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A self-similar solution for the porous medium equation in a two-component domain

Ján Filo* and Volker Pluschke

Abstract

We solve a particular system of nonlinear ODEs defined on the two different components of the real line connected by the nonlinear contact condition

$$w' = h' \,, \qquad h = \psi(w) \qquad \text{at the point } x = 0\,.$$

We show that, for a prescribed power-law nonlinearity ψ and using the solution (w,h), a self-similar solution to the porous medium equation in the two-component domain can be constructed. We provide the rigorous proof of a theorem in which the formula for the free boundary already presented in [8] is given.

1 Introduction

Let $0 < m, \sigma < 1$ be given. The main concern of this paper is to deal with the following system of nonlinear ODEs (see [8, Section 6])

$$\left.\begin{array}{ll} w''(x) + x(w^m(x))' - \dfrac{m}{1-m} w^m(x) = 0 & \text{for } x < 0\,, \\ h''(x) + \alpha x (h^\sigma(x))' - \dfrac{\omega\sigma}{1-m} h^\sigma(x) = 0 & \text{for } x > 0\,, \\ w'(0) = h'(0)\,, \quad h(0) = M w^\omega(0)\,, & \\ w(x) - (-a\,x)^{1/(1-m)} = \mathcal{O}\left((-a\,x)^{m/(1-m)}\right) & \text{as } x \to -\infty\,, \\ h(x) \to 0 \text{ as } x \to \infty\,, & \end{array}\right\} \tag{1.1}$$

where a, M are given positive constants,

$$\alpha = \frac{1 - m\sigma}{(1+\sigma)(1-m)} \,, \quad \omega = \frac{m+1}{\sigma+1} \tag{1.2}$$

and

$$f = \mathcal{O}(g) \quad \text{means} \quad \limsup_{x\to-\infty} \left| \frac{f(x)}{g(x)} \right| < \infty\,. \tag{1.3}$$

We prove that there is a weak solution (w,h) of this system such that w is a smooth decreasing function with $w(0) > 0$, $w'(0) < 0$ and h is a positive smooth decreasing function on $[0,\zeta)$ that is identically zero for all $x \geq \zeta$, i.e.

$$h(x) > 0 \quad \text{for} \quad x \in [0,\zeta) \qquad \text{and} \qquad h(x) = 0 \quad \forall\, x \geq \zeta\,. \tag{1.4}$$

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The reason why we study the system (1.1) with such particular choice of exponents (1.2) is the following. Assuming, that (w,h) is our solution of (1.1), functions

$$u(x,t)=t^{1/(1-m)}w\left(\frac{x}{t}\right) \quad \text{and} \quad v(x,t)=t^{\omega/(1-m)}h\left(\frac{x}{t^\alpha}\right) \tag{1.5}$$

are appropriately defined solutions of the system of nonlinear PDEs

$$\left.\begin{array}{lll} (u^m)_t-u_{xx}=0 & -\infty<x<0\,, & 0<t<T\,, \\ (v^\sigma)_t-v_{xx}=0 & 0<x<\infty\,, & 0<t<T\,, \\ u_x(0,t)=v_x(0,t) & & 0<t<T\,, \\ v(0,t)=M\,u^\omega(0,t) & & 0<t<T\,, \\ u(x,0)=(-ax)^{1/(1-m)} & -\infty<x<0\,, & \\ v(x,0)=0 & 0<x<\infty\,. & \end{array}\right\} \tag{1.6}$$

These solutions (1.5) are invariant under the dilation scaling (self-similarity)

$$\begin{array}{lll} u(x,t) &=& \lambda^{1/(1-m)}\,u\left(\dfrac{x}{\lambda},\dfrac{t}{\lambda}\right) \quad \text{for } x\le 0\,,\ t>0, \\ v(x,t) &=& \lambda^{\omega/(1-m)}\,v\left(\dfrac{x}{\lambda^\alpha},\dfrac{t}{\lambda}\right) \quad \text{for } x\ge 0\,,\ t>0, \end{array} \qquad \text{for all } \lambda>0, \tag{1.7}$$

and they can provide us with useful information about the behaviour of the interface (or free boundary)

$$I[v]=\{(x,t)\in[0,\infty)\times[0,\infty):\ x=\xi(t)\}\ ,\quad \xi(t)\ =\ \inf\{x\ge 0:\ v(x,t)=0\}\,.$$

Our main result is the rigorous proof of the fact, presented already in [8], that the interface in this case has the explicit solution

$$\xi(t)\ =\ \zeta t^\alpha \tag{1.8}$$

for ζ given by (1.4), i.e.

$$v(x,t)>0 \quad \text{on} \quad [0,\xi(t)) \quad \text{and} \quad v(x,t)\ =\ 0 \quad \text{on} \quad [\xi(t),\infty)\,,\ t>0\,. \tag{1.9}$$

The asymptotic condition for w as $x\to-\infty$ is a stronger version (and, in fact, our result) of the more common condition

$$\frac{w(x)}{(-ax)^{1/(1-m)}}\to 1 \quad \textit{as } x\to-\infty \tag{1.10}$$

which is required that u given by (1.5) approaches the initial condition in (1.6).

If we consider more general contact condition

$$v(0,t)=\psi(u(0,t))$$

for some other continuous increasing function ψ in (1.6) which is done for our auxiliary problems to (1.1) on a bounded domain, then we are loosing the relation between (1.1) and (1.6) through (1.5). The behaviour of the interface for more general (even for another power nonlinearity) contact condition is an open problem. Also the uniqueness of (1.1) remains open and the question arises if some further restrictions are needed in order to prove it.

Additional information about self-similar solutions and further references concerning the porous medium equation can be found in [1]-[3], [5], [13],[14]. The motivation for studying the free boundary problem with such a contact condition can be found in [10], see also [7].

The paper is organized as follows. Initially, in Section 2, we define an approximation of problem (1.1) on a bounded domain (Problem (P)) as well as a suitable approximation of the contact condition (Problem (P_L)). In Section 3 we derive comparison principles for these two problems and we find explicit sub- and supersolutions, which are useful for the original problem (1.1), too. In Section 4 we prove existence of a solution: first to problems (P) and (P_L) and then, by an approximation process, to (1.1). These proofs are essentially based on the results of Section 3. Finally, in Section 5 we show that (1.5) is a (weak) solution of the porous medium system (1.6) and we derive properties (1.8),(1.9) of the interface which are obtained from our knowledge concerning the behaviour of (w, h).

Let us finish this section by introducing some notation. We write u^m instead of $|u|^m \operatorname{sign} u$. Moreover, we set $[w]_+ := \max\{w, 0\}$ for the positive part of w and $\operatorname{sign}^+(w) := \max\{\operatorname{sign}(w), 0\}$. A function f is said to be increasing (nondecreasing) if $x < y$ implies $f(x) < f(y)$ $(f(x) \le f(y))$. If the last inequality is reversed, we obtain the definition of a decreasing (nonincreasing) function. We call a function $f : [a,b] \to \mathbb{R}$ piecewise monotone, if $[a,b]$ may be subdivided into a finite number of subintervals where f is nondecreasing or nonincreasing. $H^1(a,b) = W^{1,2}(a,b)$ is the usual Sobolev space, see e.g. [6]. In the following we use the notation $V = H^1(-l,0) \times H^1(0,l)$ with its closed subspaces $\tilde{V} := \{\varphi = (\varphi^-, \varphi^+) \in V : \varphi^-(0) = \varphi^+(0)\}$, $V^\circ := \{\varphi = (\varphi^-, \varphi^+) \in V : \varphi^-(-l) = \varphi^+(l) = 0\}$, and $\tilde{V}^\circ = \tilde{V} \cap V^\circ$. $\tilde{V}$ and $\tilde{V}^\circ$ may be identified with $H^1(-\ell,\ell)$ and $\mathring{H}^1(-\ell,\ell)$, respectively.

2 Approximation

In what follows we shall approximate problem (1.1) in two steps. First we consider our system on a bounded domain, i.e. we investigate

$$D^- w := -w''(x) - x(w^m(x))' + \frac{m}{1-m}\, w^m(x) = 0 \qquad \text{for} \quad -\ell < x < 0\,, \tag{2.1}$$

$$D^+ h := -h''(x) - \alpha x(h^\sigma(x))' + \frac{\omega\sigma}{1-m}\, h^\sigma(x) = 0 \qquad \text{for} \quad 0 < x < \ell\,, \tag{2.2}$$

with boundary conditions

$$w(-\ell) = w_\ell > 0\,, \qquad h(\ell) = 0 \tag{2.3}$$

and the general contact conditions

$$w'(0) = h'(0)\,, \qquad h(0) = \psi(w(0)) \tag{2.4}$$

for some continuous and increasing function $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(0) = 0$.

Condition (2.4_2) generalizes the condition $h(0) = M w^\omega(0)$, which seems to be the only form to get self-similar solutions to the porous medium system (1.6).

In the second step conditions (2.4) are replaced by the approximation

$$w'(0) + L\big(\psi(w(0)) - h(0)\big) = 0\,, \qquad -h'(0) + L\big(h(0) - \psi(w(0))\big) = 0 \tag{2.5}$$

for some $L > 0$ that we finally send to $+\infty$. In the following we refer to (2.1)-(2.4) as **Problem (P)** and to (2.1)-(2.3),(2.5) as **Problem ($\mathbf{P_L}$)**, respectively.

Now we introduce the notion of a solution, subsolution, and supersolution, resp., to these problems in a weak sense.

Definition 2.1 *(a) A couple $(w, h) \in V$ is called (weak) solution of Problem (P) if it satisfies the relation*

$$\int_{-\ell}^{0}\Big(w'(\varphi^-)' + w^m(x\varphi^-)' + \frac{m}{1-m}\,w^m\varphi^-\Big)dx + \int_{0}^{\ell}\Big(h'(\varphi^+)' + \alpha h^\sigma(x\varphi^+)' + \frac{\omega\sigma}{1-m}h^\sigma\varphi^+\Big)dx + g\big(\varphi^+(0) - \varphi^-(0)\big) = 0 \quad (2.6)$$

for some $g \in \mathbb{R}$ and for all $\varphi = (\varphi^-, \varphi^+) \in V^\circ$ as well as the conditions

$$w(-\ell) = w_\ell\,, \qquad h(\ell) = 0 \quad (2.7)$$

and

$$h(0) = \psi(w(0)). \quad (2.8)$$

(b) A couple $(w, h) \in V$ is called subsolution (supersolution) of Problem (P) if it satisfies (a) for all nonnegative $\varphi \in V^\circ$ with relation symbol "$\le$" ("$\ge$") instead of "$=$" in relations (2.6) and (2.7). We write then ($2.6_\le$) and ($2.7_\le$) (resp. ($2.6_\ge$) and ($2.7_\ge$)) for these inequalities.

Let us explain the meaning of the number g. If (w', h') has a trace at $x = 0$, e.g. if $(w, h) \in C^2[-\ell, 0] \times C^2[0, \ell]$, then integration by parts in relation ($2.6_\le$) leads to $w'(0) \le g \le h'(0)$ for a subsolution and, analogously, to $w'(0) \ge g \ge h'(0)$ for a supersolution. If (w, h) is a solution of Problem (P) the g is just the derivative $w'(0) = h'(0) = g$. In that case we may also test relation (2.6) with $\varphi \in \tilde{V}^\circ$ (see next proposition), then g disappears. However, we need the item with g in order to include condition (2.4_1) in a proper way into definitions of weak sub- or supersolution of our problems.

Proposition 2.1 *The following two assertions are equivalent:*

(i) A couple $(w, h) \in V$ fulfils relation (2.6) for all $\varphi \in V^\circ$ and some $g \in \mathbb{R}$;
(ii) A couple $(w, h) \in V$ fulfils relation

$$\int_{-\ell}^{0}\Big(w'(\varphi^-)' + w^m(x\varphi^-)' + \frac{m}{1-m}\,w^m\varphi^-\Big)dx + \int_{0}^{\ell}\Big(h'(\varphi^+)' + \alpha h^\sigma(x\varphi^+)' + \frac{\omega\sigma}{1-m}h^\sigma\varphi^+\Big)dx = 0 \quad (2.9)$$

for all $\varphi \in \tilde{V}^\circ$.

Moreover, the constant g in (i) is uniquely determined by

$$g = \lim_{\delta\to 0+}\frac{1}{\delta}\int_{-\delta}^{0} w'(x)dx = \lim_{\delta\to 0+}\frac{1}{\delta}\int_{0}^{\delta} h'(x)dx.$$

Proof: see Appendix.

We conclude this section with the corresponding definition of a weak solution, subsolution, and supersolution, respectively, of Problem (P_L), which is obvious by the above considerations.

Definition 2.2 *A couple $(w, h) \in V$ is called solution (subsolution, supersolution) of Problem (P_L) if the conditions of Definition 2.1 excluding (2.8) hold with $g = L\big(h(0) - \psi(w(0))\big)$.*

3 Comparison principle on bounded intervals

3.1 Comparison principles for Problems (P) and (P_L)

We obtain comparison results for Problems (P) and (P_L) as an immediate consequence of the following main theorem of this section.

Theorem 3.1 *Assume two nonnegative and piecewise monotone couples $(\underline{w}, \underline{h}) \in V$ and $(\overline{w}, \overline{h}) \in V$ fulfilling relations ($2.6_{\le}$),($2.7_{\le}$) with $g = \underline{g}$ and ($2.6_{\ge}$),($2.7_{\ge}$) with $g = \overline{g}$, respectively. Then it holds*

$$\frac{1}{1-m}\int_{-\ell}^{0} [\underline{w}^m - \overline{w}^m]_+ \, dx + \left(\frac{\omega\sigma}{1-m} + \alpha\right)\int_0^{\ell} [\underline{h}^\sigma - \overline{h}^\sigma]_+ \, dx + (\underline{g} - \overline{g})\left(\text{sign}^+(\underline{h}(0) - \overline{h}(0)) - \text{sign}^+(\underline{w}(0) - \overline{w}(0))\right) \le 0. \quad (3.1)$$

Before we prove the theorem we state an auxiliary assertion.

Lemma 3.1 *Let $w \in H^1(a, b)$ be nonnegative and piecewise monotone. Then $(w^m)' \in L_1(a, b)$ for every $m \in (0, 1)$.*

Proof: We carry out the proof for two subintervals $[a, b] = [a, \xi] \cup [\xi, b]$ where w is nonincreasing on $[a, \xi]$ and nondecreasing on $[\xi, b]$. For $\varepsilon > 0$ let $w_\varepsilon(x) = \max\{w(x), \varepsilon\}$ and

$$f_\varepsilon(x) = (w_\varepsilon(x)^m)' = \begin{cases} \frac{m}{w(x)^{1-m}}\, w'(x) & \text{for } w(x) > \varepsilon, \\ 0 & \text{for } 0 \le w(x) \le \varepsilon. \end{cases}$$

Then it holds

$$\int_a^\xi |f_\varepsilon(x)|\, dx = \int_a^\xi (-w_\varepsilon(x)^m)' dx = w_\varepsilon(a)^m - w_\varepsilon(\xi)^m \le w(a)^m =: c_1.$$

Moreover, by definition, $\big(|f_\varepsilon|\big) = \big(-f_\varepsilon\big)$ is a monotone increasing sequence with

$$-f_\varepsilon(x) \to -f(x) = \begin{cases} \frac{m}{w(x)^{1-m}}\, (-w'(x)) & \text{for } w(x) > 0, \\ 0 & \text{for } w(x) = 0. \end{cases}$$

as $\varepsilon \searrow 0$ for a.e. $x \in [a, \xi]$. Hence, the monotone convergence theorem implies

$$f_\varepsilon \to f \quad \text{in } L_1(a, \xi) \qquad \text{as } \varepsilon \searrow 0.$$

By means of

$$\int_\xi^b |f_\varepsilon(x)|\,dx = \int_\xi^b (w_\varepsilon(x)^m)'dx = w_\varepsilon(b)^m - w_\varepsilon(\xi)^m \le w(b)^m =: c_2$$

in the same way we obtain

$$f_\varepsilon \to f \quad \text{in } L_1(\xi, b) \qquad \text{as } \varepsilon \searrow 0,$$

hence $f \in L_1(a,b)$ and $f = (w^m)'$. □

The example $w(x) = x^s\,(2 + \cos\frac{1}{x})$, $x \in (0,1)$, $s > \frac{3}{2}$, $0 < m \le \frac{1}{s}$, shows that Lemma 3.1 is false if piecewise monotonicity is omitted.

Proof of Theorem 3.1: Let $\eta_\delta : \mathbb{R} \to [0,1]$ be a smooth and nondecreasing approximation of $\operatorname{sign}^+(\cdot)$ with

$$\eta_\delta(\xi) \begin{cases} = 0 & \text{for } \xi \le 0 \\ \in [0,1] & \text{for } 0 < \xi < \delta \\ = 1 & \text{for } \xi \ge \delta \end{cases}, \qquad \delta > 0. \tag{3.2}$$

Since $\underline{w}(-\ell) - \overline{w}(-\ell) \le 0$, $\underline{h}(\ell) - \overline{h}(\ell) \le 0$ due to $(2.7_\le)$,$(2.7_\ge)$, the function $\varphi = \big(\eta_\delta(\underline{w} - \overline{w}), \eta_\delta(\underline{h} - \overline{h})\big)$ is an admissible test function for $(2.6_\le)$ and $(2.6_\ge)$. We take the difference of these relations and obtain

$$\begin{aligned} &\int_{-\ell}^0 \Big[(\underline{w}' - \overline{w}')^2\eta_\delta'(\underline{w} - \overline{w}) + (\underline{w}^m - \overline{w}^m)\big(x\eta_\delta(\underline{w} - \overline{w})\big)' \\ &\qquad\qquad + \frac{m}{1-m}(\underline{w}^m - \overline{w}^m)\,\eta_\delta(\underline{w} - \overline{w})\Big]dx \\ &+ \int_0^\ell \Big[(\underline{h}' - \overline{h}')^2\eta_\delta'(\underline{h} - \overline{h}) + \alpha(\underline{h}^\sigma - \overline{h}^\sigma)\big(x\eta_\delta(\underline{h} - \overline{h})\big)' \\ &\qquad\qquad + \frac{\omega\sigma}{1-m}(\underline{h}^\sigma - \overline{h}^\sigma)\,\eta_\delta(\underline{h} - \overline{h})\Big]dx \\ &+ (\underline{g} - \overline{g})\big(\eta_\delta(\underline{h} - \overline{h}) - \eta_\delta(\underline{w} - \overline{w})\big) \le 0. \end{aligned} \tag{3.3}$$

We transform the second items within the integrals. By monotonicity assumption and due to Lemma 3.1 we have $(\underline{w}^m - \overline{w}^m)' \in L_1(-\ell, 0)$, $(\underline{h}^\sigma - \overline{h}^\sigma)' \in L_1(0,\ell)$. Hence, we may apply integration by parts and obtain

$$\begin{aligned} &\int_{-\ell}^0 (\underline{w}^m - \overline{w}^m)\big(x\eta_\delta(\underline{w} - \overline{w})\big)'dx \\ &\qquad = \quad (\underline{w}^m - \overline{w}^m)\,x\,\eta_\delta(\underline{w} - \overline{w})\Big|_{x=-\ell}^{x=0} - \int_{-\ell}^0 (\underline{w}^m - \overline{w}^m)'\,x\,\eta_\delta(\underline{w} - \overline{w})\,dx \\ &\qquad = \quad -\int_{-\ell}^0 x\,(\underline{w}^m - \overline{w}^m)'\eta_\delta(\underline{w}^m - \overline{w}^m)\,dx \\ &\qquad\qquad + \int_{-\ell}^0 x\,(\underline{w}^m - \overline{w}^m)'\big(\eta_\delta(\underline{w}^m - \overline{w}^m) - \eta_\delta(\underline{w} - \overline{w})\big)dx. \end{aligned}$$

Here we used $(2.7_{\leq})$,$(2.7_{\geq})$ again. The last integral $I_\delta(\underline{w},\overline{w})$ tends to 0 as $\delta \to 0$ by the Lebesgue theorem on dominated convergence since $\underline{w}^m - \overline{w}^m$ and $\underline{w} - \overline{w}$ have the same sign. Let now

$$\Phi_\delta(w) = \int_{-\infty}^{w} \eta_\delta(\xi)\, d\xi\,,$$

then we arrive at

$$\begin{aligned}
&\int_{-\ell}^{0} (\underline{w}^m - \overline{w}^m)\big(x\eta_\delta(\underline{w} - \overline{w})\big)' dx \\
&\qquad = \quad -\int_{-\ell}^{0} x\,\frac{d}{dx}\Phi_\delta(\underline{w}^m - \overline{w}^m)\, dx + I_\delta(\underline{w},\overline{w}) \qquad (3.4)\\
&\qquad = \quad x\,\Phi_\delta(\underline{w}^m - \overline{w}^m)\Big|_{x=-\ell}^{x=0} + \int_{-\ell}^{0} \Phi_\delta(\underline{w}^m - \overline{w}^m)\, dx + I_\delta(\underline{w},\overline{w}) \\
&\qquad \to \quad \int_{-\ell}^{0} [\underline{w}^m - \overline{w}^m]_+\, dx \qquad \text{as } \delta \to 0.
\end{aligned}$$

In the same way we obtain

$$\int_0^\ell \alpha(\underline{h}^\sigma - \overline{h}^\sigma)\big(x\eta_\delta(\underline{h} - \overline{h})\big)' dx \;\to\; \alpha \int_0^\ell [\underline{h}^\sigma - \overline{h}^\sigma]_+\, dx \qquad \text{as } \delta \to 0. \qquad (3.5)$$

The first items in the integrals of (3.3) are nonnegative and may be omitted. Then the limit process $\delta \to 0$ in (3.3) by means of (3.4) and (3.5) immediately leads to (3.1). □

Corollary 3.1 *Let $(\underline{w},\underline{h})$ be a subsolution of Problem (P) and $(\overline{w},\overline{h})$ be a supersolution of Problem (P) in the sense of Definition 2.1 being nonnegative and piecewise monotone. Then*

$$\underline{w}(x) \leq \overline{w}(x) \quad \forall x \in [-\ell, 0] \qquad \textit{and} \qquad \underline{h}(x) \leq \overline{h}(x) \quad \forall x \in [0, \ell]\,.$$

This is obvious since $\underline{h}(0) = \psi(\underline{w}(0))$, $\overline{h}(0) = \psi(\overline{w}(0))$, hence $(\underline{g} - \overline{g})\big(\text{sign}^+(\underline{h}(0) - \overline{h}(0)) - \text{sign}^+(\underline{w}(0) - \overline{w}(0))\big) = 0$ in (3.1) because of monotonicity of ψ.

Corollary 3.2 *Let $(\underline{w},\underline{h})$ be a subsolution of Problem (P_L) and $(\overline{w},\overline{h})$ be a supersolution of Problem (P_L) in the sense of Definition 2.2 being nonnegative and piecewise monotone. Then*

$$\underline{w}(x) \leq \overline{w}(x) \quad \forall x \in [-\ell, 0] \qquad \textit{and} \qquad \underline{h}(x) \leq \overline{h}(x) \quad \forall x \in [0, \ell]\,.$$

In that case we have $\underline{g} = L\big(\underline{h}(0) - \psi(\underline{w}(0))\big)$, $\overline{g} = L\big(\overline{h}(0) - \psi(\overline{w}(0))\big)$, and the last item in (3.1) is

$$s_3 = L\Big([\underline{h}(0) - \overline{h}(0)] - [\psi(\underline{w}(0)) - \psi(\overline{w}(0))]\Big)\Big(\text{sign}^+[\underline{h}(0) - \overline{h}(0)] - \text{sign}^+[\underline{w}(0) - \overline{w}(0)]\Big).$$

If $\text{sign}[\underline{h}(0) - \overline{h}(0)] = \text{sign}[\underline{w}(0) - \overline{w}(0)]$ then $s_3 = 0$, otherwise $s_3 > 0$ by the assumptions on ψ again, which yields the assertion of Corollary 3.2.

The next corollary concerns the case if we consider the one-sided equations (2.1) or (2.2) separately on $(-\ell, 0)$ or $(0, \ell)$, respectively, with Dirichlet boundary conditions

$$w(-\ell) = w_\ell\,, \quad w(0) = w_0 \qquad \text{and} \qquad h(0) = h_0\,, \quad h(\ell) = 0, \qquad (3.6)$$

resp., or Dirichlet-Neumann conditions

$$w(-\ell) = w_\ell\,, \quad w'(0) = g_w\,, \tag{3.7}$$
$$h(\ell) = 0\,, \quad -h'(0) = g_h\,, \tag{3.8}$$

respectively. Then we obtain a comparison principle e.g. on $[-\ell, 0]$ by choosing $\varphi^+(x) \equiv 0$ in the proof of Theorem 3.1. All items with $\underline{h}, \overline{h}$ in (3.1) disappear, moreover the last item on the left-hand side of (3.1) disappears in case of the Dirichlet problem (3.6) because of $\varphi^-(0) = 0$.

Defining a subsolution and a supersolution to (2.1),(3.6) or (2.1),(3.7), resp., in an analogue way to Definition 2.1, we formulate the result for the left-hand side $[-\ell, 0]$.

Corollary 3.3 *Let $\underline{w}, \overline{w} \in H^1(-\ell, 0)$ be nonnegative and piecewise monotone on $[-\ell, 0]$. If $\underline{w}$ is a subsolution and $\overline{w}$ is a supersolution, resp., to (2.1),(3.6), then $\underline{w}(x) \le \overline{w}(x)$ for all $x \in [-\ell, 0]$.*
If $\underline{w}$ is a subsolution and $\overline{w}$ is a supersolution, resp., to (2.1),(3.7), then it holds

$$\int_{-\ell}^{0} [\underline{w}^m - \overline{w}^m]_+ \, dx - (\underline{g_w} - \overline{g_w}) \, \mathrm{sign}^+(\underline{w}(0) - \overline{w}(0)) \le 0. \tag{3.9}$$

An analogue result holds for $\underline{h}, \overline{h} \in H^1(0, \ell)$.

Obviously, a solution of our problems is a sub- and supersolution, too. In order to apply these corollaries to a solution, however, we have to verify the monotonicity assumption.

Lemma 3.2 *Let w, h be nonnegative weak solutions of (2.1) and (2.2), respectively. Then $w: [-\ell, 0] \to \mathbf{R}_+$ and $h: [0, \ell] \to \mathbf{R}_+$ are piecewise monotone functions.*

Proof: Suppose first that $w \in H^1(-\ell, 0)$, $w(-\ell) > 0$ is a weak nonnegative solution of the nonlinear ODE (2.1) and we set $U = (-\ell, \iota)$, where

$$\iota = \sup\{x \in (-\ell, 0] : \; w(x) > 0\}\,.$$

Observe that since $w > 0$ on U,

$$-w'' + b(x)w' + c(x)w = 0\,, \; b(x) = -mxw^{m-1}(x)\,, \; c(x) = \frac{m}{1-m} w^{m-1}$$

and we can apply the known results about interior regularity of second-order elliptic equations (see e.g. [6]) to conclude that

$$w \in C^\infty(U) \;\text{ and (2.1) holds in the classical sense on } U\,.$$

(i) Let $\iota = 0$ and $w(0) > 0$. Then $w \in C^\infty([-\ell, 0])$ and it follows from the classical one-dimensional maximum principle (see e.g. [12]) that $w(x)$ can not attain a local positive maximum in $(-\ell, 0)$. Therefore w is monotone on $[-\ell, 0]$ or it is first nonincreasing and then nondecreasing.

(ii) Let $\iota = 0$ and $w(0) = 0$. As $w(x)$ can not attain a local positive maximum in $(-\ell, 0)$, it is nonincreasing.

(iii) Let $\iota < 0$ and $w(0) = 0$. Then it is not difficult to see that w is nonincreasing on $[-\ell, \iota]$ and it has to be identically zero on $[\iota, 0]$.

(iv) Let $\iota < 0$ and $w(0) > 0$. Define $\lambda = \inf\{x \in (\iota, 0) : \; w(x) > 0\}$. Then w is nonincreasing on $[-\ell, \iota]$, it is nondecreasing on $[\lambda, 0]$ and it is possibly zero on $[\iota, \lambda]$.

If $w(-\ell) = 0$, then due to the fact, that w cannot attain a local positive maximum on $(-\ell, 0)$, it is a nondecreasing function.

For the solution h of (2.2) holds an analogue result as for w. □

In Section 4 we prove existence of nonnegative solutions to Problems (P) and (P_L). Moreover, in Theorems 4.1 and 4.2 it is shown that there are no solutions where nonnegativity is violated. Therefore, as a consequence of Lemma 3.2 and Corollaries 3.1, 3.2 we obtain

Theorem 3.2 *A weak solution (w, h) to Problem (P) or Problem (P_L), respectively, is unique and bounded by*

$$\begin{aligned} 0 \le w(x) \le w_\ell \qquad & \textit{for } -\ell \le x \le 0, \\ 0 \le h(x) \le \psi(w_\ell) \qquad & \textit{for } \quad 0 \le x \le \ell. \end{aligned} \tag{3.10}$$

It only remains to remark that $(\overline{w}, \overline{h}) = (w_\ell, \psi(w_\ell))$ is a supersolution to Problem (P) as well as to Problem (P_L).

3.2 Explicit sub- and supersolutions and some consequences

By means of the comparison result of the above section it is possible to derive better estimates of our solution than boundedness (3.10). Since Problem (P_L) only was an auxiliary problem to our main concern, we restrict us to Problem (P). First, one easily computes that

$$\omega(x) = \left(\frac{a^2}{1-m} - ax \right)^{1/(1-m)} \tag{3.11}$$

is a solution of equation (2.1) and

$$w(x) = \left(\frac{b^2}{1-m} - ax \right)^{1/(1-m)} \tag{3.12}$$

is a subsolution to equation (2.1) if $0 \le b \le a$ and a supersolution to (2.1) if $b \ge a$. Using this function we are able to exclude the (trivial) case that the solution of Problem (P) has the form $(w, h) = (w(x), 0)$ with $w(0) = 0$.

Lemma 3.3 *If (w, h) is a nonnegative solution of Problem (P) with $w_\ell > 0$ then $h(0) = \psi(w(0)) > 0$, $w'(0) = h'(0) < 0$, w is decreasing on $[-\ell, 0]$ and h is nonincreasing on $[0, \ell]$.*

Proof: 1. Assume first that $w(0) = 0$. Then also $h(0) = 0$ and thus, since h may not attain any positive maximum in $(0, \ell)$, $h(x) = 0 \; \forall x \in [0, \ell]$. Consequently, it is easy to see that $g = 0$ in relation (2.6). Moreover, for positive, sufficiently small a, the function ω defined by (3.11) is a subsolution to (2.1),(3.7) satisfying

$$\omega(-\ell) \le w(-\ell) \qquad \text{and} \qquad \underline{g_\omega} = \omega'(0) < 0 = g =: \overline{g_w} \; .$$

Therefore, due to the validity of comparison principle for monotone solutions, Corollary 3.3, ω can be compared with w on $[-\ell, 0]$ to conclude that

$$\omega(x) \le w(x) \qquad \forall x \in [-\ell, 0]$$

- a contradiction to $w(0) = 0 < \omega(0)$.

2. It follows therefore that $w(0) > 0$. Then (3.12) is a subsolution to (2.1), (3.6) for sufficiently small positive a, b, and Corollary 3.3 implies $w(x) \geq \underline{w}(x) > 0$ for all $x \in [-\ell, 0]$. Hence w has no zeros and (w, h) is a classical solution of (2.1),(2.2) on $[-\ell, 0] \times [0, \zeta)$ for some positive $\zeta \leq \ell$. Moreover, by the classical maximum principle, w and h are nonincreasing functions and even decreasing on every subinterval where w and h are positive.

3. We must now verify $w'(0) < 0$.

For this, let $w'(0) = 0$. Then also $h(0) > 0$, $h'(0) = 0$ and h is nonincreasing on $[0, \ell]$. Therefore it attains its maximum at $x = 0$, but from the classical one-dimensional maximum principle, see e.g. [12, Theorem 1.2], it follows that

$$h'(0) < 0 \qquad \text{a contradiction to} \qquad h'(0) = 0\,.$$

If $w'(0) > 0$, then also $h'(0) > 0$ and the contradiction follows immediately.

This observation confirms the assertion. □

Our next goal is to derive a supersolution which shows us that $h(x) \equiv 0$ for all $x \geq \overline{\zeta}$ and some $\overline{\zeta}$ with $0 < \overline{\zeta} < \ell$.

Let us remind the Barenblatt solution

$$h_B(x) = \left[C - D(\sigma)x^2\right]_+^{\frac{1}{1-\sigma}}, \qquad D = D(\sigma) = \frac{\sigma(1-\sigma)}{2(1+\sigma)}\,, \; b > 0 \tag{3.13}$$

solving the ordinary differential equation

$$h' + \frac{\sigma}{1+\sigma}\, x\, h^\sigma = 0 \qquad \text{and therefore} \qquad \left(h' + \frac{\sigma}{1+\sigma}\, x\, h^\sigma\right)' = 0$$

which is derived from porous medium equation by looking for a self-similar solution. Unfortunately, it is not a solution of our equation (2.2) but it is a supersolution. To demonstrate this, choose $\ell > \sqrt{C/D}$ and write equation (2.2) in divergence form

$$D^+ h = -(h' + \alpha x h^\sigma)' + (\alpha + \beta) h^\sigma = 0.$$

Then we obtain

$$\begin{aligned}
&\int_0^\ell \big((h_B' + \alpha x h_B^\sigma)\varphi' + (\alpha+\beta) h_B^\sigma \varphi\big)\, dx \\
&= \underbrace{\int_0^\ell \Big(h_B' + \frac{\sigma}{\sigma+1} x h_B^\sigma\Big)\varphi'\, dx}_{=0} + \int_0^\ell \Big(\alpha - \frac{\sigma}{\sigma+1}\Big) x h_B^\sigma \varphi'\, dx + \int_0^\ell (\alpha+\beta) h_B^\sigma \varphi\, dx \\
&= \int_0^\ell \left[\Big(\frac{\sigma}{\sigma+1} - \alpha\Big)\, x \big(h_B^\sigma\big)' + \Big(\beta + \frac{\sigma}{\sigma+1}\Big)\, h_B^\sigma\right] \varphi(x)\, dx \geq 0
\end{aligned}$$

for all nonnegative $\varphi \in H^1(0, \ell)$ since $\alpha \geq \frac{\sigma}{\sigma+1}$, $(h_B^\sigma)' \leq 0$, and $h_B^\sigma \geq 0$ for all $x \in [0, \ell]$. This is $(2.6_{\geq})$ if $g = 0$ and $\varphi^- \equiv 0$. Now we conclude

Lemma 3.4 *For any nonnegative solution (w, h) of Problem (P) with $0 < h(0) < D(\sigma)\ell^2$ it holds*

$$\zeta = \sup\{x \in [0, \ell] : \; h(x) > 0\} \in (0, \ell). \tag{3.14}$$

Proof: The Barenblatt solution h_B given by (3.13) is a supersolution to (2.2),(3.6) if $C \geq h(0)$. Then comparison due to Corollary 3.3 yields $0 \leq h(x) \leq h_B(x)$ for $x \in [0, \ell]$ and thus $h(x) \equiv 0$ for all $x \geq \sqrt{h(0)/D(\sigma)}$. □

Lemma 3.4 raises the question of a bound for $h(0)$. Since our main concern is to solve problem (1.6) on the whole real line by approximation by a sequence of solutions of Problem (P) for $\ell \to \infty$ the bound of (w, h) given in Theorem 3.2 is too rough. We need a bound that is independent of ℓ. For this reason we first look for a suitable supersolution to Problem (P). Although the function ω given by (3.11) is a solution of equation (2.1) and h_B is a supersolution to equation (2.2), respectively, the couple (ω, h_B) is not a supersolution to Problem (P) since $\omega'(0) < h'_B(0) = 0$. However, we may construct a supersolution to Problem (P) based on these functions.

Lemma 3.5 *There are numbers $\rho > a/m$ and $b > a$ independent of ℓ such that $(\overline{w}, h_B)$ with*

$$\overline{w}(x) = \begin{cases} \left(\frac{b^2}{1-m} - ax\right)^{1/(1-m)} & \text{if } -\infty < x \leq -\rho, \\ \left(\frac{b^2}{1-m} + \frac{a}{2}(\frac{x^2}{\rho} + \rho)\right)^{1/(1-m)} & \text{if } -\rho \leq x \leq 0, \end{cases} \tag{3.15}$$

and h_B defined by (3.13) with $C = \psi(b^2/(1-m) + a\rho/2)^{1-\sigma}$ is a supersolution to Problem (P) if the boundary value fulfils the relation $w_\ell \leq \overline{w}(-\ell)$.

Proof: Fix some $\rho > a/m$. Note that $\overline{w}(-\ell) \geq w_\ell$ and $h_B(\ell) \geq 0$ independent of $x = \ell$. Moreover, $\overline{w}'(0) = h'_B(0) = 0$ and $h_B(0) = \psi(\overline{w}(0))$. Hence, it remains to verify the integral inequality (2.6$_\geq$) with $g = 0$.

First for $-\rho \leq x \leq 0$ one easy calculates

$$D^-\overline{w}(x) \geq \left(-\frac{ma^2}{(1-m)^2} - \frac{ma}{1-m}\rho + \left(m - \frac{a}{\rho}\right)\frac{\overline{w}^{1-m}}{1-m}\right)\overline{w}^{2m-1}.$$

Defining some $\lambda \in (0, 1)$ by $\lambda\rho = a/m$, by means of $\overline{w}^{1-m} \geq b^2/(1-m)$ we obtain

$$D^-\overline{w}(x) \geq \left(-\frac{ma^2}{(1-m)^2} - \frac{ma}{1-m}\rho + (1-\lambda)\frac{mb^2}{(1-m)^2}\right)\overline{w}^{2m-1}.$$

For sufficiently large $b > a$ this yields $D^-\overline{w}(x) \geq 0$ for all $x \in [-\rho, 0]$. Moreover, if $-\ell \leq x \leq -\rho$, $\overline{w}(x)$ coincides with (3.12) which is a supersolution in the case $b > a$ and $D^+h_B(x) \geq 0$ holds on $[0, \ell]$ in a weak sense as shown above. Consequently, since $\overline{w} \in C^1([-\ell, 0])$ and $\overline{w}'(0) = h'_B(0) = 0$, the couple $(\overline{w}, h_B)$ is a supersolution in the sense of Definition 2.1. □

Obviously, since $h_B(0) = \psi(\overline{w}(0))$ the couple $(\overline{w}, h_B)$ defined in Lemma 3.5 is a supersolution to Problem (P$_L$) for all $L > 0$, too. We summarize the results of this subsection.

Theorem 3.3 *Let (w, h) be the weak solution to Problem (P) or Problem (P$_L$), respectively, with boundary value w_ℓ between $(a\ell)^{1/(1-m)}$ and $\overline{w}(-\ell)$. Then it is bounded by*

$$\begin{aligned} (-ax)^{1/(1-m)} \leq w(x) \leq \overline{w}(x) \qquad & \text{for } -\ell \leq x \leq 0, \\ 0 \leq h(x) \leq h_B(x) \qquad & \text{for } \quad 0 \leq x \leq \ell, \end{aligned}$$

where the functions $\overline{w}$ and h_B are defined by (3.15) and (3.13), respectively, and are independent of ℓ. Moreover, there is a number $\ell_0 > 0$ such that

$$\emptyset \neq \operatorname{supp} h = [0, \zeta] \subset [0, \ell_0] \qquad \forall \ell > \ell_0.$$

Proof: The first assertion follows from Corollary 3.1 or Corollary 3.2, respectively, since $((-ax)^{1-m}, 0)$ is a subsolution (cf. (3.12) with $b = 0$) and $(\overline{w}, h_B)$ is a supersolution (cf. Lemma 3.5). The second assertion is a consequence of Lemma 3.3 and Lemma 3.4 since $0 < h(0) \leq h_B(0)$ is bounded independently on ℓ. □

4 Existence

The aim of this section is to prove existence of a solution of problem (1.1). The way to this end is subdivided into three steps.

Step 1: We solve Problem (P_L) on $[-\ell, 0] \times [0, \ell]$. This is done by reformulating it as a fixed point problem based on independent third boundary value problems for (2.1) on $[-\ell, 0]$ and (2.2) on $[0, \ell]$, respectively. The main tool to apply Schauders fixed point theorem is the comparison result from Corollary 3.3 which yields the required properties of the fixed point mapping.

Step 2: We solve Problem (P) by a limit process $L \to \infty$ in (P_L). Since the item $L\big(h(0) - \psi(w(0))\big)^2$ will be bounded uniformly with respect to L we obtain $h(0) - \psi(w(0)) \to 0$ as $L \to \infty$.

Step 3: The solution of problem (1.1) on the real line is constructed by a limit process $\ell \to \infty$ in Problem (P). The difficulty to overcome is the dependence on ℓ of the a priori bounds for the approximates in $H^1(-\ell, 0) \times H^1(0, \ell)$ since $w(x)$ will be unbounded as $x \to -\infty$. The asymptotic boundary condition imposed on w in (1.1) then easily follows from the estimate in Theorem 3.3.

4.1 Existence of solutions of Problems (P_L) and (P)

We start with a solution of Problem (P_L). Consider first the decoupled boundary value problems

$$D^- w = 0 \quad \text{for } -\ell < x < 0\,, \qquad w(-\ell) = w_\ell, \quad w'(0) + L\psi(w(0)) = f_w \tag{4.1}$$

and

$$D^+ h = 0 \quad \text{for } 0 < x < \ell\,, \qquad h(\ell) = 0, \quad -h'(0) + L\,h(0)) = f_h \tag{4.2}$$

for given $f_w, f_h \geq 0$, $L > 0$.

Proposition 4.1 *For given real numbers $w_\ell, f_w, f_h \geq 0$ and $L > 0$ there are nonnegative (weak) solutions $w \in H^1(-\ell, 0)$ and $h \in H^1(0, \ell)$ of problems (4.1) and (4.2) fulfilling the relations*

$$\int_{-\ell}^{0} \Big(w'\varphi' + w^m (x\varphi)' + \frac{m}{1-m}\, w^m \varphi \Big) dx + L\psi(w(0))\varphi(0) = f_w \varphi(0) \tag{4.3}$$

for all $\varphi \in H^1(-\ell, 0)$ *with* $\varphi(-\ell) = 0$ *and*

$$\int_0^\ell \Big(h'\varphi' + \alpha h^\sigma (x\varphi)' + \frac{\omega\sigma}{1-m} h^\sigma \varphi \Big) dx + Lh(0)\varphi(0) = f_h \varphi(0) \tag{4.4}$$

for all $\varphi \in H^1(0, \ell)$ *with* $\varphi(\ell) = 0$*, respectively, as well as the boundary condition* $w(-\ell) = w_\ell$*,* $h(\ell) = 0$*.*

Proof: We intend to build a solution of (4.1) and (4.2) as the limit of certain finite dimensional approximations (Galerkin method) following step by step the way of [6, Chapter 9.1], for example, but we omit more details here. Nonnegativity of w and h can be shown by the argument given in the proof of Theorem 4.1 (ii). Then $w(0) < 0$, $w'(0) \leq 0$ or $h(0) < 0$, $h'(0) \geq 0$, respectively, provide a contradiction to $f_w, f_h \geq 0$. □

We construct the fixed point mapping which is equivalent to Problem (P_L). Let $w_\ell \geq 0$, $L > 0$ be fixed and (w, h) be the tuple of solutions of (4.1) and (4.2) corresponding to given data f_w and f_h, respectively. Then Proposition 4.1 defines an operator

$$\mathcal{S} : \mathbb{R}^2 \to H^1(-\ell, 0) \times H^1(0, \ell) \qquad \text{with } \mathcal{S}(f_w, f_h) = (\mathcal{S}^- f_w, \mathcal{S}^+ f_h) = (w, h).$$

Defining the modified trace operator

$$\mathcal{T} : H^1(-\ell, 0) \times H^1(0, \ell) \to \mathbb{R}^2 \qquad \text{with } \mathcal{T}(w, h) = (Lh(0), L\psi(w(0))),$$

we have the composition $\mathcal{A} = \mathcal{T}\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 4.1 *Let* (f_w^*, f_h^*) *be a fixed point of the operator* $\mathcal{A}$*. Then* $(w, h) = \mathcal{S}(f_w^*, f_h^*)$ *is a solution of Problem* (P_L)*.*

Proof: w fulfils relation (4.3) with $f_w = f_w^* = Lh(0)$ and h fulfils (4.4) with $f_h = f_h^* = L\psi(w(0))$. Test now (4.3) with $\varphi = \varphi^-$ and (4.4) with $\varphi = \varphi^+$ for given $(\varphi^-, \varphi^+) \in V^\circ$, the sum of these relations fulfils the conditions of Definition 2.2. □

We want to apply Schauders fixed point theorem. Therefore we have to show that $\mathcal{A}$ is a continuous operator mapping a convex compact set $K \subset \mathbb{R}^2$ into itself. Choose

$$K = \{(f_w, f_h) \in \mathbb{R}^2 : 0 \leq f_w \leq L\,\psi(w_\ell), 0 \leq f_h \leq L\,\psi(w_\ell)\}.$$

Lemma 4.2 *Let* $w_\ell \geq 0$*. Then* $\mathcal{A} : K \to K$*.*

Proof: Obviously, $\overline{w} = w_\ell$ is a supersolution to (2.1),(3.7) with $\overline{g} = 0$, and $\underline{w} = w = \mathcal{S}^- f_w$ is a subsolution with $\underline{g} = f_w - L\psi(w(0))$, respectively. Then estimate (3.9) from Corollary 3.3 reads as

$$\int_{-\ell}^0 [w(x)^m - w_\ell^m]_+ \, dx - \big(f_w - L\psi(w(0))\big) \operatorname{sign}^+(w(0) - w_\ell) \leq 0.$$

Since $f_w \leq L\,\psi(w_\ell)$ this implies

$$\int_{-\ell}^0 [w(x)^m - w_\ell^m]_+ \, dx + L\,\big(\psi(w(0)) - \psi(w_\ell)\big) \operatorname{sign}^+(w(0) - w_\ell) \leq 0,$$

which yields $w(x) \le w_\ell$ for all $x \in [-\ell, 0]$ due to monotonicity of ψ.

The analogue to (3.9) for (2.2),(3.8) with $\overline{h} = \psi(w_\ell)$, $\underline{h} = h = \mathcal{S}^+ f_h$, $\overline{g} = 0$ and $\underline{g} = f_h - Lh(0)$ is

$$\int_0^\ell [h(x)^\sigma - \psi(w_\ell)^\sigma]_+ \, dx - \big(f_h - Lh(0)\big) \operatorname{sign}^+(h(0) - \psi(w_\ell)) \le 0.$$

Since $f_h \le L\,\psi(w_\ell)$ it follows $h(x) \le \psi(w_\ell)$ for all $x \in [0, \ell]$. Nonnegativity of w and h is obtained from Proposition 4.1 since $(f_w, f_h) \in K$ are nonnegative. This implies $0 \le w(0) \le w_\ell$ and $0 \le h(0) \le \psi(w_\ell)$, which yields $\mathcal{T}(w, h) \in K$. □

Lemma 4.3 *Let $w_\ell \ge 0$. Then $\mathcal{A} : K \to K$ is continuous.*

Proof: Let (f_w, f_h) and $(\widetilde{f_w}, \widetilde{f_h})$ be given and $(w, h) = \mathcal{S}(f_w, f_h)$, $(\tilde{w}, \tilde{h}) = \mathcal{S}(\widetilde{f_w}, \widetilde{f_h})$. Application of Corollary 3.3 first to $\underline{w} = w$, $\overline{w} = \tilde{w}$ and then to $\underline{w} = \tilde{w}$, $\overline{w} = w$ yields

$$\int_{-\ell}^0 [w^m - \tilde{w}^m]_+ \, dx - \Big((f_w - \widetilde{f_w}) - L\big(\psi(w(0)) - \psi(\tilde{w}(0))\big) \Big) \operatorname{sign}^+(w(0) - \tilde{w}(0)) \le 0,$$
$$\int_{-\ell}^0 [\tilde{w}^m - w^m]_+ \, dx - \Big((\widetilde{f_w} - f_w) - L\big(\psi(\tilde{w}(0)) - \psi(w(0))\big) \Big) \operatorname{sign}^+(\tilde{w}(0) - w(0)) \le 0.$$

Since ψ is an increasing function we have $L(\psi(w_1(0)) - \psi(w_2(0))) \operatorname{sign}^+(w_1(0) - w_2(0)) = L\,[\psi(w_1(0)) - \psi(w_2(0))]_+$, hence the sum of the two inequalities leads to

$$\int_{-\ell}^0 |w(x)^m - \tilde{w}(x)^m| \, dx + L\,|\psi(w(0)) - \psi(\tilde{w}(0))| \le |f_w - \widetilde{f_w}|.$$

In the same way an analogue inequality for h

$$\int_0^\ell |h(x)^\sigma - \tilde{h}(x)^\sigma| \, dx + L\,|h(0)) - \tilde{h}(0)| \le |f_h - \widetilde{f_h}|$$

follows. This means continuous dependence of $\mathcal{A}(f_w, f_h)$ on the data (f_w, f_h) and it concludes the proof. □

Theorem 4.1 *For given $w_\ell \ge 0$ there is a unique weak solution (w, h) of Problem (P_L) which is nonnegative on $[-\ell, 0] \times [0, \ell]$.*

Proof: (i) Existence: Owing to Lemma 4.2 and Lemma 4.3 we can apply Schauders fixed point theorem which yields existence of $(f_w^*, f_h^*) \in K$ with $\mathcal{A}(f_w^*, f_h^*) = (f_w^*, f_h^*)$. Then $(w, h) = \mathcal{S}(f_w^*, f_h^*)$ is a solution of Problem (P_L) due to Lemma 4.1.

(ii) Nonnegativity: Assume there is a solution of Problem (P_L) with $w(x_0) < 0$ for some $x_0 \in (-\ell, 0]$. By the argument given in the proof of Lemma 3.2 w cannot have a negative local minimum in $(-\ell, 0)$. Hence, since $w(-\ell) = w_\ell \ge 0$, $\min_{-\ell \le x \le 0} w(x) = w(0) < 0$ and $w'(0) \le 0$. Contact condition (2.5) then implies $h(0) \le \psi(w(0)) < 0$ and $h'(0) \le 0$. Since $h(\ell) = 0$ and also h cannot have a negative minimum, the only possible situation is $\min_{0 \le x \le \ell} h(x) = h(0) < 0$ with $h'(0) = 0$, $h''(0) \ge 0$. But this is a contradiction to equation (2.2) at $x = 0$.

If, on the other hand, $w(x) \geq 0$ for all $x \in [-\ell, 0]$ and $h(x_0) < 0$ for some $x_0 \in [0, \ell)$, then again $\min_{0 \leq x \leq \ell} h(x) = h(0) < 0$ and $h'(0) \geq 0$. Condition (2.5) now implies $\psi(w(0)) \leq h(0) < 0$ which is a contradiction to $w(0) \geq 0$, too.

(iii) Uniqueness follows from Corollary 3.2 in combination with Lemma 3.2. □

We come to step 2.

Theorem 4.2 *For given $w_\ell \geq 0$ there is a unique weak solution (w, h) of Problem (P) which is nonnegative on $[-\ell, 0] \times [0, \ell]$.*

Before we prove this theorem we need some a priori estimates for the solution of Problem (P_L).

Lemma 4.4 *Let (w, h) be a solution of Problem (P_L). Then there are constants $c_i = c_i(\ell, w_\ell)$, $i = 1, 2, 3$, independent of $L > 0$ such that*

$$\int_{-\ell}^{0} |w'(x)|^2 \, dx \leq c_1(\ell, w_\ell), \tag{4.5}$$

$$\int_{0}^{\ell} |h'(x)|^2 \, dx \leq c_2(\ell, w_\ell), \tag{4.6}$$

$$L \left| h(0) - \psi(w(0)) \right|^2 \leq c_3(\ell, w_\ell). \tag{4.7}$$

Proof: (i) We derive (4.5). Choose test function $\varphi = (w(x) + \tau(x), 0)$ with perturbation $\tau(x) = \frac{x}{\ell} w_\ell - (1 + \frac{x}{\ell}) w(0)$ and test relation (2.6) with it. Since $\varphi^-(0) = \varphi^+(0)$ the last item in (2.6) disappears and we obtain

$$\int_{-\ell}^{0} \Big(w'(w + \tau)' + w^m (x(w + \tau))' + \frac{m}{1 - m} \, w^m (w + \tau) \Big) dx = 0.$$

Owing to (3.10) w is bounded by w_ℓ, moreover τ and τ' are bounded since $0 \leq w(0) \leq w_\ell$, hence

$$\int_{-\ell}^{0} |w'(x)|^2 \, dx \leq c(\ell, w_\ell) + \int_{-\ell}^{0} |w'(x)\tau'(x)| dx + \ell \, w_\ell^m \int_{-\ell}^{0} |w'(x)| \, dx.$$

Now Youngs inequality finishes the proof of (4.5).

(ii) To prove (4.6) and (4.7) we test (2.6) corresponding to Problem (P_L) (cf. Definition 2.2) with $\varphi(x) = \big((1 + \frac{x}{\ell}) \psi(w(0)), h(x) \big)$ which yields

$$\int_{-\ell}^{0} \Big(w' \frac{1}{\ell} \psi(w(0)) + w^m \big(1 + \frac{2x}{\ell}\big) \psi(w(0)) + \frac{m}{1 - m} \, w^m \big(1 + \frac{x}{\ell}\big) \psi(w(0)) \Big) dx$$
$$+ \int_{0}^{\ell} \Big(|h'|^2 + \alpha h^\sigma (xh)' + \frac{\omega \sigma}{1 - m} h^{\sigma + 1} \Big) dx + L \left(h(0) - \psi(w(0)) \right)^2 = 0.$$

Using again boundedness of w and h by (3.10), Youngs inequality and the already proved estimate (4.5) it is easy to see that (4.6) and (4.7) hold. □

Proof of Theorem 4.2: We choose a sequence (L_n) with $L_n \to \infty$ and investigate the behavior of the corresponding solutions (w_n, h_n) of Problem (P_{L_n}). $((w_n, h_n))_{n \in \mathbb{N}}$ is bounded in V due to (4.5) and (4.6). V is compactly embedded in $C([-\ell,0]) \times C([0,\ell])$. Hence, there is a subsequence $((w_{n_k},h_{n_k}))_{k\in\mathbb{N}}$ of $((w_n,h_n))_{n\in\mathbb{N}}$ with

$$(w_{n_k},h_{n_k}) \rightharpoonup (w,h) \qquad \text{in } V = H^1(-\ell,0)\times H^1(0,\ell) \tag{4.8}$$

and

$$(w_{n_k},h_{n_k}) \to (w,h) \qquad \text{in } C([-\ell,0])\times C([0,\ell]). \tag{4.9}$$

Now test relations (2.6) corresponding to Problems $(\mathrm{P}_{L_{n_k}})$ with $\varphi \in \tilde{V}^\circ$, i.e. $\varphi^-(0) = \varphi^+(0)$. This yields

$$\int_{-\ell}^0 \Big(w_{n_k}'(\varphi^-)' + w_{n_k}^m (x\varphi^-)' + \frac{m}{1-m}\, w_{n_k}^m \varphi^- \Big) dx \\ + \int_0^\ell \Big(h_{n_k}'(\varphi^+)' + \alpha h_{n_k}^\sigma (x\varphi^+)' + \frac{\omega\sigma}{1-m} h_{n_k}^\sigma \varphi^+ \Big) dx = 0.$$

Convergence properties (4.8) and (4.9) allow to go to the limit $k\to\infty$ in this relation, hence (w,h) fulfils

$$\int_{-\ell}^0 \Big(w'(\varphi^-)' + w^m (x\varphi^-)' + \frac{m}{1-m}\, w^m \varphi^- \Big) dx \\ + \int_0^\ell \Big(h'(\varphi^+)' + \alpha h^\sigma (x\varphi^+)' + \frac{\omega\sigma}{1-m} h^\sigma \varphi^+ \Big) dx = 0$$

for all $\varphi \in \tilde{V}^\circ$ which is (2.6) for this class of test functions. Moreover, owing to (4.9), $w(-\ell) = w_\ell$ and $h(\ell) = 0$. To check the contact condition (2.8) we see from (4.7) that $h_{n_k}(0) - \psi(w_{n_k}(0)) \to 0$ as $k\to\infty$. Since on the other hand $h_{n_k}(0) \to h(0)$ and $\psi(w_{n_k}(0)) \to \psi(w(0))$ due to (4.9) this yields $h(0) = \psi(w(0))$. To conclude that (w,h) is a solution of Problem (P) it only remains to refer to Proposition 2.1 which states existence of a g such that (2.6) is fulfilled for all $\varphi \in V^\circ$.

Finally, nonnegativity holds since the solutions (w_n,h_n) of Problem (P_{L_n}) are nonnegative and uniqueness is given by Corollary 3.1 in combination with Lemma 3.2. □

Remark 4.1 *Since the solution (w,h) of Problem (P) is unique the convergence properties (4.8) and (4.9) hold for the whole sequence (w_n,h_n). Hence, the (unique) solution of Problem (P) may be approximated (uniformly with respect to x) by solutions (w_n,h_n) of Problem (P_{L_n}) for an arbitrary choice of $L_n\to\infty$.*

4.2 The contact problem on the real line

We continue with step 3 and return to problem (1.1) in this way. First, we have to define the proper notion of a solution.

Definition 4.1 *A couple (w,h) is called (weak) solution of Problem (1.1) if*

(i) $(w,h) \in V$ *for any fixed* $\ell > 0$*;*
(ii) it satisfies the relation

$$\int_{-\infty}^{0} \Big(w'(\varphi^-)' + w^m (x\varphi^-)' + \frac{m}{1-m}\, w^m \varphi^- \Big) dx \\ + \int_0^\infty \Big(h'(\varphi^+)' + \alpha h^\sigma (x\varphi^+)' + \frac{\omega\sigma}{1-m} h^\sigma \varphi^+ \Big) dx = 0 \quad (4.10)$$

for all test functions $\varphi = (\varphi^-, \varphi^+)$ *with bounded support which belong to* $\tilde{V}$ *for any positive* ℓ*;*
(iii) it holds the contact condition

$$h(0) = M\, w^\omega(0) \tag{4.11}$$

with $M, \omega > 0$ *and*
(iv) the asymptotic conditions

$$w(x) - (-ax)^{1/(1-m)} = \mathcal{O}\Big((-ax)^{m/(1-m)} \Big) \quad \text{as } x \to -\infty, \tag{4.12}$$

$$h(x) \to 0 \qquad \text{as } x \to \infty \tag{4.13}$$

are fulfilled.

The idea to come to a solution of (1.1) is to observe the solution (w,h) of Problem (P) as $\ell \to \infty$. In order to obtain condition (4.12) for the limit we restrict to $\ell > \rho$ and prescribe the boundary value

$$w_\ell = w(-\ell) = \left(\frac{b^2}{1-m} + a\ell \right)^{1/(1-m)} \tag{4.14}$$

where the constants ρ and b are given in Lemma 3.5.

Choose now a sequence $\ell_\nu := \nu \to \infty$, $\nu \in \mathbb{N}$, and denote by (w_ν, h_ν) the solution of problem (P) with $\ell = \ell_\nu$. Remember the assertion of Theorem 3.3 that $\operatorname{supp} h \subset [0, \ell_0]$ for all $\ell > \ell_0$ then

$$h_\nu(x) \equiv 0 \qquad \forall x \in [\ell_0, \nu] \tag{4.15}$$

if $\nu > \ell_0$.

Hence we fix some $R > \max\{\rho, \ell_0\} > 0$ and restrict to $\nu \geq R$. We show that the corresponding sequence (w_ν, h_ν) is nonincreasing w.r.t. ν and bounded on $[-R, R]$. In order to conclude that the limit is a solution of our equations we need some a priori estimates else. Since every test function in Definition 4.1 vanishes outside a compact set it is enough to derive local a priori estimates on $[-R, R]$.

Lemma 4.5 *Let* $R > \max\{\rho, \ell_0\}$ *be fixed and* $\nu \geq R$*. Then there are constants* C *and* $C(R)$ *independent of* ν *such that*

$$0 \leq w_\nu(x) \leq C(R), \quad -R \leq x \leq 0, \quad \text{and} \quad 0 \leq h_\nu(x) \leq C, \qquad 0 \leq x \leq \nu, \tag{4.16}$$

$$\int_{-R}^{0} |w_\nu'(x)|^2\, dx \leq C(R), \qquad \text{and} \qquad \int_0^\nu |h_\nu'(x)|^2\, dx \leq C. \tag{4.17}$$

Proof: Estimate (4.16) follows from Theorem 3.3. For the proof of (4.17) in a first step we test (2.6) with

$$\varphi^-(x) = \begin{cases} w(x) + \tau(x) & \text{for } -R \le x \le 0, \\ 0 & \text{for } -\nu \le x \le -R, \end{cases} \qquad \tau(x) = \frac{x}{R} w(-R) - (1 + \frac{x}{R}) w(0),$$
$$\varphi^+(x) = 0 \quad \text{for } 0 \le x \le \nu.$$

The same manipulations as in the proof of Lemma 4.4 by means of $0 \le w(x) \le C(R)$ lead to the first inequality of (4.17). Secondly, we test with $\varphi = (0, h - \tau)$ where $\tau(x) = h(0)[1 - \frac{x}{\ell_0}]_+$. This is an admissible test function since $h(x) = 0$ for all $x \ge \ell_0$. We obtain

$$\int_0^{\ell_0} \Big(h'(h-\tau)' + \alpha h^\sigma (x(h-\tau))' + \frac{\omega\sigma}{1-m} h^\sigma (h-\tau) \Big) dx = 0.$$

This implies the second inequality of (4.17) by means of Youngs inequality and (4.16). □

Theorem 4.3 *There is a solution of problem (1.1) in the sense of Definition 4.1 which is the limit of a nonincreasing sequence* $\big((w_\nu, h_\nu)\big)_{\nu \in \mathbb{N}}$ *of solutions of Problem (P) with* $\ell = \nu$ *and boundary data (4.14).*

Proof: Let R, ν be chosen as above and compare $(w_{\nu+1}, h_{\nu+1})$ and (w_ν, h_ν). We have $h_{\nu+1}(\nu) = h_\nu(\nu) = 0$ due to (4.15) and $w_{\nu+1}(\nu) \le w_\nu(\nu)$ due to (4.15) and the estimate from Theorem 3.3 for $(w_{\nu+1}, h_{\nu+1})$. Then Corollary 3.1 and Lemma 3.2 applied to the interval $[-\nu, \nu]$ imply $w_{\nu+1}(x) \le w_\nu(x)$ for all $x \in [-\nu, 0]$ and $h_{\nu+1}(x) \le h_\nu(x)$ for all $x \in [0, \nu]$. Hence,

$$(w_\nu, h_\nu) \searrow (w, h) \qquad \text{pointwise on } [-R, R] \quad \text{nonincreasingly as } \nu \to \infty. \tag{4.18}$$

By Lebesgue's Dominated Convergence Theorem the convergence also holds in $L_1(-R, 0) \times L_1(0, R)$. Owing to the strong convergence of the whole sequence in this larger function space the estimates of Lemma 4.5 and the compact embedding $H(a, b) \subset C([a, b])$ yield

$$(w_\nu, h_\nu) \rightharpoonup (w, h) \qquad \text{in } H^1(-R, 0) \times H^1(0, R) \tag{4.19}$$

and

$$(w_\nu, h_\nu) \to (w, h) \qquad \text{in } C([-R, 0]) \times C([0, R]) \tag{4.20}$$

as $\nu \to \infty$ which is condition (i) of Definition 4.1.

Fix now a test function $\varphi \in \tilde{V}$ with compact support and choose $R > \max\{\rho, \ell_0\}$ large enough such that $\operatorname{supp} \varphi \subset [-R, 0] \times [0, R]$. Testing relation (2.6) for (w_ν, h_ν), $\nu > R$, with this test function we obtain

$$\int_{-R}^0 \Big(w_\nu' (\varphi^-)' + w_\nu^m (x\varphi^-)' + \frac{m}{1-m} w_\nu^m \varphi^- \Big) dx$$
$$+ \int_0^R \Big(h_\nu' (\varphi^+)' + \alpha h_\nu^\sigma (x\varphi^+)' + \frac{\omega\sigma}{1-m} h_\nu^\sigma \varphi^+ \Big) dx = 0.$$

Passing to the limit by means of (4.19),(4.20) this yields relation(4.10).

Condition (4.11) holds since it holds for every approximate (w_ν, h_ν). Moreover, (4.13) is fulfilled since (4.15) holds for all h_ν and convergence goes on in $C([0,R])$. Finally, due to Theorem 3.3 every w_ν is estimated independently of ν by

$$(-ax)^{1/(1-m)} \le w_\nu(x) \le \left(\frac{b^2}{1-m} - ax\right)^{1/(1-m)} \qquad \text{for } -\nu \le x \le -\rho$$

with constants some $\rho > a/m$ and $b > a$. The locally uniform convergence (4.20) implies the same estimate for the limit w on $(-\infty, -\rho]$ and thus

$$\begin{aligned} 0 \le w(x) - (-ax)^{1/(1-m)} &\le \left(\frac{b^2}{1-m} - ax\right)^{1/(1-m)} - \Big(-ax\Big)^{1/(1-m)} \\ &\le \frac{b^2}{(1-m)^2}\left(\frac{b^2}{1-m} - ax\right)^{m/(1-m)} \end{aligned} \tag{4.21}$$

which implies the asymptotic condition (4.12). □

Remark 4.2 *From the proof we see that the solution constructed in Theorem 4.3 is bounded by the bounds given in Theorem 3.3, i.e.*

$$\begin{aligned} (-ax)^{1/(1-m)} \le w(x) \le \overline{w}(x) \qquad & \textit{for } -\infty < x \le 0, \\ 0 \le h(x) \le h_B(x) \qquad & \textit{for } \quad 0 \le x < \infty, \end{aligned} \tag{4.22}$$

where the functions $\overline{w}$ and h_B are defined by (3.15) and (3.13), respectively. More than required in (4.13), these estimates together with Lemma 3.3 show that there is a number $\zeta > 0$ such that $\operatorname{supp} h = [0, \zeta]$.

Remark 4.3 *Replacing boundary condition (4.14) by $w_\ell = (a\ell)^{1/(1-m)}$ in the same way we obtain a nondecreasing sequence (w_ν, h_ν) converging to a solution (w, h) of problem (1.1). It fulfils the estimate (4.22), too.*

Unfortunately, we could not verify the question of uniqueness of a solution of problem (1.1). However, in view of the results of Section 3, we may conclude that the set of solutions is an ordered set without points of intersection. Moreover, the bounds given in Remark 4.2 hold for all possible solutions. Actually, all possible solutions are even enclosed between the two solutions given in Theorem 4.3 and Remark 4.3. We finish this section with the proof of these facts.

Corollary 4.1 *Let (w_1, h_1) and (w_2, h_2) be two solutions of problem (1.1). Then*

$$\begin{aligned} w_1(-x) \le w_2(-x) \quad \wedge \quad h_1(x) \le h_2(x) \qquad \forall x \in [0,\infty) \qquad & \textit{or} \\ w_1(-x) \ge w_2(-x) \quad \wedge \quad h_1(x) \ge h_2(x) \qquad \forall x \in [0,\infty). & \end{aligned}$$

If there is a point $x_0 \in \mathbb{R}$ with $w_1(x_0) = w_2(x_0)$ or $h_1(x_0) = h_2(x_0) > 0$, respectively, then $(w_1, h_1) \equiv (w_2, h_2)$.

Proof: Let (w_1, h_1), (w_2, h_2) be two solutions of Problem (1.1) and $w_1(x_1) > w_2(x_1)$, $w_1(x_2) < w_2(x_2)$ (or $h_1(x_1) > h_2(x_1)$, $h_1(x_2) < h_2(x_2)$, respectively) for some $x_1, x_2 \in \mathbb{R}$. Let now $\ell \geq \max\{|x_1|, |x_2|\}$ be large enough such that $h_1(\ell) = h_2(\ell) = 0$, and observe e.g. $w_1(-\ell) \leq w_2(-\ell)$. Corollary 3.1 together with Lemma 3.2 imply $w_1(x) \leq w_2(x)$ for all $x \in [-\ell, 0]$ and $h_1(x) \leq h_2(x)$ for all $x \in [0, \ell]$, which is a contradiction at x_1.

Let now $w_1(x_0) = w_2(x_0)$ (or $h_1(x_0) = h_2(x_0) > 0$, resp.) for some $x_0 \in \mathbb{R}$. Since the solutions cannot intersect and every solution of problem (1.1) is a classical solution at all points where it is positive, furthermore we have $w_1'(x_0) = w_2'(x_0)$ (or $h_1'(x_0) = h_2'(x_0)$, resp.). Then the classical uniqueness theorem yields $(w_1, h_1) \equiv (w_2, h_2)$. □

For the next corollary we denote the solution of problem (1.1) obtained as a limit of a nonincreasing sequence (w_ν, h_ν) due to Theorem 4.3 by $(w_{\max}, h_{\max})$ and the limit of the nondecreasing sequence (w_ν, h_ν) due to Remark 4.3 by $(w_{\min}, h_{\min})$, respectively. Obviously, due to construction and Corollary 4.1 it holds $(w_{\min}, h_{\min}) \leq (w_{\max}, h_{\max})$ (where this ordering of pairs means the ordering of components due to Corollary 4.1).

Corollary 4.2 *Every solution (w, h) of problem (1.1) fulfils the estimate (4.22) and, moreover, $(w_{\min}, h_{\min}) \leq (w, h) \leq (w_{\max}, h_{\max})$.*

Proof: We carry out the proof of the lower estimates for any solution (w, h).
1. We prove $w(x) \geq (-ax)^{1/(1-m)}$. Suppose there is $x_0 < 0$ with $w(x_0) < (-ax)^{1/(1-m)}$. Then we choose $\tilde{a} < a$ such that

$$w(x_0) < (-\tilde{a}x_0)^{1/(1-m)}. \tag{4.23}$$

Moreover, because of the asymptotic condition (4.12) which implies (1.10) and Remark 4.2, there is a $\ell > 0$ it such that $w(x) > (-\tilde{a}x)^{1/(1-m)}$ for all $x \leq -\ell$ and $h(x) = 0$ for all $x \geq \ell$.

Consider now (w, h) and $(\underline{w}, \underline{h}) = ((-\tilde{a}x)^{1/(1-m)}, 0)$ on $[-\ell, \ell]$. Then $(\underline{w}, \underline{h})$ is a subsolution and (w, h) is a supersolution, resp., since $\underline{w}(-\ell) < w(\ell)$ and $\underline{h}(\ell) = h(\ell) = 0$. Hence Corollary 3.1 yields $\underline{w}(x) \leq w(x)$ for all $x \in [-\ell, 0]$ which is a contradiction to (4.23).

2. Suppose there is a solution (w, h) with $w(x) \geq (-ax)^{1/(1-m)}$ for all $x \in (-\infty, 0]$ and $w(x_0) < w_{\min}(x_0)$ for some $x_0 \leq 0$. Owing to uniform convergence (4.20) there is a solution (w_ν, h_ν) of Problem (P) constructed due to Remark 4.3 with $w_\nu(x_0) > w(x_0)$ and $w_\nu(-\nu) = (a\nu)^{1/(1-m)} \leq w(-\nu)$ which is a contradiction to Corollary 3.1 again. The same idea is used if $0 < h(x_0) < h_{\min}(x_0)$ for some $x_0 \geq 0$ or for the upper estimate by $(w_{\max}, h_{\max})$, respectively.

3. The same argument as in step 1 with $\tilde{a} > a$ applies to the upper bound $(\overline{w}, h_B)$ since it depends continuously and increasingly on a for all $x \leq \sqrt{C/D(\sigma)}$. □

5 Self-similar solutions

The main aim of this section is to prove that the couple (u, v) given by (1.5) is the solution of (1.6) in the sense of the following

Definition 5.1 *A couple (u, v) is called a solution of Problem (1.6) if*
(i) $(u, v) \in L^\infty(0, T; V)$ *for any fixed* $\ell, T > 0$ *;*

(ii) the integral identity

$$\int_0^\infty \int_{-\infty}^0 \Big((-ax)^{m/(1-m)} - u^m)\varphi_t^- \; + \; u_x \varphi_x^- \Big) \, dx \, dt$$
$$+ \int_0^\infty \int_0^\infty \big(-v^\sigma \varphi_t^+ \; + \; v_x \varphi_x^+ \big) \, dx \, dt = 0 \quad (5.1)$$

holds for all sufficiently smooth test functions $\varphi = (\varphi^-, \varphi^+)$ *with bounded support, i.e.* $\varphi \in L^1(0,T;V)$ *and* $\varphi_t \in L^1$ *for any positive* T, ℓ *and* $\varphi^-(0,t) = \varphi^+(0,t)$ *;*
(iii) the contact condition

$$v(0,t) = M u^\omega(0,t)$$

is satisfied for any $t \geq 0$*.*

As

$$u_x(x,t) \; = \; t^{m/(1-m)} \, w' \left(\frac{x}{t} \right)$$

and

$$\int_{-\ell}^0 |u_x(x,t)|^2 \, dx \; = \; t^{(1+m)/(1-m)} \int_{-\ell/t}^0 |w'(y)|^2 dy \, ,$$

the next proposition seems to be useful.

Proposition 5.1

$$\int_{-\ell}^0 |w'(y)|^2 dy \; = \; \mathcal{O} \left(\ell^{(1+m)/(1-m)} \right) \qquad as \quad \ell \to \infty \, .$$

Proof of Proposition 5.1: Let $0 < \delta \ll 1$ and $1 \ll \ell$ be given. Note that $\varphi = (\chi_\delta w, 0)$, where

$$\chi_\delta(x) = \begin{cases} 0 & \text{for } x \leq -\ell - 1 \\ x + \ell + 1 & \text{for } -\ell - 1 \leq x \leq -\ell \\ 1 & \text{for } -\ell \leq x \leq -\delta \\ -\delta^{-1} x & \text{for } -\delta \leq x \leq 0 \end{cases}$$

is the admissible test function in the weak formulation (2.6) of our problem. Performing tedious but straightforward manipulations we get

$$\begin{aligned} w^2(-\ell) + 2 \int_{-\ell-1}^0 & \left(|w'|^2 + \frac{2m}{1-m^2} |w|^{m+1} \right) \chi_\delta(x) \, dx \\ & + \quad \frac{2m}{(m+1)\delta} \int_{-\delta}^0 (-x) w^{m+1}(x) \, dx \; + \; \frac{w^2(-\delta) - w^2(0)}{\delta} \\ & = \quad w^2(-\ell - 1) \; + \; \frac{2m}{m+1} \int_{-\ell-1}^{-\ell} (-x) w^{m+1}(x) \, dx \; . \end{aligned}$$

Omitting nonnegative terms on the left–hand side of the above equality and letting $\delta \to 0$ we arrive at

$$w^2(-\ell) \; + \; \frac{4m}{1-m^2} \int_{-\ell}^0 w^{m+1}(x) \, dx \; + \; 2 \int_{-\ell}^0 |w'|^2(x) \, dx$$
$$\leq \; w^2(-\ell - 1) \; + \; \frac{2m}{m+1} \int_{-\ell-1}^{-\ell} (-x) w^{m+1}(x) \, dx \; . \quad (5.2)$$

Let us now recall the estimate (4.22), i.e.

$$(-ax)^{1/(1-m)} \leq w(x) \leq \left(\frac{b^2}{1-m} - ax\right)^{1/(1-m)}$$

in which the first inequality holds for any $x \leq 0$ and the second one for all sufficiently large $|x|$. With the assistance of (4.22) we can now estimate the first two terms on the left hand side from below and both terms on the right hand side from above to obtain

$$(a\ell)^{1/(1-m)} + \frac{2m}{a(m+1)}(a\ell)^{2/(1-m)} + 2\int_{-\ell}^{0} |w'|^2(x)\,dx \leq \left(a\ell + a + \frac{b^2}{1-m}\right)^{1/(1-m)} + \frac{2m}{a(m+1)}a(\ell+1)\left(a\ell + a + \frac{b^2}{1-m}\right)^{(m+1)/(1-m)}. \quad (5.3)$$

Next, (5.3) easily yields

$$2\int_{-\ell}^{0} |w'|^2(x)\,dx \leq \left(a\ell + a + \frac{b^2}{1-m}\right)^{1/(1-m)} - (a\ell)^{1/(1-m)} + \frac{2m}{a(m+1)}\left[\left(a\ell + a + \frac{b^2}{1-m}\right)^{2/(1-m)} - (a\ell)^{2/(1-m)}\right] \quad (5.4)$$

and the required estimate of Proposition (5.1) follows easily. □

We are now ready to formulate the main result of this section.

Theorem 5.1 *There is a solution (u, v) of problem (1.6) in the sense of Definition 5.1. This solution is given by (1.5) where (w, h) solves problem (1.1). It has the following interface (free boundary)*

$$I[v] = \{(x,t) \in [0,\infty) \times [0,\infty) : \; x = \xi(t) = \zeta t^{\alpha}\}$$

with $\zeta = \inf\{x \geq 0 \; : \; h(x) = 0\} > 0$.

Proof: 1. The item (i) of Definition 5.1 follows directly from Proposition 5.1.
2. Let $\ell \gg 1$ and $T \gg 1$ be arbitrary but from now on fixed. Assume a sufficiently smooth function $g = g(y,t)$ such that its support lies in $[-\ell, 0] \times [0, T]$ and a sufficiently smooth function $f = f(y,t)$ such that its support lies in $[0, \ell] \times [0, T]$, $g(0,t) = f(0,t)$. With the help of g and f we construct test functions

$$\varphi^-(x) = g(tx,t) \quad \text{for} \quad x \leq 0 \qquad \text{and} \qquad \varphi^+(x) = f(t^{\alpha}x, t) \quad \text{for} \quad x \geq 0$$

with some fixed $t \in (0,T)$, that we insert as test functions into the identity (4.10).

As $(w^m)' \in L^1(-\ell/t, 0)$ and $(h^{\sigma})' \in L^1(0, \ell/t)$, see Lemma 3.1, we easily arrive at

$$\int_{-\ell/t}^{0} \Big(w'(x) g_y(tx,t)t - x(w^m)'(x) g(tx,t) + \frac{m}{1-m}\, w^m(x) g(tx,t)\Big)\,dx + \int_{0}^{\ell/t^{\alpha}} \Big(h'(x) f_y(t^{\alpha}x,t)t^{\alpha} - \alpha x(h^{\sigma})'(x) f(t^{\alpha},t) + \frac{\omega\sigma}{1-m} h^{\sigma}(x) f(t^{\alpha}x,t)\Big)\,dx = 0$$

Setting

$$x = \frac{y}{t} \quad \text{for} \quad x \le 0 \qquad \text{and} \qquad x = \frac{y}{t^\alpha} \quad \text{for} \quad x \ge 0$$

and recalling (1.5), see also [8], we obtain

$$t^{-m/(1-m)} \int_{-\ell}^{0} \Big((u^m)_t \, g \, + \, u_y g_y \Big)(y,t)\, dy \\ + t^{1-\alpha-\omega\sigma/(1-m)} \int_0^\ell \Big((v^\sigma)_t f \, + \, v_y f_y \Big)(y,t)\, dy \, = \, 0$$

As

$$1 \, - \, \alpha \, - \, \frac{\omega\sigma}{1-m} \, = \, -\frac{m}{1-m} \, ,$$

taking $0 < \delta \ll 1$ we get, after some manipulations,

$$- \int_{-\ell}^{0} u^m(y,\delta) g(y,\delta) dy \, + \int_\delta^T \int_{-\ell}^{0} \Big(-u^m g_t + u_y g_y \Big)(y,t)\, dy\, dt \\ - \int_0^\ell v^\sigma(y,\delta) f(y,\delta)\, dy + \int_\delta^T \int_0^\ell \Big(-v^\sigma f_t + v_y f_y \Big)(y,t)\, dy\, dt \, = \, 0 \, . \quad (5.5)$$

Since $u^m(y,\delta) = \delta^{m/(1-m)} w^m(y/\delta)$ and there exists a positive constant C such that

$$\left| \delta^{1/(1-m)} w\left(\frac{y}{\delta}\right) - (-ay)^{1/(1-m)} \right| \, \le \, C \, (a\ell)^{m/(1-m)} \, \delta$$

holds for any $-\ell \le y \le -\delta\rho$ due to (4.21) (as well as for $-\delta\rho \le y \le 0$ since $|w(y/\delta) - (-ay/\delta)^{1/(1-m)}| \le C$), we see that

$$\int_{-\ell}^{0} u^m g(y,\delta) dy \to \int_{-\ell}^{0} (-ay)^{m/(1-m)} g(y,0) dy = - \int_0^T \int_{-\ell}^{0} (-ay)^{m/(1-m)} g_t(y,t)\, dydt$$

as $\delta \to 0$. Therefore we can let $\delta \to 0$ in (5.5) to obtain (5.1) with $\varphi = (\varphi^-, \varphi^+) = (g, f)$. □

Appendix

Proof of Proposition 2.1:

The implication $(i) \Rightarrow (ii)$ is obvious since the item $g(\varphi^+(0) - \varphi^-(0))$ disappears if $\varphi \in \tilde{V}^\circ$. It remains to prove

$(ii) \Rightarrow (i)$: Let first $w \in H^1(-\ell, 0)$ and $\phi \in C[-\ell, 0]$. Then the limit

$$\lim_{\delta \to 0+} \frac{1}{\delta} \int_{-\delta}^{0} w'(x)\phi(x)\, dx \\ = \lim_{\delta \to 0+} \left\{ \frac{1}{\delta} \int_{-\delta}^{0} w'(x) dx \, \phi(0) + \frac{1}{\delta} \int_{-\delta}^{0} w'(x) \big[\phi(x) - \phi(0)\big] dx \right\} = g_w \, \phi(0)$$

exists with the Steklov average

$$g_w := \lim_{\delta \to 0+} \frac{1}{\delta} \int_{-\delta}^{0} w'(x) dx$$

since

$$\left|\frac{1}{\delta}\int_{-\delta}^{0} w'(x)\big[\phi(x)-\phi(0)\big]dx\right| \le \frac{1}{\delta}\int_{-\delta}^{0}|w'(x)|dx \max_{x\in[-\delta,0]}|\phi(x)-\phi(0)| \to 0$$

as $\delta \to 0$. Analogously, it is

$$\lim_{\delta\to 0+}\frac{1}{\delta}\int_{0}^{\delta} h'(x)\phi(x)\,dx = g_h\,\phi(0) \qquad \text{with} \quad g_h := \lim_{\delta\to 0+}\frac{1}{\delta}\int_0^{\delta} h'(x)dx$$

for $h \in H^1(0,\ell)$ and $\phi \in C[0,\ell]$.

Consider now a couple (w,h) fulfilling relation (2.9), an arbitrary test function $\phi \in V^\circ \subset C[-\ell,0]\times C[0,\ell]$ and the cut-off function $\chi_\delta(x) := \max\left\{1-\frac{|x|}{\delta},0\right\}$. We test (2.9) with $\varphi = (1-\chi_\delta)\phi = \phi - \chi_\delta\phi \in \tilde{V}^\circ$ and obtain

$$\begin{aligned}
&\int_{-\ell}^{0}\Big(w'(\phi^-)' + w^m(x\phi^-)' + \frac{m}{1-m}\,w^m\phi^-\Big)dx \\
&+\int_0^{\ell}\Big(h'(\phi^+)' + \alpha h^\sigma(x\phi^+)' + \frac{\omega\sigma}{1-m}h^\sigma\phi^+\Big)dx \\
&-\frac{1}{\delta}\int_{-\delta}^{0} w'(x)\phi^-(x)dx + \frac{1}{\delta}\int_0^{\delta} h'(x)\phi^+(x)dx \\
&-\int_{-\delta}^{0}\Big(w'(\phi^-)' + w^m(x\phi^-)' + \frac{m}{1-m}\,w^m\phi^-\Big)\chi_\delta(x)dx \\
&-\int_0^{\ell}\Big(h'(\phi^+)' + \alpha h^\sigma(x\phi^+)' + \frac{\omega\sigma}{1-m}h^\sigma\phi^+\Big)\chi_\delta(x)dx \\
&+\frac{1}{\delta}\int_{-\delta}^{0} w^m x\phi^-\,dx - \frac{1}{\delta}\int_0^{\delta}\alpha h^\sigma x\phi^+\,dx \\
&= 0.
\end{aligned}$$

The last four integrals tend to zero as $\delta \to 0$, hence the considerations at the beginning of the proof yield

$$\begin{aligned}
&\int_{-\ell}^{0}\Big(w'(\phi^-)' + w^m(x\phi^-)' + \frac{m}{1-m}\,w^m\phi^-\Big)dx \\
&\qquad +\int_0^{\ell}\Big(h'(\phi^+)' + \alpha h^\sigma(x\phi^+)' + \frac{\omega\sigma}{1-m}h^\sigma\phi^+\Big)dx - g_w\phi^-(0) + g_h\phi^+(0) = 0.
\end{aligned}$$

If we choose for a moment $\phi \in \tilde{V}^\circ$ with $\phi(0) \neq 0$ then in addition (w,h) fulfils relation (2.9). Together with our above result this implies

$$-g_w\phi^-(0) + g_h\phi^+(0) = (g_h - g_w)\phi(0) = 0,$$

hence $g_w = g_h =: g$. This yields relation (2.6) which concludes the proof. □

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