal{P}_1)$ with respect to $e \in \operatorname{int} K$ if there does not exist any $x \in M$ such that

$$f(x) - f(\bar{x}) + \varepsilon e \in -\operatorname{int} K.$$

For more details on the recent developments in the study of approximate and weakly approximate efficient points in vector optimization see for example [11] and [10] and the references there in. We will now demonstrate that under natural conditions an ε-weakly minimal solution is also a modified ε-*KKT* point of the convex problem $(\mathcal{P}_1)$.

Theorem 3.7 *Let us consider the problem $(\mathcal{P}_1)$ under $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$, (S). Let $e \in \operatorname{int} K$ and $\varepsilon > 0$ be given and further let $\bar{x} \in M$ be an ε-weakly minimal solution with respect to $e \in \operatorname{int} K$. Then $\bar{x}$ is a modified μ-KKT point for a constant $\mu > 0$ (which depends on e).*

Proof. Since $\bar{x}$ is a weakly minimal solution with respect to $e \in \operatorname{int} K$ it is clear that the following system

$$(f(x) - f(\bar{x}) + \varepsilon e, g(x)) \in -\operatorname{int}(K \times Q), \quad x \in X$$

has no solutions. Hence by using the generalized Gordan's Theorem of the alternative there exists $0 \neq (y^*, z^*) \in (K^* \times Q^*)$ such that

$$\langle y^*, f(x) - f(\bar{x}) + \varepsilon e\rangle + \langle z^*, g(x)\rangle \geq 0, \quad \forall x \in X.$$

Using the fact that there exists $\hat{x}$ such that $-g(\hat{x}) \in \operatorname{int} Q$ it is routine matter to show that $y^* \neq 0$ and thus without loss of generality we can have $\|y^*\|_* = 1$. Hence $\langle y^*, e\rangle > 0$ since $e \in \operatorname{int} K$. Let us denote $\mu := \varepsilon\langle y^*, e\rangle$. Thus from the above inequality we have for all $x \in X$

$$\langle y^*, f(x)\rangle + \langle z^*, g(x)\rangle + \mu \geq \langle y^*, f(\bar{x})\rangle. \tag{12}$$

Now setting $x = \bar{x}$ in (12) we have $\langle z^*, g(\bar{x})\rangle \geq -\mu$. Moreover, noting that $\langle z^*, g(\bar{x})\rangle \leq 0$ we can rewrite (12) as

$$\langle y^*, f(x)\rangle + \langle z^*, g(x)\rangle + \mu \geq \langle y^*, f(\bar{x})\rangle + \langle z^*, g(\bar{x})\rangle.$$

This shows, taking into account the notation in (8), that $L(x, y^*, z^*) \geq L(\bar{x}, y^*, z^*) - \mu$ for all $x \in X$ which shows that $\bar{x}$ is an $\mu-$minimum of $L(x, y^*, z^*)$ over X. Thus using the Ekeland's variational principle we conclude that there exists $\hat{x} \in X$ such that $||\bar{x} - \hat{x}|| \leq \sqrt{\mu}$ and further $\hat{x}$ is a solution of the following problem

$$\min_{x \in X} L(x, y^*, z^*) + \sqrt{\mu}||x - \hat{x}||.$$

Hence from standard rules of Convex Analysis we know that

$$0 \in \partial L(\hat{x}, y^*, z^*) + \sqrt{\mu}\mathbb{B},$$

where $\mathbb{B}$ is the closed unit ball in X. Now again using the sum rule for subdifferentials we have

$$0 \in \partial(y^* \circ f)(\hat{x}) + \partial(z^* \circ g)(\hat{x}) + \sqrt{\mu}\mathbb{B}.$$

Further using Lemma 3.1 we have

$$0 \in y^* \circ \partial_K f(\hat{x}) + z^* \circ \partial_Q g(\hat{x}) + \sqrt{\mu}\mathbb{B}.$$

Hence there exists $\hat{A} \in \partial_K f(\hat{x})$ and $\hat{B} \in \partial_Q g(\hat{x})$ such that

$$||\hat{A}^T y^* + \hat{B}^T z^*||_* \leq \sqrt{\mu},$$

and so the result follows. □

Our following result essentially demonstrates that if we have a sequence of modified ε-KKT points of the convex problem $(\mathcal{P}_1)$ and if the sequence converges to a point, say $\bar{x}$, then $\bar{x}$ is a weakly minimal solution for the problem $(\mathcal{P}_1)$ under certain natural assumptions.

Theorem 3.8 *Let us consider the convex problem* $(\mathcal{P}_1)$ *under* $(A_{X,Y,Z})$, $(A_{f,K})$, $(A_{g,Q})$, (S). *Let* $\bar{x} \in \limsup_{\varepsilon\to 0} \hat{T}_{mod}(\varepsilon)$. *Then* $\bar{x}$ *is a weakly minimal solution of the convex problem* $(\mathcal{P}_1)$.

Proof. From the hypothesis it is clear that there exist a sequence (x_n), with $x_n \to \bar{x}$ and a sequence (ε_n) with $\varepsilon_n \to 0$ such that $x_n \in \hat{T}_{mod}(\varepsilon_n)$. This means, by definition, that $x_n \in M$ for all $n \in \mathbb{N}$ and since g is continuous and Q is a closed and convex cone it is clear that $\bar{x} \in M$. Now from the definition of modified ε-KKT point we have that for each n there exists x'_n such that $||x'_n - x_n|| \leq \sqrt{\varepsilon_n}$ and there exists $y^*_n \in K^*$, $\|y^*_n\|_* = 1$, $z^*_n \in Q^*$, $A'_n \in \partial_K f(x'_n)$ and $B'_n \in \partial_Q g(x'_n)$ such that

$$||(A'_n)^T y^*_n + (B'_n)^T z^*_n||_* \leq \sqrt{\varepsilon_n} \tag{13}$$

and

$$\langle z^*_n, g(x_n)\rangle \geq -\varepsilon_n. \tag{14}$$

Now, as $x_n \to \bar{x}$ it is clear that $x'_n \to \bar{x}$ and then using the proof style of Theorem 3.5 we can show that the sequence (z^*_n) is bounded and without loss of generality we can assume that $z^*_n \to z^*$ with $z^* \in Q^*$. Further, the sequence $\{y^*_n\}$ is bounded since $||y^*_n||_* = 1$ for all n. Hence again without loss of generality we assume that $y^*_n \to y^*$ with $y^* \in K^*$. Now using Lemma 3.1 we have that $\xi_n = (A'_n)^T y^*_n \in \partial(y^*_n \circ f)(x'_n)$ and $\psi_n = (B'_n)^T z^*_n \in \partial(z^*_n \circ g)(x'_n)$. Again using [15, Lemma 3.7] we have without loss of generality that $\xi_n \to \xi$ and $\psi_n \to \psi$ with $\xi \in \partial(y^* \circ f)(\bar{x})$ and $\psi \in \partial(z^* \circ g)(\bar{x})$. Thus as $n \to 0$ from (13) and (14) we have $\xi + \psi = 0$ and $\langle z^*, g(\bar{x})\rangle \geq 0$. Further, $z^* \in Q^*$ and $g(\bar{x}) \in -Q$ implies $\langle z^*, g(\bar{x})\rangle \leq 0$, whence $\langle z^*, g(\bar{x})\rangle = 0$. From Lemma 3.1 we have $A \in \partial_K f(\bar{x})$ and $B \in \partial_Q g(\bar{x})$ such that $\xi = A^T y^*$ and $\psi = B^T z^*$. Now we can apply the sufficiency part of Theorem 3.3 to get the conclusion. □

4 The nonsmooth and nonconvex case

The aim of this section is to study some set-valued optimization problems from the point of view of stability of KKT points.

Let us consider the set-valued map $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$, where X, Y and Z are, in the sequel, finite dimensional spaces. Therefore, in this section we assume again $(A_{X,Y,Z})$.

We consider the following problem

$$(\mathcal{P}_2) \quad \min F(x), \text{ subject to } x \in X, 0 \in G(x) + Q.$$

In fact, we shall also consider a simplified version of problem $(\mathcal{P}_2)$ where G is not a set-valued map but a single-valued one denoted by g. Then the problem above is represented as

$$(\mathcal{P}'_2) \quad \min F(x), \text{ subject to } x \in X, g(x) \in -Q.$$

A point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ is called Pareto minimum point for problem $(\mathcal{P}_2)$ if

$$(F(M'') - \overline{y}) \cap -K = \{0\},$$

where $M'' := \{x \in X \mid 0 \in G(x) + Q\}$.
The definition for Pareto solutions for $(\mathcal{P}'_2)$ is obtained by replacing M'' in the above definition by $\{x \in X \mid g(x) \in -Q\}$.

The main tools we use in this section are the generalized differentiability concepts introduced by Mordukhovich (see [15, Chapters 1, 3]). It is well known that these objects enjoy rich calculus in Asplund spaces. However, as we already said, we work here on finite dimensional vector spaces. The definitions that appear below are from the comprehensive monograph of Mordukhovich [15] and we present them here for the sake of completeness. Also for the simplicity of the exposition we shall give the definitions in the finite dimensional setting.

Definition 4.1 *Let X be a finite dimensional normed space and $M'' \subset X$ be a non-empty closed subset of X and let $x \in M''$. The basic (or limiting, or Mordukhovich) normal cone to M'' at x is*

$$N(M'', x) := \{x^* \in X \mid \exists x_n \stackrel{M''}{\to} x, x_n^* \to x^*, x_n^* \in N_F(M'', x_n)\}$$

where $N_F(M'', z)$ denotes the Fréchet normal cone to M'' at a point $z \in M''$, given as

$$N_F(M'', z) := \{x^* \in X \mid \limsup_{u \in M'', u \to z} \frac{x^*(u - z)}{\|u - z\|} \leq 0\}.$$

Definition 4.2 *Let $F : X \rightrightarrows Y$ be a set-valued map, where X and Y are finite dimensional normed spaces and $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$. Then the coderivative of F at the $(\bar{x}, \bar{y}) \in \operatorname{Gr} F$ is the set-valued map $D^* : Y \rightrightarrows X$ given as*

$$D^*F(x, y)(y^*) := \{x^* \in X \mid (x^*, -y^*) \in N(\operatorname{Gr} F, (x, y))\}.$$

A set-valued mapping H acting between two finite dimensional normed vector spaces X and Y is said to be inner-semicompact at $\overline{x} \in X$ if for every sequence $x_k \to \overline{x}$ there is a sequence $y_k \in H(x_k)$ which has a convergent subsequence. For example if H is uniformly bounded at $\overline{x}$ then H is inner-semicompact at $\overline{x}$.

Finally, we present the notion of a Lipschitz-like set-valued map.

Definition 4.3 *A set-valued map $F : X \rightrightarrows Y$, where X and Y are finite dimensional normed spaces, is called Lipschitz-like at a point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ if there exist a positive number $L > 0$ and some neighborhoods U of $\overline{x}$ and V of $\overline{y}$ s.t. for every $x, u \in U$ one has:*

$$F(x) \cap V \subset F(u) + L\,\|x - u\|\,\mathbb{B},$$

where $\mathbb{B}$ denotes the closed unit ball of X.

We start by considering the problem $(\mathcal{P}_2)$.

Theorem 4.4 *[6, Theorem 4.5] Assume $(A_{X,Y,Z})$. Consider the problem $(\mathcal{P}_2)$ and let $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ be a Pareto minimum of $(\mathcal{P}_2)$. Assume that F is Lipschitz-like at $(\overline{x}, \overline{y})$ and $G(\overline{x})$ is compact. Further assume that the mapping $x \mapsto G(x) \cap (-Q)$ is inner-semicompact at $\overline{x}$. Moreover, assume that the following qualification condition hold true: For any $z \in G(\overline{x}) \cap (-Q)$ one has*

$$N(-Q, y) \cap \ker D^*G(\overline{x}, z) = \{0\}. \tag{15}$$

Then there exist $y^* \in K^*$ with $||y^*||_* = 1$ such that the set

$$\begin{aligned} L^\circ_{y^*} = \{&z^* \in Z^* \mid \exists \overline{z} \in G(\overline{x}) \cap (-Q), \quad \text{with} \quad z^* \in N(-Q, \overline{z}) \quad \text{such that} \\ &0 \in D^*F(\overline{x}, \overline{y})(y^*) + D^*G(\overline{x}, \overline{z})(z^*)\}. \end{aligned} \tag{16}$$

is nonempty and bounded.

Definition 4.5 *Let $\varepsilon \geq 0$. A feasible point $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ of problem $(\mathcal{P}_2)$ is said to be a $\varepsilon-KKT$ point for $(\mathcal{P}_2)$ if there exist $y^* \in K^*, ||y^*||_* = 1$, $z^* \in \mathbb{R}^k$, $\overline{z} \in G(\overline{x}) \cap -Q$ with $z^* \in N(-Q, \overline{z})$ and $x^* \in X^*$, $\|x^*\|_* \leq \varepsilon$ s.t.*

$$x^* \in D^*F(\overline{x}, \overline{y})(y^*) + D^*G(\overline{x}, \overline{z})(z^*).$$

Denote by T° the set-valued map which assigns to every $\varepsilon \in \mathbb{R}_+$ the subset of $X \times Y$ of all $\varepsilon - KKT$ points of $(\mathcal{P}_2)$. In this nonsmooth set-valued case the stability result we have is the following. When $\varepsilon = 0$ in Definition 4.5, $(\overline{x}, \overline{y})$ is called a KKT point for $(\mathcal{P}_2)$.

Theorem 4.6 *Assume $(A_{X,Y,Z})$. Suppose that F is Lipschitz-like at $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, G has closed graph, $(G(x_n))$ is uniformly bounded around $\overline{x}$ and for any $z \in G(\overline{x}) \cap -Q$*

$$N(-Q, z) \cap \ker D^*G(\overline{x}, z) = \{0\}.$$

If $\varepsilon \geq 0$ and $(\overline{x}, \overline{y}) \in \operatorname{Limsup}_{\mu \to \varepsilon} T^\circ(\mu)$, then $(\overline{x}, \overline{y}) \in T^\circ(\varepsilon)$.

Proof. Let $\varepsilon \in \mathbb{R}_+$ and take $(\overline{x}, \overline{y}) \in \operatorname{Limsup}_{\mu \to \varepsilon} T^\circ(\mu)$. Thus, there exist some sequences $(\mu_n) \to \varepsilon$, $(x_n, y_n) \subset \operatorname{Gr} F$, $(x_n, y_n) \to (\overline{x}, \overline{y})$, $(y_n^*), \|y_n^*\|_* = 1, (z_n^*) \subset Z^*$, $z_n \in G(x_n) \cap -Q$ with $z_n^* \in N(-Q, z_n)$, (x_n^*), $\|x_n^*\|_* \leq \mu_n$ s.t.

$$x_n^* \in D^*F(x_n, y_n)(y_n^*) + D^*G(x_n, z_n)(z_n^*). \tag{17}$$

First, we show that (z_n^*) is bounded. Suppose that this is not the case: There exists a subsequence (denoted (z_n^*) as well) s.t. $\|z_n^*\|_* \to \infty$. It is clear that $z_n^* \neq 0$ for n large enough, whence, by division with $\|z_n^*\|$, one gets

$$\|z_n^*\|_*^{-1} x_n^* \in D^*F(x_n, y_n)(\|z_n^*\|_*^{-1} y_n^*) + D^*G(x_n, z_n)(\|z_n^*\|_*^{-1} z_n^*),$$

i.e., there exist $(u_n^*) \subset X^*$ s.t.

$$\begin{aligned} u_n^* &\in D^*F(x_n, y_n)(\|z_n^*\|_*^{-1} y_n^*) \\ -u_n^* + \|z_n^*\|_*^{-1} x_n^* &\in D^*G(x_n, z_n)(\|z_n^*\|_*^{-1} z_n^*). \end{aligned}$$

Note that in finite dimensions the graph of the set-valued mapping $N(\Omega, \cdot)$ is closed (see [17]). Now we employ the hypotheses in order to deduce, by moving to subsequences if needed, that $\|z_n^*\|_*^{-1} y_n^* \to 0$, $\|z_n^*\|_*^{-1} x_n^* \to 0$, $u_n^* \to u^* \in X^*$ (since F is Lipschitz-like at $(\overline{x}, \overline{y})$, (u_n^*) is bounded, see [15, Theorem 1.43]) and $z_n \to z \in G(\overline{x}) \cap -Q$ (since $(G(x_n))_n$ is uniformly bounded), $\|z_n^*\|_*^{-1} z_n^* \to z^* \in N(-Q, z) \setminus \{0\}$, and

$$\begin{aligned} u^* &\in D^*F(\overline{x}, \overline{y})(0) \\ -u^* &\in D^*G(\overline{x}, z)(z^*). \end{aligned}$$

Therefore,

$$0 \in D^*F(\overline{x}, \overline{y})(0) + D^*G(\overline{x}, z)(z^*).$$

One uses again the fact that F is Lipschitz-like at $(\overline{x}, \overline{y})$ to observe that $D^*F(\overline{x}, \overline{y})(0) = \{0\}$, whence

$$0 \in D^*G(\overline{x}, z)(z^*)$$

in contradiction with the qualification condition we have imposed. Therefore, for the second step of the proof, we know that in relation (17) the sequence (z_n^*) is bounded so it is convergent (on a subsequence, if necessary) towards an element $z^* \in N(-Q,z)$ for an $z \in G(\overline{x}) \cap -Q$. Similarly, (y_n^*) is convergent towards an element y^* of norm 1 from K^*. Moreover (x_n^*) is bounded too, whence it is convergent to some $x^* \in X^*$ with $\|x^*\|_* \leq \varepsilon$. Passing to the limit (as above) in (17) we get

$$x^* \in D^*F(\overline{x},\overline{y})(y^*) + D^*G(\overline{x},z)(z^*)$$

i.e., $(\overline{x},\overline{y}) \in T^\circ(\varepsilon)$. The proof is complete. $\square$

We have two corollaries.

Corollary 4.7 *Assume $(A_{X,Y,Z})$. Suppose that $(x_n,y_n) \subset \operatorname{Gr} F$ are $\varepsilon_n - KKT$ points for $(\mathcal{P}_2)$ where $(\varepsilon_n) \to 0$ s.t. $\varepsilon_n > 0$ and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ s.t. $(x_n,y_n) \to (\overline{x},\overline{y})$. Suppose that F is Lipschitz-like at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, G has closed graph, $(G(x_n))$ is uniformly bounded and for any $z \in G(\overline{x}) \cap -Q$*

$$N(-Q,z) \cap \ker D^*G(\overline{x},z) = \{0\}.$$

Then $(\overline{x},\overline{y})$ is a KKT point for $(\mathcal{P}_2)$.

Corollary 4.8 *Assume $(A_{X,Y,Z})$. Suppose that $(x_n,y_n) \subset \operatorname{Gr} F$ and $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ are KKT points for $(\mathcal{P}_2)$ s.t. $(x_n,y_n) \to (\overline{x},\overline{y})$. Suppose that F is Lipschitz-like at $(\overline{x},\overline{y}) \in \operatorname{Gr} F$, G has closed graph, $(G(x_n))$ is uniformly bounded and for any $z \in G(\overline{x}) \cap -Q$*

$$N(-Q,z) \cap \ker D^*G(\overline{x},z) = \{0\}.$$

Then $(\overline{x},\overline{y})$ is a KKT points for $(\mathcal{P}_2)$.

We turn now our attention to the problem $(\mathcal{P}_2')$. If we adapt [6, Theorem 4.4] to the case of finite dimensional spaces we get the following result.

Theorem 4.9 *Assume $(A_{X,Y,Z})$ and let $(\overline{x},\overline{y})$ be a Pareto minimum point for $(\mathcal{P}_2')$. Suppose that F is Lipschitz-like at $(\overline{x},\overline{y})$ and g is locally Lipschitz at $\overline{x}$. If*

$$z^* \in Q^*, 0 \in D^*g(\overline{x})(z^*) \Rightarrow z^* = 0, \tag{18}$$

then there exists $y^* \in K^*, \|y^*\|_* = 1$ s. t.

$$L_{y^*}^{\circ\prime} := \{z^* \in Q^* \mid 0 \in D^*F(\overline{x},\overline{y})(y^*) + D^*g(\overline{x})(z^*)\}. \tag{19}$$

is non-empty and bounded.

Definition 4.10 *Let $\varepsilon \geq 0$. A feasible point $(\overline{x},\overline{y}) \in \operatorname{Gr} F$ of problem $(\mathcal{P}_2')$ is said to be a $\varepsilon - KKT$ point for $(\mathcal{P}_2')$ if there exists $y^* \in K^*, ||y^*||_* = 1$, $z^* \in Q^*$ and $x^* \in X^*$, $\|x^*\|_* \leq \varepsilon$ s.t.*

$$x^* \in D^*F(\overline{x},\overline{y})(y^*) + D^*g(\overline{x})(z^*).$$

Denote by $T^{\circ\prime}$ the set-valued map which assigns to every $\varepsilon \in \mathbb{R}_+$ the subset of $X \times Y$ of all $\varepsilon - KKT$ points of $(\mathcal{P}_2')$. If $\varepsilon = 0$ in Definition 4.10, then $(\overline{x},\overline{y})$ is called a KKT point for $(\mathcal{P}_2')$. Our stability results for this problem uses the same hypotheses as above.

Theorem 4.11 *Assume* $(A_{X,Y,Z})$*. Suppose that* F *is Lipschitz-like at* $(\overline{x},\overline{y}) \in \operatorname{Gr} F$*, and* g *is locally Lipschitz at* $\overline{x}$ *and the following implication holds:*

$$z^* \in Q^*, 0 \in D^*g(\overline{x})(z^*) \Rightarrow z^* = 0.$$

If $\varepsilon \geq 0$ *and* $(\overline{x},\overline{y}) \in \operatorname{Limsup}_{\mu\to\varepsilon} T^{\circ\prime}(\mu)$*, then* $(\overline{x},\overline{y}) \in T^{\circ\prime}(\varepsilon)$*.*

Proof. Let $\varepsilon \in \mathbb{R}_+$ and take $(\overline{x},\overline{y}) \in \operatorname{Limsup}_{\mu\to\varepsilon} T^{\circ\prime}(\mu)$. Thus, there exist some sequences $(\mu_n) \to \varepsilon$, $(x_n, y_n) \subset \operatorname{Gr} F$, $(x_n, y_n) \to (\overline{x},\overline{y})$ and $(y_n^*), \|y_n^*\|_* = 1, (z_n^*) \subset Q^*$, s.t. (x_n^*), $\|x_n^*\|_* \leq \mu_n$ s.t.

$$x_n^* \in D^*F(x_n, y_n)(y_n^*) + D^*g(x_n)(z_n^*). \tag{20}$$

We claim that (z_n^*) is bounded. Suppose that this is not the case: There exists a subsequence (denoted (z_n^*)) s.t. $\|z_n^*\|_* \to \infty$. It is clear that $z_n^* \neq 0$ for n large enough, whence, by division with $\|z_n^*\|_*$, one gets

$$\|z_n^*\|_*^{-1} x_n^* \in D^*F(x_n, y_n)(\|z_n^*\|_*^{-1} y_n^*) + D^*g(x_n)(\|z_n^*\|_*^{-1} z_n^*),$$

i.e., there exist $(u_n^*) \subset X^*$ s.t.

$$\begin{aligned} u_n^* &\in D^*F(x_n, y_n)(\|z_n^*\|_*^{-1} y_n^*) \\ -u_n^* + \|z_n^*\|_*^{-1} x_n^* &\in D^*g(x_n)(\|z_n^*\|_*^{-1} z_n^*). \end{aligned}$$

Now we have that $\|z_n^*\|_*^{-1} y_n^* \to 0$, $\|z_n^*\|_*^{-1} x_n^* \to 0$, $u_n^* \to u^* \in X^*$ (since F is Lipschitz-like at $(\overline{x},\overline{y})$, (u_n^*) is bounded, see [15, Theorem 1.43]) and $g(x_n) \to g(\overline{x})$ (since g is continuous), $\|z_n^*\|_*^{-1} z_n^* \to z^* \in Q^*$, and

$$\begin{aligned} u^* &\in D^*F(\overline{x},\overline{y})(0), \\ -u^* &\in D^*g(\overline{x})(z^*), \end{aligned}$$

whence

$$0 \in D^*F(\overline{x},\overline{y})(0) + D^*g(\overline{x})(z^*).$$

One uses again the fact that F is Lipschitz-like at $(\overline{x},\overline{y})$ to observe that $D^*F(\overline{x},\overline{y})(0) = \{0\}$, whence

$$0 \in D^*g(\overline{x}, z)(z^*).$$

This is a contradiction to the qualification condition we have imposed. Now, since the sequence (z_n^*) is bounded it is convergent (on a subsequence, if necessary) towards an element $z^* \in Q^*$. Similarly, (y_n^*) is convergent towards an element y^* with $||y^*||_* = 1$ from K^* and x_n^* is convergent to an element x^* with $\|x^*\|_* \leq \varepsilon$. Passing to the limit (as above) in (17) we get

$$x^* \in D^*F(\overline{x},\overline{y})(y^*) + D^*g(\overline{x})(z^*)$$

and this is the conclusion. □

Again, we record some corollaries in two particular cases.

Corollary 4.12 *Assume* $(A_{X,Y,Z})$. *Suppose that* $(x_n, y_n) \subset \operatorname{Gr} F$ *are* $\varepsilon_n - KKT$ *points for* $(\mathcal{P}_2')$ *where* $(\varepsilon_n) \to 0$ *s.t.* $\varepsilon_n > 0$ *and* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *s.t.* $(x_n, y_n) \to (\overline{x}, \overline{y})$. *Suppose that* F *is Lipschitz-like at* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, *and* g *is locally Lipschitz at* $\overline{x}$ *and the following implication holds:*

$$z^* \in Q^*, 0 \in D^* g(\overline{x})(z^*) \Rightarrow z^* = 0.$$

Then $(\overline{x}, \overline{y})$ *is a* KKT *point for* $(\mathcal{P}_2')$.

Corollary 4.13 *Assume* $(A_{X,Y,Z})$. *Suppose that* $(x_n, y_n) \subset \operatorname{Gr} F$ *are* KKT *points for* $(\mathcal{P}_2')$ *and* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$ *s.t.* $(x_n, y_n) \to (\overline{x}, \overline{y})$. *Suppose that* F *is Lipschitz-like at* $(\overline{x}, \overline{y}) \in \operatorname{Gr} F$, *and* g *is locally Lipschitz at* $\overline{x}$ *and the following implication holds:*

$$z^* \in Q^*, 0 \in D^* g(\overline{x})(z^*) \Rightarrow z^* = 0.$$

Then $(\overline{x}, \overline{y})$ *is a* KKT *point for* $(\mathcal{P}_2')$.

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