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Report No. 11 (2009)

Editors:
Professors of the Institute for Mathematics, Martin-Luther-University Halle-Wittenberg.

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Jan Prüss and Gieri Simonett

Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider the motion of two superposed immiscible, viscous, incompressible, capillary fluids that are separated by a sharp interface which needs to be determined as part of the problem. Allowing for gravity to act on the fluids, we prove local well-posedness of the problem. In particular, we obtain well-posedness for the case where the heavy fluid lies on top of the light one, that is, for the case where the Rayleigh-Taylor instability is present. Additionally we show that solutions become real analytic instantaneously.

Mathematics Subject Classification (2000). Primary: 35R35. Secondary: 35Q10, 76D03, 76D45, 76T05.

Keywords. Navier-Stokes equations, free boundary problem, surface tension, gravity, Rayleigh-Taylor instability, well-posedness, analyticity.

1. Introduction and Main Results

We consider a free boundary problem describing the motion of two immiscible, viscous, incompressible capillary fluids, fluid_1 and fluid_2, occupying the regions

$$\Omega_i(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i (y - h(t,x)) > 0,\ t \geq 0\}, \quad i = 1,2.$$

The fluids, thus, are separated by the interface

$$\Gamma(t) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = h(t,x) : x \in \mathbb{R}^n,\ t \geq 0\},$$

called the free boundary, which needs to be determined as part of the problem. The motion of the fluids is governed by the incompressible Navier-Stokes equations where surface tension on the free boundary is included. In addition, we also allow

The research of GS was partially supported by NSF, Grant DMS-0600870.

for gravity to act on the fluids. The governing equations then are given by the system

$$\left\{\begin{aligned} \rho\big(\partial_t u + (u|\nabla)u\big) - \mu\Delta u + \nabla q &= 0 && \text{in} \quad \Omega(t)\\ \operatorname{div} u &= 0 && \text{in} \quad \Omega(t)\\ -[\![S(u,q)\nu]\!] &= \sigma\kappa\nu + [\![\rho]\!]\gamma_a y && \text{on} \quad \Gamma(t)\\ [\![u]\!] &= 0 && \text{on} \quad \Gamma(t)\\ V &= (u|\nu) && \text{on} \quad \Gamma(t)\\ u(0) &= u_0 && \text{in} \quad \Omega_0\\ \Gamma(0) &= \Gamma_0\,. \end{aligned}\right. \tag{1.1}$$

Here ρ and μ are given by

$$\rho = \rho_1\chi_{\Omega_1(t)} + \rho_2\chi_{\Omega_2(t)}, \quad \mu = \mu_1\chi_{\Omega_1(t)} + \mu_2\chi_{\Omega_2(t)},$$

with χ the indicator function, where the constants ρ_i and μ_i denote the densities and viscosities of the respective fluids. The constant $\sigma > 0$ denotes the surface tension, and γ_a is the acceleration of gravity. Moreover, $S(u,q)$ is the stress tensor defined by

$$S(u,q) = \mu_i\big(\nabla u + (\nabla u)^{\mathsf{T}}\big) - qI \quad \text{in} \quad \Omega_i(t),$$

where $q = \tilde{q} + \rho\gamma_a y$ denotes the modified pressure incorporating the potential of the gravity force, and

$$[\![v]\!] = (v_{|_{\Omega_2(t)}} - v_{|_{\Omega_1(t)}})|_{\Gamma(t)}$$

denotes the jump of the quantity v, defined on the respective domains $\Omega_i(t)$, across the interface $\Gamma(t)$. Finally, $\kappa = \kappa(t,\cdot)$ is the mean curvature of the free boundary $\Gamma(t)$, $\nu = \nu(t,\cdot)$ is the unit normal field on $\Gamma(t)$, and $V = V(t,\cdot)$ is the normal velocity of $\Gamma(t)$. Here we use the convention that $\nu(t,\cdot)$ points from $\Omega_1(t)$ into $\Omega_2(t)$, and that $\kappa(x,t)$ is negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$.

Given are the initial position $\Gamma_0 = \operatorname{graph}(h_0)$ of the interface, and the initial velocity

$$u_0 : \Omega_0 \to \mathbb{R}^{n+1}, \quad \Omega_0 := \Omega_1(0) \cup \Omega_2(0).$$

The unknowns are the velocity field $u(t,\cdot) : \Omega(t) \to \mathbb{R}^{n+1}$, the pressure field $q(t,\cdot) : \Omega(t) \to \mathbb{R}$, and the free boundary $\Gamma(t)$, where $\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$.

Our main result shows that problem (1.1) admits a unique local smooth solution, provided that $\|\nabla h_0\|_\infty := \sup_{x\in\mathbb{R}^n} |\nabla h_0(x)|$ is sufficiently small.

Theorem 1.1. *Let $p > n+3$. Then given $\beta > 0$, there exists $\eta = \eta(\beta) > 0$ such that for all initial values*

$$(u_0, h_0) \in W_p^{2-2/p}(\Omega_0, \mathbb{R}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n), \quad [\![u_0]\!] = 0,$$

satisfying the compatibility conditions

$$[\![\mu D(u_0)\nu_0 - \mu(\nu_0|D(u_0)\nu_0)\nu_0]\!] = 0, \quad \operatorname{div} u_0 = 0 \ \textit{ on } \ \Omega_0,$$

with $D(u_0) := (\nabla u_0 + (\nabla u_0)^{\mathsf{T}})$, *and the smallness-boundedness condition*

$$\|\nabla h_0\|_\infty \le \eta, \qquad \|u_0\|_\infty \le \beta,$$

there is $t_0 = t_0(u_0, h_0) > 0$ *such that problem* (1.1) *admits a classical solution* (u, q, Γ) *on* $(0, t_0)$. *The solution is unique in the function class described in Theorem* 4.2. *In addition,* $\Gamma(t)$ *is a graph over* $\mathbb{R}^n$ *given by a function* $h(t)$ *and* $\mathcal{M} = \bigcup_{t\in(0,t_0)} \big(\{t\} \times \Gamma(t)\big)$ *is a real analytic manifold, and with*

$$\mathcal{O} := \{(t, x, y) : \ t \in (0, t_0), \ x \in \mathbb{R}^n, y \neq h(t, x)\},$$

the function $(u, q) : \mathcal{O} \to \mathbb{R}^{n+2}$ *is real analytic.*

Remarks 1.2. (a) More precise statements for the transformed problem will be given in Section 4. Due to the restriction $p > n + 3$ we obtain

$$h \in C(J; BUC^2(\mathbb{R}^n)) \cap C^1(J; BUC^1(\mathbb{R}^n)),$$

where $J = [0, t_0]$. In particular, the normal of $\Omega_1(t)$, the normal velocity of $\Gamma(t)$, and the mean curvature of $\Gamma(t)$ are well-defined and continuous, so that (1.1) makes sense pointwise. For u we obtain

$$u \in BUC(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}), \quad \nabla u \in BUC(\mathcal{O}, \mathbb{R}^{(n+1)^2}).$$

Also interesting is the fact that the surface pressure jump is analytic on $\mathcal{M}$ as well.

(b) It is possible to relax the assumption $p > n + 3$. In fact, $p > (n + 3)/2$ turns out to be sufficient. In order to keep the arguments simple, we impose here the stronger condition $p > n + 3$.

(c) It is well-known that the situation where gravity is acting on two superposed immiscible fluids - with the heavier fluid lying above a fluid of lesser density - leads to an instability, the Rayleigh-Taylor instability. In this case, small disturbances of the equilibrium situation $(u, h) = (0, 0)$ can cause instabilities, where the heavy fluid moves down under the influence of gravity, and the light material is displaced upwards, leading to vortices. Our results show that problem (1.1) is also well-posed in this case, provided $\|\nabla h_0\|_\infty$ is small enough, yielding smooth solutions for a short time. In the forthcoming publication [29] we will give a rigorous proof showing that the equilibrium solution $(u, h) = (0, 0)$ is L_p-unstable. To the best of our knowledge these are the first rigorous results concerning the Navier-Stokes equations subject to the Rayleigh-Taylor instability.

(d) If $\gamma_a = 0$ then it is shown in [28] that problem (1.1) admits a solution with the same regularity properties on an arbitrary fixed time interval $[0, t_0]$, provided that $\|u_0\|_{W_p^{2-2/p}(\Omega_0)}$ and $\|h_0\|_{W_p^{3-2/p}(\mathbb{R}^n)}$ are sufficiently small (depending on t_0).

(e) We point out that in Theorem 1.1 we only need a smallness condition on the sup-norm of ∇h_0 (relative to the vertical component of the velocity). In case of a more general geometry, this condition can always be achieved by a judicious choice of a reference manifold.

The motion of a layer of viscous, incompressible fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity (in which case Ω_0 is a strip, bounded above by Γ_0 and below by a fixed surface Γ_b) has been considered by Allain [1], Beale [7], Beale and Nishida [8], Tani [35], by Tani and Tanaka [36], and by Shibata and Shimizu [32]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions is proved in [7] for $\sigma > 0$, and in [36] for $\sigma \geq 0$, and the asymptotic decay rate for $t \to \infty$ is studied in [8]. We also refer to [9], where in addition the presence of a surfactant on the free boundary and in one of the bulk phases is considered.

In case that $\Omega_1(t)$ is a bounded domain, $\gamma_a = 0$, and $\Omega_2(t) = \emptyset$, one obtains the *one-phase* Navier-Stokes equations with surface tension, describing the motion of an isolated volume of fluid. For an overview of the existing literature in this case we refer to the recent publications [28, 31, 32, 33].

Results concerning the *two-phase problem* (1.1) with $\gamma_a = 0$ in the $3D$-case are obtained in [11, 12, 13, 34]. In more detail, Densiova [12] establishes existence and uniqueness of solutions (of the transformed problem in Lagrangian coordinates) with $v \in W_2^{s,s/2}$ for $s \in (5/2, 3)$ in case that one of the domains is bounded. Tanaka [34] considers the two-phase Navier-Stokes equations with thermo-capillary convection in bounded domains, and he obtains existence and uniqueness of solutions with $(v, \theta) \in W_2^{s,s/2}$ for $s \in (7/2, 4)$, with θ denoting the temperature.

In order to prove our main result we transform problem (1.1) into a problem on a fixed domain. The transformation is expressed in terms of the unknown height function h describing the free boundary. Our analysis proceeds with establishing maximal regularity results for an associated linear problem. relying on the powerful theory of maximal regularity, in particular on the H^∞-calculus for sectorial operators, the Dore-Venni theorem, and the Kalton-Weis theorem, see for instance [2, 14, 16, 22, 23, 26, 30].

Based on the linear estimates we can solve the nonlinear problem by the contraction mapping principle. Analyticity of solutions is obtained as in [28] by the implicit function theorem in conjunction with a scaling argument, relying on an idea that goes back to Angenent [4, 5] and Masuda [24]; see also [17, 18, 20].

The plan for this paper is as follows. Section 2 contains the transformation of the problem to a half-space and the determination of the proper underlying linear problem. In Section 3 we analyze this linearization and prove the crucial maximal regularity result in an L_p-setting. Section 4 is then devoted to the nonlinear problem and contains the proof of our main result. Finally we collect and prove in an appendix some of the technical results used in order to estimate the nonlinear terms.

2. The transformed problem

The nonlinear problem (1.1) can be transformed to a problem on a fixed domain by means of the transformations

$$\begin{aligned} v(t,x,y) &:= (u_1,\dots,u_n)(t,x,y+h(t,x)),\\ w(t,x,y) &:= u_{n+1}(t,x,y+h(t,x)),\\ \pi(t,x,y) &:= q(t,x,y+h(t,x)), \end{aligned}$$

where $t\in J=[0,a]$, $x\in\mathbb{R}^n$, $y\in\mathbb{R}$, $y\neq 0$. With a slight abuse of notation we will in the sequel denote the transformed velocity again by u, that is, we set $u=(v,w)$. With this notation we obtain the transformed problem

$$\left\{\begin{aligned} \rho\partial_t u-\mu\Delta u+\nabla\pi &= F(u,\pi,h) && \text{in } \dot{\mathbb{R}}^{n+1}\\ \operatorname{div} u &= F_d(u,h) && \text{in } \dot{\mathbb{R}}^{n+1}\\ -[\![\mu\partial_y v]\!]-[\![\mu\nabla_x w]\!] &= G_v(u,[\![\pi]\!],h) && \text{on } \mathbb{R}^n\\ -2[\![\mu\partial_y w]\!]+[\![\pi]\!]-\sigma\Delta h-[\![\rho]\!]\gamma_a h &= G_w(u,h) && \text{on } \mathbb{R}^n\\ [\![u]\!] &= 0 && \text{on } \mathbb{R}^n\\ \partial_t h-\gamma w &= -(\gamma v|\nabla h) && \text{on } \mathbb{R}^n\\ u(0)=u_0,\ h(0)&=h_0, \end{aligned}\right. \tag{2.1}$$

for $t>0$, where $\dot{\mathbb{R}}^{n+1}=\{(x,y)\in\mathbb{R}^n\times\mathbb{R}: y\neq 0\}$.

The nonlinear functions have been computed in [28] and are given by:

$$\begin{aligned} F_v(v,w,\pi,h) &= \mu\{-2(\nabla h|\nabla_x)\partial_y v+|\nabla h|^2\partial_y^2 v-\Delta h\partial_y v\}+\partial_y\pi\nabla h\\ &\quad+\rho\{-(v|\nabla_x)v+(\nabla h|v)\partial_y v-w\partial_y v\}+\rho\partial_t h\partial_y v,\\ F_w(v,w,h) &= \mu\{-2(\nabla h|\nabla_x)\partial_y w+|\nabla h|^2\partial_y^2 w-\Delta h\partial_y w\}\\ &\quad+\rho\{-(v|\nabla_x)w+(\nabla h|v)\partial_y w-w\partial_y w\}+\rho\partial_t h\partial_y w,\\ F_d(v,h) &= (\nabla h|\partial_y v) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} G_v(v,w,[\![\pi]\!],h) &= -[\![\mu(\nabla_x v+(\nabla_x v)^{\mathsf T})]\!]\nabla h+|\nabla h|^2[\![\mu\partial_y v]\!]+(\nabla h|\,[\![\mu\partial_y v]\!])\nabla h\\ &\quad-[\![\mu\partial_y w]\!]\nabla h+\{[\![\pi]\!]-\sigma(\Delta h-G_\kappa(h))\}\nabla h,\\ G_w(v,w,h) &= -(\nabla h|\,[\![\mu\nabla_x w]\!])-(\nabla h|\,[\![\mu\partial_y v]\!])+|\nabla h|^2[\![\mu\partial_y w]\!]-\sigma G_\kappa(h) \end{aligned} \tag{2.3}$$

with

$$G_\kappa(h)=\frac{|\nabla h|^2\Delta h}{(1+\sqrt{1+|\nabla h|^2})\sqrt{1+|\nabla h|^2}}+\frac{(\nabla h|\nabla^2 h\nabla h)}{(1+|\nabla h|^2)^{3/2}}, \tag{2.4}$$

where $\nabla^2 h$ denotes the Hessian matrix of all second order derivatives of h.

Before studying solvability results for problem (2.1) let us first introduce suitable function spaces. Let $\Omega\subseteq\mathbb{R}^m$ be open and X be an arbitrary Banach space. By $L_p(\Omega;X)$ and $H^s_p(\Omega;X)$, for $1\le p\le\infty$, $s\in\mathbb{R}$, we denote the X-valued Lebegue and the Bessel potential spaces of order s, respectively. We will also frequently make use of the fractional Sobolev-Slobodeckij spaces $W^s_p(\Omega;X)$, $1\le p<\infty$, $s\in\mathbb{R}\setminus\mathbb{Z}$, with norm

$$\|g\|_{W^s_p(\Omega;X)}=\|g\|_{W^{[s]}_p(\Omega;X)}+\sum_{|\alpha|=[s]}\Big(\int_\Omega\int_\Omega\frac{\|\partial^\alpha g(x)-\partial^\alpha g(y)\|^p_X}{|x-y|^{m+(s-[s])p}}\,dx\,dy\Big)^{1/p},\tag{2.5}$$

where $[s]$ denotes the largest integer smaller than s. Let $a\in(0,\infty]$ and $J=[0,a]$. We set

$$_0W^s_p(J;X):=\begin{cases}\{g\in W^s_p(J;X): g(0)=g'(0)=\ldots=g^{(k)}(0)=0\},\\ \text{if}\quad k+\frac{1}{p}<s<k+1+\frac{1}{p},\ k\in\mathbb{N}\cup\{0\},\\ W^s_p(J;X),\quad\text{if}\quad s<\frac{1}{p}.\end{cases}$$

The spaces $_0H^s_p(J;X)$ are defined analogously. Here we remind that $H^k_p=W^k_p$ for $k\in\mathbb{Z}$ and $1<p<\infty$, and that $W^s_p=B^s_{pp}$ for $s\in\mathbb{R}\setminus\mathbb{Z}$.
For $\Omega\subset\mathbb{R}^m$ open and $1\le p<\infty$, the homogeneous Sobolev spaces $\dot H^1_p(\Omega)$ of order 1 are defined as

$$\begin{aligned}&\dot H^1_p(\Omega):=(\{g\in L_{1,\mathrm{loc}}(\Omega):\|\nabla g\|_{L_p(\Omega)}<\infty\},\|\cdot\|_{\dot H^1_p(\Omega)})\\ &\|g\|_{\dot H^1_p(\Omega)}:=\big(\sum_{j=1}^m\|\partial_j g\|^p_{L_p(\Omega)}\big)^{1/p}.\end{aligned}\tag{2.6}$$

Then $\dot H^1_p(\Omega)$ is a Banach space, provided we factor out the constant functions and equip the resulting space with the corresponding quotient norm, see for instance [21, Lemma II.5.1]. We will in the sequel always consider the quotient space topology without change of notation. In case that Ω is locally Lipschitz, it is known that $\dot H^1_p(\Omega)\subset H^1_{p,\mathrm{loc}}(\overline\Omega)$, see [21, Remark II.5.1], and consequently, any function in $\dot H^1_p(\Omega)$ has a well-defined trace on $\partial\Omega$.

For $s\in\mathbb{R}$ and $1<p<\infty$ we also consider the homogeneous Bessel-potential spaces $\dot H^s_p(\mathbb{R}^n)$ of order s, defined by

$$\begin{aligned}&\dot H^s_p(\mathbb{R}^n):=(\{g\in\mathcal{S}'(\mathbb{R}^n):\dot I^s g\in L_p(\mathbb{R}^n)\},\|\cdot\|_{\dot H^s_p(\mathbb{R}^n)}),\\ &\|g\|_{\dot H^s_p(\mathbb{R}^n)}:=\|\dot I^s g\|_{L_p(\mathbb{R}^n)},\end{aligned}\tag{2.7}$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the space of all tempered distributions, and $\dot I^s$ is the Riesz potential given by

$$\dot I^s g:=(-\Delta)^{s/2}g:=\mathcal{F}^{-1}(|\xi|^s\mathcal{F}g),\quad g\in\mathcal{S}'(\mathbb{R}^n).$$

By factoring out all polynomials, $\dot H^s_p(\mathbb{R}^n)$ becomes a Banach space with the natural quotient norm. For $s\in\mathbb{R}\backslash\mathbb{Z}$, the homogeneous Sobolev-Slobodeckij spaces $\dot W^s_p(\mathbb{R}^n)$ of fractional order can be obtained by real interpolation as

$$\dot W^s_p(\mathbb{R}^n):=(\dot H^k_p(\mathbb{R}^n),\dot H^{k+1}_p(\mathbb{R}^n))_{s-k,p},\quad k<s<k+1,$$

where $(\cdot,\cdot)_{\theta,p}$ is the real interpolation method. It follows that

$$\dot{I}^s \in \mathrm{Isom}(\dot{H}_p^{t+s}(\mathbb{R}^n), \dot{H}_p^t(\mathbb{R}^n)) \cap \mathrm{Isom}(\dot{W}_p^{t+s}(\mathbb{R}^n), \dot{W}_p^t(\mathbb{R}^n)), \quad s,t \in \mathbb{R}, \tag{2.8}$$

with $\dot{W}_p^k = \dot{H}_p^k$ for $k \in \mathbb{Z}$. We refer to [6, Section 6.3] and [37, Section 5] for more information on homogeneous functions spaces. In particular, it follows from parts (ii) and (iii) in [37, Theorem 5.2.3.1] that the definitions (2.6) and (2.7) are consistent if $\Omega = \mathbb{R}^n$, $s = 1$, and $1 < p < \infty$. We note in passing that

$$\Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|g(x)-g(y)|^p}{|x-y|^{n+sp}}\,dx\,dy\Big)^{1/p}, \quad \Big(\int_0^\infty t^{(1-s)p}\|\frac{d}{dt}P(t)g\|^p_{L_p(\mathbb{R}^n)}\frac{dt}{t}\Big)^{1/p} \tag{2.9}$$

define equivalent norms on $\dot{W}_p^s(\mathbb{R}^n)$ for $0 < s < 1$, where $P(\cdot)$ denotes the Poisson semigroup, see [37, Theorem 5.2.3.2 and Remark 5.2.3.4]. Moreover,

$$\gamma_\pm \in \mathcal{L}(\dot{W}_p^1(\mathbb{R}^{n+1}_\pm), \dot{W}_p^{1-1/p}(\mathbb{R}^n)), \tag{2.10}$$

where $\gamma_\pm$ denotes the trace operators, see for instance [21, Theorem II.8.2].

3. The Linearized Two-Phase Stokes Problem with Free boundary

It turns out that, unfortunately, the nonlinear term $(\gamma v|\nabla h)$ occurring in (2.1) cannot be made small in the norm of $\mathbb{F}_4(a)$, defined below in (4.2), by merely taking $\|\nabla h\|_\infty$ small. This can, however, be achieved for the modified term $(b - \gamma v|\nabla h)$, provided b is properly chosen so that $b(0) = \gamma v_0$. As a consequence, we now need to consider the modified linear problem

$$\left\{\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= f && \text{in } \dot{\mathbb{R}}^{n+1}\\ \operatorname{div} u &= f_d && \text{in } \dot{\mathbb{R}}^{n+1}\\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= g_v && \text{on } \mathbb{R}^n\\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] &= g_w + \sigma\Delta h + [\![\rho]\!]\gamma_a h && \text{on } \mathbb{R}^n\\ [\![u]\!] &= 0 && \text{on } \mathbb{R}^n\\ \partial_t h - \gamma w + (b(t,x)|\nabla)h &= g_h && \text{on } \mathbb{R}^n\\ u(0) = u_0,\ h(0) &= h_0. \end{aligned}\right. \tag{3.1}$$

Here we mention that the simpler case where $b = 0$ and $\gamma_a = 0$ was studied in [28, Theorem 5.1]. We obtain the following maximal regularity result.

Theorem 3.1. *Let $p > n + 3$ be fixed, and assume that ρ_j and μ_j are positive constants for $j = 1, 2$, and set $J = [0, a]$. Suppose*

$$b_0 \in \mathbb{R}^n, \quad b_1 \in W_p^{1-1/2p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^n)),$$

and set $b(\cdot) = b_0 + b_1(\cdot)$. Then the Stokes problem with free boundary (3.1) *admits a unique solution (u, π, h) with regularity*

$$\begin{aligned}
&u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})),\\
&\pi \in L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
&[\![\pi]\!] \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)),\\
&h \in W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n))
\end{aligned} \tag{3.2}$$

if and only if the data $(f, f_d, g, g_h, u_0, h_0)$ satisfy the following regularity and compatibility conditions:

(a) $f \in L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$,

(b) $f_d \in H^1_p(J; \dot{H}^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1}))$,

(c) $g = (g_v, g_w) \in W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1}))$,

(d) $g_h \in W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n))$,

(e) $u_0 \in W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})$, $h_0 \in W^{3-2/p}_p(\mathbb{R}^n)$,

(f) $\operatorname{div} u_0 = f_d(0)$ *in* $\dot{\mathbb{R}}^{n+1}$ *and* $[\![u_0]\!] = 0$ *on* $\mathbb{R}^n$ *if* $p > 3/2$,

(g) $-[\![\mu \partial_y v_0]\!] - [\![\mu \nabla_x w_0]\!] = g_v(0)$ *on* $\mathbb{R}^n$ *if* $p > 3$.

The solution map $[(f, f_d, g, g_h, u_0, h_0) \mapsto (u, \pi, h)]$ is continuous between the corresponding spaces.

If $b_1 \equiv 0$ then the result is true for all $p \in (1, \infty)$, $p \neq 3/2, 3$.

Proof. (i) Since $\mathbb{F}_4(a)$, defined by

$$\mathbb{F}_4(a) := W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),$$

is a multiplication algebra for $p > n+3$, the operator $[h \mapsto (b|\nabla)h]$ maps the space

$$\mathbb{E}_4(a) := W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n))$$

continuously into $\mathbb{F}_4(a)$ with bound $|b_0| + C_a \|b_1\|_{\mathbb{F}_4(a)}$, see Lemma 5.5(a).
As in the proof of [28, Theorem 5.1] it suffices to consider the reduced problem

$$\left\{\begin{aligned}
\rho \partial_t u - \mu \Delta u + \nabla \pi &= 0 && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\
\operatorname{div} u &= 0 && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\
-[\![\mu \partial_y v]\!] - [\![\mu \nabla_x w]\!] &= 0 && \text{on} \quad \mathbb{R}^n\\
-2[\![\mu \partial_y w]\!] + [\![\pi]\!] &= \sigma \Delta h + [\![\rho]\!] \gamma_a h && \text{on} \quad \mathbb{R}^n\\
[\![u]\!] &= 0 && \text{on} \quad \mathbb{R}^n\\
\partial_t h - \gamma w + (b(t,x)|\nabla) h &= \tilde{g}_h && \text{on} \quad \mathbb{R}^n\\
u(0) = 0,\ h(0) &= 0,
\end{aligned}\right. \tag{3.3}$$

where the function $\tilde{g}_h \in {}_0\mathbb{F}_4(a)$ is defined in a similar way as in formula (5.5) in [28]. This can be accomplished by choosing $h_1 := h_{1,b} \in \mathbb{E}_4(a)$ such that

$$h_1(0) = h_0, \quad \partial_t h_1(0) = g_h(0) + \gamma w_0 - (b(0)|\nabla h_0),$$

and then setting $\tilde{g}_h := \tilde{g}_{h,b} := g_h + \gamma w_1 - (b|\nabla h_1) - \partial_t h_1$, where w_1 has the same meaning as in step (i) of the proof of [28, Theorem 5.1].

(ii) We first consider the reduced problem (3.3) for the case where $b \equiv b_0$ is constant. The corresponding boundary symbol $s_{b_0}(\lambda, \xi)$ is given by

$$s_{b_0}(\lambda, \xi) = \lambda + \big(\sigma|\xi| - [\![\rho]\!]\gamma_a/|\xi|\big)k(z) + i(b_0|\xi), \tag{3.4}$$

where we use the same notation as in the proof of [28, Theorem 5.1]. Here we remind that k has the following properties: k is holomorphic in $\mathbb{C} \setminus \mathbb{R}_-$ and

$$k(0) = \frac{1}{2(\mu_1 + \mu_2)}, \quad zk(z) \to \frac{1}{\rho_1 + \rho_2} \quad \text{for} \quad |z| \to \infty, \tag{3.5}$$

uniformly in $z \in \bar{\Sigma}_\vartheta$ for $\vartheta \in [0, \pi)$ fixed. In particular there is a constant $N = N(\vartheta)$ such that

$$|k(z)| + |zk(z)| \le N, \quad z \in \Sigma_\vartheta. \tag{3.6}$$

In the following we fix $\beta > 0$. For further analysis it will be convenient to introduce the related extended symbol

$$\tilde{s}(\lambda, \tau, \zeta) := \lambda + \sigma\tau k(z) + i\tau\zeta - [\![\rho]\!]\gamma_a k(z)/\tau, \tag{3.7}$$

where $(\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta$ with η sufficiently small, $z := \lambda/\tau^2$, and $\zeta \in U_{\beta,\delta}$ with $U_{\beta,\delta} := \{\zeta \in \mathbb{C} : |\mathrm{Re}\,\zeta| < \beta + 1, \ |\mathrm{Im}\,\zeta| < \delta\}$ and $\delta \in (0, 1]$. Clearly $\tilde{s}(\lambda, |\xi|, (b_0|\xi/|\xi|)) = s_{b_0}(\lambda, \xi)$ for $(\lambda, \xi) \in \Sigma_\eta \times \dot{\mathbb{R}}^n$.

We are going to show that for every fixed $\beta > 0$ there are positive constants λ_0, δ, $\eta = \eta(\beta)$, and $c_j = c_j(\beta, \lambda_0, \delta, \eta)$ such that

$$c_0\big[|\lambda| + |\tau|\big] \le |\tilde{s}(\lambda, \tau, \zeta)| \le c_1\big[|\lambda| + |\tau|\big], \tag{3.8}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ with $|\lambda| \ge \lambda_0$. The upper estimate is easy to obtain: fixing $\vartheta \in (\pi/2, \pi)$ and $\lambda_0 > 0$, it follows from (3.6) and the identity $k(z)/\tau = zk(z)\tau/\lambda$ that

$$|\tilde{s}(\lambda, \tau, \zeta)| \le |\lambda| + \big(\sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0\big)|\tau| \le c_1\big[|\lambda| + |\tau|\big] \tag{3.9}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$, where $|\lambda| \ge \lambda_0$ and $\eta \in (0, \eta_0)$ with $\eta_0 := (\vartheta - \pi/2)/3$.

In order to obtain a lower estimate we proceed as follows. Suppose first that $\beta, \lambda_0 > 0$ are fixed and η_0 is as above. Then we obtain

$$\begin{aligned} |\tilde{s}(\lambda, \tau, \zeta)| &\ge |\lambda| - \big(\sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0\big)|\tau| \\ &\ge (1/2)|\lambda| + (m/4)|\tau| = c_0(\beta, \lambda_0)[|\lambda| + |\tau|], \end{aligned} \tag{3.10}$$

provided $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$, $\eta \in (0, \eta_0)$, and $|\lambda| \ge \lambda_0$ as well as $|\lambda| \ge m|\tau|$ with

$$(m/4) \ge \sigma N + (\beta + 2) + |[\![\rho]\!]|\gamma_a N/\lambda_0.$$

Next we will derive an estimate from below in case that $|\lambda| \le M|\tau|^2$ with M a positive constant. From (3.5) follows that there are constants $H, L, R > 0$, depending on M, such that

$$L \le \mathrm{Re}\,(\sigma k(z)) \le R, \quad |\mathrm{Im}\,(\sigma k(z))| \le H, \tag{3.11}$$

whenever $(\lambda,\tau)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta$, for $\eta\in(0,\eta_0)$ and $|\lambda|\le M|\tau|^2$, where $z=\lambda/\tau^2$. By choosing δ small enough we obtain from (3.11) and the definition of $U_{\beta,\delta}$

$$0<L-\delta\le \mathrm{Re}\,(\sigma k(z)+i\zeta)\le R+\delta,\quad |\mathrm{Im}\,(\sigma k(z)+i\zeta)|\le H+(\beta+1)$$

provided $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$, $\eta\in(0,\eta_0)$ and $|\lambda|\le M|\tau|^2$, where $z=\lambda/\tau^2$. By choosing η small enough we conclude that there is $\alpha=\alpha(M,\beta,\delta,\eta)\in(0,\pi/2)$ such that

$$\tau(\sigma k(z)+i\zeta)\in\Sigma_\alpha \tag{3.12}$$

whenever $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$ and $|z|\le M$ with $z=\lambda/\tau^2$. We can additionally assume that η is chosen so that $\psi:=\pi/2-\alpha-\eta>0$. This implies

$$\begin{aligned}|\tilde s(\lambda,\tau,\zeta)|&\ge c(\psi)\big[|\lambda|+|\tau|\,|\sigma k(z)+i\zeta|\big]-|\tau||[\![\rho]\!]|\gamma_a N/\lambda_1\\ &\ge c(\psi)\min(1,L-\delta)\big[|\lambda|+|\tau|\big]-|\tau||[\![\rho]\!]|\gamma_a N/\lambda_1\\ &\ge c_0(M,\beta,\lambda_1)\big[|\lambda|+|\tau|\big],\end{aligned}\tag{3.13}$$

provided $(\lambda,\tau,\zeta)\in\Sigma_{\pi/2+\eta}\times\Sigma_\eta\times U_{\beta,\delta}$, $|\lambda|\le M|\tau|^2$ and $|\lambda|\ge\lambda_1$, where λ_1 is chosen big enough.

Noting that the curves $|\lambda|=m|\tau|$ and $|\lambda|=M|\tau|^2$ intersect at $(m/M,m^2/M)$ we obtain (3.8) by choosing $\lambda_0:=\max(\lambda_1,m^2/M)$.

(iii) In the following, we fix $\beta>0$ and we assume that $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$. Let then S_{b_0} be the operator corresponding to the symbol s_{b_0}. It is clear that S_{b_0} is bounded from ${}_0\mathbb{E}_4(a)$ to ${}_0\mathbb{F}_4(a)=:X$ and it remains to prove that it is boundedly invertible. For this we use the $\mathcal{H}^\infty$-calculus and similar arguments as in [27, Section 4] and [28, Section 5]. First we note that D_n admits an $\mathcal{R}$-bounded $\mathcal{H}^\infty$-calculus in X with angle 0; this follows from [14, Theorem 4.11]. Therefore by the estimates obtained in (3.8), the operator family

$$\{(\lambda+D_n^{1/2})\tilde s^{-1}(\lambda,D_n^{1/2},\zeta):(\lambda,\zeta)\in\Sigma_{\pi/2+\eta}\times U_{\beta,\delta},\ |\lambda|\ge\lambda_0\}$$

is $\mathcal{R}$-bounded. Since $G=\partial_t$ is in $\mathcal{H}^\infty(X)$ with angle $\pi/2$, the theorem of Kalton and Weis [22, Theorem 4.4] implies that the operator family

$$\{(G+D_n^{1/2})\tilde s^{-1}(G,D_n^{1/2},\zeta):\zeta\in U_{\beta,\delta}\}$$

is bounded and holomorphic on $U_{\beta,\delta}$. Finally, we employ the Dunford calculus for the bounded linear operator $R_{b_0}:=(b_0|R)$, where R denotes the Riesz operator with symbol $\xi/|\xi|$, $\xi\in\dot{\mathbb{R}}^n$. The operator R_{b_0} is bounded and its spectrum is $\sigma(R_{b_0})=[-|b_0|,|b_0|]$, as e.g. the Mikhlin theorem shows. Since the operator family

$$\{(G+D_n^{1/2})\tilde s^{-1}(G,D_n^{1/2},\zeta):\zeta\in U_{\beta,\delta}\}$$

is bounded and holomorphic in a neighborhood of $\sigma(R_{b_0})$, the classical Dunford calculus shows that the operator

$$(G+D_n^{1/2})\tilde s^{-1}(G,D_n^{1/2},R_{b_0})$$

is bounded in X, uniformly for all $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$. This shows that $S_{b_0}:{}_0\mathbb{E}_4(a)\to{}_0\mathbb{F}_4(a)$ is boundedly invertible, uniformly for all $b_0\in\mathbb{R}^n$ with $|b_0|\le\beta$.

We emphasize that the bound for the operator $S_{b_0}^{-1} : {}_0\mathbb{F}_4(a) \to {}_0\mathbb{E}_4(a)$ depends only on the parameters ρ_j, μ_j, σ, γ_a, p and β, for $|b_0| \le \beta$.

(iv) By means of a perturbation argument the result for constant b can be extended to variable $b = b_0 + b_1(t,x)$. In fact, given $\beta > 0$ there exists a number $\eta > 0$ such that the solution operator S_b^{-1} exists and is bounded uniformly, provided $|b_0| \le \beta$ and $\|b_1\|_\infty + \|b_1\|_{\mathbb{F}_4(a)} \le 2\eta$. This follows easily from the estimate

$$\|(b_1|\nabla h)\|_{{}_0\mathbb{F}_4(a)} \le c_0(\|b_1\|_\infty + \|b_1\|_{\mathbb{F}_4(a)})\|h\|_{{}_0\mathbb{E}_4(a)},$$

see Lemma 5.5(c).

(v) In the general case we use a localization technique, similar to [3, Section 9]. For this purpose we first decompose J into subintervals $J_k = [k\delta, (k+1)\delta]$ of length $\delta > 0$ and solve the problem successively on these subintervals. Since $b \in BUC(J; C_0(\mathbb{R}^n, \mathbb{R}^n))$, given any $\eta > 0$ we may choose $\delta > 0$ and $\varepsilon > 0$ such that

$$|b(t,x) - b(s,y)| \le \eta \quad \text{for all} \quad (t,x), (s,y) \in J \times \mathbb{R}^n$$

with $|t-s| \le \delta$ and $|x-y|_\infty \le \varepsilon$. Let $\{U_j := x_j + (\varepsilon/2)Q : j \in \mathbb{N}\}$ be an enumeration of the open covering $\{(\varepsilon/2)(z/2 + Q) : z \in \mathbb{Z}^n\}$ of $\mathbb{R}^n$, where $Q = (-1,1)^n$. Clearly,

$$|b(t,x) - b(s,y)| \le \eta, \quad s,t \in J_k, \quad x,y \in U_j. \tag{3.14}$$

Let ϕ be a smooth cut-off function with support contained in $(\varepsilon/2)Q$ such that $\phi \equiv 1$ on $(\varepsilon/4)Q$. Define

$$\phi_j := (\tau_{x_j}\phi)\Big(\sum_{k\in\mathbb{N}}(\tau_{x_k}\phi)^2\Big)^{-1/2}, \quad j \in \mathbb{N},$$

where $(\tau_{x_j}\phi)(x) := \phi(x - x_j)$. Consequently, ϕ_j is a smooth cut-off function with $\mathrm{supp}(\phi_j) \subset U_j$ and $\sum_j \phi_j^2 \equiv 1$. For a function space $\mathfrak{F}(J;\mathbb{R}^n) \subset L_p(J; L_p(\mathbb{R}^n))$ we define

$$\begin{aligned} r(h_j) &:= \sum_j \phi_j h_j, && (h_j) \in \mathfrak{F}(J;\mathbb{R}^n)^{\mathbb{N}}, \\ r^c h &:= (\phi_j h), && h \in \mathfrak{F}(J;\mathbb{R}^n). \end{aligned}$$

Similarly as in [3, Section 9] one shows that

$$r \in \mathcal{L}(\ell_p(\mathfrak{F}(J;\mathbb{R}^n)), \mathfrak{F}(J;\mathbb{R}^n)),\ r^c \in \mathcal{L}(\mathfrak{F}(J;\mathbb{R}^n), \ell_p(\mathfrak{F}(J;\mathbb{R}^n))), \quad rr^c = I, \tag{3.15}$$

for $\mathfrak{F}(J;\mathbb{R}^n) \in \{\mathbb{F}_4(a), \mathbb{E}_4(a)\}$.

Let θ be a smooth cut-off function with $\mathrm{supp}(\theta) \subset (\varepsilon/2)Q$ such that $\theta \equiv 1$ on $\mathrm{supp}(\phi)$ and let $\theta_j := \tau_{x_j}\theta$. Define

$$b_{j,k}(t,x) := \theta_j(x)\left(b(t,x) - b(k\delta, x_j)\right), \quad (t,x) \in J \times \mathbb{R}^n.$$

It follows that

$$\|b_{j,k}\|_{BC(J_k\times\mathbb{R}^n)} + \|b_{j,k}\|_{\mathbb{F}_4(J_k)} \le c_0\eta, \quad k = 0,\dots,m, \quad j \in \mathbb{N}, \tag{3.16}$$

provided δ is chosen small enough. Indeed, the estimates for $\|b_{j,k}\|_{BC(J_k,\mathbb{R}^n)}$ follow immediately from (3.14), while the estimates for $\|b_{j,k}\|_{\mathbb{F}_4(J_k)}$ can be shown by approximating b by functions that have better time regularity and by carefully estimating the products $\|\theta_j(b-b(k\delta,x_j))\|_{\mathbb{F}_4(J_k)}$.

We now concentrate on the first interval $J_0=[0,\delta]$. Let $L\in\mathcal{L}({}_0\mathbb{E}_4(a),{}_0\mathbb{F}_4(a))$ denote the operator with symbol $\sigma\tau k(z)$, i.e.

$$L:=(\sigma D_n^{1/2}-[\![\rho]\!]\gamma_a D_n^{-1/2})k(GD_n^{-1}):=L_1+L_2.$$

If follows from (3.16) and step (iii) that the operator

$$S_j:=G+L+(b(0,x_j)+b_{j,0}|\nabla):{}_0\mathbb{E}_4(\delta)\to{}_0\mathbb{F}_4(\delta)$$

is invertible. Moreover, there is a constant C_0, depending only on $\sup_j|b(0,x_j)|$ - and therefore only on $\|b\|_{BC(J\times\mathbb{R}^n)}$ - such that $\|S_j^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4(\delta),{}_0\mathbb{E}_4(\delta))}\le C_0,\ j\in\mathbb{N}$.

(vi) Suppose that for a given $g\in{}_0\mathbb{F}_4(\delta)$ we have a solution $h\in{}_0\mathbb{E}_4(\delta)$ of

$$Gh+Lh+(b|\nabla)h=g.$$

Multiplying this equation by ϕ_j, using that $b\,\partial^\alpha\phi_j=(b(0,x_j)+b_{j,0})\,\partial^\alpha\phi_j$ and $rr^c=I$ this yields

$$S_j\phi_jh-[L,\phi_j]h-(b|\nabla\phi_j)h=(S_j-[L,\phi_j]r-(b|\nabla\phi_j)r)\,r^ch=r^cg,$$

where $[\cdot,\cdot]$ denotes the commutator. We now interpret this equation as an equation in $\ell_p({}_0\mathbb{F}_4(\delta))$. It follows from step (iv) that $(S_j)\in\mathrm{Isom}(\ell_p({}_0\mathbb{E}_4(\delta)),\ell_p({}_0\mathbb{F}_4(\delta)))$ and

$$\|(S_j^{-1})\|_{\mathcal{L}(\ell_p({}_0\mathbb{F}_4(\delta)),\ell_p({}_0\mathbb{E}_4(\delta)))}\le C_0. \tag{3.17}$$

We shall show below in step (vi) that the commutators satisfy

$$([L,\phi_j]+(b|\nabla\phi_j))\in\mathcal{L}({}_0\mathbb{F}(a),\ell_p({}_0\mathbb{F}_4(a))). \tag{3.18}$$

Assuming this property, it follows from (3.15) that

$$\|(([L,\phi_j]+(b|\nabla\phi_j))r(h_j))\|_{\ell_p({}_0\mathbb{F}_4(\delta))}\le C\|(h_j)\|_{\ell_p({}_0\mathbb{F}_4(\delta))}\le C\delta^\alpha\|(h_j)\|_{\ell_p({}_0\mathbb{E}_4(\delta))}$$

for some α depending only on p and n. Therefore, choosing δ small enough we can conclude that $(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)\in\mathrm{Isom}(\ell_p({}_0\mathbb{E}_4(\delta)),\ell_p({}_0\mathbb{F}_4(\delta)))$ with

$$\|\,(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)^{-1}\,\|\le 2C_0.$$

Let $T_b:=r\,(S_j-([L,\phi_j]+(b|\nabla\phi_j))r)^{-1}\,r^c$. Then $T_b\in\mathcal{L}({}_0\mathbb{F}_4(\delta),{}_0\mathbb{E}_4(\delta))$ is a left inverse of $S_b:=G+L+(b|\nabla)$. Hence

$$\|h\|_{{}_0\mathbb{E}_4(\delta)}=\|T_bS_bh\|_{{}_0\mathbb{E}_4(\delta)}\le 2C_0\|r\|\,\|r^c\|\,\|S_bh\|_{{}_0\mathbb{E}_4(\delta)},\quad h\in{}_0\mathbb{E}_4(\delta). \tag{3.19}$$

Replacing b by ρb, $\rho\in[0,1]$, we have a continuous family $\{S_{\rho b}\}$ of operators $S_{\rho b}$ which all satisfy the a-priori estimate (3.19) uniformly in $\rho\in[0,1]$. Since S_0 is an isomorphism, we can infer from a homotopy argument that S_b is an isomorphism as well. Repeating successively these arguments for the intervals J_k, including the reduction from step (i), proves the assertion of the corollary.

(vii) We still have to verify the estimate in (3.18). Since the covering $\{U_j: j\in\mathbb{N}\}$ has finite multiplicity, one obtains

$$\|((\partial^\alpha\phi_j)g)\|_{\ell_p({}_0\mathbb{F}_4(a))}\le C(\alpha)\|g\|_{{}_0\mathbb{F}_4(a)},\quad g\in{}_0\mathbb{F}_4(a). \tag{3.20}$$

This together with Proposition 5.5(b) shows that

$$\|(b|\nabla\phi_j)h\|_{\ell_p({}_0\mathbb{F}_4(a)} \le C\|bh\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|b\|_\infty + \|b\|_{\mathbb{F}_4(a)})\|h\|_{{}_0\mathbb{F}_4(a)}.$$

The estimates for the commutators $[L,\phi_j]$ are more involved. The operator $A = GD_n^{-1}$ with canonical domain is sectorial and admits a bounded $\mathcal{H}^\infty$-calculus with angle $\pi/2$ in ${}_0H^s_p(J;K^r_p(\mathbb{R}^n))$, for $K \in \{H,W\}$, and also in ${}_0W^s_p(J;K^r_p(\mathbb{R}^n))$ by real interpolation. Hence fixing $\theta \in (0,\pi/2)$, the following resolvent estimate holds in these spaces:

$$\|z(z-A)^{-1}\| \le M, \quad \text{for all } z \in -\Sigma_\theta.$$

The function $k(z)$ is holomorphic in $\mathbb{C}\setminus(-\infty,-2\delta_0]$ for some $\delta_0 > 0$ and behaves like $1/z$ as $|z| \to \infty$. Choose the contour

$$\Gamma = (\infty,\delta_0]e^{i\psi} \cup \delta_0 e^{i[\psi,2\pi-\psi]} \cup [\delta_0,\infty)e^{-i\psi},$$

where $\pi > \psi > \pi-\theta$. Then we have the Dunford integral

$$k(A) = \frac{1}{2\pi i}\int_\Gamma k(z)(z-A)^{-1}dz,$$

which is absolutely convergent. This shows that $k(A)$ is bounded, as is $Ak(A)$ thanks to $A \in \mathcal{H}^\infty$, thus $A^{1/2}k(A)$ is bounded as well. Therefore the identity $k(A)D_n^{-1/2} = G^{-1/2}A^{1/2}k(A)$ shows that L_2 is bounded since $G^{-1/2}$ is, and (3.18) follows for $[L_2,\phi_j]$. For the commutator $[L_1,\phi_j]$ we obtain

$$[L_1,\phi_j] = \sigma[k(A)D_n^{1/2},\phi_j] = \sigma[k(A),\phi_j]D_n^{1/2} + \sigma k(A)[D_n^{1/2},\phi_j].$$

Using the Dunford integral for $k(A)$ this yields

$$[k(A),\phi_j] = \frac{1}{2\pi i}\int_\Gamma k(z)[(z-A)^{-1},\phi_j]dz = \frac{1}{2\pi i}\int_\Gamma k(z)(z-A)^{-1}[A,\phi_j](z-A)^{-1}dz,$$

hence with

$$\begin{aligned}[A,\phi_j] &= GD_n^{-1}[\phi_j,D_n]D_n^{-1} = A(\Delta\phi_j + 2(\nabla\phi_j|\nabla))D_n^{-1}\\ &= A(\Delta\phi_j D_n^{-1} + 2i(\nabla\phi_j|R)D_n^{-1/2}),\end{aligned}$$

we have

$$[k(A),\phi_j]D_n^{1/2} = \frac{1}{2\pi i}\int_\Gamma k(z)A(z-A)^{-1}\{-\Delta\phi_j G^{-1/2}A^{1/2} + 2i(\nabla\phi_j|R)\}(z-A)^{-1}dz.$$

Let $h \in {}_0\mathbb{F}_4$ be given. Then we obtain from

$$\|k(z)A(z-A)^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4)} \le C/|z|, \quad \|A^{1/2}(z-A)^{-1}\|_{\mathcal{L}({}_0\mathbb{F}_4)} \le C/|z|^{1/2}, \quad z \in \Gamma,$$

from (3.20), and from Minkowski's inequality for integrals

$$\begin{aligned}&\Big\|\Big(\int_\Gamma k(z)A(z-A)^{-1}\Delta\phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h\,dz\Big)\Big\|_{\ell_p({}_0\mathbb{F}_4)}\\ &\le C\int_\Gamma \frac{1}{|z|}\|(\Delta\phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h)\|_{\ell_p({}_0\mathbb{F}_4)}\,|dz|\\ &\le C\int_\Gamma \frac{1}{|z|^{3/2}}\|h\|_{{}_0\mathbb{F}_4}\,|dz| \le C\|h\|_{{}_0\mathbb{F}_4}\end{aligned}$$

where we also used that $G^{-1/2}$ is bounded on compact intervals. In the same way we can estimate the second term in the integral representation of $[k(A),\phi_j]D^{1/2}$, this time using the fact that R is bounded.

To estimate the commutators $[D_n^{1/2},\phi_j]$ in ${}_0\mathbb{F}_4$ note that

$$\begin{aligned}(D_n)^{1/2} = D_n(D_n)^{-1/2} &= \frac{1}{\sqrt{\pi}} D_n \int_0^\infty e^{-D_n t} t^{-\frac12}\,dt \\ &= \frac{1}{\sqrt{\pi}}\Big(D_n \int_0^1 e^{-D_n t} t^{-\frac12}\,dt + D_n \int_1^\infty e^{-D_n t} t^{-\frac12}\,dt\Big) \\ &=: \frac{1}{\sqrt{\pi}}(T_1+T_2),\end{aligned}$$

with $e^{-D_n t}$ denoting the bounded analytic semigroup generated by the Laplacian in $H_p^s(\mathbb{R}^n)$ which extends by real interpolation to $W_p^s(\mathbb{R}^n)$, and then canonically to ${}_0\mathbb{F}_4$. Thus by (3.20) there is a constant $C>0$ such that for $h\in {}_0\mathbb{F}_4$ we have

$$\begin{aligned}\|(\phi_j T_2 h)\|_{\ell_p({}_0\mathbb{F}_4)} &= \|(\phi_j \int_1^\infty D_n e^{-D_n t} t^{-\frac12} h\,dt)\|_{\ell_p({}_0\mathbb{F}_4)} \\ &\le C\|\int_1^\infty D_n e^{-D_n t} t^{-\frac12} h\,dt\|_{{}_0\mathbb{F}_4} \\ &\le C\int_1^\infty t^{-\frac32}\,dt\,\|h\|_{{}_0\mathbb{F}_4} \le C\|h\|_{{}_0\mathbb{F}_4}, \\ \|(T_2\phi_j h)\|_{\ell_p({}_0\mathbb{F}_4)} &= \|\int_1^\infty D_n e^{-D_n t} t^{-\frac12}(\phi_j h)\,dt\|_{\ell_p({}_0\mathbb{F}_4)} \\ &\le C\|(\phi_j h)\|_{\ell_p({}_0\mathbb{F}_4)} \le C\|h\|_{{}_0\mathbb{F}_4}.\end{aligned}$$

Hence $\|([\phi_j,T_2]h)\|_{\ell_p({}_0\mathbb{F}_4)} \le C\|h\|_{{}_0\mathbb{F}_4}$.

We consider next the commutator $[T_1,\phi_j]$. Let $k_t(x) = (2\pi t)^{-n/2}\exp(-|x|^2/4t)$ denote the Gaussian kernel. Then for fixed $t>0$, the operator $D_n e^{-D_n t}$ is the convolution with kernel $-\Delta k_t(x)$, which is of class C^∞. It is not difficult to see that there are constants $C,c>0$ such that

$$|\Delta k_t(x)| \le Ct^{-(n+2)/2}\, e^{-c|x|^2/t}, \quad x\in\mathbb{R}^n,\ t>0. \tag{3.21}$$

Choosing a cut-off function $\chi\in C^\infty(\mathbb{R}^n)$ with $\chi\equiv 1$ in $B_\rho(0)$, $\mathrm{supp}\,(\chi)\subset B_{2\rho}(0)$ and $0\le\chi\le 1$ elsewhere, we set

$$-\Delta k_t(x) = -(1-\chi(x))\Delta k_t(x) - \chi(x)\Delta k_t(x) =: k_{3,t}(x)+k_{4,t}(x), \quad x\in\mathbb{R}^n,\ t>0,$$

and we denote by T_l the convolution operators with kernels $\int_0^1 k_{l,t}t^{-1/2}dt$, $l=3,4$. For the kernel of T_3 we obtain from (3.21) the estimate

$$\begin{aligned}|\int_0^1 k_{3,t}(x)t^{-1/2}\,dt| &\le C\int_0^1 e^{-c|x|^2/t} t^{-(n+3)/2}\,dt \\ &\le Ce^{-c_1|x|^2}\int_1^\infty e^{-c_2|x|^2 s} s^{(n-1)/2} ds \le Ce^{-c_1|x|^2},\end{aligned}$$

as $k_{3,t}(x) = 0$ for $|x| \le \rho$. Thus this kernel is in $L_1(\mathbb{R}^n)$ and hence we may estimate the commutator $[T_3, \phi_j]$ in the same way as $[T_2, \phi_j]$.

For the remaining commutator $[T_4, \phi_j]$ note that

$$\partial^\alpha [T_4, \phi_j] = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} [T_4, \partial^\beta \phi_j] \partial^{\alpha-\beta}.$$

This shows that it is enough to estimate the commutator $[T_4, \phi_j]$ in $L_p(\mathbb{R}^n)$, as it then extends to $H_p^m(\mathbb{R}^n)$ and by interpolation to $W_p^s(\mathbb{R}^n)$, and then canonically to ${}_0\mathbb{F}_4$. Next we observe that for $x, y \in \mathbb{R}^n$

$$\partial^\alpha \phi_j(y) - \partial^\alpha \phi_j(x) = \partial^\alpha \phi_j'(x)(y-x) + r_{j,\alpha}(x,y),$$

where $|r_{j,\alpha}(x,y)| \le C|x-y|^2$, with some constant C independent of j and $|\alpha| \le 2$. Therefore

$$\begin{aligned}
[T_4, \phi_j]h(x) &= \int_0^1 \int_{\mathbb{R}^n} (\phi_j(y) - \phi_j(x)) k_{4,t}(x-y) h(y)\, dy\, t^{-\frac{1}{2}}\, dt \\
&= -\phi_j'(x) \int_{\mathbb{R}^n} \int_0^1 (y-x) k_{4,t}(x-y) t^{-\frac{1}{2}}\, dt\, h(y)\, dy + \\
&\qquad + \int_{\mathbb{R}^n} \int_0^1 r_j(x,y) k_{4,t}(x-y) t^{-\frac{1}{2}}\, dt\, h(y)\, dy \\
&=: T_{5,j}h(x) + T_{6,j}h(x).
\end{aligned}$$

We observe that the support of the kernel $k_{4,t}$ is contained in $B_{2\rho}(0)$, and consequently we may replace h by $\psi_j h$, where ψ_j is a cut-off function which equals 1 on $\mathrm{supp}(\phi_j) + B_{2\rho}(0)$. In the following we fix a smooth cut-off function ψ which equals 1 on $\mathrm{supp}(\phi) + B_{2\rho}(0)$ and then set $\psi_j := \tau_{x_j}\psi$. We then have

$$\|(T_{l,j}h)\|_{\ell_p(L_p)} = \|(T_{l,j}\psi_j h)\|_{\ell_p(L_p)} \le \sup_k \|T_{l,k}\|_{\mathcal{L}(L_p)} \|(\psi_j h)\|_{\ell_p(L_p)} \le C\|h\|_{L_p},$$

provided we can show that the operators $T_{l,k}$ are L_p-bounded with bound independent of $k \in \mathbb{N}$ for $l = 5, 6$.

The operators $T_{l,j}$ satisfy

$$\begin{aligned}
T_{5,j}h = \phi_j'(q * h) \quad &\text{with} \quad q(x) = \chi(x) \int_0^1 x \Delta k_t(x) t^{-\frac{1}{2}}\, dt,\ x \in \mathbb{R}^n \\
|T_{6,j}h| \le r * |h| \quad &\text{with} \quad r(x) = C\chi(x) \int_0^1 |x|^2 |\Delta k_t(x)| t^{-\frac{1}{2}}\, dt,\ x \in \mathbb{R}^n.
\end{aligned}$$

The Fourier transform of q is given by $\widehat{q}(\xi) = C\widehat{\chi} * \int_0^1 \nabla_\xi (|\xi|^2 e^{-t|\xi|^2}) t^{-1/2} dt$ and we verify that

$$\sup_{\alpha \le (1,\dots,1)} \sup_{\xi \in \mathbb{R}^n} |\xi|^{|\alpha|} |\partial^\alpha \widehat{q}(\xi)| \le M$$

for some $M < \infty$. It thus follows from Mikhlin's multiplier theorem that

$$\|T_{5,j}h\|_{L_p} \le C\|\phi_j'\|_\infty \|h\|_{L_p} \le C\|h\|_{L_p}.$$

Finally, in order to estimate $T_{6,j}$ we infer from (3.21) that

$$r(x) \le C \int_0^1 |x|^2 e^{-c|x|^2/t}\, t^{-\frac{n+3}{2}}\, dt \le Ce^{-c_1|x|^2}|x|^{-(n-1)} \int_1^\infty e^{-c_2 s} s^{(n-1)/2} ds$$

for $x \in \mathbb{R}^n$. It follows that $r \in L_1(\mathbb{R}^n)$ which implies by Young's inequality $\|T_{6,j}h\|_p \le C\|h\|_p$ with a uniform constant C. □

Remarks 3.2. (a) We mention that the proof for the estimate of $[D_n^{1/2}, \phi_j]$ follows the ideas of [15, Lemma 6.4].

(b) If $\rho_2 \le \rho_1$, i.e. the light fluid lies above the heavy one, then the estimate (3.8) can be improved in the following sense: for every $\beta > 0$ and $\lambda_0 > 0$ there are positive constants δ, $\eta = \eta(\beta)$ and $c_j = c_j(\beta, \lambda_0, \delta, \eta)$ such that

$$c_0\big[|\lambda| + |\tau|\big] \le \tilde{s}(\lambda, \tau, \zeta) \le c_1\big[|\lambda| + |\tau|\big] \tag{3.22}$$

for all $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ and $|\lambda| \ge \lambda_0$. For this we observe that estimates (3.9) and (3.10) certainly also hold in case that $\rho_2 \le \rho_1$. On the other hand, given $M > 0$ we conclude as in (3.11) that $L \le \mathrm{Re}\,((\rho_1 - \rho_2)\gamma_a k(z)) \le R$ and $|\mathrm{Im}\,((\rho_1 - \rho_2)\gamma_a k(z))| \le H$, with appropriate positive constants L, R, H. This shows that there exists $\alpha = \alpha(M, \eta) \in (0, \pi/2)$ such that

$$(\rho_1 - \rho_2)\gamma_a k(z)/\tau \in \Sigma_\alpha, \quad (\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta, \quad |z| \le M \tag{3.23}$$

with $\eta \in (0, \eta_0)$ chosen small enough, where we can assume that α coincides with the angle in (3.12). Combining (3.12) and (3.23) yields

$$\begin{aligned}
|\tilde{s}(\lambda, \tau, \zeta)| &\ge c(\psi)\big[|\lambda| + |\tau(\sigma k(z) + i\zeta) + (\rho_1 - \rho_2)\gamma_a k(z)/\tau|\big] \\
&\ge c(\psi)c(\alpha)\big[|\lambda| + |\tau(\sigma k(z) + i\zeta)| + |(\rho_1 - \rho_2)\gamma_a k(z)/\tau|\big] \\
&\ge c_0(M, \beta, \delta, \eta)\big[|\lambda| + |\tau|\big]
\end{aligned}$$

provided $(\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta,\delta}$ and $|\lambda| \le M|\tau|^2$. Noting again that the curves $|\lambda| = m|\tau|$ and $|\lambda| = M|\tau|^2$ intersect at $(m/M, m^2/M)$ we obtain (3.22) by choosing M big enough.

(c) If $\rho_2 \le \rho_1$ we can conclude from the lower estimate in (3.22) that the function $\tilde{s}$ does not have zeros in $\overline{\Sigma}_{\pi/2} \times \mathbb{R}_+ \times [-\beta, \beta]$. This holds in particular true for the symbol $s(\lambda, \tau) := \tilde{s}(\lambda, \tau, 0)$, indicating that there are no instabilities in case that the light fluid lies on top of the heavy one.

(d) If $\rho_2 > \rho_1$ then it is shown in [29] that the symbol s has for each $\tau \in (0, \tau_*)$ with $\tau_* := ((\rho_2 - \rho_1)\gamma_a/\sigma)^{1/2}$ a zero $\lambda = \lambda(\tau) > 0$, pertinent to the Rayleigh-Taylor instability.

(e) Further mapping properties of the boundary symbol $s(\lambda, \tau) := \tilde{s}(\lambda, \tau, 0)$ and the associated operator S in case that $\gamma_a = 0$ have been derived in [27]. In particular, we have investigated the singularities and zeros of s, and we have studied the mapping properties of S in case of low and high frequencies, respectively.

4. The nonlinear problem

In this section we prove existence und uniqueness of solutions for the nonlinear problem (2.1), and we show additionally that solutions immediately regularize and are real analytic in space and time. In order to facilitate this task, we first introduce some notation. We set

$$\begin{aligned}
\mathbb{E}_1(a) &:= \{u \in H^1_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) : [\![u]\!] = 0\},\\
\mathbb{E}_2(a) &:= L_p(J; \dot{H}^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{E}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}_4(a) &:= W^{2-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap H^1_p(J; W^{2-1/p}_p(\mathbb{R}^n))\\
&\qquad \cap W^{1/2-1/2p}_p(J; H^2_p(\mathbb{R}^n)) \cap L_p(J; W^{3-1/p}_p(\mathbb{R}^n)),\\
\mathbb{E}(a) &:= \{(u, \pi, q, h) \in \mathbb{E}_1(a) \times \mathbb{E}_2(a) \times \mathbb{E}_3(a) \times \mathbb{E}_4(a) : [\![\pi]\!] = q\}.
\end{aligned} \tag{4.1}$$

The space $\mathbb{E}(a)$ is given the natural norm

$$\|(u, \pi, q, h)\|_{\mathbb{E}(a)} = \|u\|_{\mathbb{E}_1(a)} + \|\pi\|_{\mathbb{E}_2(a)} + \|q\|_{\mathbb{E}_3(a)} + \|h\|_{\mathbb{E}_4(a)}$$

which turns it into a Banach space. Moreover, we set

$$\begin{aligned}
\mathbb{F}_1(a) &:= L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})),\\
\mathbb{F}_2(a) &:= H^1_p(J; H^{-1}_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\dot{\mathbb{R}}^{n+1})),\\
\mathbb{F}_3(a) &:= W^{1/2-1/2p}_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W^{1-1/p}_p(\mathbb{R}^n, \mathbb{R}^{n+1})),\\
\mathbb{F}_4(a) &:= W^{1-1/2p}_p(J; L_p(\mathbb{R}^n)) \cap L_p(J; W^{2-1/p}_p(\mathbb{R}^n)),\\
\mathbb{F}(a) &:= \mathbb{F}_1(a) \times \mathbb{F}_2(a) \times \mathbb{F}_3(a) \times \mathbb{F}_4(a).
\end{aligned} \tag{4.2}$$

The generic elements of $\mathbb{F}(a)$ are the functions (f, f_d, g, g_h).

Let $b \in \mathbb{F}_4(a)^n$ be a given function. Then we define the nonlinear mapping

$$N_b(u, \pi, q, h) := \big(F(u, \pi, h), F_d(u, h), G(u, q, h), (b - \gamma v|\nabla h)\big) \tag{4.3}$$

for $(u, \pi, q, h) \in \mathbb{E}(a)$, where, as before, $u = (v, w)$, $F = (F_v, F_w)$ and $G = (G_v, G_w)$. We will now study the mapping properties of N_b and we will derive estimates for the Fréchet derivative of N_b.

Proposition 4.1. *Suppose $p > n + 3$ and $b \in \mathbb{F}_4(a)^n$. Then*

$$N_b \in C^\omega(\mathbb{E}(a), \mathbb{F}(a)), \quad a > 0. \tag{4.4}$$

Let $DN_b(u, \pi, q, h)$ denote the Fréchet derivative of N_b at $(u, \pi, q, h) \in \mathbb{E}(a)$. Then $DN_b(u, \pi, q, h) \in \mathcal{L}({}_0\mathbb{E}(a), {}_0\mathbb{F}(a))$, and for any number $a_0 > 0$ there is a positive *constant* $M_0 = M_0(a_0, p)$ *such that*

$$
\begin{aligned}
&\|DN_b(u,\pi,q,h)\|_{\mathcal{L}({}_0\mathbb{E}(a),\,{}_0\mathbb{F}(a))}\\
&\le M_0\big[\|b-\gamma v\|_{BC(J;BC)\cap\mathbb{F}_4(a)} + \|(u,\pi,q,h)\|_{\mathbb{E}(a)}\big]\\
&+ M_0\big[\big(\|\nabla h\|_{BC(J;BC^1)} + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{BC(J;BC)}\big)\|u\|_{\mathbb{E}_1(a)}\big]\\
&+ M_0\big[P(\|\nabla h\|_{BC(J;BC)})\|\nabla h\|_{BC(J;BC)} + Q\big(\|\nabla h\|_{BC(J;BC^1)}, \|h\|_{\mathbb{E}_4(a)}\big)\|h\|_{\mathbb{E}_4(a)}\big]
\end{aligned}
$$

for all $(u,\pi,q,h)\in\mathbb{E}(a)$ *and all* $a\in(0,a_0]$. *Here,* P *and* Q *are fixed polynomials with coefficients equal to one.*

Proof. The proof of the proposition is relegated to the end of the appendix. □

Given $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$ we define

$$\Theta_{h_0}(x,y) := (x, y + h_0(x)), \quad (x,y)\in\mathbb{R}^n\times\mathbb{R}. \tag{4.5}$$

Letting $\Omega_{h_0,i} := \{(x,y)\in\mathbb{R}^n\times\mathbb{R} : (-1)^i(y-(h_0(x))>0\}$ and $\Omega_{h_0} := \Omega_{h_0,1}\cup\Omega_{h_0,2}$ we obtain from Sobolev's embedding theorem that

$$\Theta_{h_0}\in \mathrm{Diff}^2(\dot{\mathbb{R}}^{n+1},\Omega_{h_0})\cap \mathrm{Diff}^2(\mathbb{R}^{n+1}_-,\Omega_{h_0,1})\cap \mathrm{Diff}^2(\mathbb{R}^{n+1}_+,\Omega_{h_0,2}),$$

i.e., Θ_{h_0} yields a C^2-diffeomrphism between the indicated domains. The inverse transformation obviously is given by $\Theta_{h_0}^{-1}(x,y) = (x, y-h_0(x))$. It then follows from the chain rule and the transformation rule for integrals that

$$\Theta_{h_0}^*\in \mathrm{Isom}(H_p^k(\dot{\mathbb{R}}^{n+1}), H_p^k(\Omega_{h_0})), \quad [\Theta_{h_0}^*]^{-1} = \Theta_*^{h_0}, \quad k=0,1,2,$$

where we use the notation

$$
\begin{aligned}
\Theta_{h_0}^* u &:= u\circ\Theta_{h_0}, \quad u:\Omega_{h_0}\to\mathbb{R}^m,\\
\Theta_*^{h_0} v &:= v\circ\Theta_{h_0}^{-1}, \quad v:\dot{\mathbb{R}}^{n+1}\to\mathbb{R}^m.
\end{aligned}
$$

We are now ready to prove our main result of this section.

Theorem 4.2. (*Existence of solutions for the nonlinear problem* (2.1)).

(a) *For every* $\beta>0$ *there exists a constant* $\eta=\eta(\beta)>0$ *such that for all initial values*

$$(u_0,h_0)\in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+1})\times W_p^{3-2/p}(\mathbb{R}^n) \quad \textit{with} \quad [\![u_0]\!]=0,$$

satisfying the compatibility conditions

$$[\![\mu D(\Theta_*^{h_0}u_0)\nu_0 - \mu(\nu_0|D(\Theta_*^{h_0}u_0)\nu_0)\nu_0]\!]=0, \quad div(\Theta_*^{h_0}u_0)=0, \tag{4.6}$$

and the smallness-boundedness condition

$$\|\nabla h_0\|_\infty\le\eta, \quad \|u_0\|_\infty\le\beta, \tag{4.7}$$

there is a number $t_0=t_0(u_0,h_0)$ *such that the nonlinear problem* (2.1) *admits a unique solution* $(u,\pi,[\![\pi]\!],h)\in\mathbb{E}_1(t_0)$.

(b) *The solution has the additional regularity properties*

$$(u,\pi)\in C^\omega((0,t_0)\times\dot{\mathbb{R}}^{n+1},\mathbb{R}^{n+2}), \quad [\![\pi]\!], h\in C^\omega((0,t_0)\times\mathbb{R}^n). \tag{4.8}$$

In particular, $\mathcal{M}=\bigcup_{t\in(0,t_0)}\big(\{t\}\times\Gamma(t)\big)$ *is a real analytic manifold.*

Proof. The proof of this result proceeds in a similar way as the proof of Theorem 6.3 in [28].

For a given function $b \in \mathbb{F}_4(a)^n$ we consider the nonlinear problem

$$\left\{\begin{aligned} \rho\partial_t u - \mu\Delta u + \nabla\pi &= F(u,\pi,h) && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\ \operatorname{div} u &= F_d(u,h) && \text{in} \quad \dot{\mathbb{R}}^{n+1}\\ -[\![\mu\partial_y v]\!] - [\![\mu\nabla_x w]\!] &= G_v(u,[\![\pi]\!],h) && \text{on} \quad \mathbb{R}^n\\ -2[\![\mu\partial_y w]\!] + [\![\pi]\!] - \sigma\Delta h &= G_w(u,h) && \text{on} \quad \mathbb{R}^n\\ [\![u]\!] &= 0 && \text{on} \quad \mathbb{R}^n\\ \partial_t h - \gamma w + (b|\nabla h) &= (b-\gamma v|\nabla h) && \text{on} \quad \mathbb{R}^n\\ u(0) = u_0,\ h(0) &= h_0, \end{aligned}\right. \tag{4.9}$$

which clearly is equivalent to (2.1).

In order to economize our notation we set $z := (u,\pi,q,h)$ for $(u,\pi,q,h) \in \mathbb{E}(a)$. With this notation, the nonlinear problem (2.1) can be restated as

$$L_b z = N_b(z), \quad (u(0),h(0)) = (u_0,h_0), \tag{4.10}$$

where L_b denotes the linear operator on the left-hand side of (4.9), and N_b correspondently denotes the nonlinear mapping on the right-hand site of (4.9).

It is convenient to first introduce an auxiliary function $z^* = z^*_b \in \mathbb{E}(a)$ which resolves the compatibility conditions and the initial conditions in (4.10), and then to solve the resulting reduced problem

$$L_b z = N_b(z+z^*) - L_b z^* =: K_b(z), \quad z \in {}_0\mathbb{E}(a), \tag{4.11}$$

by means of a fixed point argument.

(i) Suppose that (u_0,h_0) satisfies the (first) compatibility condition in (4.6), and let

$$[\![\pi_0]\!] := \theta^*_{h_0}\{[\![\mu(\nu_0|D(\Theta^{h_0}_* u_0)\nu_0)]\!] + \sigma\kappa\},$$

where $\theta_{h_0} := \Theta_{h_0}|_{\mathbb{R}^{n+1}\times\{0\}}$. Here we observe that $\theta^*_{h_0}[\![\omega]\!] = [\![\Theta^*_{h_0}\omega]\!]$ for any function $\omega : \Omega_{h_0} \to \mathbb{R}^m$ which has one-sided limits. It is then clear from the definition in (2.3)–(2.4) that the following compatibility conditions hold:

$$\begin{aligned} -[\![\mu\partial_y v_0]\!] - [\![\mu\nabla_x w_0]\!] &= G_v(u_0,[\![\pi_0]\!],h_0) && \text{on} \quad \mathbb{R}^n\\ -2[\![\mu\partial_y w_0]\!] + [\![\pi_0]\!] - \sigma\Delta h_0 &= G_w(u_0,h_0) && \text{on} \quad \mathbb{R}^n \end{aligned} \tag{4.12}$$

where, as before, $u_0 = (v_0,w_0)$. Next we introduce special functions $(0,f^*_d,g^*,g^*_h) \in \mathbb{F}(a)$ which resolve the necessary compatibility conditions. First we set

$$c^*(t) := \begin{cases} \mathcal{R}_+ e^{-tD_{n+1}}\mathcal{E}_+(v_0|\nabla h_0) & \text{in} \quad \mathbb{R}^{n+1}_+,\\ \mathcal{R}_- e^{-tD_{n+1}}\mathcal{E}_-(v_0|\nabla h_0) & \text{in} \quad \mathbb{R}^{n+1}_-, \end{cases} \tag{4.13}$$

where $\mathcal{E}_\pm \in \mathcal{L}(W^{2-2/p}_p(\mathbb{R}^{n+1}_\pm), W^{2-2/p}_p(\mathbb{R}^{n+1}))$ is an appropriate extension operator and $\mathcal{R}_\pm$ is the restriction operator. Due to $(v_0|\nabla h_0) \in W^{2-2/p}_p(\dot{\mathbb{R}}^{n+1})$ we obtain

$$c^* \in H^1_p(J; L_p(\mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\dot{\mathbb{R}}^{n+1})).$$

Consequently,

$$f_d^* := \partial_y c^* \in \mathbb{F}_2(a) \quad \text{and} \quad f_d^*(0) = F_d(v_0, h_0). \tag{4.14}$$

Next we set

$$g^*(t) := e^{-D_n t} G(u_0, [\![\pi_0]\!], h_0). \quad g_h^*(t) := e^{-D_n t}(b(0) - \gamma v_0 | \nabla h_0). \tag{4.15}$$

It then follows from (4.14) and [19, Lemma 8.2] that $(0, f_d^*, g^*, g_h^*) \in \mathbb{F}(a)$. (4.12) and the second condition in (4.6) show that the necessary compatibility conditions of Theorem 3.1 are satisfied and we can conclude that the linear problem

$$L_b z^* = (0, f_d^*, g^*, g_h^*), \quad (u^*(0), h^*(0)) = (u_0, h_0), \tag{4.16}$$

has a unique solution $z^* = z_b^* \in \mathbb{E}(a)$. With the auxiliary function z^* now determined, we can focus on the reduced equation (4.11), which can be converted into the fixed point equation

$$z = L_b^{-1} K_b(z), \quad z \in {}_0\mathbb{E}(a). \tag{4.17}$$

Due to the choice of (f_d^*, g^*, g_h^*) we have $K_b(z) \in {}_0\mathbb{F}(a)$ for any $z \in {}_0\mathbb{E}(a)$, and it follows from Proposition 4.1 that

$$K_b \in C^\omega({}_0\mathbb{E}(a), {}_0\mathbb{F}(a)).$$

Consequently, $L_b^{-1} K_b : {}_0\mathbb{E}(a) \to {}_0\mathbb{E}(a)$ is well defined and smooth.

(ii) An inspection of the proof of Theorem 3.1 shows that given $\beta > 0$ we can find a positive number $\delta_0 = \delta_0(b)$ such that

$$L_b^{-1} \in \mathcal{L}({}_0\mathbb{F}(a), {}_0\mathbb{E}(a)), \quad \|L_b^{-1}\|_{\mathcal{L}({}_0\mathbb{F}(a), {}_0\mathbb{E}(a))} \le M, \quad a \in [0, \delta_0], \tag{4.18}$$

whenever $b \in \mathbb{F}_4(a)^n$ and $\|b\|_{BC[0,a], BC(\mathbb{R}^n))} \le \beta$. It should be pointed out that the bound M is universal for all functions $b \in \mathbb{F}_4(a)^n$ with $\|b\|_\infty \le \beta$, whereas the number $\delta_0 = \delta(b)$ may depend on b.

(iii) We will now fix a pair of initial values $(u_0, h_0) \in W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}) \times W_p^{3-2/p}(\mathbb{R}^n)$ satisfying (4.6) and (4.7) with

$$\eta := 1/(16 M_0 M), \tag{4.19}$$

where the constants M_0 and M are given in (4.1) and (4.18), respectively. We choose

$$b(t) := e^{-D_n t} \gamma v_0, \quad t \ge 0. \tag{4.20}$$

Then $b \in \mathbb{F}_4(a)^n$ and $\|b\|_{BC([0,a]; BC(\mathbb{R}^n))} \le \|\gamma v_0\|_{BC(\mathbb{R}^n)} \le \beta$ for any $a > 0$, as $\{e^{-D_n t} : t \ge 0\}$ is a contraction semigroup on $BUC(\mathbb{R}^n)$. Hence the estimate (4.18) holds true for this (and any other choice) of initial values. It should be pointed out once more that the bound M is universal for all initial values u_0 with $\|v_0\|_\infty \le \beta$ - and hence for $b(t) := e^{-D_n t} \gamma v_0$ - whereas the number δ_0 may depend on γv_0.

We note in passing that $g_h^* = 0$ for this particular choice of the function b. Without loss of generality we can assume that $M_0, M \geq 1$. We shall show that $L_b^{-1}K_b$ is a contraction on a properly defined subset of ${}_0\mathbb{E}(a)$ for $a \in (0, \delta_0]$ chosen sufficiently small. For $r > 0$ and $a \in (0, \delta_0]$ we set

$$ {}_0\mathbb{B}_{\mathbb{E}(a)}(z^*, r) := \{z \in \mathbb{E}_1(a) : z - z^* \in {}_0\mathbb{E}(a),\ \|z - z^*\|_{\mathbb{E}(a)} < r\}. $$

We remark that a and r are independent parameters that can be chosen as we please. Let then $r_0 > 0$ be fixed. It is not difficult to see that there exists a number $R_0 = R_0(u_0, h_0, \delta_0, r_0)$ such that

$$ \begin{aligned} &\|\nabla(h + h^*)\|_{BC(J;BC^1)} + \|h + h^*\|_{\mathbb{E}_4(a)} + \|u + u^*\|_{BC(J;BC)} \\ &\qquad + Q\big(\|\nabla(h + h^*)\|_{BC(J;BC^1)}, \|h + h^*\|_{\mathbb{E}_4(a)}\big) \leq R_0 \end{aligned} $$

for all $u \in {}_0\mathbb{B}_{\mathbb{E}_1(a)}(0, r)$ and $h \in {}_0\mathbb{B}_{\mathbb{E}_4(a)}(0, r)$, with $a \in (0, \delta_0]$ and $r \in (0, r_0]$ arbitrary, where $z^* = (u^*, \pi^*, q^*, h^*)$ is the solution of equation (4.16) and where Q is defined in Proposition 4.1. Let $M_1 := M_0(1 + R_0)$. It then follows from Proposition 4.1 and (4.18) that

$$ \begin{aligned} &\|D(L_b^{-1}K_b)(z)\|_{{}_0\mathbb{E}(a)} \\ &\qquad \leq M_1 M\big[\|b - \gamma(v + v^*)\|_{BC(J;BC) \cap \mathbb{F}_4(a)} + \|z + z^*\|_{\mathbb{E}(a)}\big] \\ &\qquad + M_0 M\big[P\big(\|\nabla(h + h^*)\|_{BC(J;BC)}\big)\|\nabla(h + h^*)\|_{BC(J;BC)}\big] \end{aligned} \tag{4.21} $$

for all $z \in {}_0\mathbb{B}_{\mathbb{E}(a)}(0, r)$ and $a \in (0, \delta_0]$.

(iv) For (u_0, h_0) fixed, the norm of z^* in $\mathbb{E}(a)$ (which involves various integral expressions evaluated over the interval $(0, a)$) can be made as small as we like by choosing $a \in (0, \delta_0]$ small. Let then $a_1 \in (0, \delta_0]$ be fixed so that

$$ \begin{aligned} &\|\nabla h^*\|_{BC([0,a_1],BC)} \leq 2\eta, \\ &M_1 M\big(\|b - \gamma v^*\|_{BC([0,a_1];BC) \cap \mathbb{F}_4(a_1)} + \|z^*\|_{\mathbb{E}(a_1)}\big) \leq 1/8. \end{aligned} \tag{4.22} $$

Since $(\nabla h^*, b - \gamma v^*) \in {}_0BC([0, \delta_0], BC(\mathbb{R}^n))$ and $\|\nabla h^*(0)\|_\infty = \|\nabla h_0\|_\infty \leq \eta$, the estimates in (4.22) certainly hold for a_1 sufficiently small.
In a next step we choose $2r_1 \in (0, r_0]$ so that

$$ \begin{aligned} &\|\nabla h\|_{{}_0BC([0,a_1],BC(\mathbb{R}^n))} \leq \eta, \\ &M_1 M\big(\|\gamma v\|_{{}_0BC([0,a_1];BC) \cap {}_0\mathbb{F}_4(a_1)} + \|z\|_{{}_0\mathbb{E}(a_1)}\big) \leq 1/8, \end{aligned} \tag{4.23} $$

for all $h \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$, $v \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$, and $z \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1)$. It follows from Proposition 5.1 that (4.23) can indeed be achieved. Combining (4.19)–(4.23) gives

$$ \|D(L_b^{-1}K_b)(z)\|_{{}_0\mathbb{E}(a)} \leq 1/2, \quad z \in {}_0\mathbb{B}_{\mathbb{E}(a_1)}(0, 2r_1) \tag{4.24} $$

showing that $L_b^{-1}K_b : {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1) \to {}_0\mathbb{E}(a_1)$ is a contraction, where ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1)$ denotes the closed ball in ${}_0\mathbb{E}(a_1)$ with center at 0 and radius r_1.

It remains so show that $L_b^{-1}K_b$ maps ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0, r_1)$ into itself. From (4.24) and the mean value theorem follows

$$\begin{aligned}\|L_b^{-1}K_b(z)\|_{{}_0\mathbb{E}(a_1)} &\le \|L_b^{-1}K_b(z)-L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}+\|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}\\ &\le r_1/2+\|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}, \quad z\in {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0,r_1).\end{aligned}$$

Here we observe that the norm of $L_b^{-1}K_b(0)=L_b^{-1}(K(z^*)-(0,f_d^*,g^*,g_h^*))$ in ${}_0\mathbb{E}(a_1)$ can be made as small as we wish by choosing a_1 small enough. We may assume that a_1 was already chosen so that $\|L_b^{-1}K_b(0)\|_{{}_0\mathbb{E}(a_1)}\le r_1/2$.

(v) We have shown in (iv) that the mapping

$$L_b^{-1}K_b : {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0,r_1)\to {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0,r_1)$$

is a contraction. By the contraction mapping theorem $L_b^{-1}K_b$ has a unique fixed point $\hat z\in {}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(0,r_1)\subset {}_0\mathbb{E}(a_1)$ and it follows immediately from (4.10)–(4.11) that $\hat z+z^*$ is the (unique) solution of the nonlinear problem (2.1) in ${}_0\overline{\mathbb{B}}_{\mathbb{E}(a_1)}(z^*,r_1)$. Setting $t_0=a_1$ gives the assertion in part (a) of the Theorem.

(vi) The proof that (u,π,q,h) is analytic in space and time proceeds exactly in the same way as in steps (vi)–(vii) of the proof of Theorem 6.3 in [28], with the only difference that here $g_h^*=g_{h,\lambda,\nu}^*=0$, and that the operator D_ν in formula (6.30) of [28] is to be replaced by $D_{\lambda,\nu}$, defined by

$$\mathcal{D}_{\lambda,\nu}h:=(\lambda b_{\lambda,\nu}-\nu|\nabla h),\qquad b_{\lambda,\nu}(t,x):=b(\lambda t,x+t\nu). \tag{4.25}$$

We note that $D_{1,0}=(b|\nabla\cdot)$. In the same way as in [25, Lemma 8.2] one obtains that

$$[(\lambda,\nu)\mapsto b_{\lambda,\nu}]:(1-\delta,1+\delta)\times\mathbb{R}^n\to\mathbb{F}_4(a) \tag{4.26}$$

is real analytic. The remaining arguments are now the same as in [28], and this completes the proof of Theorem 4.2. □

Proof of Theorem 1.1: Clearly, the compatibility conditions of Theorem 1.1 are satisfied if and only if (4.6) is satisfied. Moreover, the smallness-boundedness condition of Theorem 1.1 is equivalent to (4.7), where we have slightly abused notation by using the same symbol for u_0 and its transformed version $\Theta_{h_0}^*u_0$.

Theorem 4.2 yields a unique solution $(v,w,\pi,[\pi],h)\in\mathbb{E}(t_0)$ which satisfies the additional regularity properties listed in part (b) of the theorem. Setting

$$(u,q)(t,x,y)=(v,w,\pi)(t,x,y-h(t,x)),\quad (t,x,y)\in\mathcal{O},$$

we then conclude that $(u,q)\in C^\omega(\mathcal{O},\mathbb{R}^{n+2})$ and $[q]\in C^\omega(\mathcal{M})$. The regularity properties listed in Remark 1.2(a) are implied by Proposition 5.1(a),(c). Finally, since $\pi(t,x,y)$ is defined for every $(t,x,y)\in\mathcal{O}$, we can conclude that

$$q(t,\cdot)\in\dot H_p^1(\Omega(t))\subset UC(\Omega(t))$$

for every $t\in(0,t_0)$. □

5. Appendix

In this section we state and prove some technical results that were used above.

Proposition 5.1. *Suppose $p > n+3$. Then the following embeddings hold:*

(a) $\mathbb{E}_1(a) \hookrightarrow BC(J; W_p^{2-2/p}(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1})) \hookrightarrow BC(J; BC^1(\dot{\mathbb{R}}^{n+1}, \mathbb{R}^{n+1}))$ *and there is a constant $C_0 = C_0(p)$ such that*

$$\|u\|_{{}_0BC(J;W_p^{2-2/p})} + \|u\|_{{}_0BC(J;BC^1)} \le C_0 \|u\|_{{}_0\mathbb{E}_1(a)}$$

for all $u \in {}_0\mathbb{E}_1(a)$ and all $a \in (0,\infty)$.

(b) $\mathbb{E}_3(a) \hookrightarrow BC(J; BC(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*

$$\|g\|_{{}_0BC(J;BC)} \le C_0 \|g\|_{{}_0\mathbb{E}_3(a)}$$

for all $g \in {}_0\mathbb{E}_3(a)$ and all $a \in (0,\infty)$.

(c) $\mathbb{F}_4(a) \hookrightarrow BC(J; W_p^1(\mathbb{R}^n)) \cap BC(J; BC^1(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*

$$\|g\|_{{}_0BC(J;W_p^1)} + \|g\|_{{}_0BC(J;BC^1)} \le C_0 \|g\|_{{}_0\mathbb{F}_4(a)}$$

for all $g \in {}_0\mathbb{F}_4(a)$ and all $a \in (0,\infty)$.

(d) $\mathbb{E}_4(a) \hookrightarrow BC^1(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^2(\mathbb{R}^n))$ *and there exists a constant $C_0 = C_0(p)$ such that*

$$\|h\|_{{}_0BC^1(J;BC^1)} + \|h\|_{{}_0BC(J;BC^2)} \le C_0 \|h\|_{{}_0\mathbb{E}_4(a)}$$

for all $h \in {}_0\mathbb{E}_4(a)$ and all $a \in (0,\infty)$.

(e) $\partial_j \in \mathcal{L}(\mathbb{E}_4(a), \mathbb{E}_3(a)) \cap \mathcal{L}(\mathbb{E}_4(a), \mathbb{F}_4(a))$ *for $j = 1, \cdots, n$. Moreover, for every given $a_0 > 0$ there is a constant $C_0 = C_0(a_0, p)$ such that*

$$\|\partial_j h\|_{\mathbb{E}_3(a)} + \|\partial_j h\|_{\mathbb{F}_4(a)} \le C_0 \|h\|_{\mathbb{E}_4(a)}$$

for all $h \in \mathbb{E}_4(a)$ and all $a \in (0, a_0]$.

Proof. We refer to [25, Proposition 6.2] for a proof of (a)-(b). The assertion in (c) can established in the same way, using that $\mathbb{F}_4(a) \hookrightarrow BC(J; W_p^{2-3/p}(\mathbb{R}^n))$, see [19, Remark 5.3(d)]. In order to show that the embedding constant in (d) does not depend on $a \in (0, a_0]$ we define an extension operator in the following way: for $h \in {}_0BC^1([0,a]; X)$, with X an arbitrary Banach space, we first set $\tilde{h}(t) := 0$ for $t \le 0$, so that $\tilde{h} \in BC^1((-\infty, a]; X)$, and then define

$$(\mathcal{E}h)(t) := \begin{cases} h(t) & \text{if} \quad 0 \le t \le a, \\ 3\tilde{h}(2a-t) - 2\tilde{h}(3a-2t) & \text{if} \quad a \le t. \end{cases} \tag{5.1}$$

A moment of reflection shows that $\mathcal{E}h \in {}_0BC^1([0,\infty); X)$, and that $\mathcal{E}h$ is an extension of h. It is evident that the norm of $\mathcal{E} : {}_0BC^1([0,a]; X) \to {}_0BC^1([0,\infty); X)$ is independent of $a \in [0, a_0]$. The assertion follows now by the same arguments as in the proof of [25, Proposition 6.2].

Let $a_0 > 0$ be fixed. In order to establish part (e) it suffices to show that there is a constant $C_0 = C(a_0, p, r)$ such that

$$\|g\|_{W_p^r([0,a];X)} \le C_0 \|g\|_{H_p^1([0,a];X)}, \quad a \in (0,a], \tag{5.2}$$

where X is an arbitrary Banach space and $r \in [0,1]$. This follows from Hardy's inequality as follows: for $r \in (0,1)$ fixed we have

$$\begin{aligned}
\frac{1}{2}\langle g\rangle^p_{W_p^r([0,a];X)} &= \int_0^a \int_s^a \frac{\|g(t)-g(s)\|_X^p}{(t-s)^{1+rp}}\,dt\,ds \\
&= \int_0^a \int_0^{a-s} \frac{\|g(s+\tau)-g(s)\|_X^p}{\tau^{1+rp}}\,d\tau\,ds \\
&\le \int_0^a \int_0^{a-s} \frac{1}{\tau^{1+rp}} \left(\int_0^\tau \|\partial g(s+\sigma)\|_X\,d\sigma\right)^p d\tau\,ds \\
&\le c(r,p) \int_0^a \int_0^{a-s} \frac{1}{\tau^{1-(1-r)p}} \|\partial g(s+\tau)\|_X^p\,d\tau\,ds \\
&= c(r,p) \int_0^a \frac{1}{\tau^{1-(1-r)p}} \int_0^{a-\tau} \|\partial g(s+\tau)\|_X^p\,ds\,d\tau \\
&\le c(r,p) \int_0^a \frac{1}{\tau^{1-(1-r)p}}\,d\tau\ \|\partial g\|^p_{L_p([0,a];X)}
\end{aligned}$$

where ∂g is the derivative of g, and this readily yields (5.2). □

Our next result will be important in order to derive estimates for the nonlinearities in (2.1).

Lemma 5.2. *Suppose $p > n+3$. Let $a_0 \in (0,\infty)$ be given. Then*

(a) *$\mathbb{E}_3(a)$ is a multiplication algebra and we have the following estimate*

$$\|g_1 g_2\|_{\mathbb{E}_3(a)} \le (\|g_1\|_\infty + \|g_1\|_{\mathbb{E}_3(a)})(\|g_2\|_\infty + \|g_2\|_{\mathbb{E}_3(a)}) \tag{5.3}$$

for all $(g_1, g_2) \in \mathbb{E}_3(a) \times \mathbb{E}_3(a)$ and all $a > 0$.

(b) *There exists a constant $C_0 = C_0(a_0, p)$ such that*

$$\|g_1 g_2\|_{{}_0\mathbb{E}_3(a)} \le C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{E}_3(a)})\|g_2\|_{{}_0\mathbb{E}_3(a)} \tag{5.4}$$

for all $(g_1, g_2) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_3(a)$ and all $a \in (0, a_0]$.

(c) *There exists a constant $C_0 = C_0(a_0, p)$ such that*

$$\|g\partial_j h\|_{{}_0\mathbb{E}_3(a)} \le C_0 \|g\|_{\mathbb{E}_3(a)} \|h\|_{{}_0\mathbb{E}_4(a)}, \quad j = 1, \cdots, n, \tag{5.5}$$

for all $(g,h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0, a_0]$.

(d) *Suppose $(g,\psi) \in \mathbb{E}_3(a) \times \mathbb{E}_3(a)$ and let $\beta(t,x) := \sqrt{1+\psi^2(t,x)}$. Then $\frac{g}{\beta^k} \in \mathbb{E}_3(a)$ for $k \in \mathbb{N}$ and the following estimate holds*

$$\left\|\frac{g}{\beta^k}\right\|_{\mathbb{E}_3(a)} \le (1 + \|\psi\|_{\mathbb{E}_3(a)})^k (\|g\|_\infty + \|g\|_{\mathbb{E}_3(a)}). \tag{5.6}$$

Proof. The assertions in (a)-(b) follow from (the proof of) Proposition 6.6.(ii) and (iv) in [25].

(c) To economize our notation we set $r = 1/2 - 1/2p$ and $\theta = 1 - 1/p$.
Suppose that $(g, h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$. We first observe that

$$\|g\partial_j h\|_{W_p^r(J;L_p)} \le \big(\|g\|_{L_p(J;L_p)} + \langle g\rangle_{W_p^r(J;L_p)}\big)\|\partial_j h\|_{{}_0BC(J;L_\infty)} + \left(\int_0^a\int_0^a \|g(s)(\partial_j h(t) - \partial_j h(s))\|_{L_p}^p \,\frac{dt\,ds}{|t-s|^{1+rp}}\right)^{1/p}.$$

Using Hölder's inequality, and the fact that $(1 - r - 1/p) = r > 0$, we obtain the estimate

$$\begin{aligned}
&\int_0^a\int_0^a \|g(s)(\partial_j h(t) - \partial_j h(s))\|_{L_p}^p \,\frac{dt\,ds}{|t-s|^{1+rp}}\\
&\le \int_0^a\int_0^a \|g(s)\|_{L_p}^p \left(\left|\int_s^t \|\partial_t\partial_j h(\tau)\|_{L_\infty}\,d\tau\right|\right)^p \,\frac{dt\,ds}{|t-s|^{1+rp}}\\
&\le \int_0^a \|g(s)\|_{L_p}^p \left(\int_0^a \frac{dt}{|t-s|^{1-(1-r-1/p)p}}\right) ds \int_0^a \|\partial_t\partial_j h(\tau)\|_{L_p}^p\,d\tau\\
&\le C_0(a_0,p)\|g\|_{L_p(J;L_p)}^p \|\partial_t\partial_j h\|_{L_p(J;L_\infty)}^p
\end{aligned} \tag{5.7}$$

for $a \in (0, a_0]$. Hence we conclude that

$$\begin{aligned}
\|g\partial_j h\|_{{}_0W_p^r(J;L_p)} &\le C_0\|g\|_{W_p^r(J;L_p)}\big(\|\partial_j h\|_{{}_0BC(J;L_\infty)} + \|\partial_t\partial_j h\|_{L_p(J;L_\infty)}\big)\\
&\le C_0\|g\|_{\mathbb{E}_3(a)}\|h\|_{{}_0\mathbb{E}_4(a)}
\end{aligned} \tag{5.8}$$

uniformly in $a \in (0, a_0]$. It is easy to verify that

$$\begin{aligned}
\|g\partial_j h\|_{L_p(J;W_p^\theta)} &\le \|g\|_{L_p(J;W_p^\theta)}\|\partial_j h\|_{{}_0BC(J;L_\infty)} + \|g\|_{L_p(J;L_\infty)}\|\partial_j h\|_{{}_0BC(J;W_p^\theta)}\\
&\le C_0\|g\|_{\mathbb{E}_3(a)}\|h\|_{{}_0\mathbb{E}_4(a)}.
\end{aligned} \tag{5.9}$$

Combining the estimates (5.8)–(5.9) yields (5.5).

(d) As in the proof of Proposition 6.6.(v) in [25] we obtain

$$\begin{aligned}
\|g/\beta\|_{W_p^r(J;L_p)} &\le \|1/\beta\|_\infty(\|g\|_{L_p(J;L_p)} + \langle g\rangle_{W_p^r(J;L_p)}) + \|g\|_\infty\langle 1/\beta\rangle_{W_p^r(J;L_p)}\\
&\le (1 + \langle 1/\beta\rangle_{W_p^r(J;L_p)})(\|g\|_\infty + \|g\|_{W_p^r(J;L_p)}).
\end{aligned}$$

Thus it remains to estimate the term $\langle 1/\beta\rangle_{W_p^r(J;L_p)}$. Using that $\beta^2(t,x) - \beta^2(s,x) = \psi^2(t,x) - \psi^2(s,x)$ one easily verifies that

$$\left|\frac{1}{\beta(s,x)} - \frac{1}{\beta(t,x)}\right| = \left|\frac{\beta^2(t,x) - \beta^2(s,x)}{\beta(s,x)\beta(t,x)(\beta(t,x) + \beta(s,x))}\right| \le |\psi(t,x) - \psi(s,x)|$$

and this yields $\langle 1/\beta\rangle_{W_p^r(J;L_p)} \le \langle\psi\rangle_{W_p^r(J;L_p)}$. Consequently,

$$\|g/\beta\|_{W_p^r(J;L_p)} \le (1 + \|\psi\|_{W_p^r(J:L_p)})(\|g\|_\infty + \|g\|_{W_p^r(J;L_p)}).$$

A similar argument shows that

$$\|g/\beta\|_{L_p(J;W_p^\theta)} \le (1+\|\psi\|_{L_p(J;W_p^\theta)})(\|g\|_\infty + \|g\|_{L_p(J;W_p^\theta)}).$$

Combining the last two estimates gives (5.6) for $k=1$. The general case then follows by induction. □

Corollary 5.3. *Suppose $p>n+3$. Let $a_0\in(0,\infty)$ and $k\in\mathbb{N}$ with $k\ge 1$ be given.*

(a) *There exists a constant $C_0=C_0(a_0,p,k)$ such that*

$$\|(g_1\cdots g_k)\bar g\|_{{}_0\mathbb{E}_3(a)} \le C_0\prod_{i=1}^k\big(\|g_i\|_\infty+\|g_i\|_{\mathbb{E}_3(a)}\big)\|\bar g\|_{{}_0\mathbb{E}_3(a)}$$

for all functions $g_i\in\mathbb{E}_3(a)$, $1\le i\le k$, $\bar g\in{}_0\mathbb{E}_3(a)$, and all $a\in(0,a_0]$.

(b) *There exists a constant $C_0=C_0(a_0,p,k)$ such that*

$$\begin{aligned}&\|g(\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h)\|_{{}_0\mathbb{E}_3(a)}\\ &\le C_0\big(\|\nabla h\|_\infty^k+\|h\|_{\mathbb{E}_4(a)}\|\nabla h\|_\infty^{k-1}+\|\nabla h\|^k_{BC(J;W_p^{1-1/p})}\big)\|g\|_{\mathbb{E}_3(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}\\ &\le C_0\big(\|\nabla h\|^k_{BC(J;BC^1)}+\|h\|_{\mathbb{E}_4(a)}\|\nabla h\|_\infty^{k-1}\big)\|g\|_{\mathbb{E}_3(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}\end{aligned}$$

for $h\in\mathbb{E}_4(a)$, $\bar h\in{}_0\mathbb{E}_4(a)$, $a\in(0,a_0]$, $1\le j\le n$, and $\ell_i\in\{1,\cdots n\}$ with $i=1,\cdots,k$.

Proof. (a) follows from (5.4) by iteration.

(b) The first line in (5.8) shows that

$$\begin{aligned}&\|g(\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h)\|_{W_p^r(J;L_p)}\\ &\le C_0\|g\|_{W_p^r(J;L_p)}(\|\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h\|_{{}_0BC(J;L_\infty)}+\|\partial_t(\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h)\|_{L_p(J;L_\infty)}).\end{aligned}$$

Next we note that

$$\|\partial_{\ell_1}h\cdots\partial_{\ell_k}h(\partial_t\partial_j\bar h)\|_{L_p(J;L_\infty)}\le\|\nabla h\|_\infty^k\|\partial_t\partial_j\bar h\|_{L_p(J;L_\infty)},$$

and

$$\|\partial_{\ell_1}h\cdots(\partial_t\partial_{\ell_i}h)\cdots\partial_{\ell_k}h\partial_j\bar h\|_{L_p(J;L_\infty)}\le\|\partial_t\partial_{\ell_i}h\|_{L_p(J;L_\infty)}\|\nabla h\|_\infty^{k-1}\|\partial_j\bar h\|_{{}_0BC(J;L_\infty)}.$$

Proposition 6.1(d) now implies the assertion for $\|\cdot\|_{W_p^r(J;L_p)}$. On the other hand we have by (5.9) for $\theta=1-1/p$

$$\begin{aligned}&\|g(\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h)\|_{L_p(J;W_p^\theta)}\\ &\le\|g\|_{L_p(J;W_p^\theta)}\|\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h\|_\infty+\|g\|_{L_p(J;L_\infty)}\|\partial_{\ell_1}h\cdots\partial_{\ell_k}h\partial_j\bar h\|_{{}_0BC(J;W_p^\theta)}\\ &\le C_0\|g\|_{\mathbb{E}_3(a)}\big(\|\nabla h\|_\infty^k+\|\nabla h\|^k_{BC(J;W_p^{1-1/p})}\big)\|h\|_{{}_0\mathbb{E}_4(a)}\end{aligned}$$

since $W_p^\theta(\mathbb{R}^n)$ is a multiplication algebra. The last inequality then follows from Sobolov's embedding theorem. □

Remark 5.4. It can be shown that the estimate in (5.5) can be improved as follows: For every $a_0 \in (0,\infty)$ there is a constant $C_0 = C_0(a_0,p) > 0$ and a constant $\theta = \theta(p) > 0$ such that

$$\|g\partial_j h\|_{{}_0\mathbb{E}_3(a)} \le C_0\, a^\theta \|g\|_{\mathbb{E}_3(a)} \|h\|_{{}_0\mathbb{E}_4(a)}$$

holds for all $(g,h) \in \mathbb{E}_3(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0,a_0]$. In the same way, the constant C_0 in Corollary 5.3(b) can be replaced by $C_0 a^\theta$.

Lemma 5.5. *Suppose $p > n+3$. Let $a_0 \in (0,\infty)$ be given. Then*

(a) *$\mathbb{F}_4(a)$ is a multiplication algebra and we have the estimate*

$$\|g_1 g_2\|_{\mathbb{F}_4(a)} \le C_a \|g_1\|_{\mathbb{F}_4(a)} \|g_2\|_{\mathbb{F}_4(a)}$$

for all $(g_1,g_2) \in \mathbb{F}_4(a) \times \mathbb{F}_4(a)$, where the constant C_a depends on a.

(b) *There exists a constant $C_0 = C_0(a_0,p)$ such that*

$$\|g_1 g_2\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{F}_4(a)}) \|g_2\|_{{}_0\mathbb{F}_4(a)} \tag{5.10}$$

for all $(g_1,g_2) \in \mathbb{F}_4(a) \times {}_0\mathbb{F}_4(a)$ and all $a \in (0,a_0]$.

(c) *There exists a constant $C_0 = C_0(a_0,p)$ such that*

$$\|g\partial_j h\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|g\|_\infty + \|g\|_{\mathbb{F}_4(a)}) \|h\|_{{}_0\mathbb{E}_4(a)}, \quad j = 1,\cdots,n, \tag{5.11}$$

for all $(g,h) \in \mathbb{F}_4(a) \times {}_0\mathbb{E}_4(a)$ and $a \in (0,a_0]$.

Proof. Here we equip $\mathbb{F}_4(a)$ with the (equivalent) norm

$$\|g\|_{\mathbb{F}_4(a)} = \|g\|_{W_p^{1-1/2p}(J;L_p)} + \sum_{i=1}^n \|\partial_i g\|_{L_p(J;W_p^{1-1/p})}\,. \tag{5.12}$$

(a) This follows from Proposition 5.1(c) by similar arguments as in the proof of Proposition 6.6(ii) and (iv) in [25].

(b) It follows from part (a) and Proposition 5.1(c) that

$$\|g_1 g_2\|_{{}_0W_p^r(J;L_p)} \le C_0(\|g_1\|_\infty + \|g_1\|_{W_p^r(J;L_p)}) \|g_2\|_{{}_0\mathbb{F}_4(a)}, \quad (g_1,g_2) \in \mathbb{F}_4(a) \times {}_0\mathbb{F}_4(a)$$

where $r = 1 - 1/2p$. Next, observe that again by Proposition 5.1(c)

$$\begin{aligned}\|(\partial_i g_1) g_2\|_{L_p(J;W_p^\theta)} &\le \|\partial_i g_1\|_{L_p(J;W_p^\theta)} \|g_2\|_{{}_0BC(J;L_\infty)} + \|\partial_i g_1\|_{L_p(J;L_\infty)} \|g_2\|_{{}_0BC(J;W_p^\theta)} \\ &\le C_0 \|g_1\|_{\mathbb{F}_4(a)} \|g_2\|_{{}_0\mathbb{F}_4(a)}\end{aligned}$$

where $\theta = 1 - 1/p$. Moreover,

$$\begin{aligned}\|g_1 \partial_i g_2\|_{L_p(J;W_p^\theta)} &\le \|g_1\|_{L_p(J;W_p^\theta)} \|\partial_i g_2\|_{{}_0BC(J;L_\infty)} + \|g_1\|_\infty \|\partial_i g_2\|_{L_p(J;W_p^\theta)} \\ &\le C_0(\|g_1\|_\infty + \|g_1\|_{\mathbb{F}_4(a)}) \|g_2\|_{{}_0\mathbb{F}_4(a)}.\end{aligned}$$

The estimates above in conjunction with (5.12) yields (5.10).

(c) follows from (b) by setting $g_2 = \partial_j h$ and from Proposition 5.1(e), which certainly is also true for ${}_0\mathbb{E}_4(a)$. □

Proof of Proposition 4.1**:**

It follows as in the proof of [28, Proposition 6.2] that $N_b \in C^\omega(\mathbb{E}(a), \mathbb{F}(a))$, and moreover, that $DH_b(z) \in \mathcal{L}({}_0\mathbb{E}(a), {}_0\mathbb{F}(a))$ for $z \in \mathbb{E}(a)$. It thus remains to prove the estimates stated in the proposition.

Without always writing this explicitly, all the estimates derived below will be uniform in $a \in (0, a_0]$, for $a_0 > 0$ fixed. Moreover, all estimates will be uniform for $(\bar{u}, \bar{\pi}, \bar{q}, \bar{h}) \in {}_0\mathbb{E}(a)$.

(i) Without changing notation we consider here the extension of h from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ defined by $h(t, x, y) = h(t, x)$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in J$. With this interpretation we have

$$\|\partial h\|_{\infty, J\times\mathbb{R}^{n+1}} = \|\partial h\|_{\infty, J\times\mathbb{R}^n}, \quad h \in \mathbb{E}(a), \quad \partial \in \{\partial_j, \Delta, \partial_t\}, \tag{5.13}$$

where $\|\cdot\|_{\infty,U}$ denotes the sup-norm for the set $U \subset J \times \mathbb{R}^{n+1}$. Next we observe that

$$\begin{aligned} &BC(J; BC(\mathbb{R}^{n+1})) \cdot L_p(J; L_p(\mathbb{R}^{n+1})) \hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \\ &BC(J; L_p(\mathbb{R}^{n+1})) \cdot L_p(J; BC(\mathbb{R}^{n+1})) \hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \\ &BC(J; BC(\mathbb{R}^{n+1})) \cdot BC(J; BC(\mathbb{R}^{n+1})) \hookrightarrow BC(J; BC(\mathbb{R}^{n+1})), \end{aligned} \tag{5.14}$$

that is, multiplication is continuous and bilinear in the indicated function spaces (with norm equal to 1).

Let us first consider the term $F_1(u, h) := |\nabla h|^2 \partial_y^2 u$ appearing in the definition of F. Its Fréchet derivative at (u, h) is given by

$$DF_1(u, h)[\bar{u}, \bar{h}] = |\nabla h|^2 \partial_y^2 \bar{u} + 2(\nabla h|\nabla \bar{h})\partial_y^2 u.$$

Suppose $(\bar{u}, \bar{h}) \in {}_0\mathbb{E}_1(a) \times {}_0\mathbb{E}_4(a)$. From (5.13), the first and third line in (5.14) and Proposition 5.1(d) follows the estimate

$$\|DF_1(u, h)[\bar{u}, \bar{h}]\|_{{}_0\mathbb{F}(a)} \le C_0 \|\nabla h\|_\infty (\|\nabla h\|_\infty + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)})$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$. It is important to note that the constant C_0 does not depend on the length of the interval $J = (0, a)$ for $a \in (0, a_0]$.

Next, let us take a closer look at the term $F_2(u, h) := \Delta h \partial_y u$ in the definition of F. The Fréchet derivative is $DF_2(u, h)[\bar{u}, \bar{h}] = \Delta h \partial_y \bar{u} + \Delta \bar{h} \partial_y u$. We infer from (5.13), the first and second line in (5.14), and Proposition 5.1 that

$$\begin{aligned} \|DF_2(u, h)[\bar{u}, \bar{h}]\|_{{}_0\mathbb{F}(a)} &\le (\|\Delta h\|_{L_p(J;L_\infty)} + \|\partial_y u\|_{L_p(J;L_p)})\cdot \\ &\qquad (\|\partial_y \bar{u}\|_{{}_0BC(J;L_p)} + \|\Delta \bar{h}\|_{{}_0BC(J;L_\infty)}) \\ &\le C_0(\|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar{u}\|_{{}_0\mathbb{E}_1(a)} + \|\bar{h}\|_{{}_0\mathbb{E}_4(a)}) \end{aligned}$$

for all $(u, h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.

The derivative of $F_3(u, h) := (u|\nabla h)\partial_y u$, where $\nabla h := (\nabla h, 0)$, is given by

$$DF_3(u, h)[\bar{u}, \bar{h}] = (\bar{u}|\nabla h)\partial_y u + (u|\nabla h)\partial_y \bar{u} + (u|\nabla \bar{h})\partial_y u$$

and it follows from (5.13)–(5.14) and Proposition 5.1(a),(d) that there is a constant $C_0 > 0$ such that

$$\|DF_3(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{F}(a)} \le C_0(\|\nabla h\|_\infty + \|u\|_\infty)\|u\|_{\mathbb{E}_1(a)}(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})$$

for all $(u,h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.
Let us also consider the term $F_4(u,h) := \partial_t h \partial_y u$. Observing that

$$DF_4(u,h)[\bar u,\bar h] = \partial_t h \partial_y \bar u + \partial_t \bar h \partial_y u,$$

that $\partial_t : \mathbb{E}_4(a) \to \mathbb{F}_4(a)$ is linear and continuous and

$$\mathbb{F}_4(a) \hookrightarrow L_p(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^1(\mathbb{R}^n)) \tag{5.15}$$

we conclude from (5.13)–(5.15) and Proposition 5.1(a),(c) that there is a constant $C_0 = C_0(a_0)$ such that

$$\begin{aligned}\|DF_4(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{F}(a)} &\le (\|\partial_t h\|_{L_p(J;L_\infty)} + \|\partial_y u\|_{L_p(J;L_p)}) \\ &\qquad (\|\partial_y \bar u\|_{{}_0BC(J;L_p)} + \|\partial_t \bar h\|_{{}_0BC(J;L_\infty)}) \\ &\le C_0(\|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}$$

for all $(u,h) \in \mathbb{E}_1(a) \times \mathbb{E}_4(a)$.

The derivative of $F_5(\pi,h) := \partial_y \pi \nabla h$ is given by

$$DF_5(\pi,h)[\bar\pi,\bar h] = \partial_y \bar\pi \nabla h + \partial_y \pi \nabla \bar h.$$

It follows from (5.13)–(5.14) and Proposition 5.1(d) that there exists C_0 such that

$$\begin{aligned}\|DF_5(\pi,h)[\bar\pi,\bar h]\|_{{}_0\mathbb{F}(a)} &\le (\|\nabla h\|_\infty + \|\partial_y \pi\|_{L_p(J;L_p)}) \\ &\qquad (\|\partial_y \bar\pi\|_{L_p(J;L_p)} + \|\nabla h\|_{{}_0BC(J;L_\infty)}) \\ &\le C_0(\|\nabla h\|_\infty + \|\pi\|_{\mathbb{E}_2(a)})(\|\bar\pi\|_{{}_0\mathbb{E}_2(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}$$

for all $(\pi,h) \in \mathbb{E}_2(a) \times \mathbb{E}_4(a)$. The remaining terms in the definition of F can be analyzed in the same way. Summarizing we have shown that there is a constant C_0 such that

$$\begin{aligned}&\|DF(u,\pi,h)[\bar u,\bar\pi,\bar h]\|_{{}_0\mathbb{F}_1(a)} \\ &\quad \le C_0\big[\|\nabla h\|_\infty + \|\nabla h\|_\infty^2 + \|(u,\pi,h)\|_{\mathbb{E}(a)} \\ &\qquad\quad + (\|\nabla h\|_\infty + \|u\|_\infty)\|u\|_{\mathbb{E}_1(a)}\big]\,\|(\bar u,\bar\pi,\bar h)\|_{{}_0\mathbb{E}(a)}\end{aligned} \tag{5.16}$$

for all $(u,\pi,h) \in \mathbb{E}(a)$ and all $a \in (0,a_0]$.

(ii) We will now consider the nonlinear function $F_d(u,h) = (\nabla h|\partial_y v)$, stemming from the transformed divergence. Since $h(x,y) := h(x)$ does not depend on y we have

$$F_d(u,h) = (\nabla h|\partial_y u) = \partial_y(\nabla h|u). \tag{5.17}$$

We note that

$$\partial_y \in \mathcal{L}\big(H^1_p(J; L_p(\mathbb{R}^{n+1})), H^1_p(J; H^{-1}_p(\mathbb{R}^{n+1}))\big). \tag{5.18}$$

The norm of this linear mapping does not depend on the length of the interval $J = [0,a]$. It is easy to see that multiplication is continuous in the following function spaces:

$$\begin{aligned} &H^1_p(J;BC(\mathbb{R}^{n+1}))\cdot H^1_p(J;L_p(\mathbb{R}^{n+1}))\hookrightarrow H^1_p(J;L_p(\mathbb{R}^{n+1}))\\ &BC(J;BC^1(\dot{\mathbb{R}}^{n+1}))\cdot L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1}))\hookrightarrow L_p(J;H^1_p(\dot{\mathbb{R}}^{n+1})).\end{aligned}\tag{5.19}$$

The derivative of F_d at $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ is given by

$$DF_d(u,h)[\bar u,\bar h] = (\nabla h|\partial_y\bar u) + (\nabla\bar h|\partial_y u) = \partial_y((\nabla h|\bar u) + (\nabla\bar h|u)).$$

We want to derive a uniform estimate for $DF_d(u,h)[\bar u,\bar h]$ which does not depend on the length of the interval $J=[0,a]$. We conclude from (5.13)–(5.15) that

$$\begin{aligned} &\|(\nabla h|\bar u)\|_{{}_0H^1_p(J;L_p)} \sim \|(\nabla h|\bar u)\|_{L_p(J;L_p)} + \|(\partial_t\nabla h|\bar u)\|_{L_p(J;L_p)} + \|(\nabla h|\partial_t\bar u)\|_{L_p(J;L_p)}\\ &\le \|\nabla h\|_\infty\|\bar u\|_{L_p(J;L_p)} + \|\partial_t\nabla h\|_{L_p(J;L_\infty)}\|\bar u\|_{{}_0BC(J;L_p)} + \|\nabla h\|_\infty\|\partial_t\bar u\|_{L_p(J;L_p)}\\ &\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)})\|\bar u\|_{{}_0H^1_p(J;L_p)}.\end{aligned}$$

Similar arguments also yield $\|(\nabla\bar h|u)\|_{{}_0H^1_p(J;L_p)}\le C_0\|u\|_{H^1_p(J;L_p)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}$. These estimates in combination with (5.18) show that there is a constant C_0 such that

$$\begin{aligned} &\|(\nabla h|\partial_y\bar u) + (\partial_y u|\nabla\bar h)\|_{{}_0H^1_p(J;H^{-1}_p)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$, where C_0 is uniform in $a\in(0,a_0]$. Observing that

$$\|(\nabla h|\partial_y\bar u)\|_{L_p(J;L_p)} + \Sigma_{j=1}^{n+1}\|(\partial_j\nabla h|\partial_y\bar u) + (\nabla h|\partial_j\partial_y\bar u)\|_{L_p(J;L_p)}$$

defines an equivalent norm for $\|(\nabla h|\partial_y\bar u)\|_{L_p(J;H^1_p)}$, we infer once more from (5.13)–(5.14) and Propostion 5.1 that

$$\begin{aligned} &\|(\nabla h|\partial_y\bar u)\|_{L_p(J;H^1_p)} \le C_0(\|h\|_{\mathbb{E}_4(a)} + \|\nabla h\|_\infty)\|\bar u\|_{{}_0\mathbb{E}_1(a)}\\ &\|(\nabla\bar h|\partial_y u)\|_{L_p(J;H^1_p)} \le C_0\|u\|_{L_p(J;H^2_p)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}.\end{aligned}$$

Summarizing we have shown that there exists a constant C_0 such that

$$\begin{aligned} &\|DF_d(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{F}_2(a)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}\tag{5.20}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ and $a\in(0,a_0]$.

(iii) We remind that

$$[\![\mu\partial_i\,\cdot]\!]\in\mathcal{L}\big(H^1_p(J;L_p(\dot{\mathbb{R}}^{n+1}))\cap L_p(J;H^2_p(\dot{\mathbb{R}}^{n+1})),\mathbb{E}_3(a)\big)\tag{5.21}$$

where $[\![\mu\partial_i u]\!]$ denotes the jump of the quantity $\mu\partial_i u$ with u a generic function from $\dot{\mathbb{R}}^{n+1}\to\mathbb{R}$, and where $\partial_i=\partial_{x_i}$ for $i=1,\dots,n$ and $\partial_{n+1}=\partial_y$.

The mapping $G(u,q,h)$ is made up of terms of the form

$$[\![\mu\partial_i u_k]\!]\partial_j h,\quad [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h,\quad q\partial_j h,\quad \Delta h\partial_j h,\quad G_\kappa(h),\quad G_\kappa(h)\partial_j h$$

where u_k denotes the k-th component of a function $u \in \mathbb{E}_1(a)$. It follows from Lemma 5.2(a) and (5.21) that the mappings

$$\begin{aligned}&(u,h) \mapsto [\![\mu\partial_i u_k]\!]\partial_j h,\ [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h : \mathbb{E}_1(a)\times\mathbb{E}_4(a) \to \mathbb{E}_3(a),\\ &(q,h)\mapsto q\partial_j h : \mathbb{E}_3(a)\times\mathbb{E}_4(a)\to\mathbb{E}_3(a), \quad h\mapsto \Delta h\partial_j h:\mathbb{E}_4(a)\to\mathbb{E}_3(a)\end{aligned}$$

are multilinear and continuous. Let us now take a closer look at the term $G_1(u,h) := [\![\mu\partial_i u_k]\!]\partial_j h$. Its Fréchet derivative is given by

$$DG_1(u,h)[\bar u,\bar h] = \partial_j h[\![\mu\partial_i \bar u_k]\!] + [\![\mu\partial_i u_k]\!]\partial_j \bar h.$$

Setting $g_1 = \partial_j h$ and $g_2 := [\![\mu\partial_i \bar u_k]\!]$ we obtain from (5.4) and (5.21) the estimate

$$\|\partial_j h[\![\mu\partial_i\bar u_k]\!]\|_{{}_0\mathbb{E}_3(a)} \le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)})\|\bar u\|_{{}_0\mathbb{E}_1(a)}.$$

On the other hand, setting $g := [\![\mu\partial_i u_k]\!]$ we conclude from (5.5) and (5.21) that

$$\|[\![\mu\partial_i u_k]\!]\partial_j\bar h\|_{{}_0\mathbb{E}_3(a)} \le C_0\|u\|_{\mathbb{E}_1(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}.$$

Consequently,

$$\begin{aligned}&\|DG_1(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{E}_3(a)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|u\|_{\mathbb{E}_1(a)})(\|\bar u\|_{{}_0\mathbb{E}_1(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}\tag{5.22}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$, and all $a\in(0,a_0]$.

Given $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ let $G_2(u,h) := [\![\mu\partial_i u_k]\!]\partial_j h\partial_l h$. The Fréchet derivative of G_2 is given by

$$DG_2(u,h)[\bar u,\bar h] = \partial_j h\partial_l h[\![\mu\partial_i\bar u_k]\!] + [\![\mu\partial_i u_k]\!]\partial_j h\partial_j\bar h + [\![\mu\partial_i u_k]\!]\partial_l h\partial_j \bar h.$$

From Corollary 5.3(a),(b) and (5.21) follows that there is a constant C_0 such that

$$\begin{aligned}\|DG_2(u,h)[\bar u,\bar h]\|_{{}_0\mathbb{E}_3(a)} &\le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)})^2\|\bar u\|_{{}_0\mathbb{E}_1(a)}\\ &\quad + C_0\big(\|\nabla h\|_{BC(J;BC^1)} + \|h\|_{\mathbb{E}_4(a)}\big)\|u\|_{\mathbb{E}_1(a)}\|\bar h\|_{{}_0\mathbb{E}_4(a)}\end{aligned}\tag{5.23}$$

for all $(u,h)\in\mathbb{E}_1(a)\times\mathbb{E}_4(a)$ and all $a\in(0,a_0]$.

The terms $G_3(q,h) := q\partial_j h$ and $G_4(h) := \Delta h\partial_j h$ can be analyzed in the same way as the term G_1, yielding the following estimates

$$\begin{aligned}&\|DG_3(q,h)[\bar q,\bar h]\|_{{}_0\mathbb{E}_3(a)}\\ &\qquad\le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|q\|_{\mathbb{E}_3(a)})(\|\bar q\|_{{}_0\mathbb{E}_3(a)} + \|\bar h\|_{{}_0\mathbb{E}_4(a)})\end{aligned}\tag{5.24}$$

as well as

$$\|DG_4(h)\bar h\|_{{}_0\mathbb{E}_3(a)} \le C_0(\|\nabla h\|_\infty + \|\nabla h\|_{\mathbb{E}_3(a)} + \|\nabla^2 h\|_{\mathbb{E}_3(a)})\|\bar h\|_{{}_0\mathbb{E}_4(a)}.\tag{5.25}$$

Let us now consider the term

$$G_5(h) = \frac{1}{(1+\beta)\beta}(\partial_j h)^2\Delta h, \quad \beta(t,x) := \sqrt{1+|\nabla h(t,x)|^2},$$

appearing in the definition of G_κ. The Fréchet derivative of G_5 at h is given by

$$DG_5(h)\bar h = -\Big(\frac{1}{(1+\beta)^2\beta^2} + \frac{1}{(1+\beta)\beta^3}\Big)(\partial_j h)^2\Delta h\partial_k h\partial_k \bar h + \frac{1}{(1+\beta)\beta}\big(2\partial_j h\Delta h\partial_j\bar h + (\partial_j h)^2\Delta\bar h\big).$$

Before continuing, we note that the term $1/(1+\beta)$ can be treated in exactly the same way as $1/\beta$, as a short inspection of the proof of Lemma 5.2(d) shows. It follows then from Corollary 5.3(a)–(b) and from (5.6) that there is a constant C_0 such that

$$\|DG_5(h)\bar h\|_{{}_0\mathbb{E}_3(a)} \le C_0\big[P(\|\nabla h\|_\infty)+Q(\|\nabla h\|_{BC(J;BC^1)}, \|h\|_{\mathbb{E}_4(a)})\big]\|\bar h\|_{{}_0\mathbb{E}_4(a)} \tag{5.26}$$

for all $h\in\mathbb{E}_4(a)$ and all $a\in(0,a]$, where P and Q are polynomials with coefficients equal to one and vanishing zero-order terms. Analogous arguments can be used for the remaining terms $(\nabla h|\nabla^2 h\nabla h)/\beta^3$ and $G_\kappa(h)\partial_j h$ appearing in G, yielding the same estimate as in (5.26).

(iv) It remains to consider the nonlinear term $H_b(v,h) := (b-\gamma v|\nabla h)$. The Fréchet derivative is given by $DH_b(v,h)[\bar v,\bar h] = -(\nabla h|\gamma\bar v) + (b-\gamma v|\nabla\bar h)$. From Lemma 5.5(b) with $g_1=\partial_j h$ and $g_2=\gamma\bar v_k$, where $\bar v_k$ denotes the k-th component of $\bar v$, follows $\|(\nabla h|\gamma\bar v)\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)})\|\bar v\|_{{}_0\mathbb{E}_1(a)}$. Lemma 5.5(c) with $g=(b-\gamma v)_k$ and $h=\bar h$ implies

$$\|(b-\gamma v|\nabla\bar h)\|_{{}_0\mathbb{F}_4(a)} \le C_0(\|b-\gamma v\|_\infty + \|b-\gamma v\|_{\mathbb{F}_4(a)})\|\bar h\|_{{}_0\mathbb{E}_4(a)}.$$

We have, thus, shown that

$$\|DH_b(v,h)\| \le C_0(\|\nabla h\|_\infty + \|h\|_{\mathbb{E}_4(a)} + \|b-\gamma v\|_{BC(J;BC)\cap\mathbb{F}_4(a)}). \tag{5.27}$$

Combining the estimates in (5.16), (5.20) and (5.22)–(5.27) yields the assertions of Proposition 4.1. ☐

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