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Abstract In order to show necessary conditions for approximate solutions of vector-valued optimization problems in general spaces we introduce an axiomatic approach for a scalarization scheme. Several examples illustrate this scalarization scheme. Using an Ekeland-type variational principle by Isac [12] and suitable scalarization techniques we prove the optimality conditions under different assumptions concerning the ordering cone and under certain differentiability assumptions for the objective function.

1 Introduction

The aim of our paper is to present necessary conditions for approximate solutions of vector-valued optimization problems in Banach spaces using an Ekeland-type variational principle by Isac [12] under different differentiability properties of the objective function. In the proofs of the assertions a nonlinear scalarization technique plays an important role. We will use an axiomatic approach for the scalarization scheme. In order to apply the variational principle in partially ordered spaces one needs additional assumptions for the ordering cone. Furthermore, the differentiability properties require certain assumptions concerning the ordering cone and the objective function. So a discussion of corresponding ordering and topological assumptions is important for our assertions.

In this paper we will be mainly concerned with the following vector minimization problem (VP) given as

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$$V - \min f(x), \quad \text{subject to} \quad x \in S,$$

where (X,d) is a complete metric space and Y is a locally convex space, $S \subseteq X$, $K \subset Y$ is a proper (i.e., $\{0\} \neq K$, $K \neq Y$) pointed closed convex cone which induces a partial order on Y (i.e., $y^1 \leq_K y^2 \Longleftrightarrow y^2 \in y^1 + K \quad (y^1, y^2 \in Y)$), $f : S \to Y$. We describe the solution concepts for the vector optimization problem (VP) with respect to the ordering cone K in Section 3.

In order to show necessary optimality conditions for the problem (VP) using an Ekeland-type variational principle and differential calculus one needs certain assumptions concerning the spaces, the ordering cone and the objective function. In the assertions of our paper we suppose some of the following assumptions with respect to the spaces:

$(A_{space}1)$ (X,d) is a complete metric space and Y is a locally convex space.
$(A_{space}2)$ X and Y are Banach spaces.
$(A_{space}3)$ X is an Asplund space and $Y = \mathbb{R}^n$.

Remark 1. A Banach space X is said to be an Asplund space (cf. Phelps [23, Def. 1.22]) if every continuous convex function defined on a non-empty open convex subset D of X is Fréchet differentiable at each point of some dense G_δ subset of D. If the dual space X^* of the Banach space X is separable, then X is an Asplund space. Every reflexive Banach space is an Asplund space. The sequence space c_0, and furthermore, the spaces l^p, $L^p[0,1]$ for $1 < p < \infty$ are examples for Asplund spaces. The space l^1 is not an Asplund space.

Concerning the objective function we have different assumptions with respect to the derivatives (see Section 5) that we will use:

$(A_{map}1)$ The vector-valued directional derivative $Df(x,h)$ of $f : X \to Y$ at $x \in X$ in direction $h \in X$ exists for all $x, h \in X$ (cf. Definition 8).
$(A_{map}2)$ $f : X \to Y$ is strictly differentiable at $x \in X$ (cf. Definition 7).
$(A_{map}3)$ $f : X \to Y$ is locally Lipschitz at $x \in X$ (cf. Definition 4).

Furthermore, in order to apply an Ekeland-type variational principle (see Theorem 1) we suppose that (X,d) and Y fulfill $(A_{space}1)$, consider $f : X \to Y$ and formulate the following assumption $(A_{map}4)$ with respect to a closed normal (cf. Definition 1) cone $K \subset Y$ and $k^0 \in K \setminus \{0\}$:

$(A_{map}4)$ For every $u \in X$ and for every real number $\alpha > 0$ the set

$$\left\{ x \in X \mid f(x) - f(u) + k^0 \alpha d(u,x) \in -K \right\}$$

is closed.

Most of our results are related to the non-convex case, however under certain convexity assumptions we get stronger results. Consider the proper pointed closed convex cone K in Y and a non-empty convex subset S of X. The function $f : S \to Y$ is called convex if for all $x^1, x^2 \in X$ and for all $\lambda \in [0,1]$ holds

$$f(\lambda x^1 + (1-\lambda)x^2) \in \lambda f(x^1) + (1-\lambda) f(x^2) - K.$$

$(A_{map}5)$ The function $f : S \to Y$ is convex.

Our paper is organized as follows: In Section 2 we give an overview on cone properties that are important for deriving existence results in infinite dimensional spaces. Especially, we give several examples in general spaces for cones having the Daniell property and for cones with non-empty interior. Solution concepts for vector optimization problems and an Ekeland-type variational principle by Isac [12] under the assumptions $(A_{space}1)$ and $(A_{map}4)$ are presented in Section 3. An axiomatic scalarization scheme that is important for deriving optimality conditions is introduced in Section 4. We present several examples for scalarizing functionals having some of the properties supposed in the scalarization scheme. In Section 5 we recall differentiability properties of vector-valued functions. We show necessary optimality conditions for vector optimization problems under assumptions $(A_{space}2)$ and $(A_{map}4)$ in Section 6. For the case of vector optimization problems where a Lipschitz objective function takes its values in a finite dimensional space we prove necessary conditions for approximate solutions in Section 7 under assumptions $(A_{space}3)$ and $(A_{map}3)$ using the subdifferential calculus by Mordukhovich [18].

2 Properties of cones

In the following we give a survey of some properties of cones in ordered topological spaces; they are compiled in this way in order to make the choice of Y as made in Section 5 plausible.

In order to prove existence results for solutions of optimization problems in infinite dimensional spaces where the solution concept is given by a partial order induced by a closed pointed and convex cone one needs additional assumptions concerning the connections between topology and order (cf. Isac [11]).

First, we recall some corresponding cone properties (that the cone is normal, well-based, nuclear, Daniell property), compare Peressini [22], Isac [11], Isac, Bulavsky, Kalashnikov [14], Jahn [17], Hyers, Isac, Rassias [16], Göpfert, Riahi, Tammer, Zalinescu [9]. In many important cases the ordering cone has not such a property, for instance the usual ordering cone in the space of continuous functions has not a bounded base and the Daniell property is not given. In Figure 1 we give an overview on such additional cone properties and corresponding relations for the case that Y is a Banach space, C and K are proper convex cones in Y. As usual, we denote by

$$K^* := \{y^* \in Y^* \mid y^*(y) \geq 0 \ \forall y \in K\}$$

the continuous dual cone of K, and by

$$K^{\#} := \{y^* \in K^* \mid y^*(y) > 0 \ \forall y \in K \setminus \{0\}\}$$

the quasi-interior of K^*.

In order to study connections between *topology* and *order* we say that a non-empty subset A of the linear space Y is **full** with respect to the convex cone $K \subset Y$ if

$$A = (A+K) \cap (A-K).$$

Definition 1. Let (Y, τ) be a topological linear space and let $K \subset Y$ be a convex cone. Then K is called **normal** (relative to τ) if the origin $0 \in Y$ has a neighborhood base formed by **full** sets w.r.t. K.

Definition 2. Let Y be a Hausdorff topological vector space and $K \subset Y$ a proper convex cone.

(i) K is called **based** if there exists a convex set B, such that $K = \mathbb{R}_+ B$ and $0 \notin cl\, B$.
(ii) K is called **well-based** if there exists a bounded convex set B, such that $K = \mathbb{R}_+ B$ and $0 \notin cl\, B$.
(iii) Let the topology of Y be defined by a family $\mathscr{P}$ of seminorms. K is called **supernormal** or **nuclear** if for each $p \in \mathscr{P}$ there exists $y^* \in Y^*$, such that $p(y) \leq \langle y, y^* \rangle$ for all $y \in K$; it holds $y^* \in K^*$ in this case.
(iv) K is said to be **Daniell** if any non increasing net having a lower bound converges to its infimum.

$$\begin{array}{ccc}
K \text{ well-based} & \overset{K=cl\,K,\; Y=\mathbb{R}^n}{\Longleftarrow} & K \text{ pointed} \\
\Updownarrow \; Y \text{ normed} & & \Uparrow \\
K \text{ nuclear} & \Longrightarrow & K \text{ normal} \\
\Downarrow \; K \text{ complete} & & \\
K \text{ Daniell} & \Longleftarrow & K \text{ compact base.}
\end{array}$$

Fig. 1 Cone properties.

Now, we give a few examples of Daniell cones.

Example 1. First, we recall the following result (cf. Peressini [22], Proposition 3.1, p. 90, 91): If $\{x_\alpha\}_{\alpha \in A}$ is a net which is increasing (decreasing) in a topological vector space (Y, τ) ordered by a closed convex cone K and if x_0 is a cluster point of $\{x_\alpha\}$, then $x_0 = \sup_{\alpha \in A} x_\alpha$ ($x_0 = \inf_{\alpha \in A} x_\alpha$). We recall that a convex cone is regular if any decreasing (increasing) net which has a lower bound (upper bound) is convergent. By the result, cited above we have that any regular cone is Daniell.

Example 2. If $(Y, ||\cdot||)$ is a Banach lattice, that is Y is a Banach space, vector lattice and the norm is absolute, i.e., $||x|| = ||\, |x|\, ||$ for any $x \in Y$, then the cone $Y_+ = \{y \in Y \mid y \geq 0\}$ is Daniell if Y has weakly compact intervals.

Example 3. Finally, a convex cone with a weakly compact base is a Daniell cone.

Proposition 1. *(Isac [11]):*

Let $(Y,\mathscr{P})$ be an Hausdorff locally convex space and $K\subset Y$ a proper convex cone. Then

$$K \text{ well-based} \implies K \text{ nuclear} \implies K \text{ normal}.$$

If Y is a normed space, then

$$K \text{ nuclear} \implies K \text{ well-based}.$$

Remark 2. Among the classical Banach spaces their usual positive cones are well-based only in l^1 and $L^1(\Omega)$, but l^1 is not an Asplund space.

Relations between supernormal (nuclear) cones, Pareto efficiency and geometrical aspects of Ekeland's principle are derived by Isac, Bulavsky and Kalashnikov [14].

Let Y be a topological vector space over $\mathbb{R}$. Assume (Y,K) is at the same time a vector lattice with the lattice operations $x\mapsto x^+$, $x\mapsto x^-$, $x\mapsto |x|$, $(x,y)\mapsto \sup\{x,y\}$ and $(x,y)\mapsto \inf\{x,y\}$.

Definition 3. A set $A\subset Y$ is called *solid*, if $x\in A$ and $|y|\leq|x|$ implies $y\in A$. The space Y is called locally solid, if it possesses a neighborhood of 0 consisting of solid sets.

Lemma 1 ([26]). *The following properties are equivalent:*

(i) Y *is locally solid.*
(ii) K *is normal, and the lattice operations are continuous.*

In order to derive optimality conditions in general spaces (cf. Section 6) there is often the assumption that the (natural) ordering cone has a non-empty interior. Now, we give some examples of convex cones with non-empty interior.

Example 4. Any closed convex cone K in the Euclidean space $(\mathbb{R}^n,\langle\cdot,\cdot\rangle)$ such that K is self-adjoint (i.e., $K=K^{**}$) has a non-empty interior.

Example 5. We consider the space of continuous functions $C[a,b]$ with the norm $||x||=\sup\{|x(t)|\mid t\in[a,b]\}$. The cone of positive functions in $C[a,b]$

$$K_{C[a,b]} := \{x\in C[a,b]\mid x(t)\geq 0\ \forall t\in[a,b]\}$$

has a non-empty interior.

Example 6. Let $Y=l^2(\mathbb{N},\mathbb{R})$ with the well-known structure of a Hilbert space. The convex cone

$$K_{l^2} := \{x=\{x_i\}_{i\geq 0}\mid x_0\geq 0 \text{ and } \sum_{i=1}^{\infty} x_i^2 \leq x_0^2\}$$

has a non-empty interior

$$\text{int}\, K_{l^2} := \{x = \{x_i\}_{i\geq 0} \mid x_0 > 0 \text{ and } \sum_{i=1}^{\infty} x_i^2 < x_0^2\}.$$

Example 7. Let l^∞ be the space of bounded sequences of real numbers, equipped with the norm $||x|| = \sup_{n\in\mathbb{N}}\{|x_n|\}$. The cone

$$K_{l^\infty} := \{x = \{x_n\}_{n\in\mathbb{N}} \mid x_n \geq 0 \text{ for any } n \in \mathbb{N}\}$$

has a non-empty interior (cf. Peressini [22], p. 186).

Example 8. Let $C^1[a,b]$ be the real vector space formed by all real continuously differentiable functions defined on $[a,b]$ $(a,b \in \mathbb{R}, a < b)$, equipped with the norm

$$||f||_1 := \{\int_a^b (f(t))^2\,\mathrm{d}t + \int_a^b (f'(t))^2\,\mathrm{d}t\}^{1/2}$$

for any $f \in C^1[a,b]$. Using a Sobolev's imbedding theorem, we can show that the natural ordering cone

$$K_{C^1} := \{f \in C^1[a,b] \mid f \geq 0\}$$

has a non-empty interior. The proof is based on some technical details (cf. Da Silva [4]).

Example 9. About the locally convex spaces, we put in evidence the following result. If (Y,τ) is a real locally convex space, then for every closed convex cone $K \subset Y$, with non-empty interior, there exists a continuous norm $||\cdot||$ on Y such that K has a non-empty interior in the normed space $(Y, ||\cdot||)$.

Furthermore, in order to show optimality conditions one has sometimes both assumptions: that the ordering cone has a non-empty interior and has the Daniell property. So it is important to ask for examples in infinite dimensional spaces, where the ordering cone has both properties.

Example 10. (see Jahn [17]) Consider the real linear space $L_\infty(\Omega)$ of all (equivalence classes of) essentially bounded functions $f : \Omega \to \mathbb{R}$ $(\emptyset \neq \Omega \subset \mathbb{R}^n)$ equipped with the norm $||\cdot||_{L_\infty(\Omega)}$ given by

$$||f||_{L_\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega}\{|f(x)|\} \text{ for all } f \in L_\infty(\Omega).$$

The ordering cone

$$K_{L_\infty(\Omega)} := \{f \in L_\infty(\Omega) \mid f(x) \geq 0 \text{ almost everywhere on } \Omega\}$$

has a non-empty interior and is weak* Daniell.

3 An Ekeland-type variational principle for vector optimization problems

Concerning the vector optimization problem (VP) we use the following (approximate) solution concepts: Assume $(A_{space}1)$. Let us consider $A \subset Y$, a pointed closed convex cone $K \subset Y$, $\varepsilon \geq 0$ and $k^0 \in K \setminus \{0\}$.

- A point $y_0 \in A$ is said to be an εk^0*-minimal point of* A *with respect to* K, if there exists no other point $y \in A$ such that $y - y_0 \in -\varepsilon k^0 - (K \setminus \{0\})$. We denote this by $y_0 \in \varepsilon k^0 - Eff(A,K)$, where $\varepsilon k^0 - Eff(A,K)$ is the set of εk^0 -minimal points of A with respect to the ordering cone K. A point $x_0 \in S$ is called an εk^0 *-efficient point* of (VP), if $f(x_0) \in \varepsilon k^0 - Eff(f(S),K)$. Is $x_0 \in S$ an εk^0 -efficient point of (VP) with $\varepsilon = 0$ we say that x_0 is an efficient point of (VP) and we write $f(x_0) \in Eff(f(S),K)$.
- A point $y_0 \in A$ is said to be an εk^0 *-properly minimal element of* A *with respect to* K, if there is a closed normal cone $B \subset Y$ with $K \setminus \{0\} \subset \text{int}\, B$ such that $y_0 \in \varepsilon k^0 - Eff(A,B)$. The set of εk^0 -properly minimal elements of A with respect to K is denoted by $\varepsilon k^0 - pEff(A,K)$. A point $x_0 \in S$ is called an εk^0 *-properly efficient* point for (VP), if $f(x_0) \in \varepsilon k^0 - Eff(f(S),B)$ where B is a closed normal cone with $K \setminus \{0\} \subset \text{int}\, B$. Is $x_0 \in S$ an εk^0 -properly efficient point of (VP) with $\varepsilon = 0$ we say that x_0 is a properly efficient point of (VP) and we write $f(x_0) \in pEff(f(S),K)$.

We will apply a vector-valued variational principle of Ekeland's type in order to show necessary conditions for approximately efficient solutions of the vector optimization problem (VP). There are many vector-valued variants of Ekeland's variational principle (and equivalent assertions) with different assumptions concerning the ordering cone in Y and concerning the properties of the objective function $f : X \to Y$ (cf. [12], [13], [19], [27], [28]). Here we recall the variational principle by Isac [12, Theorems 4 and 7], [16, Theorem 8.4] that is shown for the case that the ordering cone K in Y is normal without assuming that the interior of K is non-empty.

Theorem 1. *(Isac [12]) Assume* $(A_{space}1)$, K *is a closed normal cone,* $k^0 \in K \setminus \{0\}$. *Furthermore, suppose that for* $f : X \to Y$ *the assumption* $(A_{map}4)$ *with respect to* K *and* k^0 *is fulfilled. If* $\varepsilon > 0$ *is an arbitrary real number and* $x_0 \in X$ *is an element with* $f(x_0) \leq_K f(x) + \varepsilon k^0$ *for all* $x \in X$ *then there exists* $x_\varepsilon \in X$ *such that* $f(x_\varepsilon) \leq_K f(x_0)$, $d(x_\varepsilon, x_0) \leq \sqrt{\varepsilon}$ *and, moreover,*

$$f(x) + k^0 \sqrt{\varepsilon} d(x, x_\varepsilon) \leq_K f(x_\varepsilon) \quad \Longrightarrow \quad x = x_\varepsilon. \tag{1}$$

Remark 3. The assertion (1) in Theorem 1 means that x_ε is an efficient element of the perturbed objective function $f_{\sqrt{\varepsilon}k^0}(x) := f(x) + k^0 \sqrt{\varepsilon} d(x, x_\varepsilon)$ with respect to K, i.e., $f_{\sqrt{\varepsilon}k^0}(x_\varepsilon) \in Eff(f_{\sqrt{\varepsilon}k^0}(X), K)$.

4 Nonlinear scalarization scheme

In order to prove optimality conditions we will introduce an axiomatic approach for scalarization by means of (in general nonlinear) functionals. We consider a linear topological space Y, a proper set $K \subset Y$ and a scalarizing functional $\varphi : Y \to \mathbb{R} \cup \{\pm\infty\}$ having some of the following properties:

$(A_\varphi 1)$ The functional φ is K-monotone, i.e., $y, w \in Y$, $y \in w - K$ implies $\varphi(y) \leq \varphi(w)$.
$(A_\varphi 1')$ The functional φ is strictly K-monotone, i.e., $y, w \in Y$, $y \in w - (K \setminus \{0\})$ implies $\varphi(y) < \varphi(w)$.
$(A_\varphi 2)$ The functional φ is convex.
$(A_\varphi 2')$ The functional φ is sublinear.
$(A_\varphi 2'')$ The functional φ is linear.
$(A_\varphi 3)$ The functional φ enjoys the translation property

$$\forall s \in \mathbb{R},\ \forall y \in Y :\ \varphi\left(y + sk^0\right) = \varphi(y) + s. \tag{2}$$

$(A_\varphi 4)$ The functional φ is lower continuous.
$(A_\varphi 4')$ The functional φ is continuous.

Examples for functionals satisfying the axioms given above are listed in the following:

Example 11. Assume that B is a closed proper subset of Y and $K \subset Y$ is a proper set with $B + K \subset B$. Let $k^0 \in Y \setminus \{0\}$ such that $B + [0, +\infty)k^0 \subset B$. Consider the functional $\varphi := \varphi_{B,k^0} : Y \to \mathbb{R} \cup \{\pm\infty\}$, defined by

$$\varphi(y) := \inf\left\{t \in \mathbb{R} \mid y \in tk^0 - B\right\}. \tag{3}$$

We use the convention $\inf \emptyset = +\infty$. Then it holds $\operatorname{dom} \varphi = \mathbb{R}k^0 - B$.

If $B = K$ is a proper closed convex cone and $k^0 \in \mathit{int}\, K$ the functional (3) fulfills $(A_\varphi 1)$, $(A_\varphi 2')$, $(A_\varphi 3)$ and $(A_\varphi 4')$.
Moreover, if B is a proper closed convex subset of Y with $B + (K \setminus \{0\}) \subset \operatorname{int} B$, B does not contain lines parallel to k^0 (i.e., $\forall y \in Y,\ \exists t \in \mathbb{R}\ :\ y + tk^0 \notin D$) and $\mathbb{R}k^0 - B = Y$ the functional (3) is finite-valued and fulfills $(A_\varphi 1')$, $(A_\varphi 2)$, $(A_\varphi 3)$ and $(A_\varphi 4)$. These properties of the functional (3) are shown in [9, Theorem 2.3.1].

Example 12. The scalarizing functional by Pascoletti and Serafini [21] for a vector optimization problem (VP)

$$\mathrm{V} - \min_{x \in S \subset \mathbb{R}^n} f(x) = (f_1(x), \ldots, f_m(x))^T$$

(where $Y = \mathbb{R}^m$, $K = \mathbb{R}^m_+$, S convex and $f_i : S \to \mathbb{R}$ convex for all $i = 1, \ldots, m$) given by

$$\min\ t \tag{4}$$

subject to the constraints

$$f(x) \in a + tr - K,$$

$$x \in S,\ t \in \mathbb{R},$$

(with parameters $a \in \mathbb{R}^m$ and $r \in \operatorname{int} \mathbb{R}^m_+$) satisfies the axioms $(A_\varphi 1)$, $(A_\varphi 2)$, $(A_\varphi 3)$, $(A_\varphi 4')$ (cf. [9, Theorem 2.3.1]).

Example 13. The following functional was introduced by Hiriart-Urruty [15]: Assume that Y is a normed space. For a non-empty set $A \subset Y$, $A \neq Y$, the oriented distance function $\Delta_A : Y \to \mathbb{R}$ is given as $\Delta_A(y) = d_A(y) - d_{Y \setminus A}(y)$ (where $d_A(y) = \inf\{||a - y|| \mid a \in A\}$ is the distance function to a set A). It is well known that this function has the following properties (see [29, Proposition 3.2]):
(i) Δ_A is Lipschitzian of rank 1.
(ii) If A is convex, then Δ_A is convex and if A is a cone, then Δ_A is positively homogeneous.
(iii) Assume that A is a closed convex cone. If $y_1, y_2 \in Y$ with $y_1 - y_2 \in A$, then $\Delta_A(y_1) \leq \Delta_A(y_2)$.

The functional Δ_A satisfies the axioms $(A_\varphi 1)$, $(A_\varphi 2)$, $(A_\varphi 4')$ if A is a closed convex cone.

Example 14. Certain nonlinear functionals are used in financial mathematics in order to express a risk measure (for example a valuation of risky investments) with respect to an acceptance set $B \subset Y$. Artzner, Delbean, Eber and Heath [1] (compare Heyde [10]) introduced coherent risk measures. Risk measures are functionals $\mu : Y \to \mathbb{R} \cup \{\pm\infty\}$, where Y is a vector space of random variables. In the papers by Artzner, Delbean, Eber and Heath [1] and Rockafellar, Uryasev and Zabarankin [24] the following properties of coherent risk measures μ are supposed:

$(P1)$ $\mu(y + tk^0) = \mu(y) - t$,

$(P2)$ $\mu(0) = 0$ and $\mu(\lambda y) = \lambda \mu(y)$ for all $y \in Y$ and $\lambda > 0$,

$(P3)$ $\mu(y^1 + y^2) \leq \mu(y^1) + \mu(y^2)$ for all $y^1, y^2 \in Y$,

$(P4)$ $\mu(y^1) \leq \mu(y^2)$ if $y^1 \geq y^2$.

The sublevel set $L_\mu(0) =: B$ of μ to the level 0 is a convex cone and corresponds to the acceptance set. It can be shown that a coherent risk measure admits a representation as

$$\mu(y) = \inf\{t \in \mathbb{R} \mid y + tk^0 \in B\}. \tag{5}$$

It can be seen that a coherent risk measure can be identified with the functional $\varphi_{B,k^0}(-y)$ (see (3)) by

$$\varphi_{B,k^0}(y) = \mu(-y).$$

We get corresponding properties $(A_\varphi 1)$, $(A_\varphi 2')$, $(A_\varphi 3)$ for the functional $\mu(-y)$ like in Example 11 for the functional φ_{B,k^0} depending from the properties of the set B, i.e., of the acceptance set B in Mathematical Finance.

Examples for coherent risk measures are the **conditional value at risk** (cf. [7], Section 4.4, Definition 4.43) and the **worst-case risk measure** (cf. Example 16).

Example 15. (**Value at Risk**) Let Ω be a fixed set of scenarios. A financial position is described by a mapping $x : \Omega \to \mathbb{R}$ and x belongs to a given class $\mathscr{X}$ of financial positions. Assume that $\mathscr{X}$ is the linear space of bounded measurable functions containing the constants on some measurable space (Ω, A). Furthermore, let P be a probability measure on (Ω, A). A position x is considered to be acceptable if the probability of a loss is bounded by a given level $\lambda \in (0,1)$, i.e., if $P[x < 0] \leq \lambda$. The corresponding monetary risk measure $V@R_\lambda$, defined by

$$V@R_\lambda(x) := \inf\{m \in \mathbb{R} \mid P(m + x < 0) \leq \lambda\}$$

is called *Value at Risk*. $V@R_\lambda$ is the smallest amount of capital which, if added to x and invested in the risk-free asset, keeps the probability of a negative outcome below the level λ.

$V@R_\lambda$ is positively homogeneous but in general it is not convex (cf. Föllmer and Schied [7], Example 4.11), this means that $(A_\varphi 2)$ and $(A_\varphi 2')$ are not fulfilled.

Example 16. (**Worst-case risk measure**) Consider the *worst-case risk measure* ρ_{max} defined by

$$\rho_{max}(x) := -\inf_{w \in \Omega} x(w) \text{ for all } x \in \mathscr{X},$$

where Ω is a fixed set of scenarios, $x : \Omega \to \mathbb{R}$ and x belongs to a given class $\mathscr{X}$ of financial positions. Assume that $\mathscr{X}$ is the linear space of bounded measurable functions containing the constants on some measurable space (Ω, A). The value $\rho_{max}(x)$ is the least upper bound for the potential loss which occur in any scenario. ρ_{max} is a coherent risk measure (cf. Föllmer and Schied [7], Example 4.8) such that we get the properties mentioned in Example 14.

5 Differentiability properties of vector-valued functions

In this section we suppose that assumption $(A_{space}2)$ is fulfilled and consider $f : X \to Y$. Furthermore, assume that $K \subset Y$ is a proper pointed closed convex cone.

First of all, we introduce a concept of a vector-valued local Lipschitz property for $f : X \to Y$.

Definition 4 ([26]). $f : X \to Y$ is called **locally Lipschitz** at $x \in X$, if there is a function $P : X \times \mathbb{R} \to K$ such that

$$\Big| \frac{f(u+th) - f(u)}{t} \Big| \leq P(h,\varepsilon) \qquad \forall u \in U(x,\varepsilon),\ t \in (0,\varepsilon) \tag{6}$$

for all sufficiently small $\varepsilon > 0$. Therein, P is supposed to be continuous in h, and $\lim_{h\to 0} P(h,\varepsilon) = 0$ for each $\varepsilon > 0$.

This property is a basis for the definition of a directional derivative, which follows the idea of the Clarke directional derivative of real-valued functions. First, we recall Clarke's generalized directional derivative:

Definition 5 ([3]). Let X be a Banach space and let f be Lipschitz near a given point x and let v be any other vector in X. A mapping $f^\circ : X \to Y$ defined by

$$f^\circ(x,v) := \lim_{t\downarrow 0} \sup_{y\to x} \frac{f(y+tv) - f(y)}{t}$$

is called Clarke's generalized directional derivative of f at x in direction v.

Definition 6. Clarke's tangent cone (contingent cone) is defined by

$$\mathscr{T}(S,x) := \{h \in X \mid d_S^\circ(x,h) = 0\},$$

where $d_S(x) := \inf\{||y - x|| \mid y \in S\}$ is the distance function to a non-empty set $S \subset X$, X is a Banach space and $x \in X$.

This cone can be described also in the following way:

$$\mathscr{T}(S,x) := \{h \in X \mid \forall \{x_n\}_{n\in N} \subseteq S,\ x_n \to x, \forall \{t_n\}_{n\in N} \in (0,+\infty),\ t_n \to 0$$

$$\exists \{h_n\}_{n\in N} \subseteq X :\ h_n \to h,\ x_n + t_n h_n \in S\ \forall n \in N\}.$$

Furthermore, we study strictly differentiable mappings:

Definition 7 ([3]). $f : X \to Y$ is called strictly differentiable at $x \in X$ if there is a linear continuous mapping $D_S f(x) : X \to Y$ such that for each $h \in X$, for each sequence $\{t_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+$ and for each sequence $\{x_n\}_{n\in\mathbb{N}} \in X$ with $x_n \to x$ and $t_n \to 0$ the following holds

$$D_S f(x)(h) = \lim_{n\to\infty} \frac{f(x_n + t_n h) - f(x_n)}{t_n},$$

provided the convergence is uniform for h in compact sets.

Remark 4. If f is Lipschitz near x the convergence is uniform for h in compact sets. Definition 7 is a certain "Hadamard type strict derivative".

Definition 8 ([26]). We define the vector-valued directional derivative $Df(x,h)$ of f at $x \in X$ in direction $h \in X$ by $Df(x,\cdot) : X \to Y$,

$$Df(x,h) := \lim_{\varepsilon \downarrow 0} \sup_{u \in U(x,\varepsilon),\ t \in (0,\varepsilon)} \frac{f(u+th) - f(u)}{t}.$$

Remark 5. In the following we assume that certain directional derivatives exist. In order to have sufficient conditions for the existence of the directional derivative one can suppose that Y is a Daniell locally convex vector lattice and f is locally Lipschitz.

Using the vector-valued directional derivative we introduce the subdifferential of $f : X \to Y$:

Definition 9. The subdifferential of $f : X \to Y$ at the point $x \in X$ is defined by

$$\partial f(x) := \{L \in \mathscr{L}(X,Y) \mid L(h) \leq_K Df(x,h)\ \forall h \in X\},$$

where $\mathscr{L}(X,Y)$ denotes the space of linear continuous operators from X to Y.

Under certain conditions on a set $D \subset Y$, we can conclude from the derivatives being an element of D that certain differential quotients are elements of D as well:

Lemma 2 ([26]). *Let $D \subset Y$ be such that*

(i) $\quad$ *int* $D \neq \emptyset$
(ii) $\quad$ *int* $D - K \subset$ *int* D.

Assume $Df(x,h) \in$ int D for $x,h \in X$. Then there is a real number $\varepsilon(h) > 0$ such that

$$\frac{f(u+th) - f(u)}{t} \in \text{int } D \qquad \forall u \in U(x,\varepsilon(h)),\ t \in (0,\varepsilon(h)). \tag{7}$$

What is more, also with small perturbations of the direction h an estimation for the differential quotient can be given.

Lemma 3. *Assume that $D \subset Y$ satisfies the conditions (i) and (ii) from Lemma 2, $x,h \in X$. Moreover, suppose that f is locally Lipschitz at $x \in X$, the vector-valued directional derivative $Df(x,h)$ exists and $Df(x,h) \in$ int D. Then, for each neighborhood V of 0 in Y satisfying $Df(x,h) + V \subset$ int D there is a real number $\varepsilon(h) > 0$ and a neighborhood U' of h such that*

$$\frac{f(u+th') - f(u)}{t} \in Df(x,h) + V - K \qquad \forall u \in U(x,\varepsilon(h)),\ h' \in U',\ t \in (0,\varepsilon(h)). \tag{8}$$

In particular this implies

$$\frac{f(u+th') - f(u)}{t} \in \text{int } D \qquad \forall u \in U(x,\varepsilon(h)),\ h' \in U',\ t \in (0,\varepsilon(h)). \tag{9}$$

Proof: There is a neighborhood $V \subset Y$ of 0 such that $Df(x,h) + V \subset$ *int* D. Without loss of generality, we assume V to be solid.

Choose a solid neighborhood V' of 0 such that $V' + V' \subset V$; furthermore, choose $\varepsilon_0 > 0$ such that

$$\sup_{u\in U(x,\varepsilon_0),\ t\in(0,\varepsilon_0)} \frac{f(u+th)-f(u)}{t} \in Df(x,h)+V',\ \text{hence} \tag{10}$$

$$\frac{f(u+th)-f(u)}{t} \in Df(x,h)+V'-K \tag{11}$$
$$\forall u\in U(x,\varepsilon_0),\ t\in(0,\varepsilon_0).$$

Finally, fix $U'\in\mathscr{U}(h)$ such that for each $h'\in U'$ holds

$$P(h'-h,\varepsilon_0)\in V',$$

with P being the function corresponding to (6). Now, define $\varepsilon(h):=\min\{\frac{\varepsilon_0}{2},\ \frac{\varepsilon_0}{2\|h\|}\}$. For each $u\in U(x,\varepsilon(h))$ and each $t\in(0,\varepsilon)$ we have $u+th\in u(x,\varepsilon_0)$. Hence, the vector-valued local Lipschitz property of f yields for these u and t

$$\left|\frac{f(u+th')-f(u+th)}{t}\right| \le P(h'-h,\varepsilon_0)\in V'. \tag{12}$$

The solidity of V' now leads to

$$\frac{f(u+th')-f(u+th)}{t}\in V'. \tag{13}$$

For each $u\in U(x,\varepsilon(h))$, $t\in(0,\varepsilon(h))$ and $h'\in U'$ we thus have derived

$$\frac{f(u+th')-f(u)}{t} = \underbrace{\frac{f(u+th)-f(u)}{t}}_{\in Df(x,h)+V'-K} + \underbrace{\frac{f(u+th')-f(u+th)}{t}}_{\in V'}$$
$$\subset Df(x,h)+V-K\subset \mathit{int}\, D.$$

□

Lemma 4. *Assume that $g:X\to Y$ is sublinear and $f:X\to Y$ convex. Then it holds:*

(i) $Dg(x,h)\le g(h)$ *for all* $x\in X$ *and all* $h\in X$.
(ii) $Dg(0,h)=g(h)$ *for all* $h\in X$.
(iii) $\partial f(x)=\partial Df(x,\cdot)(0)$.

Proof: (i). By the sublinearity of g we have for all $x,h\in X$ and $t\in(0,1)$

$$g(u+th)\le g(u)+g(th),\qquad \text{also}$$
$$\frac{g(u+th)-g(u)}{t}\le\frac{g(th)}{t}=g(h);$$

the last equality holds because g is positively homogeneous. Consequently, it follows that
$\sup_{u\in U(x,\varepsilon),\ t\in(0,\varepsilon)}\frac{g(u+th)-g(u)}{t}\le g(h)$ for $\varepsilon<1$. For the limes (which is guaranteed to exist) this implies

$$\lim_{\varepsilon \downarrow 0} \sup_{u \in U(x,\varepsilon),\ t \in (0,\varepsilon)} \frac{g(u+th)-g(u)}{t} \leq g(h).$$

(ii) For $x = 0$ the supremum is attained at $u = 0$.

(iii) Set $\hat{g} = Df(x,\cdot)$. Then $\hat{g}$ is subadditive and positively homogeneous (cf. Staib [26, Lemma 1.2.6]. Hence $D\hat{g}(0,h) = \hat{g}(h)$ by (ii). Thus, $D\hat{g}(0,h) = Df(x,h)$ holds according to the definition of $\hat{g}$, and the assertion follows.

□

In Section 7 we will show necessary conditions for approximately efficient elements of a vector optimization problem using the Mordukhovich subdifferential. Here we recall the corresponding definition.

Definition 10. [18] Let S be a non-empty subset of X and let $\alpha \geq 0$. Given $x \in \mathrm{cl}\ S$ the non-empty set

$$N^F_\alpha(S,x) = \left\{ x^* \in X^* : \limsup_{y \to x, y \in S} \frac{\langle x^*, y-x \rangle}{\|y-x\|} \leq \alpha \right\}$$

is called the set of Fréchet α-normals to S at x. When $\alpha = 0$, then the above set is a cone, called the set of Fréchet normals and denoted by $N^F(S,x)$.
Let $x_0 \in \mathrm{cl}\, S$. The non-empty cone

$$N_L(S,x_0) = \limsup_{x \to x_0, \alpha \downarrow 0} N^F_\alpha(S,x)$$

is called the limiting normal cone or the Mordukhovich normal cone to S at x_0.

Definition 11. [18] Let $f : X \to \mathbb{R} \cup \{\pm\infty\}$ be a given proper function and $x_0 \in \mathrm{dom} f$. The set

$$\partial_L f(x_0) = \{x^* \in X^* : (x^*, -1) \in N_L(\mathrm{epi} f, (x_0, f(x_0))\}$$

is called the limiting subdifferential or the Mordukhovich subdifferential of f at x_0. If $x_0 \notin \mathrm{dom} f$, then we set $\partial_L f(x_0) = \emptyset$.

Remark 6. If $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is convex, then $\partial_L f(x)$ coincides with the Fenchel subdifferential $\partial f(x)$.

6 Necessary optimality conditions for vector optimization problems in general spaces based on directional derivatives

In this section we derive necessary conditions for approximate solutions of the vector optimization problem (VP). Under the assumption that the ordering cone K

has a non-empty interior we show in Theorem 2 necessary conditions for approximately efficient elements of (VP). Furthermore, in Theorem 3 we derive necessary conditions for approximately efficient points of (VP) without the assumption that $\text{int}\, K \neq \emptyset$. Here $\varepsilon > 0$ and $k^0 \in K \setminus \{0\}$ are fixed arbitrarily and represent an admissible error of the approximate solutions.

Lemma 5. *Suppose that $K \subset Y$ is a pointed closed convex cone with* $\text{int}\, K \neq \emptyset$*. Fix an arbitrary $c > 0$, $k^0 \in \text{int}\, K$ and set $D = -ck^0 - K$. Then it holds* $\text{int}\, D - K \subset \text{int}\, D$.

Proof: Fix an arbitrary $y \in \text{int}\, D - K$. This means $y = y_1 - y_2$ with certain $y_1 \in \text{int}\, D$ and $y_2 \in K$, where again $y_1 = -ck^0 - y_3$ with an $y_3 \in \text{int}\, K$. Hence we have $y = -ck^0 - (y_2 + y_3)$. Now, by the convexity of K we conclude $(y_2 + y_3) \in \text{int}\, K$ and consequently $y \in \text{int}\, D$.

□

Theorem 2. *Consider the vector optimization problem (VP). Suppose that $K \subset Y$ is a closed normal cone, int $K \neq \emptyset$ and $k^0 \in K \setminus \{0\}$. Assume $(A_{space}2)$, $(A_{map}1)$, and $(A_{map}4)$ with respect to K and k^0. Furthermore, suppose that $S \subseteq X$ is closed. If $\varepsilon > 0$ is an arbitrary real number and $x_0 \in S$ is an element with $f(x_0) \leq_K f(x) + \varepsilon k^0$ for all $x \in S$ then there exists an element $x_\varepsilon \in S$ with $f(x_\varepsilon) \leq_K f(x_0)$ and*

(i) $\|x_0 - x_\varepsilon\| \leq \sqrt{\varepsilon}$;
(ii) $Df(x_\varepsilon, h) \notin -\sqrt{\varepsilon} k^0 \|h\| - \text{int}\, K$ *for all* $h \in \mathscr{T}(S, x_\varepsilon)$.
(iii) $C \subset \mathscr{T}(S, x_\varepsilon)$*, C is a convex cone, implies the existence of $y^* \in K^*$, $y^* \neq 0$, satisfying*

$$y^* \circ Df(x_\varepsilon, h) \geq -\sqrt{\varepsilon}\, y^*(k^0) \text{ for all } h \in C \text{ with } \|h\| = 1.$$

(iv) *Assume $(A_{map}5)$ and $S = X$. For C as above there is an element $y^* \in K^* \setminus \{0\}$ such that*

$$0 \in \overline{y^* \circ \partial f(x_\varepsilon)}^{w^*} - C^* + \sqrt{\varepsilon}\, y^*(k^0) B^0_{X^*}$$

(where ∂ is the usual convex subdifferential and $B^0_{X^*}$ is the unit ball in X^*).
If the order intervals in Y are weakly compact, there holds even

$$0 \in y^* \circ \partial f(x_\varepsilon) - C^* + \sqrt{\varepsilon}\, y^*(k^0) B^0_{X^*}.$$

Proof: The assumptions $(A_{space}2)$, $(A_{map}4)$ and that K is a closed normal cone are fulfilled. Furthermore, since S is a closed set in a Banach space it is a complete metric space endowed with the distance given by the norm such that the assumptions of Theorem 1 are fulfilled. Choose $x_\varepsilon \in S$ according to Theorem 1; this directly implies (i).
(ii) Furthermore, for the element $x_\varepsilon \in S$ the following holds

$$f_{\sqrt{\varepsilon}k^0}(x_\varepsilon) \in Eff(f_{\sqrt{\varepsilon}k^0}[S], K), \quad \text{where} \quad f_{\sqrt{\varepsilon}k^0}(x) := f(x) + \sqrt{\varepsilon} k^0 \|x - x_\varepsilon\|$$

taking into account Theorem 1. This means

$$f(x) + \sqrt{\varepsilon} k^0 \|x - x_\varepsilon\| \notin f(x_\varepsilon) - K \setminus \{0\} \qquad \forall x \in S.$$

Fix an $h \in \mathscr{T}(S, x_\varepsilon)$. Then there are sequences $h_n \to h$, $t_n \downarrow 0$ such that $x_\varepsilon + t_n h_n \in S$. For these we have

$$f(x_\varepsilon + t_n h_n) + \sqrt{\varepsilon} k^0 t_n \|h_n\| - f(x_\varepsilon) \notin -K \setminus \{0\},$$

hence

$$\frac{f(x_\varepsilon + t_n h_n) - f(x_\varepsilon)}{t_n} \notin -\sqrt{\varepsilon} k^0 \|h_n\| - K \setminus \{0\}. \tag{14}$$

Assume now $Df(x_\varepsilon, h) \in -\sqrt{\varepsilon} k^0 \|h\| - \text{int } K$; this means

$$Df(x_\varepsilon, h) = -\sqrt{\varepsilon} k^0 \|h\| - y_1$$

with an $y_1 \in \text{int } K$.

Choose an neighborhood V of 0 in Y in such a way that $y_1 + 2V \subset \text{int } K$. According to Lemma 3 with $D := -\sqrt{\varepsilon} k^0 \|h\| - K$ there is a number $\varepsilon(h) > 0$ and a neighborhood U' of h, such that (in particular, with $u = x_\varepsilon$) it holds

$$\frac{f(x_\varepsilon + t h') - f(x_\varepsilon)}{t} \in Df(x_\varepsilon, h) + V - K = -\sqrt{\varepsilon} k^0 \|h\| - y_1 + V - K$$

for all $t \in (0, \varepsilon(h))$ and $h' \in U'$. For sufficiently large indices n this implies

$$\frac{f(x_\varepsilon + t_n h_n) - f(x_\varepsilon)}{t_n} \in -\sqrt{\varepsilon} k^0 \|h\| - y_1 + V - K.$$

Finally, choose n large enough to satisfy

$$-\sqrt{\varepsilon} k^0 \|h\| = -\sqrt{\varepsilon} k^0 \|h_n\| + v_n \quad \text{with a } v_n \in V,$$

which is possible because of $h_n \to h$. This, however, means

$$\frac{f(x_\varepsilon + t_n h_n) - f(x_\varepsilon)}{t_n} \in -\underbrace{\sqrt{\varepsilon} k^0 \|h_n\| - y_1 + 2V}_{\subset \text{int } D} - K \subset \text{int } D,$$

contradicting (14).

(iii) Let B^0 denote the unit ball in X. The set $Df(x_\varepsilon, C \cap B^0) + K$ is convex. Since $-\sqrt{\varepsilon} k^0 \|h\| - \text{int } K \supset -\sqrt{\varepsilon} k^0 - \text{int } K$ for elements h with $\|h\| \leq 1$, we have $(Df(x_\varepsilon, C \cap B^0) + K) \cap (-\sqrt{\varepsilon} k^0 - \text{int } K) = \emptyset$ by (ii); this means

$$\left[(Df(x_\varepsilon, C \cap B^0) + \sqrt{\varepsilon} k^0 + K\right] \cap - \text{int } K = \emptyset.$$

By a separation argument we find an element $y^* \in Y^*$ with $y^* \neq 0$ and an $\beta \in \mathbb{R}$ satisfying

$$y^*(y) \geq \beta \qquad \forall y \in Df(x_\varepsilon, C \cap B^0) + \sqrt{\varepsilon}k^0 + K \tag{15}$$
$$y^*(y) < \beta \qquad \forall y \in -\text{int } K. \tag{16}$$

Since $0 \in \text{cl int } (-K) = -K$, from (16) follows that $\beta \geq 0$.

Now assume that $y^*(y) > 0$ for an element $y \in -\text{int } K$. For a certain positive multiple $cy \in -\text{int } K$ of y this implies $y^*(cy) > \beta$, contradicting (16). Hence, $y^*(y) \leq 0$ for each $y \in -\text{int } K$; this inequality even holds for each $y \in -\text{cl int } K$ because of the continuity of y^*. This means $y^* \in K^* \setminus \{0\}$.

In the following we exploit (15):
Let $h \in C$, $\|h\| = 1$. With $y \in Df(x_\varepsilon, h) + \sqrt{\varepsilon}k^0 + v$ ($v \in K$ arbitrary) we also have $y^*(y) \geq 0$. Hence

$$y^*(Df(x_\varepsilon, h) + \sqrt{\varepsilon}k^0 + v) \geq 0;$$

in particular, with $v = 0$ we get

$$y^* \circ Df(x_\varepsilon, h) \geq -\sqrt{\varepsilon}\, y^*(k^0).$$

(iv). For C as in (iii) choose $y^* \in K^*$ according to (iii); this is,

$$y^* \circ Df(x_\varepsilon, h) \geq -\sqrt{\varepsilon}\, y^*(k^0)\|h\|$$

for all $h \in C$. Define $p(h) := y^* \circ Df(x_\varepsilon, h) + \sqrt{\varepsilon}\, y^*(k^0)\|h\|$ for $h \in C$ and the sets S_1 and S_2 in $X \times \mathbb{R}$ by

$$S_1 := \text{epi } (p),$$
$$S_2 := \{(h, \alpha) \in X \times \mathbb{R} : \quad h \in C,\ \alpha \leq 0\}.$$

Both S_1 as well as S_2 is convex. Furthermore, we have int $S_1 \neq \emptyset$ and int $S_1 \cap S_2 = \emptyset$. By a separation argument we conclude the existence of an $(x^*, \alpha^*) \in (X \times \mathbb{R})^* = X^* \times \mathbb{R}$, $(x^*, \alpha^*) \neq 0$ and $\beta \in \mathbb{R}$ satisfying

$$(x^*, \alpha^*)(h, \alpha) \geq \beta \qquad \forall (h, \alpha) \in S_1, \tag{17}$$
$$(x^*, \alpha^*)(h, \alpha) \leq \beta \qquad \forall (h, \alpha) \in S_2. \tag{18}$$

With $(0,0) \in S_1 \cap S_2$ we deduce $\beta = 0$, and $(0, \alpha) \in \text{int } S_1$ for $\alpha > 0$ yields $\alpha^* > 0$. Setting $\alpha = 0$ in (17) leads to $\frac{x^*}{-\alpha^*} \in C^*$. Using (18) this yields

$$\frac{x^*}{-\alpha^*}(h) \leq y^* \circ Df(x_\varepsilon, h) + \sqrt{\varepsilon}\|h\|\, y^*(k^0).$$

Since $y^* \in K^*$, $y^* \circ Df(x_\varepsilon, h)$ is a convex function in h; this is passed on to the whole right side of the above inequality. Hence, we have

$$\frac{x^*}{-\alpha^*} \in \partial(y^* \circ Df(x_\varepsilon, \cdot) + \sqrt{\varepsilon}\|\cdot\|\, y^*(k^0))(0)$$

with the usual convex subdifferential ∂. Subdifferential calculus further yields

$$\begin{aligned}\frac{x^*}{-\alpha^*} &\in \partial(y^* \circ Df(x_\varepsilon,\cdot))(0) + \partial(\sqrt{\varepsilon}\|\cdot\|\, y^*(k^0))(0)\\ &\subset \overline{y^* \circ \partial(Df(x_\varepsilon,\cdot))(0)}^{w^*} + \sqrt{\varepsilon}\, y^*(k^0) B^0_{X^*}.\end{aligned}$$

By Lemma 4 (iii), under convexity assumptions concerning f, this implies

$$\frac{x^*}{-\alpha^*} \in \overline{y^* \circ \partial f(x_\varepsilon)}^{w^*} + \sqrt{\varepsilon}\, y^*(k^0) B^0_{X^*}.$$

This yields

$$0 \in \overline{y^* \circ \partial f(x_\varepsilon)}^{w^*} + \sqrt{\varepsilon}\, y^*(k^0) B^0_{X^*} - C^*.$$

Regarding the formula when the order intervals are weakly compact: We have to show that $y^* \circ \partial f(x_\varepsilon)$ is weakly closed. This follows by an argumentation along the lines of [2, Theorem 6.3] for convex operators.

□

Remark 7. The assertions in Theorem 2 are corrections of corresponding results in [26, Theorem 2.2.1] and an extension of the results in [26, Theorem 2.2.1] to approximate solutions.

Using a closed normal cone $B \subset Y$ with $K \setminus \{0\} \subset \text{int}\, B$ like in the concept of proper efficiency we can drop the strong assumption int $K \neq \emptyset$ or the ordering cone K in Y. We will show a necessary condition under the assumption that f is strictly differentiable using the abstract nonlinear scalarizing scheme and Clarke's strict derivative $D_S f(x)$ of f at $x \in X$.

Theorem 3. *Consider the vector optimization problem (VP) with $S = X$ assuming $(A_{space}2)$, $(A_{map}2)$. Let $K \subset Y$ be a pointed closed convex cone, $k^0 \in K \setminus \{0\}$ and $B \subset Y$ a closed normal cone with $K \setminus \{0\} \subset$ int B. We suppose that $(A_{map}4)$ with respect to B and k^0 is fulfilled. If $\varepsilon > 0$ is an arbitrary real number and there exists an element $x_0 \in X$ such that $f(x_0) \leq_B f(x) + \varepsilon k^0$ for all $x \in X$ then there is an element $x_\varepsilon \in X$ with $f(x_\varepsilon) \leq_B f(x_0)$ such that*

(i) $\|x_0 - x_\varepsilon\| \leq \sqrt{\varepsilon}$.
(ii) *There exists $y^* \in K^\#$ such that*

$$||y^* \circ D_S f(x_\varepsilon)||_* \leq \sqrt{\varepsilon}.$$

Proof: Consider $x_0 \in X$ such that $f(x_0) \leq_B f(x) + \varepsilon k^0$ for all $x \in X$, where B is a closed normal cone with $K \setminus \{0\} \subset \text{int}\, B$. Because of $(A_{space}2)$ (this implies $(A_{space}1)$) and $(A_{map}4)$ with respect to B and k^0 the assumptions of Theorem 1 are fulfilled.
According to Theorem 1 we get the existence of an element $x_\varepsilon \in X$ such that (i) holds. Furthermore, for x_ε the following holds $f(x_\varepsilon) \in Eff(f_{\sqrt{\varepsilon}k^0}(X), B)$, where

$f_{\sqrt{\varepsilon}k^0}(x) := f(x) + \sqrt{\varepsilon}k^0\|x - x_\varepsilon\|$.
This means

$$f(x) + \sqrt{\varepsilon}k^0\|x - x_\varepsilon\| \notin f(x_\varepsilon) - (B \setminus \{0\}) \qquad \forall x \in X,$$

i.e.,

$$f(x) \notin f(x_\varepsilon) - \sqrt{\varepsilon}k^0\|x - x_\varepsilon\| - (B \setminus \{0\}) \qquad \forall x \in X. \tag{19}$$

Consider the functional (3) and take $\varphi(y) := \varphi_{B,k^0}(y - f(x_\varepsilon))$. For B, K and k^0 the assumptions in Example 11 are fulfilled and so we get the properties $(A_\varphi 1')$ with respect to K, $(A_\varphi 2)$ and $(A_\varphi 4')$ for φ. Assume that there exists $x \in X$ such that

$$\varphi(f(x)) + \sqrt{\varepsilon}\|x - x_\varepsilon\| < \varphi(f(x_\varepsilon)) = 0.$$

Then there exists $t < -\sqrt{\varepsilon}\|x - x_\varepsilon\|$ with $f(x) - f(x_\varepsilon) \in tk^0 - B$ and so

$$\begin{aligned} f(x) &\in f(x_\varepsilon) - \sqrt{\varepsilon}\|x - x_\varepsilon\|k^0 - (B + (-\sqrt{\varepsilon}\|x - x_\varepsilon\| - t)k^0) \\ &\subset f(x_\varepsilon) - \sqrt{\varepsilon}\|x - x_\varepsilon\|k^0 - \operatorname{int} B \\ &\subset f(x_\varepsilon) - \sqrt{\varepsilon}\|x - x_\varepsilon\|k^0 - (B \setminus \{0\}), \end{aligned}$$

a contradiction to (19). So we get

$$\varphi(f(x)) \geq \varphi(f(x_\varepsilon)) - \sqrt{\varepsilon}\|x - x_\varepsilon\| \qquad \forall x \in X.$$

Because of $(A_\varphi 2)$ and $(A_\varphi 4')$ we get that the scalarizing functional φ is locally Lipschitz. Furthermore, f is supposed to be a strictly differentiable mapping and so locally Lipschitz. Hence the composition $\varphi \circ f$ is locally Lipschitz such that we can use Clarke‘s generalized directional derivative $(\varphi \circ f)^\circ$.

Consider now for $n \in \mathbb{N}$, $t_n > 0$, $x := x_\varepsilon + t_n h_n$ with $h_n \in U$ (U is a neighborhood of $h \in X$) and $\|h\| = 1$. For these we have

$$\frac{\varphi(f(x_\varepsilon + t_n h_n)) - \varphi(f(x_\varepsilon))}{t_n} \geq -\sqrt{\varepsilon}\|h_n\|. \tag{20}$$

Taking the limits for $t_n \to 0$ and $h_n \to h$ we get for Clarke‘s generalized directional derivative

$$(\varphi \circ f)^\circ(x, h) \geq -\sqrt{\varepsilon} \quad \forall h \in X \text{ with } \|h\| = 1.$$

Using the chain rule given by [3] (Theorem 2.3.10 and Proposition 2.1.2) we get that there is an element $y^* \in \partial\varphi(f(x_\varepsilon))$ such that for all $h \in X$ with $||h|| = 1$

$$y^* \circ D_S f(x_\varepsilon)(h) \geq -\sqrt{\varepsilon}.$$

Taking into account the linearity of $D_S f(x_\varepsilon)$ we get (if we replace h by $-h$)

$$y^* \circ D_S f(x_\varepsilon)(h) \leq \sqrt{\varepsilon}$$

such that

$$||y^* \circ D_S f(x_\varepsilon)||_* \leq \sqrt{\varepsilon}.$$

Finally, we will show $y^* \in K^\#$ using the property $(A_\varphi 1')$ with respect to K of φ. Let $k \in K \setminus \{0\}$. Thus we have $\varphi(y) > \varphi(y-k)$. Since φ is a continuous convex function on the Banach Space Y one has $\partial\varphi(y) \neq \emptyset$ for each $y \in Y$. Thus we have

$$\varphi(y) > \varphi(y-k) \geq \varphi(y) + y^*(-k) \quad \forall y^* \in \partial\varphi(y).$$

This shows that $y^*(k) > 0$ for any $k \in K \setminus \{0\}$. This immediately yields that $y^* \in K^\#$. This completes the proof.

□

For problems with restrictions we get the following result:

Theorem 4. *Consider the vector optimization problem (VP) under the assumptions* $(A_{space}2)$ *and* $(A_{map}2)$. *Suppose that* $S \subseteq X$ *is closed. Let* $K \subset Y$ *be a pointed closed convex cone,* $k^0 \in K \setminus \{0\}$ *and* $B \subset Y$ *a closed normal cone with* $K \setminus \{0\} \subset int\, B$. *We suppose that* $(A_{map}4)$ *with respect to* B *and* k^0 *is fulfilled. If* $\varepsilon > 0$ *is an arbitrary real number and there exists an element* $x_0 \in S$ *such that* $f(x_0) \leq_B f(x) + \varepsilon k^0$ *for all* $x \in S$ *then there is an element* $x_\varepsilon \in S$ *with* $f(x_\varepsilon) \leq_B f(x_0)$ *such that*

(i) $\|x_0 - x_\varepsilon\| \leq \sqrt{\varepsilon}$.
(ii) *There exists* $y^* \in K^\#$ *such that*

$$y^* \circ D_S f(x_\varepsilon)(h) \geq -\sqrt{\varepsilon} \quad \forall h \in \mathscr{T}(S, x_\varepsilon) \text{ with } ||h|| = 1.$$

Proof: We follow the line of the proof of Theorem 3.

□

Remark 8. The assertions in Theorems 3 and 4 are related to the proper efficiency of the element x_ε. Especially, (19) says that x_ε is a properly efficient point of $f_{\sqrt{\varepsilon}k^0}$ over X with respect to K because B is a closed normal cone with $K \setminus \{0\} \subset \text{int}\, B$, i.e., $f_{\sqrt{\varepsilon}k^0}(x_\varepsilon) \in pEff(f_{\sqrt{\varepsilon}k^0}(X), K)$. The property $K \setminus \{0\} \subset \text{int}\, B$ implies in both theorems the strong assertion $y^* \in K^\#$ for the multiplier y^*.

Remark 9. In order to derive necessary conditions for εk^0 -efficient points of (VP) (with $\varepsilon > 0$), i.e., for $x_0 \in S$ with $f(x_0) \in \varepsilon k^0 - Eff(f(S), K)$, it would be possible to use the same procedures like in the proofs of Theorems 2, 3 and 4 using corresponding variational principles (for instance [9, Corollary 3.10.14]). The same holds for εk^0 -properly efficient points of (VP) (with $\varepsilon > 0$), i.e., for $x_0 \in S$ with $f(x_0) \in \varepsilon k^0 - pEff(f(S), K)$.

7 Vector optimization problems with finite dimensional image spaces

As seen in Theorem 2 the assumptions concerning the ordering cone K for deriving optimality conditions in general spaces are strong. Now, we will show necessary optimality conditions for vector optimization problems where the objective function f takes its values in a finite dimensional space $\mathbb{R}^n$ under weaker assumptions. Corresponding results are shown in [5], [6] and [20].

For a locally Lipschitz function f we derive Lagrangian multiplier rules for approximately efficient elements of (VP) using Mordukhovichs subdifferential calculus (see Definition 11).

Theorem 5. *Consider the vector optimization problem (VP). Assume that* $(A_{space}3)$ *and* $(A_{map}3)$ *are satisfied. Suppose that* $S \subseteq X$ *is closed. Let* $K \subset Y$ *be a pointed closed convex cone,* $k^0 \in K \setminus \{0\}$ *and* $B \subset Y$ *a closed normal cone with* $K \setminus \{0\} \subset \operatorname{int} B$*. If* $\varepsilon > 0$ *is an arbitrary real number and there exists an element* $x_0 \in S$ *such that* $f(x_0) \leq_B f(x) + \varepsilon k^0$ *for all* $x \in S$ *then there are elements* $x_\varepsilon \in S$ *with* $f(x_\varepsilon) \leq_B f(x_0)$, $u^* \in K^\#$ *with* $u^*(k^0) = 1$ *and* $x^* \in X^*$ *with* $\|x^*\| \leq 1$ *such that*

$$0 \in \partial_L(u^* \circ f)(x_\varepsilon) + u^*(k^0)\sqrt{\varepsilon}x^*(x_\varepsilon) + N_{\partial_L}(S, x_\varepsilon).$$

PROOF. Consider an element $x_0 \in S$ such that $f(x_0) \leq_B f(x) + \varepsilon k^0$ for all $x \in S$ where $B \subset Y$ is a closed normal cone with $K \setminus \{0\} \subset \operatorname{int} B$. Taking into account that $(A_{map}3)$ is fulfilled for the function f, it is continuous as well and since S is a closed set in a Asplund space it is a complete metric space endowed with the distance given by the norm such that the assumptions of Theorem 1 are fulfilled. From this variational principle we get the existence of an element $x_\varepsilon \in S$ such that $f(x_\varepsilon) \leq_B f(x_0)$. Moreover, for x_ε holds $f(x_\varepsilon) \in Eff(f_{\sqrt{\varepsilon}k^0}(S), B)$ where the perturbed objective function $f_{\sqrt{\varepsilon}k^0}$ is given by

$$f_{\sqrt{\varepsilon}k^0}(x) := f(x) + k^0\sqrt{\varepsilon}\,\|x - x_\varepsilon\|\,.$$

Now, applying Theorem 3.1 in [5] we can find $u^* \in \partial\varphi(v)$ with $u^* \in K^*$, $u^*(k^0) = 1$ (where the scalarizing function φ is given by $\varphi(y) := \varphi_{B,k^0}(y - f_{\sqrt{\varepsilon}k^0}(x_\varepsilon))$ (cf. (3)) and has the properties $(A_\varphi 1')$ with respect to K, $(A_\varphi 2)$ and $(A_\varphi 4')$) such that by the calculation rules for Mordukhovich subdifferential

$$0 \in \partial_L(u^* \circ f_{\sqrt{\varepsilon}k^0})(x_\varepsilon) + N_{\partial_L}(S, x_\varepsilon). \tag{21}$$

Consider an element $x^*_\varepsilon \in \partial_L(u^* \circ f_{\sqrt{\varepsilon}k^0})(x_\varepsilon)$ involved in (21). Because of

$$\partial_L(u^* \circ f_{\sqrt{\varepsilon}k^0})(x) = \partial_L(u^* \circ (f(\cdot) + k^0\sqrt{\varepsilon}\,\|\cdot - x_\varepsilon\|))(x),$$

by use of the rule for sums and the property that Mordukhovich subdifferential coincides in the convex case with the Fenchel subdifferential, we get

$$x^*_\varepsilon \in \partial_L(u^* \circ f)(x_\varepsilon) + u^*(k^0)\sqrt{\varepsilon}\partial \left\|\cdot - x_\varepsilon\right\|(x_\varepsilon). \tag{22}$$

From (21) and (22) and taking into account the calculation rule for the subdifferential of the norm it follows that there is $x^* \in X^*$ with $\|x^*\| \leq 1$ such that

$$0 \in \partial_L(u^* \circ f)(x_\varepsilon) + u^*(k^0)\sqrt{\varepsilon}x^*(x_\varepsilon) + N_{\partial_L}(S, x_\varepsilon)).$$

The property $u^* \in K^\#$ follows analogously like in the proof of Theorem 3.

□

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