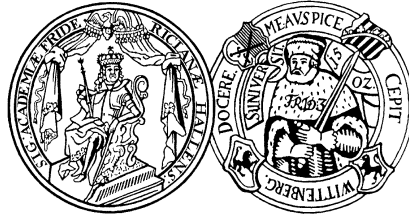# MARTIN–LUTHER–UNIVERSITÄT
HALLE–WITTENBERG
INSTITUT FÜR MATHEMATIK

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Report No. 12 (2009)

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ON THE RAYLEIGH-TAYLOR INSTABILITY FOR THE TWO-PHASE NAVIER-STOKES EQUATIONS

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ABSTRACT. The two-phase free boundary problem with surface tension and downforce gravity for the Navier-Stokes system is considered in a situation where the initial interface is close to equilibrium. The boundary symbol of this problem admits zeros in the unstable halfplane in case the heavy fluid is on top of the light one, which leads to the well-known Rayleigh-Taylor instability. Instability is proved rigorously in an L_p-setting by means of an abstract instability result due to Henry [12].

1. INTRODUCTION

Of concern is the motion of two immiscible, viscous, incompressible capillary fluids, $fluid_1$ and $fluid_2$, that occupy the regions

$$\Omega_i(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i(y - h(t,x)) > 0, \ t \geq 0\}, \quad i = 1, 2.$$

The fluids are separated by a sharp interface

$$\Gamma(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = h(t,x), \ t \geq 0\}$$

with an unknown function h that needs to be determined as part of the problem. The motion of the fluids is governed by the incompressible Navier-Stokes equations with surface tension and downforce gravity and reads as follows, where $i = 1, 2$;

$$
\begin{cases}
\rho_i\big(\partial_t u + (u \cdot \nabla)u\big) - \mu_i \Delta u + \nabla q = -\rho_i \gamma_a e_{n+1} & \text{in} \quad \Omega_i(t) \\
\operatorname{div} u = 0 & \text{in} \quad \Omega_i(t) \\
-[\![S(u,q)\nu]\!] = \sigma \kappa \nu & \text{on} \quad \Gamma(t) \\
[\![u]\!] = 0 & \text{on} \quad \Gamma(t) \\
V = u \cdot \nu & \text{on} \quad \Gamma(t) \\
u(0) = u_0 & \text{in} \quad \Omega_i(0) \\
\Gamma(0) = \Gamma_0 \, .
\end{cases}
\tag{1.1}
$$

The constants $\rho_i > 0$ and $\mu_i > 0$ denote the densities and the viscosities of the respective fluids, σ stands for the surface tension and γ_a is the acceleration of gravity. Moreover, $S(u, q)$ is the stress tensor defined by

$$S(u, q) = \mu_i\big(\nabla u + (\nabla u)^{\mathsf{T}}\big) - qI \quad \text{in} \quad \Omega_i(t),$$

and $[\![v]\!] = (v_{|\Omega_2(t)} - v_{|\Omega_1(t)})|_{\Gamma(t)}$ denotes the jump of the quantity v, defined on the respective domains $\Omega_i(t)$, across the interface $\Gamma(t)$. Finally, $\kappa = \kappa(t, \cdot)$ is the mean curvature of the free boundary $\Gamma(t)$, $\nu = \nu(t, \cdot)$ is the unit normal field on $\Gamma(t)$, and $V = V(t, \cdot)$ is the normal velocity of $\Gamma(t)$. Here we use the convention that $\nu(t, \cdot)$ points from $\Omega_1(t)$ into $\Omega_2(t)$, and that $\kappa(x, t)$ is negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$. System (1.1) comprises the *two-phase Navier-Stokes equations with surface tension subject to gravity*. In order to economize our notation, we set

$$\rho = \rho_1 \chi_{\Omega_1(t)} + \rho_2 \chi_{\Omega_2(t)}, \quad \mu = \mu_1 \chi_{\Omega_1(t)} + \mu_2 \chi_{\Omega_2(t)},$$

where χ denotes the indicator function. It is convenient to introduce the *modified pressure* $\tilde{q} := q + \rho \gamma_a y$. With this convention system (1.1) can be recast as

$$
\begin{cases}
\rho\big(\partial_t u + (u \cdot \nabla)u\big) - \mu \Delta u + \nabla \tilde{q} = 0 & \text{in} \quad \Omega(t) \\
\operatorname{div} u = 0 & \text{in} \quad \Omega(t) \\
-[\![S(u, \tilde{q})\nu]\!] = \sigma \kappa \nu + [\![\rho]\!]\gamma_a y & \text{on} \quad \Gamma(t) \\
[\![u]\!] = 0 & \text{on} \quad \Gamma(t) \\
V = u \cdot \nu & \text{on} \quad \Gamma(t) \\
u(0) = u_0 & \text{in} \quad \Omega_0 \\
\Gamma(0) = \Gamma_0\,.
\end{cases}
\qquad (1.2)
$$

Given are the initial velocity $u_0 : \Omega_0 \to \mathbb{R}^{n+1}$ with $\Omega_0 := \Omega_1(0) \cup \Omega_2(0)$ as well as the initial position $\Gamma_0 = \operatorname{graph}(h_0)$. The unknowns are the velocity field $u(t, \cdot) : \Omega(t) \to \mathbb{R}^{n+1}$, the pressure field $q(t, \cdot) : \Omega(t) \to \mathbb{R}$, and the free boundary $\Gamma(t)$, where $\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$.

In case that $\Omega_1(t)$ is a bounded domain, $\gamma_a = 0$, and $\Omega_2(t) = \emptyset$, one obtains the *one-phase* Navier-Stokes equations with surface tension, describing the motion of an isolated volume of fluid. For an overview of the existing literature in this case we refer to the recent publications [17, 19, 20, 21].

The motion of a layer of viscous, incompressible fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity (in which case Ω_0 is a strip, bounded above by Γ_0 and below by a fixed surface Γ_b) has been considered by [1, 2, 3, 20, 23, 24]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions is proved in [2] for $\sigma > 0$, and in [24] for $\sigma \geq 0$, and the asymptotic decay rate for $t \to \infty$ is studied in [3]. We also refer to [5], where in addition the presence of a surfactant on the free boundary and in one of the bulk phases is considered but gravity is neglected.

Previous results concerning the *two-phase problem* (1.2) with $\gamma_a = 0$ in the $3D$-case are obtained in [7, 8, 9, 22]. In more detail, Densiova [8] establishes existence and uniqueness of solutions (of the transformed problem in Lagrangian coordinates) with $v \in W_2^{s,s/2}$ for $s \in (5/2, 3)$ in case that one of the domains is bounded. Tanaka [22] considers the two-phase Navier-Stokes equations with thermo-capillary convection in bounded domains, and he obtains existence and uniqueness of solutions with $(v, \theta) \in W_2^{s,s/2}$ for $s \in (7/2, 4)$, with θ denoting the temperature.

Here we are interested in the situation where Γ_0 is close to a plane, say $\mathbb{R}^n$ with $n \geq 2$, i.e. Γ_0 is a graph over $\mathbb{R}^n$, given by a function h_0 that is small in an appropriate norm. Then it is natural to transform the problem to a flat fixed interface, and solve the resulting quasilinear evolution problem. Our basic well-posedness and regularity result for problem (1.2) reads as follows.

Theorem 1.1. *Fix $p > n + 3$ and let*

$$(u_0, h_0) \in W_p^{2-2/p}(\Omega_0, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n)$$

be given. Assume that the compatibility conditions

$$\operatorname{div} u_0 = 0 \quad \text{on} \quad \Omega_0, \quad [\![\mu D_0 \nu_0 - \mu(\nu_0 \cdot D_0 \nu_0)\nu_0]\!] = 0, \quad [\![u_0]\!] = 0 \text{ on } \Gamma_0, \quad (1.3)$$

are satisfied, where $D_0 = \left(\nabla u_0 + (\nabla u_0)^\mathsf{T}\right)$, and ν_0 is the unit normal field on Γ_0. Then for each $a > 0$ there exists $\eta > 0$ such that for

$$\|u_0\|_{W_p^{2-2/p}(\Omega_0)} + \|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)} < \eta,$$

there exists a unique classical solution $(u, \tilde{q}, \Gamma)$ of problem (1.2) on $(0, a)$. In addition, $\mathcal{M} = \cup_{t \in (0,a)}\left(\{t\} \times \Gamma(t)\right)$ is a real analytic manifold, and the function $(u, \tilde{q}) : \mathcal{O} \to \mathbb{R}^{n+2}$ is real analytic, where $\mathcal{O} := \cup_{t \in (0,a)}\left(\{t\} \times \Omega(t)\right)$.

Proof. This result is proved in [16] in case that $\gamma_a = 0$. The proof given there extends to the case $\gamma_a > 0$, as the additonal term $[\![\rho]\!]\gamma_a h$ on the interface is of lower order. Actually, in Section 6 we shall give a different existence proof based on the implicit function theorem. □

We mention that system (1.2) has also been analyzed in [17] for initial data that are not necessarily close to equilibrium. More precisely, it is proved in [17] that (1.2) admits unique solutions (on a possibly small time interval) that have the same regularity properties as above, provided $\|\nabla h_0\|_\infty$ is small enough.

It is the purpose of this paper to prove mathematically rigorously that the trivial solution $(u, h) = (0, 0)$ of problem (1.2) is unstable in the phase manifold $\mathcal{PM}$, to be defined below, in an L_p-setting in case that the heavy fluid overlies the lighter one, i.e. if $\rho_2 > \rho_1$. This is the *Rayleigh-Taylor instability* which is well-known in Physics and Hydrodynamics, cf. [4, 6, 11, 13] and the references given there. The Rayleigh-Taylor instability manifests itself in the way that any disturbance of the equilibrium solution $(u, h) = (0, 0)$ will grow to produce spikes of the heavy fluid moving downward and bubbles of the light fluid moving upward. The precise statement of our main result is as follows.

Theorem 1.2. *Let $p > n + 3$. Suppose that $\rho_1, \rho_2, \mu_1, \mu_2, \sigma, \gamma_a > 0$ are constants and $\rho_2 > \rho_1$. Then the trivial equilibrium $(u, \tilde{q}, h) = (0, 0, 0)$ is L_p-unstable. More precisely, there is a constant $\varepsilon_0 > 0$ such that for each $\delta > 0$ there are initial values*

$$(u_0, h_0) \in W_p^{2-2/p}(\Omega_0, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n),$$

subject to the compatibility conditions (1.3) in Theorem 1.1 with

$$\|u_0\|_{W_p^{2-2/p}} + \|h_0\|_{W_p^{3-2/p}} \leq \delta$$

such that the solution (u, h) for some $t_0 \in (0, a]$ satisfies

$$\|u(t_0)\|_{W_p^{2-2/p}} + \|h(t_0)\|_{W_p^{3-2/p}} \geq \varepsilon_0.$$

Our method depends on the proof of Theorem 1.1 presented in [16], as well as on an abstract instability result for iterates of a mapping due to Henry [12], applied here to the Poincaré map or time-one-map of the system. To verify the assumptions in Henry's result we show that the boundary symbol $s(\lambda, \tau)$ admits zeros (λ_0, τ_0) in the unstable half-plane in case $\rho_2 > \rho_1$ and prove that such a zero induces the spectral values λ_0 for the linearized operator of the problem at the trivial equilibrium.

2. Reduction to a Flat Interface and Linearization

The nonlinear problem (1.2) can be transformed to a problem on a fixed domain by means of the transformations

$$v(t, x, y) := (u_1, \ldots, u_n)(t, x, y + h(t, x)),$$
$$w(t, x, y) := u_{n+1}(t, x, y + h(t, x)),$$
$$\pi(t, x, y) := \tilde{q}(t, x, y + h(t, x)),$$

where $t \in J = [0, a]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $y \neq 0$. With a slight abuse of notation we will in the sequel denote the transformed velocity again by u, that is, we set $u = (v, w)$. With this notation we obtain the transformed problem

$$\begin{cases} \rho \partial_t u - \mu \Delta u + \nabla \pi = F(u, \pi, h) & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ \operatorname{div} u = F_d(u, h) & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ -[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = G_v(u, h) & \text{on} \quad \mathbb{R}^n \\ -2[\![\mu \partial_y w]\!] + [\![\pi]\!] - (\sigma \Delta + [\![\rho]\!]\gamma_a)h = G_w(u, h) & \text{on} \quad \mathbb{R}^n \\ [\![u]\!] = 0 & \text{on} \quad \mathbb{R}^n \\ \partial_t h - w|_{y=0} = H(u, h) & \text{on} \quad \mathbb{R}^n \\ u(0) = u_0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ h(0) = h_0, & \end{cases} \qquad (2.1)$$

for $t > 0$, where $\dot{\mathbb{R}}^{n+1} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : y \neq 0\}$. More details on this transformation, on the nonlinear right hand sides, can be found in [16]. Here we should point out, however, that the definition of G_v in this paper differs from that

in [16] in the following way: solving the second line of formula (2.7) in [16] for $[\![\pi]\!]$ and substituting the result into the expression for G_v in formula (2.8) of [16] results in

$$
\begin{aligned}
G_v(v, w, h) := & -[\![\mu(\nabla_x v + (\nabla_x v)^\mathsf{T})]\!] \nabla h + |\nabla h|^2 [\![\mu \partial_y v]\!] \\
& + \big\{ [\![\mu \partial_y w]\!] - (\nabla h \,|\, [\![\mu \nabla_x w]\!]) + |\nabla h|^2 [\![\mu \partial_y w]\!] \big\} \nabla h.
\end{aligned}
\tag{2.2}
$$

Thus the quantity $[\![\pi]\!]$ can be eliminated in the nonlinearity G_v.

The linearization of (2.1) at $(u, h) = (0, 0)$ leads to the linear inhomogeneous problem

$$
\begin{cases}
\rho \partial_t u - \mu \Delta u + \nabla \pi = f & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
\operatorname{div} u = f_d & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
-[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = g_v & \text{on} \quad \mathbb{R}^n \\
-2[\![\mu \partial_y w]\!] + [\![\pi]\!] - (\sigma \Delta h + [\![\rho]\!] \gamma_a) h = g_w & \text{on} \quad \mathbb{R}^n \\
[\![u]\!] = 0 & \text{on} \quad \mathbb{R}^n \\
\partial_t h - w|_{y=0} = g_h & \text{on} \quad \mathbb{R}^n \\
u(0) = u_0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
h(0) = h_0.
\end{cases}
\tag{2.3}
$$

We are interested in the regularity class

$$
\begin{aligned}
& u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})), \\
& \pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})),
\end{aligned}
\tag{2.4}
$$

where $J = [0, a]$. In the following, W_p^m denote as usual the Sobolev spaces if $m \in \mathbb{Z}$. For non-integer s, W_p^s are the Sobolev-Slobodeckii spaces, and H_p^s the Bessel-potential spaces. For $K \in \{H, W\}$, by $\dot{K}_p^s$ we mean the homogeneous version of K_p^s. Note that $H_p^s = W_p^s$ for integer values of s, but that in general these spaces are different. We refer to [16, Section 2] for more details.

If we assume a solution in the class (2.4), then for the right hand sides f and f_d we necessarily have $f \in L_p(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1})$ and

$$
f_d \in H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^{n+1})),
$$

since the operator div maps L_p into $\dot{H}_p^{-1}$. By trace theory we necessarily have $u_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \dot{\mathbb{R}}^{n+1})$, and the lateral trace of u belongs to

$$
Y_0 := W_p^{1-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1})),
$$

and that of $\partial_j u$ to

$$
Y_1 := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1})),
$$

see for instance [10]. Therefore $g_v \in Y_1$, and if in addition

$$
[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)),
$$

then we also have that $g_w \in Y_1$.

Concerning the regularity of the height function h we note that the equation for h lives in the trace space Y_0, hence naturally h should belong to

$$h \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)).$$

On the other hand, the equation for the normal component of the normal stress lives in Y_1, and contains the term Δh, hence h should also belong to the space $L_p(J; W_p^{3-1/p}(\mathbb{R}^n))$. These considerations lead to the following natural space for the height function h

$$h \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)).$$

This then implies $g_h \in Y_0$, as well as $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$ by trace theory. Our next theorem states that in this setting, problem (2.3) admits maximal regularity; the described regularities of the data are also sufficient. In particular, the solution map defines an isomorphism between this space of data and the solution space defined above.

Theorem 2.1. *Let $1 < p < \infty$ be fixed, $p \neq 3/2, 3$, and assume that σ, γ_a, ρ_i and μ_i are positive constants for $i = 1, 2$, and set $J = [0, a]$. Then the instationary Stokes problem with free boundary (2.3) admits a unique solution $(u, \pi, [\![\pi]\!], h)$ with regularity*

$$u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})),$$

$$\pi \in L_p(J; \dot{H}_p^1(\dot{\mathbb{R}}^{n+1})),$$

$$[\![\pi]\!] \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)),$$

$$h \in W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n))$$

if and only if the data $(f, f_d, g, g_h, u_0, h_0)$ satisfy the following regularity and compatibility conditions:

(a) $f \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$,

(b) $f_d \in H_p^1(J; \dot{H}_p^{-1}(\mathbb{R}^{n+1})) \cap L_p(J; H_p^1(\dot{\mathbb{R}}^{n+1}))$,

(c) $g = (g_v, g_w) \in W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1}))$,

(d) $g_h \in W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$,

(e) $u_0 \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$, $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$,

(f) $\operatorname{div} u_0 = f_d(0)$ in $\dot{\mathbb{R}}^{n+1}$ and $[\![u_0]\!] = 0$ on $\mathbb{R}^n$ if $p > 3/2$,

(g) $-[\![\mu \partial_y v_0]\!] - [\![\mu \nabla_x w_0]\!] = g_v(0)$ on $\mathbb{R}^n$ if $p > 3$.

The solution map $[(f, f_d, g, g_h, u_0, h_0) \mapsto (u, \pi, [\![\pi]\!], h)]$ is continuous between the corresponding spaces.

Proof. For a detailed proof of Theorem 2.1 in case $\gamma_a = 0$ we refer to [16]. The proof carries over to the case $\gamma_a > 0$ since the term $[\![\rho]\!]\gamma_a h$ is of lower order. The only change occurs in the boundary symbol $s(\lambda, |\xi|)$; see Section 3. $\square$

We also need a corresponding result for the stationary linear problem

$$
\begin{cases}
\rho\lambda_* u - \mu\Delta u + \nabla\pi = 0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
\operatorname{div} u = f_d & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
-[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = g_v & \text{on} \quad \mathbb{R}^n \\
-2[\![\mu\partial_y w]\!] + [\![\pi]\!] - (\sigma\Delta h + [\![\rho]\!]\gamma_a)h = g_w & \text{on} \quad \mathbb{R}^n \\
[\![u]\!] = 0 & \text{on} \quad \mathbb{R}^n \\
\lambda_* h - w|_{y=0} = g_h & \text{on} \quad \mathbb{R}^n,
\end{cases}
\tag{2.5}
$$

where $\lambda_* > 0$ is sufficiently large. It reads as follows.

Theorem 2.2. *Let $1 < p < \infty$ be fixed, and assume that σ, γ_a, ρ_i and μ_i are positive constants for $i = 1, 2$, and that $\lambda_* > 0$ is large enough. Then the stationary Stokes problem with free boundary (2.5) admits a unique solution (u, π, h) with regularity*

$$
u \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}; \mathbb{R}^{n+1}), \qquad \pi \in \dot{W}_p^{1-2/p}(\dot{\mathbb{R}}^{n+1}),
$$

$$
[\![\pi]\!] \in W_p^{1-3/p}(\mathbb{R}^n), \qquad\qquad h \in W_p^{3-3/p}(\mathbb{R}^n)),
$$

if and only if the data (f, f_d, g, g_h) satisfy the following regularity conditions:

(a) $f_d \in W_p^{1-2/p}(\dot{\mathbb{R}}^{n+1}) \cap \dot{H}_p^{-1}(\mathbb{R}^{n+1})$,

(b) $g = (g_v, g_w) \in W_p^{1-3/p}(\mathbb{R}^n; \mathbb{R}^{n+1})$,

(c) $g_h \in W_p^{2-3/p}(\mathbb{R}^n)$.

The solution map $[(f_d, g, g_h) \mapsto (u, \pi, [\![\pi]\!], h)]$ is continuous between the corresponding spaces.

Proof. The proof will be given at the end of the next section. $\qquad\square$

3. Zeros of the Boundary Symbol

As shown in our paper [17], the boundary symbol of the linear problem is given by

$$
s(\lambda, |\xi|) := \lambda + \frac{\sigma|\xi|^2 - [\![\rho]\!]\gamma_a}{(\rho_1 + \rho_2)\lambda/|\xi| + 4\eta_1\eta_2/(\eta_1 + \eta_2)}.
\tag{3.1}
$$

Here λ denotes the co-variable of time t and ξ that of the tangential space variable $x \in \mathbb{R}^n$, and we employed the abreviations

$$
\omega_j = \sqrt{\rho_j\lambda + \mu_j|\xi|^2}, \quad \eta_1 = \sqrt{\mu_1}\omega_1 + \mu_2|\xi|, \quad \eta_2 = \sqrt{\mu_2}\omega_2 + \mu_1|\xi|.
$$

The boundary symbol $s(\lambda, |\xi|)$ has been studied in detail in the papers [15, 16] in case $\gamma_a = 0$, and in [17] for $\gamma_a > 0$. It has been shown in [17, Remarks 3.2(b),(c)] that $s(\lambda, \tau)$ does not admit zeros $(\lambda, \tau) \neq (0, 0)$ with $\operatorname{Re}\lambda \geq 0$ and $\tau \geq 0$ in case $\rho_2 \leq \rho_1$, i.e. in the stable case. On the other hand we have the following result.

Proposition 3.1. *Suppose $\rho_2 > \rho_1 > 0$ and $\mu_1, \mu_2, \sigma, \gamma_a > 0$ are constants. Then for each $\xi \in \mathbb{R}^n$ with*

$$0 < |\xi| < \tau_* := (\gamma_a [\![\rho]\!]/\sigma)^{1/2}$$

there is $\lambda(|\xi|) > 0$ such that $s(\lambda(|\xi|), |\xi|) = 0$. Every zero of $s(\lambda, |\xi|)$ with $\mathrm{Re}\,\lambda \geq 0$ is real.

Proof. Note first that $s(\lambda, 0) \neq 0$ unless $\lambda = 0$, hence we may assume $\xi \neq 0$ below. It is convenient to the use the scaling $\zeta = \lambda/\tau^2$ where $\tau = |\xi| \in (0, \infty)$. By a slight abuse of notation we set

$$\omega_j(\zeta) = \sqrt{\rho_j \zeta + \mu_j}, \quad \eta_1(\zeta) = \sqrt{\mu_1}\omega_1(\zeta) + \mu_2, \quad \eta_2(\zeta) = \sqrt{\mu_2}\omega_2(\zeta) + \mu_1,$$

and obtain

$$s(\lambda, \tau) = \tau^2(\zeta + \psi(\tau)k(\zeta)),$$

where

$$\psi(\tau) = \frac{\sigma}{(\rho_1 + \rho_2)\tau} - \frac{(\rho_2 - \rho_1)\gamma_a}{(\rho_1 + \rho_2)\tau^3},$$

and

$$\frac{1}{k(\zeta)} = \zeta + \frac{4}{\rho_1 + \rho_2} \frac{\eta_1(\zeta)\eta_2(\zeta)}{\eta_1(\zeta) + \eta_2(\zeta)}.$$

Thus (λ, τ) is a zero of s if and only if (ζ, τ) satisfies $\zeta + \psi(\tau)k(\zeta) = 0$. It has been shown in [15] that $\arg k(\zeta) \in (-\pi/2, 0]$ if $\arg \zeta \in [0, \pi/2]$. This implies that for $\mathrm{Re}\,\zeta \geq 0$, $\zeta \neq 0$ we have $\zeta + \psi(\tau)k(\zeta) \neq 0$ if either $\tau \geq \tau_*$ or ζ is non-real. Thus we need to show that for $\tau \in (0, \tau_*)$ there is a a zero $\zeta(\tau) > 0$ of $\zeta + \psi(\tau)k(\zeta) = 0$. But the function $\Phi(\zeta) := \zeta/k(\zeta)$ is nonnegative and strictly increasing for $\zeta \geq 0$, it is zero at $\zeta = 0$ and behaves like ζ^2 as $t \to \infty$, hence $\Phi(0, \infty) \supset (0, \infty)$. On the other hand, $-\psi(0, \tau_*) = (0, \infty)$ implies that for each $\tau \in (0, \tau_*)$ there is a unique $\zeta(\tau) > 0$ such that $\Phi(\zeta(\tau)) = -\psi(\tau)$, i.e. $(\zeta(\tau), \tau)$ is a zero of the function $\zeta + \psi(\tau)k(\zeta)$, which yields the zeros $(\zeta(|\xi|)|\xi|^2, \xi)$ of the boundary symbol $s(\lambda, |\xi|)$. □

The critical value $\tau_* = \sqrt{[\![\rho]\!]\gamma_a/\sigma}$ is known as the *cutoff wave number* in the literature, cp. e.g. [11]. Note that $\Phi(\zeta) \sim \zeta^2$ as $\zeta \to \infty$, while $\Phi(\zeta) \sim \zeta/k(0)$ as $\zeta \to 0$. This gives the asymptotics

$$\zeta(\tau) \sim \sqrt{[\![\rho]\!]\gamma_a/(\rho_1 + \rho_2)}\tau^{-3/2}, \ \tau \to 0; \qquad \zeta(\tau) \sim c(\tau_* - \tau), \ \tau \to \tau_*,$$

with $c = 2\sigma k(0)/(\rho_1 + \rho_2)\tau_*^2$, which implies for $\lambda(\tau) = \tau^2 z(\tau)$

$$\lambda(\tau) \sim \sqrt{[\![\rho]\!]\gamma_a/(\rho_1 + \rho_2)}\tau^{1/2}, \ \tau \to 0; \qquad \lambda(\tau) \sim \frac{\sigma}{\mu_1 + \mu_2}(\tau_* - \tau), \ \tau \to \tau_*.$$

Thus $\lambda(\tau) \to 0$ as $\tau \to 0, \tau_*$, hence the function $[\tau \to \lambda(\tau)]$ has a positive absolute maximum in the interval $(0, \tau_*)$ which we denote by $\lambda_\infty > 0$ in the sequel.

Proof of Theorem 2.2: Necessity is obtained by trace theory. To prove sufficiency, we can use the same reductions as in [16], Sections 3-5, with the notable difference that here we only need to consider the stationary case with λ_* a fixed parameter. As in the proof of [16, Theorem 5.1] it then remains to consider the boundary symbol $s(\lambda_*, |\xi|)$. We have seen above that for $\lambda_* > \lambda_\infty$ the boundary symbol

does not vanish, and we may estimate $s(\lambda_*, \tau)$ from above and below by $c_j(1+|\tau|)$ with appropriate positive constants c_j, $j = 0, 1$. This is valid for

$$\tau \in \Sigma_\eta = \{\zeta \in \mathbb{C} : \zeta \neq 0, \ |\arg \zeta| < \eta\}$$

with sufficiently small $\eta > 0$. Therefore by Mikhlin's Fourier multiplier theorem $(1+|\xi|)/s(\lambda_*, |\xi|)$ defines a bounded linear operator in $H_p^s(\mathbb{R}^n)$ and in $W_p^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. $\square$

4. The Spectrum of the Linearization

As a base space for the functional analytic setting we use

$$X_0 = L_{p,\sigma}(\mathbb{R}^{n+1}; \mathbb{R}^{n+1}) \times W_p^{2-1/p}(\mathbb{R}^n),$$

where the subscript σ means solenoidal, and we set

$$\bar{X}_1 = H_p^2(\dot{\mathbb{R}}^{n+1}; \mathbb{R}^{n+1}) \times W_p^{3-1/p}(\mathbb{R}^n)$$

As before we use the decomposition $u = (v, w)$. Define a closed linear operator in X_0 by means of

$$A(u, h) = (-\frac{\mu}{\rho}\Delta u + \frac{1}{\rho}\nabla\pi, -w), \tag{4.1}$$

with domain $X_1 := D(A) \subset \bar{X}_1$

$$D(A) = \{(u, h) \in \bar{X}_1 \cap X_0 : \ [\![u]\!] = 0, \ [\![\mu\partial_y v + \mu\nabla_x w]\!] = 0 \text{ on } \mathbb{R}^n\}.$$

The pressure field π in (4.1) is determined as the solution of the transmission problem

$$\begin{aligned} (\frac{1}{\rho}\nabla\pi | \nabla\phi)_{L_2} &= (\frac{\mu}{\rho}\Delta u | \nabla\phi)_{L_2}, \quad \phi \in W_{p'}^1(\mathbb{R}^{n+1}), \\ [\![\pi]\!] &= 2[\![\mu\partial_y w]\!] + \sigma\Delta h + [\![\rho]\!]\gamma_a h \quad \text{on } \mathbb{R}^n. \end{aligned} \tag{4.2}$$

One should compare this operator A with [20], where the corresponding operator in the one-phase case has been introduced and analyzed.

Concerning the transmission problem, we set

$$\tilde{\pi} := \pi/\rho, \quad f := \mu\Delta u/\rho, \quad g := 2[\![\mu\partial_y w]\!] + \sigma\Delta h + [\![\rho]\!]\gamma_a h.$$

Then the solution of the transmission problem consists of two parts $\tilde{\pi} = \pi_1 + \pi_2$. π_1 is given by $\pi_1 := -\text{div}\, D_n^{-1} f$, i.e. $\nabla\pi_1 = Rf$, where R denotes the Riesz-transform with symbol $\xi \otimes \xi/|\xi|^2$. On the other hand, π_2 solves

$$\Delta\pi_2 = 0 \text{ in } \dot{\mathbb{R}}^{n+1}, \quad [\![\rho\pi_2]\!] = g - [\![\rho\pi_1]\!] =: g_0 \text{ and } [\![\partial_y\pi_2]\!] = 0 \text{ on } \mathbb{R}^n.$$

The solution of the latter problem is given by

$$\pi_2(\cdot, y) = \frac{1}{(\rho_1 + \rho_2)} \text{sign}\,(y) P(|y|) g_0,$$

where $\{P(s) : s \geq 0\}$ denotes the Poisson semigroup on $\mathbb{R}^n$. Thus $\nabla\pi \in L_p(\mathbb{R}^{n+1})$ since g and g_0 belong to $\dot{W}_p^{1-1/p}(\mathbb{R}^n)$ and $f \in L_p(\mathbb{R}^{n+1})$, see for instance formula (2.14) in [16] for the assertion for π_2.

System (2.3) can be rewritten as the abstract evolution equation

$$\frac{d}{dt}(u,h) + A(u,h) = (f_d, g_h), \quad t > 0, \quad (u(0), h(0)) = (u_0, h_0), \qquad (4.3)$$

provided $(f_d, g) = (0,0)$. Since (2.3) has maximal L_p-regularity, the abstract problem (4.3) has maximal L_p-regularity as well. In particular, $-A$ generates an analytic C_0-semigroup in X_0. Concerning the spectrum $\sigma(A)$ of A we have the following result.

Proposition 4.1. *Suppose $\rho_2 > \rho_1 > 0$ and $\mu_1, \mu_2, \sigma, \gamma_a > 0$ are constants, and let A with domain $D(A) = X_1$ be defined in X_0 as above.*
Then $\lambda_0 \in \sigma(-A)$ for each zero $(\lambda_0, |\xi_0|) \in (0, \infty) \times (0, \infty)$ of $s(\lambda, |\xi|) = 0$. In particular, $\sigma(-A) \cap \mathbb{C}_+ = [0, \lambda_\infty]$ with $\lambda_\infty > 0$ from the previous section.

Proof. (i) The idea is to show that whenever we have a zero $(\lambda_0, |\xi_0|)$ of the boundary symbol s then λ_0 is an approximate eigenvalue of $-A$, hence in $\sigma(-A)$. So suppose $s(\lambda_0, |\xi_0|) = 0$. We define functions h_ε by means of

$$h_\varepsilon(x) = e^{i\xi_0 \cdot x} \chi_\varepsilon(x), \quad x \in \mathbb{R}^n, \ \varepsilon > 0,$$

where χ is a Schwartz-function such that its Fourier-transform $\hat{\chi}$ is a test function with $\mathrm{supp}\,(\hat{\chi}) \subset \bar{B}_{\mathbb{R}^n}(0,1)$, $\chi(0) = 1$ and $\chi_\varepsilon(x) := \chi(\varepsilon x)$. This implies by means of the formula $\hat{h}_\varepsilon(\xi) = \varepsilon^{-n}\hat{\chi}((\xi - \xi_0)/\varepsilon)$ that the support of $\hat{h}_\varepsilon$ is contained in the ball $\bar{B}_{\mathbb{R}^n}(\xi_0, \varepsilon)$. In particular, for all $r \in \mathbb{R}$, $1 < p < \infty$ and $K \in \{H, W\}$, the operators $D_n := -\Delta$, $D_n^{1/2}$ and $D_n^{-1/2}$ as well as $B_0 := \lambda_0 D_n^{-1}$ act boundedly in

$$K_{p,c}^r(\mathbb{R}^n) := \{g \in K_p^r(\mathbb{R}^n) : \mathrm{supp}\,(\hat{g}) \subset \bar{B}_{\mathbb{R}^n}(\xi_0, \varepsilon)\},$$

as long as, say, $\varepsilon \leq |\xi_0|/2$. Note that the spectra of D_n, $D_n^{1/2}$ and B_0 in $K_{p,c}^r$ are contained in $[\alpha, \beta]$ for some $0 < \alpha < \beta < \infty$.

(ii) The L_p-norm of h_ε is $\|h_\varepsilon\|_p = \varepsilon^{-n/p}\|\chi\|_p$, and moreover we have

$$\|h_\varepsilon\|_{H_p^k} \leq c\varepsilon^{-n/p}\|\chi\|_{H_p^k}, \quad k \leq 4, \quad 0 < \varepsilon \leq |\xi_0|/2.$$

Hence by the interpolation inequality one sees that

$$c_0\varepsilon^{-n/p} \leq \|h_\varepsilon\|_{K_p^r} \leq c_1\varepsilon^{-n/p}, \quad 0 < \varepsilon \leq |\xi_0|/2, \quad r \in [0,4],$$

where $K \in \{H, W\}$ and c_0, c_1 are positive constants.

Next we solve the Stokes problem

$$\begin{cases} \rho\lambda_0 u - \mu\Delta u + \nabla\pi = 0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ \mathrm{div}\, u = 0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = 0 & \text{on} \quad \mathbb{R}^n \\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] = (\sigma\Delta + [\![\rho]\!]\gamma_a)h_\varepsilon & \text{on} \quad \mathbb{R}^n \\ [\![u]\!] = 0 & \text{on} \quad \mathbb{R}^n. \end{cases} \qquad (4.4)$$

This is possible since $\lambda_0 > 0$, and we obtain for the solution $(u_\varepsilon, \pi_\varepsilon)$ the estimate

$$\|u_\varepsilon\|_{H_p^2(\dot{\mathbb{R}}^{n+1})} + \|\nabla\pi_\varepsilon\|_{L_p(\dot{\mathbb{R}}^{n+1})} \leq C\|h_\varepsilon\|_{W_p^{3-1/p}(\mathbb{R}^n)} \leq Cc_1\varepsilon^{-n/p}.$$

We then have
$$\lambda_0(u_\varepsilon, h_\varepsilon) + A(u_\varepsilon, h_\varepsilon) = (0, g_\varepsilon),$$
where $g_\varepsilon = \lambda_0 h_\varepsilon + (\sigma D_n^{1/2} - [\![\rho]\!]\gamma_a D_n^{-1/2})(\rho_1 + \rho_2)^{-1}k(\lambda_0 D_n^{-1})h_\varepsilon$.

(iii) It remains to estimate g_ε. First we observe that
$$D_n h_\varepsilon(x) = |\xi_0|^2 h_\varepsilon(x) - \varepsilon r_\varepsilon(x), \quad r_\varepsilon(x) := e^{i\xi_0 \cdot x}[2i((\xi_0|\nabla)\chi)(\varepsilon x) + \varepsilon(\Delta\chi)(\varepsilon x),$$
hence
$$D_n^{-1} h_\varepsilon(x) = |\xi_0|^{-2}h_\varepsilon(x) + \varepsilon|\xi_0|^{-2}D_n^{-1}r_\varepsilon(x),$$
and therefore
$$(\zeta - B_0)h_\varepsilon = (\zeta - \lambda_0|\xi_0|^{-2})h_\varepsilon - \varepsilon|\xi_0|^{-2}B_0 r_\varepsilon,$$
which yields
$$(\zeta - B_0)^{-1}h_\varepsilon = (\zeta - \lambda_0|\xi_0|^{-2})^{-1}h_\varepsilon + \varepsilon|\xi_0|^{-2}(\zeta - \lambda_0|\xi_0|^{-2})^{-1}(\zeta - B_0)^{-1}B_0 r_\varepsilon.$$
With $k_1(\zeta) = (\sigma - \zeta[\![\rho]\!]\gamma_a/\lambda_0)(\rho_1 + \rho_2)^{-1}k(\zeta)$, by means of Dunford's functional calculus in $K_{p,c}^r(\mathbb{R}^n)$ this yields, with a closed contour Γ in the open right half-plane surrounding $[\alpha, \beta]$,

$$
\begin{aligned}
k_1(B_0)h_\varepsilon &= \frac{1}{2\pi i}\int_\Gamma k_1(\zeta)(\zeta - B_0)^{-1}h_\varepsilon \, d\zeta \\
&= k_1(\lambda_0|\xi_0|^{-2})h_\varepsilon + \frac{\varepsilon}{2\pi i|\xi_0|^2}\int_\Gamma k_1(\zeta)(\zeta - \lambda_0|\xi_0|^{-2})^{-1}(\zeta - B_0)^{-1}B_0 r_\varepsilon \, d\zeta \\
&= k_1(\lambda_0|\xi_0|^{-2})h_\varepsilon + \varepsilon K_1 r_\varepsilon,
\end{aligned}
$$

where the operator K_1 is bounded and does not depend on ε. Here we used the fact that $k_1(\zeta)$ is bounded. Finally, we have in a similar way

$$
\begin{aligned}
D_n^{1/2}h_\varepsilon &= \frac{1}{2\pi i}\int_\Gamma \sqrt{\zeta}(\zeta - D_n)^{-1}h_\varepsilon \, d\zeta \\
&= |\xi_0|h_\varepsilon - \frac{\varepsilon}{2\pi i}\int_\Gamma \sqrt{\zeta}(\zeta - |\xi_0|^2)^{-1}(\zeta - D_n)^{-1} \, d\zeta \, r_\varepsilon \\
&= |\xi_0|h_\varepsilon + \varepsilon K_2 r_\varepsilon,
\end{aligned}
$$

with a bounded operator K_2 that is independent of ε. Thus in summary we have

$$
\begin{aligned}
g_\varepsilon &= \big(\lambda_0 + (\sigma D_n^{1/2} - [\![\rho]\!]\gamma_a D_n^{-1/2})(\rho_1 + \rho_2)^{-1}k(B_0)\big)h_\varepsilon \\
&= s(\lambda_0, |\xi_0|)h_\varepsilon + \varepsilon\big(k_1(\lambda_0/|\xi_0|^2)K_2 + K_1 D_n^{1/2}\big)r_\varepsilon \\
&= \varepsilon\big(k_1(\lambda_0/|\xi_0|^2)K_2 + K_1 D_n^{1/2}\big)r_\varepsilon,
\end{aligned}
$$

since by assumption $s(\lambda_0, |\xi_0|) = 0$. The operators K_j are bounded and $D_n^{1/2}$ is so on functions whose Fourier transform has compact support bounded away from 0, hence we obtain
$$\|g_\varepsilon\|_{W_p^{2-1/p}(\mathbb{R}^n)} \leq C\varepsilon\|r_\varepsilon\|_{W_p^{2-1/p}(\mathbb{R}^n)} \leq C\varepsilon\|h_\varepsilon\|_{W_p^{4-1/p}(\mathbb{R}^n)} \leq C\varepsilon\varepsilon^{-n/p}.$$

Scaling the functions h_ε by the factor $\varepsilon^{n/p}$ we have $\|(u_\varepsilon, h_\varepsilon)\|_{X_0} \geq c > 0$ and $(\lambda_0 + A)(u_\varepsilon, h_\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus λ_0 must belong to the spectrum of $(-A)$.

(iv) Finally, if $\lambda \in \mathbb{C}_+$ is not in $[0, \lambda_\infty]$, then $s(\lambda, \cdot)$ does not vanish. Therefore the boundary symbol can be inverted by means of Mikhlin's multiplier theorem and then solving the remaining Stokes problem we see that $\lambda + A$ is boundedly invertible. Thus such λ belong to the resolvent set of A. The proof is complete. $\square$

5. HENRY'S INSTABILITY THEOREM

For the reader's convenience we provide the statement of Henry's instability theorem, see [12, Theorem 5.1.5], for a more specialized situation which is appropriate for the problem under consideration in this paper.

Theorem 5.1. *Let X be a Banach space, $T \in C^2(B_X(0, \varrho); X)$ such that $T(0) = 0$ and the spectral radius* spr $T'(0)$ *of $T'(0)$ is greater than one. Then the origin is unstable in the sense that there is a constant $\varepsilon_0 > 0$ such that for each $\delta > 0$ there is $x_\delta \in B_X(0, \delta)$ and $N \in \mathbb{N}$ such that the sequence $x_k := T^k x_\delta \in B_X(0, \varrho)$ is well-defined for all $k \in \{1, \dots, N\}$ and $\|T^N x_\delta\| \geq \varepsilon_0$.*

As an illustration consider as in [14] and [18] the quasi-linear evolution equation

$$\dot{u} + A(u)u = F(u), \ t > 0, \quad u(0) = u_0, \tag{5.1}$$

where $X_1 \hookrightarrow X_0$ are densely embedded Banach spaces, $X_\gamma := (X_0, X_1)_{1-1/p,p}$ is a real interpolation space of order $1 - 1/p$ and power $p \in (1, \infty)$ between X_0 and X_1. The nonlinearities $(F, A) : X_\gamma \to X_0 \times \mathcal{L}(X_1, X_0)$ are of class C^2. Assume that $u_* \in X_1$ is an equilibrium of (5.1), i.e. $A(u_*)u_* = F(u_*)$. Assume further that the linearization of (5.1) has maximal L_p-regularity, i.e. the operator A_0 defined by

$$A_0 v := A(u_*)v + [A'(u_*)v]u_* - F'(u_*)v, \quad v \in X_1,$$

is $\mathcal{R}$-sectorial. Due to the results in [14] and [18], the Poincaré map of (5.1), $[u_0 \mapsto Tu_0 - u_* := u(a, u_0) - u_*]$, where $u(t, u_0)$ denotes the solution of (5.1) with $u_0 \in B_{X_\gamma}(u_*, \varrho)$, is well-defined and satisfies the assumptions of Henry's instability theorem in $X := X_\gamma$, provided $\varrho > 0$ is sufficiently small. The derivative $T'(0)$ equals $e^{-A_0 a}$, which by the spectral mapping theorem for generators of analytic C_0-semigroups has spectral radius greater than one if and only if the spectrum of the operator $-A_0$ contains points in the open right half-plane. In particular, in contrary to [14] no spectral gap is required. Therefore we may conclude that the equilibrium u_* is unstable in the natural phase space X_γ of the problem.

In the situation of the two-phase Navier-Stokes equations with surface tension and subject to gravity, things are more involved due to the inherent nonlinearity of the compatibility conditions for the transformed problem

$$\text{div } u = F_d(u, h),$$
$$-[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] = G_v(u, h), \tag{5.2}$$
$$-2[\![\mu \partial_y w]\!] + [\![\pi]\!] - (\sigma \Delta h + [\![\rho]\!]\gamma_a)h = G_w(u, h),$$

on the interface. This leads to a nonlinear phase manifold where the semi-flow lives on. The way out is to parameterize this manifold. Nevertheless, the linearization

of the time-one map will turn out to be the operator e^{-A} with A described in the previous section, which by Proposition 4.1 has spectral values in the open right half-plane. This way we will still be able to apply Henry's result, and as a consequence to obtain a rigorous proof for the Rayleigh-Taylor instability.

6. Proof of the main result

(i) In a first step we parameterize the phase manifold $\mathcal{PM}$ of system (2.1) which lies in

$$X_\gamma := \{(u,h) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}; \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n) : [\![u]\!] = 0\}$$

and is defined by

$$\mathcal{PM} := \big\{(u,h) \in X_\gamma : \operatorname{div} u = u \cdot \nabla h \text{ in } \dot{\mathbb{R}}^{n+1}, \ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = G_v(u,h)\big\}.$$

We want to parameterize $\mathcal{PM}$ locally near $(0,0)$ over the closed linear subspace $X_\gamma^0 \subset X_\gamma$, defined by

$$X_\gamma^0 := \big\{(u,h) \in X_\gamma : \operatorname{div} u = 0 \text{ in } \dot{\mathbb{R}}^{n+1}, \ [\![\mu\partial_y v]\!] + [\![\mu\nabla_x w]\!] = 0 \text{ on } \mathbb{R}^n\big\},$$

by means of an analytic map $\Phi : B_{X_\gamma^0}(0,r) \to \mathcal{PM}$ which is bijective onto its range such that $\Phi'(0) = I$. For this purpose we consider the problem

$$\begin{cases} \rho\lambda_* u - \mu\Delta u + \nabla\pi = 0 & \text{in } \dot{\mathbb{R}}^{n+1} \\ \operatorname{div} u = F_d(u + \tilde{u}, h + \tilde{h}) & \text{in } \dot{\mathbb{R}}^{n+1} \\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = G_v(u + \tilde{u}, h + \tilde{h}) & \text{on } \mathbb{R}^n \\ -2[\![\mu\partial_y(w + \tilde{w})]\!] + [\![\pi]\!] = (\sigma\Delta + [\![\rho]\!]\gamma_a)(h + \tilde{h}) + G_w(u + \tilde{u}, h + \tilde{h}) & \text{on } \mathbb{R}^n \\ [\![u]\!] = 0 & \text{on } \mathbb{R}^n \\ \lambda_* h - w|_{y=0} = H(u + \tilde{u}, h + \tilde{h}) & \text{on } \mathbb{R}^n, \end{cases}$$

with some fixed, sufficiently large $\lambda_* > 0$ and given $(\tilde{u}, \tilde{h}) \in B_{X_\gamma^0}(0,r) \subset X_\gamma^0$. We write this equation in short hand notation as $L_{\lambda_*} z = N(z + \tilde{z})$ in

$$\mathbb{Z}_\gamma := \big\{(u, \pi, [\![\pi]\!], h) : (u,h) \in X_\gamma, \ \pi \in \dot{W}_p^{1-2/p}(\dot{\mathbb{R}}^{n+1}), \ [\![\pi]\!] \in W_p^{3-2/p}(\mathbb{R}^n)\big\}$$

where $z = (u, \pi, [\![\pi]\!], h)$ and $\tilde{z} = (\tilde{u}, 0, 0, \tilde{h})$. Since the nonlinear terms are polynomial, it is not difficult to verify that N is real analytic and $N'(0) = 0$; see [16, Proposition 6.2] for a related result. L_{λ_*} is invertible by Theorem 2.2, and hence the implicit function theorem yields a unique solution $z = \mathfrak{Z}(\tilde{u}, \tilde{h}) \in \mathbb{Z}_\gamma$ near 0. The mapping

$$\mathfrak{Z} : B_{X_\gamma^0}(0,r) \to \mathbb{Z}_\gamma,$$

with r chosen small enough, is real analytic and satisfies $\mathfrak{Z}'(0) = 0$. Denoting by $P : \mathbb{Z}_\gamma \to X_\gamma$ the projection given by $Pz := (u,h)$ for $z = (u, \pi, [\![\pi]\!], h)$, we set

$$\Phi(\tilde{u}, \tilde{h}) := (\tilde{u}, \tilde{h}) + \phi(\tilde{u}, \tilde{h}) \quad \text{with} \quad \phi(\tilde{u}, \tilde{h}) := P\mathfrak{Z}(\tilde{u}, \tilde{h}). \tag{6.1}$$

Then $\Phi(B_{X_\gamma^0}(0,r)) \subset \mathcal{PM}$, $\Phi : B_{X_\gamma^0}(0,r) \to X_\gamma$ is real analytic, $\Phi'(0) = I$, and Φ is injective. Hence it remains to show local surjectivity near 0. So suppose that $(\bar{u}, \bar{h}) \in \mathcal{PM}$ has sufficiently small norm. Solving the problem

$$
\begin{cases}
\rho\lambda_* u - \mu\Delta u + \nabla\pi = 0 & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
\operatorname{div} u = F_d(\bar{u}, \bar{h}) & \text{in} \quad \dot{\mathbb{R}}^{n+1} \\
-[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] = G_v(\bar{u}, \bar{h}) & \text{on} \quad \mathbb{R}^n \\
-2[\![\mu\partial_y w]\!] + [\![\pi]\!] - (\sigma\Delta + [\![\rho]\!]\gamma_a)h = G_w(\bar{u}, \bar{h}) & \text{on} \quad \mathbb{R}^n \\
[\![u]\!] = 0 & \text{on} \quad \mathbb{R}^n \\
\lambda_* h - w|_{y=0} = H(\bar{u}, \bar{h}) & \text{on} \quad \mathbb{R}^n,
\end{cases}
$$

by means of Theorem 2.2 yields a unique solution $z = (u, \pi, [\![\pi]\!], h) \in \mathbb{Z}_\gamma$. One readily verifies that $(\tilde{u}, \tilde{h}) := (\bar{u} - u, \bar{h} - h)$ belongs to X_γ^0 and $\phi(\tilde{u}, \tilde{h}) = (u, h)$, showing surjectivity of Φ near 0. In particular, $\mathcal{PM} \subset X_\gamma$ is a real analytic manifold near $0 \in X_\gamma$. We observe that due to

$$-2[\![\mu\partial_y(w + \tilde{w})]\!] + [\![\pi]\!] = (\sigma\Delta + [\![\rho]\!]\gamma_a)(h + \tilde{h}) + G_w(u + \tilde{u}, h + \tilde{h}) \text{ on } \mathbb{R}^n$$

the last condition in (5.2) is satisfied as well.

(ii) Now we proceed as in the proof of [16, Theorem 6.3], employing the notation of that proof. For a given $(u_0, h_0) \in \mathcal{PM}$ we construct the extension $z_*(u_0, h_0) \in \mathbb{E}_1(a)$. The map

$$[(\tilde{u}_0, \tilde{h}_0)) \mapsto z_*(\Phi(\tilde{u}_0, \tilde{h}_0))] : B_{X_\gamma^0}(0, r) \to \mathbb{E}(a)$$

is real analytic. Therefore, fixing $a > 0$ the mapping

$$
\begin{aligned}
& K : {}_0\mathbb{E}(a) \times B_{X_\gamma^0}(0, r) \to {}_0\mathbb{F}(a), \\
& K(z, (\tilde{u}_0, \tilde{h}_0)) := L(z + z_*(\Phi(\tilde{u}_0, \tilde{h}_0))) - N(z + z_*(\Phi(\tilde{u}_0, \tilde{h}_0)))
\end{aligned}
\tag{6.2}
$$

is real analytic. We have $K(0,0) = 0$ as well as $D_z K(0,0) = L$. Therefore the implicit function theorem yields a real analytic map

$$\Psi : B_{X_\gamma^0}(0, r) \to {}_0\mathbb{E}(a)$$

such that $K(\Psi(\tilde{u}_0, \tilde{h}_0), (\tilde{u}_0, \tilde{h}_0)) = 0$ for all $(\tilde{u}_0, \tilde{h}_0) \in B_{X_\gamma^0}(0, r)$, with r chosen sufficiently small. Thus

$$z := \Psi(\tilde{u}_0, \tilde{h}_0) + z_*(\Phi(\tilde{u}_0, \tilde{h}_0)) \in \mathbb{E}(a) \tag{6.3}$$

is the unique solution of problem (2.1) and, moreover, the mapping

$$[(\tilde{u}_0, \tilde{h}_0) \mapsto z(\tilde{u}_0, \tilde{h}_0)] : B_{X_\gamma^0}(0, r) \to \mathbb{E}(a)) \tag{6.4}$$

is real real-analytic.

(iii) Having obtained a unique solution $z = z(\tilde{u}_0, \tilde{h}_0) \in \mathbb{E}(a)$ of (2.1), we can employ the same arguments as in steps (vi)–(vii) in the proof of [16, Theorem 6.3] to establish analyticity of the solution as stated in Theorem 1.1.

(iv) Differentiating the mapping $[(\tilde{u}_0, \tilde{h}_0) \mapsto z(\tilde{u}_0, \tilde{h}_0)]$ w.r.t. the initial value $(\tilde{u}_0, \tilde{h}_0)$ one sees that the linearization at $(0,0)$ of (2.1) is given by the Cauchy problem

$$\frac{d}{dt}(u,h) + A(u,h) = 0, \quad (u(0), h(0)) = (u_0, h_0).$$

This implies that the linearization at $(0,0)$ of the time-one-map $[(\tilde{u}_0, \tilde{h}_0)] \mapsto z(1)]$ of the nonlinear problem (2.1) is given by e^{-A}. Since $(-A)$ generates an analytic C_0-semigroup in X_0 the spectral mapping theorem yields $\mathrm{spr}\,(e^{-A}) = e^{\lambda_\infty} > 1$ by Proposition 4.1, hence we may apply Theorem 5.1 to obtain instability of the trivial solution. The proof of our main result is therefore complete. $\square$

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