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A REGULARITY THEORY FOR STOCHASTIC PROCESSES WITH STATIONARY INCREMENTS AND SPECTRAL DENSITY WITH APPLICATIONS TO ANOMALOUS DIFFUSION

STEFAN SPERLICH

Abstract. Aim of this paper is the introduction to a rigorous regularity theory with respect to stochastic processes with stationary increments and spectral density. Once provided we will conclude with applications to the problems of anomalous diffusion.

1. Introduction

Aim of the present paper is to provide a regularity theory for a satisfactory large class of random motions, that are stochastic processes with stationary increments and spectral density. The paper is structured as follows. In Section 2 we define some mathematical notations. Then, in Sections 3 and 4 we introduce a rigorous regularity theory and we conclude the paper with Section 5, where we present applications to the problems of fractional diffusion.

2. Foundations

Let X be a Banach space. For an open subset $D \subset \mathbb{R}^N$, $\mathrm{H}_p^m(D;X)$ with $m \in \mathbb{N}$ denotes the classical Sobolev space, that is the space of all functions $f : D \to X$ having distributional derivatives $\partial^\alpha f \in L_p(D;X)$ of order $0 \leq |\alpha| \leq m$. For $1 \leq p < \infty$ the norm in $\mathrm{H}_p^m(D;X)$ is given by

$$\|f\|_{\mathrm{H}_p^m(D;X)} := \left[\sum_{|\alpha| \leq m} \|\partial^\alpha f\|_p^p \right]^{\frac{1}{p}} .$$

Further, for $0 < s < 1$, we define the Bessel potential spaces $\mathrm{H}_p^{sm}(D;X)$, by means of complex interpolation via

$$\mathrm{H}_p^{sm}(D;X) := \left[L_p(D;X); \mathrm{H}_p^m(D;X) \right]_s .$$

Note further, that in case $p = 2$ and X is a Hilbert or UMD space (see e.g. Amann [1] for the definition and properties of UMD spaces) we have

$$\mathrm{H}_2^s(D;X) = W_2^s(D;X), \qquad s \geq 0,$$

where $W_p^s(D;X)$ denotes the Sobolev-Slobodeckij space. For a general definition of these spaces we refer to Triebel [8] or Runst & Sickel [7]. With $s = [s] + \{s\}$,

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where $[s]$ is an integer and $0 < \{s\} < 1$, the intrinsic norm in $W_p^s(\mathbb{R}^N;X)$ can be written as

$$\|f\|_{W_p^s(\mathbb{R}^N;X)} = \|f\|_{W_p^{[s]}(\mathbb{R}^N;X)} + \\ + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\partial^\alpha f(x) - \partial^\alpha f(y)|_X^p}{|x-y|^{N+p\{s\}}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{p}}. \quad (2.1)$$

Note, that the second term from the right hand-side of (2.1) defines a semi-norm in $W_p^s(\mathbb{R}^N;X)$, which will be abbreviated by $[f]_{W_p^s(\mathbb{R}^N;X)}$ if necessary.

If $U \subset \mathbb{R}^N$ is a subset of $\mathbb{R}^N$, then $\mathrm{H}_2^s(U;X)$ denotes the restriction of the functions $f \in \mathrm{H}_2^s(\mathbb{R}^N;X)$ to the subset U. In case $J = [0,a]$ is an interval, we denote by ${}_0\,\mathrm{H}_p^s(J;X)$ the space of all functions $f : J \to X$ belonging to $\mathrm{H}_p^s(J;X)$, such that $f|_{t=0} = 0$, whenever the trace at $t = 0$ exists.

We will further consider weighted L_2 and W_2^s spaces. For $J := [0,a]$, $a > 0$, and a number $\mu \geq 0$ they are defined canonically via

$$L_{2,\mu}(J;X) := \{f : J \to X : (\cdot)^\mu f \in L_2(J;X)\},$$
$$W_{2,\mu}^s(J;X) := \{f : J \to X : (\cdot)^\mu f \in W_2^s(J;X)\}.$$

It is easy to verify that $L_2(J;X) = L_{2,0}(J;X) \hookrightarrow L_{2,\mu_1}(J;X) \hookrightarrow L_{2,\mu_2}(J;X)$ holds if and only if $\mu_1 \leq \mu_2$. With ${}_0W_{2,\mu}^s(J;X)$ we denote the space of all $W_{2,\mu}^s(J;X)$-functions whose trace at $t = 0$ is zero, if it exists.

Thanks to Hardy et al. [4, Theorem 329] we have the useful imbedding result

Lemma 2.1. *Let V be a Banach space, $0 < \mu < 1$, and $0 < \sigma < \mu$. Then*

$${}_0W_{2,\mu}^\sigma(\mathbb{R}_+;V) \hookrightarrow L_{2,\mu-\sigma}(\mathbb{R}_+;V).$$

2.1. **Stochastic processes with stationary increments.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let J be an interval of $\mathbb{R}$. An arbitrary family $\{X(t)\}_{t\in J}$ defined on Ω, such that $X(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable for each $t \in J$ is called a stochastic process and we set $X(t,\omega) = X(t)(\omega)$ for all $t \in J$ and $\omega \in \Omega$. The functions $X(\cdot,\omega)$ are called trajectories of X.

In what follows we denote by

$$D_3(t;u,v) := \mathbb{E}[(X(u) - X(t))(X(v) - X(t))]$$

the structure function of the real-valued process X.

Definition 2.2 (Processes with stationary increments)**.** *We call the random process $X = \{X(t)\}_{t\in\mathbb{R}}$ a process with stationary increments if*

(i) the mean value of its increments depends only on the length $t-s$ of the interval $[s,t]$, i.e.

$$\mathbb{E}[X(t) - X(s)] = \mathbb{E}[X(t-s) - X(0)];$$

(ii) *for $u, v, t \in \mathbb{R}$ the structure function $D_3(t; u, v)$ of the process X depends only on the differences $u - t$ and $v - t$, i.e.*

$$D_3(t; u, v) = D_3(0; u - t, v - t) =: D_2(u - t, v - t).$$

At this point we should be more careful and say that the processes under consideration have stationary increments in the wider sense. However this refinement is unnecessary in this paper where more special processes with strictly stationary increments will not be considered at all. Definition 2.2 particularly yields that a real-valued process X with stationary increments, which in addition satisfies $\mathbb{E}[X(0)] = 0$ is characterized by a function (the mean of the increments) of one variable

$$\mathbb{E}[X(t + \tau) - X(t)] =: m(\tau) \tag{2.2}$$

and by a function $D(\cdot)$ of one variable

$$\mathbb{E}|X(t + \tau) - X(t)|^2 =: D(\tau). \tag{2.3}$$

The function $D_2(\cdot, \cdot)$ can then be obtained via the identity

$$D_2(\tau_1, \tau_2) = \frac{1}{2}[D(\tau_1) + D(\tau_2) - D(|\tau_1 - \tau_2|)]. \tag{2.4}$$

Definition 2.3 (Centered processes). *A process $X := \{X(t)\}_{t \in J}$ is called centered, if $\mathbb{E}[X(t)] = 0$ holds for all $t \in J$.*

Remark 2.4. Observe, that if the process X is centered, then

$$D(\tau) = \mathrm{Var}[X(\tau)] \quad \text{and} \quad D_2(\tau_1, \tau_2) = \mathrm{Cov}[X(\tau_1), X(\tau_2)]. \tag{2.5}$$

Looking for a general form of the function $D(\tau) = D_2(\tau, \tau)$ we follow Yaglom [9, Chapter 4] and note that a process with stationary increments admits a spectral density if there is an even and nonnegative function ϕ so that

$$D(\tau) = 4 \int_0^\infty (1 - \cos \lambda\tau)\phi(\lambda)\mathrm{d}\lambda, \tag{2.6}$$

and moreover for any $\lambda_0 > 0$

$$\int_0^{\lambda_0} \lambda^2 \phi(\lambda)\mathrm{d}\lambda + \int_{\lambda_0}^\infty \phi(\lambda)\mathrm{d}\lambda < \infty. \tag{2.7}$$

3. Regularity Theory

From now on we will exclude the case $\phi \equiv 0$, since this case merely corresponds to the trivial process $X \equiv 0$ a.s. In view on regularity results, we may classify the processes under consideration with respect to their spectral densities. The first class contains all processes X which satisfy

Hypothesis (ϕ). *X is a real-valued process with stationary increments and $X(0) = 0$ a.s. Furthermore, the spectral density ϕ of X exists and there is a number $1 < \gamma < 3$, so that*

$$\sup_{\lambda \in \mathbb{R}} |\lambda|^{\gamma} \phi(\lambda) < \infty.$$

The second class is abstractly formulated as

Hypothesis (ϕ_0). *X is a real-valued process with stationary increments and $X(0) = 0$ a.s. Furthermore, the spectral density ϕ of X exists and there are numbers $1 < \gamma_0 < 3$, $\lambda_0 > 0$ and $\theta \geq 0$, so that*

(a) $\inf_{0<\lambda<\lambda_0} \lambda^{\gamma_0} \phi(\lambda) > 0$,
(b) $\limsup_{\lambda \to 0} |\lambda|^{\gamma_0} \phi(\lambda) < \infty$,
(c) $\phi(\tau\lambda) \geq \tau^{-(\gamma_0+\theta)} \phi(\lambda)$ *for all* $\lambda \in (0, \infty)$ *and* $\tau \geq 1$.

Remark 3.1. Note the following.

(1) The restrictions $\gamma_0 < 3$ and $\theta \geq 0$ are evident since the spectral density ϕ must satisfy condition (2.7). Moreover, the restriction $\gamma < 3$ (resp. $\gamma > 1$) is nontrivial if $\lambda = 0$ is not contained in the spectrum of X (resp. the spectrum of X is bounded).
(2) In view of applications one should always be exerted to choose the number θ preferably small in order to achieve optimal regularity results (see Theorems 3.10 and 3.13 below).
(3) Suppose the process X is subject to Hypothesis (ϕ_0) with the number θ chosen to be as small as possible.
    (a) For any fixed $h > 0$, the spectral density $\phi_h(\lambda) = 2(1 - \cos\lambda h)\phi(\lambda)$ of the incremental process $X_h(t) := [X(t + h) - X(t)]$ has a singularity at frequency zero if and only if $\gamma_0 > 2$. It is frequently claimed in literature, that in this case X displays long-range dependence in the sense that the dependence between the increments $[X(1) - X(0)]$ and $[X(n + 1) - X(n)]$ decays slowly as n tends to infinity and

$$\sum_{n=1}^{\infty} \mathrm{Cov}[X(1) - X(0), X(n + 1) - X(n)] = \infty.$$

    However, we stress that this is not true in general (cf. Gubner [3]).
    (b) If $\theta > 0$, then ϕ has a significance in its behavior when $|\lambda| \to \infty$. As a consequence the correlation of consecutive small increments of X exceeds the correlation of consecutive large increments. This phenomena is called intermittency in turbulence literature (e.g. Frisch [2]).

Example 3.2. Suppose X is a process with stationary increments and $X(0) = 0$ a.s. Assume the spectral density ϕ exists and is of the form

$$\phi(\lambda) = \frac{1}{|\lambda|^{\alpha}(1 + |\lambda|^{s})^{\beta}}, \qquad 1 < \alpha < 3, \quad s \geq 0, \quad \beta \geq 0.$$

Then X satisfies Hypothesis (ϕ), whereby the number γ can be chosen in $[\alpha, \alpha + s\beta] \cap [\alpha, 3)$, since for this selection

$$|\lambda|^{\gamma}\phi(\lambda) = \frac{1}{|\lambda|^{\alpha-\gamma}(1+|\lambda|^s)^{\beta}}$$

is clearly bounded on $\mathbb{R}$. Moreover, X is due to Hypothesis (ϕ_0) with $\gamma_0 = \alpha$ and $\theta \geq s\beta$. This is apparent because

$$|\lambda|^{\gamma_0}\phi(\lambda) = \frac{1}{|\lambda|^{\alpha-\gamma_0}(1+|\lambda|^s)^{\beta}}$$

is strictly positive and bounded in a neighborhood of $\lambda = 0$ if and only if $\gamma_0 = \alpha$ and

$$\begin{aligned}\phi(\tau\lambda) = \frac{1}{(\tau\lambda)^{\gamma_0}(1+|\tau\lambda|^s)^{\beta}} &= |\tau|^{-(\gamma_0+s\beta)}\frac{1}{\lambda^{\gamma_0}(|\tau|^{-s}+|\lambda|^s)^{\beta}} \\ &\geq |\tau|^{-(\gamma_0+s\beta)}\frac{1}{\lambda^{\gamma_0}(1+|\lambda|^s)^{\beta}} = |\tau|^{-(\gamma_0+s\beta)}\phi(\lambda)\end{aligned}$$

obviously holds true for all $|\tau| \geq 1$.

As we have seen in the previous example, there are processes which are due to both, Hypotheses (ϕ) and (ϕ_0). However, this is not true in general.

The following proposition clarifies, how Hypotheses (ϕ) and (ϕ_0) are connected. For brevity we define

$$\begin{aligned} f(\lambda) &:= |\lambda|^{\gamma}\phi(\lambda), & g(\lambda) &:= |\lambda|^{\gamma_0}\phi(\lambda), \\ S_f &:= \sup_{\lambda\in\mathbb{R}} f(\lambda), & I_g &:= \inf_{0<\lambda<\lambda_0} g(\lambda). \end{aligned} \tag{3.1}$$

Note that f, g, S_f and I_g depend on the parameters γ, γ_0 and λ_0, respectively.

Proposition 3.3. *Let X satisfy Hypotheses (ϕ) and (ϕ_0), then $\gamma_0 \leq \gamma \leq \gamma_0 + \theta$ and $S_f \geq \lambda_0^{\gamma-\gamma_0} I_g$. Moreover, $\theta = 0$ is equivalent to $f = g$. If this is the case, then the remainders are nondecreasing on the half-line $(0, \infty)$ and satisfy*

$$I_g = \lim_{\lambda\to 0} f(\lambda) \leq \lim_{|\lambda|\to\infty} f(\lambda) = S_f.$$

Proof. Observe the estimate for $|\tau| \geq 1$ and $\lambda \in (0, \lambda_0)$

$$|\lambda|^{-\gamma}|\tau|^{-\gamma}S_f \geq |\lambda|^{-\gamma}|\tau|^{-\gamma}f(\tau\lambda) \geq |\lambda|^{-\gamma_0}|\tau|^{-\gamma_0-\theta}g(\lambda) \geq |\lambda|^{-\gamma_0}|\tau|^{-\gamma_0-\theta}I_g,$$

in particular

$$\frac{S_f}{I_g} \geq |\lambda|^{\gamma-\gamma_0}|\tau|^{\gamma-\gamma_0-\theta}.$$

Thus necessarily $\gamma_0 \leq \gamma \leq \gamma_0 + \theta$ and $S_f \geq \lambda_0^{\gamma-\gamma_0} I_g$. The case $\theta = 0$ corresponds to $\gamma_0 = \gamma$ and therewith $f = g$. If this is the case then f is obviously bounded and $f(\lambda) \geq I_g$ in a neighborhood of $\lambda = 0$. Moreover f satisfies the growth condition $f(\tau\lambda) \geq f(\lambda)$ for all $\lambda > 0$ and $\tau \geq 1$, thus f is nondecreasing on $(0, \infty)$ and, since f is an even function, nonincreasing on $(-\infty, 0)$. □

Corollary 3.4. *Let X satisfy Hypotheses (ϕ) and (ϕ_0) with $\theta > 0$. Then the choice $\gamma = \gamma_0$ is admissible. If, in addition, $\limsup_{\lambda\to\infty} \lambda^\theta g(\lambda) < \infty$, then any selection $\gamma \in [\gamma_0, \gamma_0 + \theta] \cap [\gamma_0, 3)$ is feasible.*

Proof. Recall, that by Proposition 3.3 we have $\gamma_0 \le \gamma \le \gamma_0 + \theta$. To justify the choice $\gamma = \gamma_0$ we have to show that $\sup_{\lambda>0} \lambda^{\gamma_0}\phi(\lambda) < \infty$. This can be seen from

$$|\lambda|^{\gamma_0-\gamma} f(\lambda) = |\lambda|^{\gamma_0}\phi(\lambda) = g(\lambda).$$

Turning to the second claim, we need to verify $\sup_{\lambda>0} \lambda^{\gamma_0+\theta}\phi(\lambda) < \infty$, which follows with

$$|\lambda|^{\gamma_0+\theta-\gamma} f(\lambda) = |\lambda|^{\gamma_0+\theta}\phi(\lambda) = |\lambda|^\theta g(\lambda). \qquad \square$$

Theorem 3.5. *The following are true.*

(i) Let X be subject to Hypothesis (ϕ). Then the estimate

$$D(\tau) \le c_\phi |\tau|^{\gamma-1}, \qquad c_\phi := c_\phi(\gamma) = 2^{4-\gamma}\int_0^\infty \frac{\sin^2(\lambda)}{\lambda^\gamma}\mathrm{d}\lambda \cdot \sup_{\lambda\in\mathbb{R}} |\lambda|^\gamma \phi(\lambda) \tag{3.2}$$

holds for all $\tau \in \mathbb{R}$. Moreover, (3.2) holds with equality if $|\lambda|^\gamma\phi(\lambda)$ is identically constant.

(ii) Let X be subject to Hypothesis (ϕ_0). Then the estimate

$$D(\tau) \ge c_{\phi_0} \cdot \min\{|\tau|^{\gamma_0-1+\theta}, |\tau|^{\gamma_0-1}\},$$
$$c_{\phi_0} := c_{\phi_0}(\gamma_0, \lambda_0) = 2^{4-\gamma_0}\int_0^{\lambda_0/2} \frac{\sin^2(\lambda)}{\lambda^{\gamma_0}}\mathrm{d}\lambda \cdot \inf_{|\lambda|<\lambda_0} |\lambda|^{\gamma_0}\phi(\lambda)$$

holds for all $\tau \in \mathbb{R}$.

Proof. It is particularly seen from (2.6) that $D(\tau) = D(-\tau)$ which entails, that this proof can be reduced to the case $\tau \ge 0$. If $\tau = 0$ then trivially $D(\tau) = 0$ so that it suffices to prove the claim for $\tau > 0$. The results then follow from the spectral representation (2.6) of the function D, because for all $\tau > 0$ it is

$$D(\tau) = 4\int_0^\infty (1 - \cos\xi\tau)\phi(\xi)\mathrm{d}\xi$$
$$= 8\int_0^\infty \sin^2\left(\frac{\xi\tau}{2}\right)\phi(\xi)\mathrm{d}\xi = \frac{16}{\tau}\int_0^\infty \sin^2(\lambda)\phi\left(\frac{2\lambda}{\tau}\right)\mathrm{d}\lambda.$$

Observe now that $\int_0^\infty \sin^2(\lambda)/\lambda^\alpha \mathrm{d}\lambda$ exists if $1 < \alpha < 3$. Then by means of notations (3.1), assertion (i) follows with

$$D(\tau) = 2^{4-\gamma}\tau^{\gamma-1}\int_0^\infty \frac{\sin^2\lambda}{\lambda^\gamma} f\left(\frac{2\lambda}{\tau}\right)\mathrm{d}\lambda \le \left[2^{4-\gamma} S_f \int_0^\infty \frac{\sin^2\lambda}{\lambda^\gamma}\mathrm{d}\lambda\right]\tau^{\gamma-1} = c_\phi\tau^{\gamma-1},$$

while for $0 < \tau < 1$ (ii) is a consequence of the growth condition (ϕ_0)(c), because

$$D(\tau) = 2^{4-\gamma_0}\tau^{\gamma_0-1}\int_0^\infty \frac{\sin^2\lambda}{\lambda^{\gamma_0}} g\left(\frac{2\lambda}{\tau}\right)\mathrm{d}\lambda$$
$$\geq \left[2^{4-\gamma_0} I_g \int_0^{\lambda_0/2} \frac{\sin^2\lambda}{\lambda^{\gamma_0}}\mathrm{d}\lambda\right]\tau^{\gamma_0-1+\theta} = c_{\phi_0}\tau^{\gamma_0-1+\theta}.$$

In case $\tau \geq 1$ assertion (ii) follows with strict positivity of the remainder g in a neighborhood of $\lambda = 0$. Then

$$D(\tau) = 2^{4-\gamma_0}\tau^{\gamma_0-1}\int_0^\infty \frac{\sin^2\lambda}{\lambda^{\gamma_0}} g\left(\frac{2\lambda}{\tau}\right)\mathrm{d}\lambda$$
$$\geq \left[2^{4-\gamma_0} I_g \int_0^{\lambda_0/2} \frac{\sin^2\lambda}{\lambda^{\gamma_0}}\mathrm{d}\lambda\right]\tau^{\gamma_0-1} = c_{\phi_0}\tau^{\gamma_0-1}. \qquad \square$$

Following the latter proof we may outline, that the growth condition (c) of Hypothesis (ϕ_0) is only involved when $|\tau| < 1$ and that the constant c_ϕ particularly depends on the parameter γ. This is crucial to remember in situations where the parameter γ is not uniquely determined by Hypothesis (ϕ). As an immediate consequence of Theorem 3.5(i) we obtain

Corollary 3.6. *Suppose X is subject to Hypothesis (ϕ) and denote by Γ the set of all feasible γ. Then the estimate*

$$D(\tau) \leq \inf\left\{c_\phi(\gamma)|\tau|^{\gamma-1} : \gamma \in \Gamma\right\}$$

holds for all $\tau \in \mathbb{R}$, with $c_\phi := c_\phi(\gamma)$ from Theorem 3.5(i).

If the process X is in particular centered and Gaussian, then we obtain L_p-estimates with the aid of the Kahane-Khinchine inequality.

Corollary 3.7. *Let X be a centered Gaussian process.*

(i) If X is subject to Hypothesis (ϕ), then for every $p \in (2,\infty)$ there is a constant $c > 0$ such that the estimate

$$\mathbb{E}|X(t) - X(s)|^p \leq c|t-s|^{\frac{p(\gamma-1)}{2}}$$

holds for all $t, s \in \mathbb{R}$.

(ii) If X is subject to Hypothesis (ϕ_0), then for every $p \in (1,2)$ there is a constant $c_0 > 0$ such that the estimate

$$\mathbb{E}|X(t) - X(s)|^p \geq c_0 \cdot \min\{|t-s|^{\frac{p(\gamma_0-1+\theta)}{2}}, |t-s|^{\frac{p(\gamma_0-1)}{2}}\}$$

holds for all $t, s \in \mathbb{R}$.

Proof. With the aid of the Kahane-Khinchine inequality, the claim (i) follows directly from Theorem 3.5(i) because

$$\mathbb{E}|X(t)-X(s)|^p \le c\left(\mathbb{E}|X(t)-X(s)|^2\right)^{p/2} = c\,(D(t-s))^{p/2} \le c|t-s|^{p\frac{\gamma-1}{2}},$$

holds for all $p>2$. Here the constant $c>0$ is generic and may depend on p. The second assertion follows in a similar manner. □

We now may take a closer look to the correlation of the increments in case X is centered. We start with the case of small consecutive increments.

Proposition 3.8. *Assume X is centered and satisfies Hypotheses (ϕ) and (ϕ_0) with $\gamma=\gamma_0+\theta$. Denote by c_ϕ and c_{ϕ_0} be the constants from Theorem 3.5 and let $0<\tau\le\frac{1}{2}$.*

(i) If $\gamma_0 < 2-\log_2(c_\phi/c_{\phi_0})-\theta$, then the increments $[X(t)-X(t-\tau)]$ and $[X(t+\tau)-X(t)]$ are negative correlated.
(ii) If $\gamma_0 > 2+\log_2(c_\phi/c_{\phi_0})-\theta$, then the increments $[X(t)-X(t-\tau)]$ and $[X(t+\tau)-X(t)]$ are positive correlated.

Proof. With the aid of identity (2.4), a direct computation verifies

$$\begin{aligned}\mathrm{Cov}[X(t)-X(t-\tau),X(t+\tau)-X(t)] &= \mathbb{E}[(X(t)-X(t-\tau))(X(t+\tau)-X(t))]\\ &= -\mathbb{E}[X(-\tau)X(\tau)] = \frac{1}{2}\left(\mathbb{E}[X(2\tau)-\mathbb{E}[X(-\tau)]^2-\mathbb{E}[X(\tau)]^2\right)\end{aligned}$$

and it is a matter of the stationarity of the increments and the assumption $\mathbb{E}[X(0)]=0$, that

$$\mathbb{E}[X(-\tau)]^2 = \mathbb{E}[X(0)-X(-\tau)]^2 = \mathbb{E}[X(\tau)-X(0)]^2 = \mathbb{E}[X(\tau)]^2.$$

Thus, we already have

$$\mathbb{E}[(X(t)-X(t-\tau))(X(t+\tau)-X(t))] = \frac{1}{2}\mathbb{E}[X(2\tau)]^2-\mathbb{E}[X(\tau)]^2$$

and we may employ Theorem 3.5 to estimate

$$\begin{aligned}\frac{c_{\phi_0}}{2}(2\tau)^{\gamma_0-1+\theta}-c_\phi\tau^{\gamma-1} &\le \mathbb{E}[(X(t)-X(t-\tau))(X(t+\tau)-X(t))]\\ &\le \frac{c_\phi}{2}(2\tau)^{\gamma-1}-c_{\phi_0}\tau^{\gamma_0-1+\theta}.\end{aligned}$$

Recall that $\gamma=\gamma_0+\theta$ by assumption. Thus

$$\begin{aligned}(c_{\phi_0}2^{\gamma_0+\theta-2}-c_\phi)\tau^{\gamma_0+\theta-1} &\le \mathbb{E}[(X(t)-X(t-\tau))(X(t+\tau)-X(t))]\\ &\le (c_\phi 2^{\gamma_0+\theta-2}-c_{\phi_0})\tau^{\gamma_0+\theta-1}.\end{aligned}$$

and the result is immediate. □

Following the same strategy one observes a result for large increments of X.

Proposition 3.9. *Assume X is centered and satisfies Hypotheses (ϕ) and (ϕ_0). Denote by c_ϕ and c_{ϕ_0} the constants from Theorem 3.5 and let $\tau\ge 1$.*

(i) If $\gamma_0 < 2 - \log_2(c_\phi/c_{\phi_0})$, then the increments $[X(t) - X(t-\tau)]$ and $[X(t+\tau) - X(t)]$ are negative correlated.
(ii) If $\gamma_0 > 2 + \log_2(c_\phi/c_{\phi_0})$, then the increments $[X(t) - X(t-\tau)]$ and $[X(t+\tau) - X(t)]$ are positive correlated.

The subsequent theorem yields first regularity results in the pathwise sense. Due to the available L_p-estimates for centered Gaussian processes the results can be enhanced in this case.

Theorem 3.10. *The following are true.*

(i) Let X be subject to Hypothesis (ϕ). If $\gamma > 2$, then X is mean-square continuous and has continuous paths almost sure. Moreover, the trajectories of X are locally Hölder-continuous of order $\alpha < \frac{\gamma-2}{2}$ with probability 1.
(ii) Let X be a centered Gaussian process subject to Hypothesis (ϕ). Then X is mean-square continuous and has continuous paths almost sure. Moreover, the trajectories of X are locally Hölder-continuous of order $\alpha < \frac{\gamma-1}{2}$ with probability 1.
(iii) Let X be subject to Hypothesis (ϕ_0). If $\theta < 3 - \gamma_0$ then X is almost surely nowhere mean-square differentiable.

Proof. Assertion (i) is immediate by Theorem 3.5(i), which in particular yields

$$\mathbb{E}[X(t) - X(s)]^2 \le c|t-s|^{1+(\gamma-2)}, \qquad t, s \in \mathbb{R}.$$

Thus the Kolmogorov-Čentsov-Theorem proves the claim. Regarding (ii) Corollary 3.7 yields, that for every $p \in (2, \infty)$ there is a constant $c > 0$ such that

$$\mathbb{E}|X(t) - X(s)|^p \le c|t-s|^{1+\frac{p(\gamma-1)-2}{2}},$$

for all $s, t \in \mathbb{R}$. Employing the Kolmogorov-Čentsov-Theorem yields the Hölder-continuity on every bounded subset of $\mathbb{R}$ of order $\alpha < \frac{\gamma-1}{2} - \frac{1}{p}$ for every $2 < p < \infty$. In case (iii) the nowhere differentiability in the $L_2(\Omega)$-sense follows with

$$\lim_{s\to t} \left| \frac{\mathbb{E}[X(t) - X(s)]^2}{|t-s|^2} \right| = \lim_{s\to t} \frac{\mathbb{E}[X(t-s)]^2}{|t-s|^2} \ge c_0 \lim_{s\to t} |t-s|^{\gamma_0-3+\theta} = \infty. \qquad \square$$

Corollary 3.11. *Let X be subject to Hypothesis (ϕ). If $\gamma > 2$, then X is centered.*

Proof. Observe that the function m given by (2.2) satisfies

$$m(\tau_1) + m(\tau_2) = m(\tau_1 + \tau_2)$$

and, since $\gamma > 2$, the function m is continuous by Theorem 3.10(i), which in turn yields $m(\tau) = c_1\tau$ where c_1 is a constant. Because

$$D(\tau) = (\mathbb{E}[X(t+\tau) - X(t)])^2 + \mathrm{Var}[X(t+\tau) - X(t)] = c_1^2\tau^2 + \mathrm{Var}[X(t+\tau) - X(t)],$$

where, as a matter of course $\mathrm{Var}[X(t+\tau) - X(t)] \ge 0$, the function $D(\tau)$ includes a term proportional to τ^2 in all the cases where $c_1 \ne 0$. Theorem 3.5 yields

$D(\tau) \leq c|\tau|^{\gamma-1}$ for all $\tau \in \mathbb{R}$ with $\gamma - 1 < 2$. Hence $c_1 = 0$, which is equivalent to $0 = m(\tau) = \mathbb{E}[X(t+\tau) - X(t)] = \mathbb{E}[X(\tau)]$ for all $\tau \in \mathbb{R}$. □

Corollary 3.12. *Let X be a Gaussian process subject to Hypothesis (ϕ). If $\gamma > 2$, then with probability 1 the trajectories of X are locally Hölder-continuous of order $\alpha < \frac{\gamma-1}{2}$.*

Proof. The result is immediate by Corollary 3.11 and Theorem 3.10(ii). □

We are now in the position to formulate first results in the $L_p(\Omega)$-sense.

Theorem 3.13. *Let $T > 0$, $J = [0, T]$, $p \in (0, \infty)$ and $0 < \sigma < 1$.*

(i) Suppose X satisfies Hypothesis (ϕ). If $2\sigma < \gamma - 1$, then $X \in {}_0W_2^\sigma(J; L_2(\Omega))$.
(ii) Suppose X satisfies Hypothesis (ϕ_0). If $2\sigma \geq \gamma_0 - 1 + \theta$, then $X \notin {}_0W_2^\sigma(J; L_2(\Omega))$.
(iii) Suppose X is a centered Gaussian process subject to Hypothesis (ϕ) and let $2 \leq q < \infty$. If $2\sigma < \gamma - 1$, then $X \in {}_0W_p^\sigma(J; L_q(\Omega))$.
(iv) Suppose X is a centered Gaussian process subject to Hypothesis (ϕ_0) and let $1 < q \leq 2$. If $2\sigma \geq \gamma_0 - 1 + \theta$, then $X \notin {}_0W_p^\sigma(J; L_q(\Omega))$.

Proof. In view of Theorem 3.5 and Corollary 3.7, assertions (iii) and (iv) will be shown directly considering the semi-norm $[\cdot]_\sigma$ in ${}_0W_p^\sigma(J; L_q(\Omega))$ and exploiting the stationarity of the increments. Assertion (iii) follows with

$$
\begin{aligned}
[X]_\sigma^p &= \int_J \int_J \frac{(\mathbb{E}|X(t) - X(s)|^q)^{\frac{p}{q}}}{|t-s|^{1+p\sigma}} \mathrm{d}s\mathrm{d}t \leq c \int_J \int_J \frac{|t-s|^{\frac{p(\gamma-1)}{2}}}{|t-s|^{1+p\sigma}} \mathrm{d}s\mathrm{d}t \\
&= c \int_J \int_J |t-s|^{\frac{p(\gamma-1)}{2} - 1 - p\sigma} \mathrm{d}s\mathrm{d}t = 2c \int_0^T \int_0^t (t-s)^{\frac{p(\gamma-1)}{2} - 1 - p\sigma} \mathrm{d}s\mathrm{d}t
\end{aligned}
$$

and the last integral is finite, if and only if $\sigma < \frac{\gamma-1}{2}$. Turning to (iv) we restrict J to $[0, 1]$ and achieve analogously

$$
[X]_\sigma^p \geq 2c_0 \int_0^1 \int_0^t (t-s)^{\frac{p(\gamma_0 - 1 + \theta)}{2} - 1 - p\sigma} \mathrm{d}s\mathrm{d}t
$$

and the right hand-side is finite if and only if $\sigma < \frac{\gamma_0 - 1 + \theta}{2}$. To prove assertions (i) and (ii) set $p = q = 2$ and repeat the above arguments. □

4. Deterministic multipliers

Throughout this section let $J = [0, T]$, $G \subset \mathbb{R}^N$, $W = L_2(J; L_2(G; L_2(\Omega)))$, and $Y = L_2(J; L_2(\partial G; L_2(\Omega)))$. Aim of this section is to deduce regularity properties of the function

$$
\zeta(t, x, \omega) := \sum_{k=1}^{\infty} b_k(t, x) X_k(t, \omega) \tag{4.1}
$$

where $(X_i)_{i\in\mathbb{N}}$ are entirely independent processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and satisfying Hypothesis **(**ϕ**)** (see page 4). The scalar functions $b_i \in L_2(J; L_2(\partial G))$, $i \in \mathbb{N}$, are supposed to be deterministic.

Turning to spatial regularity, we furnish a sufficient and necessary conditions on the multiplier $b := (b_i)_{i\in\mathbb{N}}$, so that the boundary disturbance ζ affiliates to the space

$$Y_s := L_2(J; {}_0W_2^s(\partial G; L_2(\Omega))), \qquad s \geq 0. \tag{4.2}$$

Note that Y_0 is isometrically isomorphic to the basic space Y.

Theorem 4.1. *Let $s \geq 0$, $G \subset \mathbb{R}^N$ be a domain with $C^{[s]+1}$-boundary and ζ given by (4.1).*

*(i) Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of mutually independent processes with a unique spectral density ϕ subject to Hypothesis **(**ϕ**)**. Then*

$$b \in L_{2,\frac{\gamma-1}{2}}(J; {}_0W_2^s(\partial G; \ell_2)) \quad \Longrightarrow \quad \zeta \in L_2(J; {}_0W_2^s(\partial G; L_2(\Omega))).$$

*(ii) Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of mutually independent processes with a unique spectral density ϕ subject to Hypothesis **(**ϕ_0**)**. Then*

$$\zeta \in L_2(J; {}_0W_2^s(\partial G; L_2(\Omega))) \quad \Longrightarrow \quad b \in L_{2,\frac{\gamma_0-1+\theta}{2}}(J; {}_0W_2^s(\partial G; \ell_2)).$$

Proof. Without loss of generality we set $0 < s < 1$. Starting with claim (i), the presence of Theorem 3.5(i) and a straight forward computation gives

$$\begin{aligned}\|\zeta\|_{Y_s} &= \left\| \left(\mathbb{E} \left| \sum_{k=1}^{\infty} b_k(t,x) X_k(t) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} \\ &= \left\| \left(\mathbb{E} \left[\sum_{k=1}^{\infty} |b_k(t,x)X_k(t)|^2 + \sum_{k\neq l} b_k(t,x)b_l(t,x)X_k(t)X_l(t) \right] \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))}\end{aligned}$$

and with the aid of the independence of X_i and X_j for $i \neq j$ we proceed with

$$\begin{aligned}\|\zeta\|_{Y_s} &= \left\| \left(\sum_{k=1}^{\infty} |b_k(t,x)|^2 \mathbb{E}|X_k(t)|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} \\ &\leq c \left\| \left(\sum_{k=1}^{\infty} |t^{\frac{\gamma-1}{2}} b_k(t,x)|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} = c\|b\|_{L_{2,\frac{\gamma-1}{2}}(J;{}_0W_2^s(\partial G;\ell_2))}.\end{aligned}$$

Turning to (ii) we recall that is suffices to prove the claim for $J = [0, 1]$ and obtain in similar fashion to (i)

$$\|\zeta\|_{Y_s} \geq c_0 \|b\|_{L_{2,\frac{\gamma_0-1+\theta}{2}}(J;{}_0W_2^s(\partial G;\ell_2))}.$$

□

Our next aim is to deduce conditions on b, so that the boundary disturbance ζ admits some time regularity. To this end we provide some technical tools with the subsequent lemmata.

Lemma 4.2. *Suppose X is subject to Hypotheses (ϕ) with $|\lambda|^\gamma \phi(\lambda) \equiv \text{const}$ and let $b : J \to \mathbb{R}$ a deterministic function. Then there is a constant $c > 0$, so that*

$$\begin{aligned}&\mathbb{E}\left[b(t)X(t) - b(s)X(s)\right]^2 \\ &\qquad = c\left[\left(b(t)t^{\frac{\gamma-1}{2}} - b(s)s^{\frac{\gamma-1}{2}}\right)^2 + |b(t)b(s)|\left(|t-s|^{\gamma-1} - \left[t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}}\right]^2\right)\right]\end{aligned}$$

holds for all $s, t \in J$.

Proof. The claim can be shown directly with the aid of Theorem 3.5(i).

$$\begin{aligned}\mathbb{E}\left[b(t)X(t) - b(s)X(s)\right]^2 &= \mathbb{E}\left[b^2(t)X^2(t) + b^2(s)X^2(s) - 2b(t)b(s)X(t)X(s)\right] \\ &= b^2(t)\mathbb{E}[X(t)]^2 + b^2(s)\mathbb{E}[X(s)]^2 - 2b(t)b(s)\mathbb{E}[X(t)X(s)] \\ &= c\left[b^2(t)t^{\gamma-1} + b^2(s)s^{\gamma-1} - b(t)b(s)\left(t^{\gamma-1} + s^{\gamma-1} - |t-s|^{\gamma-1}\right)\right].\end{aligned}$$

We then proceed with the elementary manipulations

$$\begin{aligned}&\mathbb{E}\left[b(t)X(t) - b(s)X(s)\right]^2 \\ &\quad = c\left[b^2(t)t^{\gamma-1} + b^2(s)s^{\gamma-1} + b(t)b(s)|t-s|^{\gamma-1} - b(t)b(s)t^{\gamma-1} - b(t)b(s)s^{\gamma-1}\right] \\ &\quad = c\left[\left(b(t)t^{\frac{\gamma-1}{2}} - b(s)s^{\frac{\gamma-1}{2}}\right)^2 + b(t)b(s)\left(|t-s|^{\gamma-1} - \left[t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}}\right]^2\right)\right]\end{aligned}$$

and the proof is complete. □

Lemma 4.3. *Suppose X is subject to Hypotheses (ϕ) and let $b : J \to \mathbb{R}$ a deterministic function. Then there is a constant $c > 0$, so that*

$$\mathbb{E}\left[b(t)X(t) - b(s)X(s)\right]^2 \le c\left[b^2(t)|t-s|^{\gamma-1} + |b(t) - b(s)|^2 s^{\gamma-1}\right]$$

holds for all $s, t \in J$.

Proof. The claim can be shown directly with the aid of Theorem 3.5(i).

$$\begin{aligned}\mathbb{E}\left[b(t)X(t) - b(s)X(s)\right]^2 &= \mathbb{E}\left[b(t)(X(t) - X(s)) + X(s)(b(t) - b(s))\right]^2 \\ &\le 2\mathbb{E}[b(t)(X(t) - X(s))]^2 + 2\mathbb{E}[X(s)(b(t) - b(s))]^2 \\ &\le c\left[b^2(t)|t-s|^{\gamma-1} + |b(t) - b(s)|^2 s^{\gamma-1}\right],\end{aligned}$$

which completes the proof. □

Theorem 4.4. *Let $G \subset \mathbb{R}^N$ be a domain with boundary of class C^1 and ζ given by (4.1).*

(i) Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of mutually independent processes with a unique spectral density ϕ subject to Hypothesis (ϕ), so that $|\lambda|^\gamma \phi(\lambda) \equiv \text{const}$ and let $0 \le 2\sigma < \gamma - 1$.

$$b \in {}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J; L_2(\partial G; \ell_2)) \quad \Longrightarrow \quad \zeta \in {}_0W^\sigma_2(J; L_2(\partial G; L_2(\Omega))).$$

(ii) Let the assumptions of (i) be valid and assume further that for all $s,t \in J$ and $x \in \partial G$ it is $(b(t,x)|b(s,x))_{\ell_2} \geq 0$, then

$$\zeta \in {}_0W_2^\sigma(J; L_2(\partial G; L_2(\Omega))) \quad \Longrightarrow \quad b \in {}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J; L_2(\partial G; \ell_2)).$$

(iii) Suppose $(X_n)_{n\in\mathbb{N}}$ is a sequence of mutually independent processes with a unique spectral density ϕ subject to Hypothesis (ϕ) and let $0 \leq 2\sigma < \gamma - 1$. Then

$$b \in {}_0W_2^\sigma(J; L_2(\partial G; \ell_2)) \quad \Longrightarrow \quad \zeta \in {}_0W_2^\sigma(J; L_2(\partial G; L_2(\Omega))).$$

Proof. Regarding assertions (i) and (iii), it is due to Lemma 2.1 and Theorem 4.1 with $s = 0$, that $\zeta \in Y$. So it suffices to compare the relevant semi-norms. In the sequel we denote by $[\cdot]_\sigma$ the semi-norm of the space $W_2^\sigma(J; L_2(\partial G; L_2(\Omega)))$ and by $\mathrm{d}\eta$ the surface measure of ∂G. Let moreover be ζ_m the m-th partial sum of ζ, that is

$$\zeta_m(t,x) = \sum_{k=1}^m b_k(t,x) X_k(t).$$

Then

$$\begin{aligned}
[\zeta_m]_\sigma^2 &= \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\mathbb{E}[\zeta_m(t,x) - \zeta_m(s,x)]^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&= \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m \mathbb{E}\left[b_k(t,x)X_k(t) - b_k(s,x)X_k(s)\right]^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t. \qquad (4.3)
\end{aligned}$$

With view on (i), Lemma 4.2 yields

$$\begin{aligned}
[\zeta_m]_\sigma^2 &= c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m (b_k(t,x)t^{\frac{\gamma-1}{2}} - b_k(s,x)s^{\frac{\gamma-1}{2}})^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t + \\
&+ c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m b_k(t,x) b_k(s,x) \left[|t-s|^{\gamma-1} - (t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}})^2\right]}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t. \qquad (4.4)
\end{aligned}$$

Let us now study the second term of the right hand-side of (4.4) separately. It is due to $|t-s|^{\gamma-1} \geq (t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}})^2$ for all $s,t \in J$ (to see this multiply with $s^{1-\gamma}$ and substitute $z = t/s$), that

$$\begin{aligned}
&\int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m b_k(t,x) b_k(s,x) \left[|t-s|^{\gamma-1} - (t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}})^2\right]}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&\qquad \leq \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m |b_k(t,x) b_k(s,x)| |t-s|^{\gamma-1}}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&\qquad \leq \frac{1}{2} \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m [b_k^2(t,x) + b_k^2(s,x)]}{|t-s|^{2+2\sigma-\gamma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&\qquad = \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m b_k^2(t,x)}{|t-s|^{2+2\sigma-\gamma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t.
\end{aligned}$$

Passing to the limit $m \to \infty$ forces the existence of constants $c_1, c_2 > 0$ (in the sequel generic) so that

$$
\begin{aligned}
[\zeta]_\sigma^2 &\le c_1 [b]^2_{{}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J;L_2(\partial G;\ell_2))} + c_2 \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^\infty b_k^2(t,x)}{|t-s|^{2+2\sigma-\gamma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&= c_1 [b]^2_{{}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J;L_2(\partial G;\ell_2))} + c_2 \int_0^T \int_0^t \frac{\|b(t,\cdot)\|^2_{L_2(\partial G;\ell_2)}}{|t-s|^{2+2\sigma-\gamma}} \mathrm{d}s\mathrm{d}t \\
&= c_1 [b]^2_{{}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J;L_2(\partial G;\ell_2))} + \frac{c_2}{\gamma-1-2\sigma} \int_0^T \|t^{\frac{\gamma-1}{2}-\sigma} b(t,\cdot)\|^2_{L_2(\partial G;\ell_2)} \mathrm{d}t \\
&= c_1 [b]^2_{{}_0W^\sigma_{2,\frac{\gamma-1}{2}}(J;L_2(\partial G;\ell_2))} + \frac{c_2}{\gamma-1-2\sigma} \|b\|^2_{L_{2,\frac{\gamma-1}{2}-\sigma}(J;L_2(\partial G;\ell_2))}
\end{aligned}
\tag{4.5}
$$

and assertion (i) is established by Lemma 2.1. Turning (ii) we stress that it suffices to prove the claim for $J = [0,1]$ and that Theorem 4.1 yields $b \in L_{2,\frac{\gamma-1}{2}}(J; L_2(\partial G;\ell_2))$. Note that $(b(s,x)|b(t,x))_{\ell_2} \ge 0$ implies the existence of a number $M > 0$ such that for every $m > M$ it is $\sum_{n=1}^m b_n(s,x) b_n(t,x) \ge 0$. Choosing $m > M$ we estimate (4.4) by

$$
\begin{aligned}
[\zeta_m]_\sigma^2 &= c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m (b_k(t,x) t^{\frac{\gamma-1}{2}} - b_k(s,x) s^{\frac{\gamma-1}{2}})^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t + \\
&\quad + c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m b_k(t,x) b_k(s,x) \left[|t-s|^{\gamma-1} - (t^{\frac{\gamma-1}{2}} - s^{\frac{\gamma-1}{2}})^2 \right]}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&\ge c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m (b_k(t,x) t^{\frac{\gamma-1}{2}} - b_k(s,x) s^{\frac{\gamma-1}{2}})^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Passing to the limit $m \to \infty$ enforces (ii). Let us conclude with the proof of assertion (iii). To this end we may employ Lemma 4.3 to estimate (4.3) by

$$
\begin{aligned}
[\zeta_m]_\sigma^2 &\le c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m b_k^2(t,x)}{|t-s|^{2+2\sigma-\gamma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t + \\
&\quad + c \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m |b_k(t,x) - b_k(s,x)|^2 s^{\gamma-1}}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t.
\end{aligned}
\tag{4.6}
$$

The first term from the right hand-side of (4.6) can be treated as presented in (4.5). The second term from the right hand-side of (4.6) can be handled as

$$
\begin{aligned}
&\int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m |b_k(t,x) - b_k(s,x)|^2 s^{\gamma-1}}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t \\
&\qquad \le T^{\gamma-1} \int\limits_J \int\limits_J \int\limits_{\partial G} \frac{\sum_{k=1}^m |b_k(t,x) - b_k(s,x)|^2}{|t-s|^{1+2\sigma}} \mathrm{d}\eta(x)\mathrm{d}s\mathrm{d}t.
\end{aligned}
$$

Passing to the limit $m \to \infty$ yields the existence of constants $c_3, c_4, c_5 > 0$ which may depend on γ, T or σ, so that

$$
[\zeta_m]_\sigma^2 \le c_3 \|b\|^2_{L_{2,\frac{\gamma-1}{2}-\sigma}(J;L_2(\partial G;\ell_2))} + c_4 [b]^2_{{}_0W_2^\sigma(J;L_2(\partial G;\ell_2))} \le c_5 [b]^2_{{}_0W_2^\sigma(J;L_2(\partial G;\ell_2))},
$$

where the last estimate is verified by Lemma 2.1 and the remark before. □

5. Anomalous diffusion

Let $\alpha \in (0,2)$, $G \subset \mathbb{R}^N$ to be a domain with boundary ∂G and $J = [0,T]$ a bounded time interval. We study the parabolic boundary problem of subdiffusion (if $\alpha < 1$), normal diffusion (if $\alpha = 1$), and superdiffusion (if $\alpha > 1$) with fractional stochastic disturbances on the boundary. This problem reads as

$$\begin{cases} \partial_t^\alpha u(t,x) - \Delta u(t,x) = 0, & t \in J, \quad x \in G, \\ \mathcal{D}u(t,x) = \psi(t,x), & t \in J, \quad x \in \partial G, \\ u(0,x) = 0, & \qquad\quad x \in G, \end{cases} \tag{5.1}$$

in the basic space

$$V = L_2(J \times G \times \Omega),$$

where the fractional derivative operator ∂^α is defined as

$$(\partial^\alpha \phi)(t) := \frac{\mathrm{d}^2}{(\mathrm{d}t)^2} \int_0^t g_{2-\alpha}(t-\tau)\phi(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}_+.$$

As usual g_κ denotes the standard kernel, i.e. $g_\kappa(\tau) = \tau^{\kappa-1}/\Gamma(\kappa)$ with $\tau \geq 0$ and $\kappa > 0$. The boundary disturbance ψ is modeled as

$$\psi(t,x,\omega) = \sum_{k=1}^{\infty} b_k(t,x) X_k(t,\omega) \tag{5.2}$$

and is suppose to satisfy

Hypothesis (ψ). *$(X_i)_{i\in\mathbb{N}}$ are entirely independent processes subject to Hypothesis (ϕ) (see page 4) with a unique spectral density ϕ and $1 < \gamma < 3$. The multiplier $b := (b_n)_{n\in\mathbb{N}}$ is deterministic.*

From time to time we will stringent our assumptions and presume

Hypothesis (ψ_0). *$(X_i)_{i\in\mathbb{N}}$ are entirely independent processes with a unique spectral density ϕ subject to Hypotheses (ϕ) (see page 4), so that $|\lambda|^\gamma \phi(\lambda) \equiv \mathrm{const.}$ Moreover, the multiplier $b := (b_n)_{n\in\mathbb{N}}$ is deterministic and for all $s,t \in J$ and $x \in \partial G$ we have $(b(t,x)|b(s,x))_{\ell_2} \geq 0$.*

Remark 5.1. Note that centered Lévy processes and fractional Brownian motions with Hurst parameter $H \in (0,1)$ satisfy Hypotheses (ψ) and (ψ_0) with $\gamma = 2$ resp. $\gamma = 2H + 1$ and $\theta = 0$. A fractional Riesz-Bessel motion $\mathcal{RB}_\eta^\beta$, $\beta \in (\frac{1}{2}, \frac{3}{2})$, $\eta \geq 0$ is due to Hypothesis (ψ) with $\gamma \in [2\beta, 2(\beta+\eta)] \cap [2\beta, 3)$, but subject to ($\psi_0$) only if $\eta = 0$.

With $Y = L_2(J \times \partial G \times \Omega)$ we denote the basic space for the boundary process ψ. As to the operator $\mathcal{D}$, we either choose $\mathcal{D} = \partial_\nu$ which links to the Neumann problem, or $\mathcal{D} = I$ to study the Dirichlet problem. As usual I denotes the identity mapping.

Such problems arise in the theory of normal and anomalous diffusion, where the boundary conditions prescribe a stochastic inflow in case $\mathcal{D} = \partial_\nu$, and a stochastic concentration on the boundary for $\mathcal{D} = I$, respectively. Another typical application of those problems is the heat conduction in materials with memory (e.g. polymeric fluids or solids).

We are seeking for conditions on the parameter γ (determined by Hypothesis (ψ)) and properties of the pointwise multiplier b, so that the solution (see Section 5.2.1 for the present concept of a solution) u of (5.1) affiliates to the space Z_δ defined by

$$Z_\delta := {}_0W_2^{\frac{\alpha\delta}{4}}(J; L_2(G; L_2(\Omega))) \cap L_2\left(J; W_2^{\min\{\frac{\delta}{2};2\}}(G; L_2(\Omega))\right), \quad \delta \geq 0. \tag{5.3}$$

It will turn out, that these spaces are appropriate solution spaces. Note that the class Z_4 appears as the maximal regularity class of type L_2 associated to problem (5.1). Moreover, the spaces Z_δ, with $\delta \geq 4$ are tailored to capture results with a higher time regularity. Higher spacial regularity is not treated in this paper, since the resulting inevitable, purely technical, compatibility conditions cannot be motivated from the view of applications. For brevity we introduce the classes $U_{\delta,\gamma}$ and $U^0_{\delta,\gamma}$ for the pointwise multiplier $b := (b_i)_{i\in\mathbb{N}}$ as

$$\begin{aligned} U_{\delta,\gamma} &:= {}_0W^{\frac{\alpha\delta}{4}}_{2,\frac{\gamma-1}{2}}(J; L_2(\partial G; \ell_2)) \cap L_{2,\frac{\gamma-1}{2}}\left(J; W_2^{\frac{\delta}{2}}(\partial G; \ell_2)\right), \quad \delta \geq 0, \\ U^0_{\delta,\gamma} &:= {}_0W_2^{\frac{\alpha\delta}{4}}(J; L_2(\partial G; \ell_2)) \cap L_{2,\frac{\gamma-1}{2}}\left(J; W_2^{\frac{\delta}{2}}(\partial G; \ell_2)\right), \quad \delta \geq 0. \end{aligned} \tag{5.4}$$

It is then readily seen, that $U^0_{\delta,\gamma} \hookrightarrow U_{\delta,\gamma}$.

5.1. **Main results.** In what follows let $\alpha \in (0,2)$ and $G \subset \mathbb{R}^N$ be either the N dimensional half-space, given by

$$\mathbb{R}^N_+ := \left\{x := (x', y) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, y > 0\right\}, \tag{5.5}$$

or a domain with compact boundary ∂G of class $C^{[2/\alpha]+1}$, if not indicated otherwise.

Theorem 5.2. *Assume Hypothesis (ψ) holds. Let $0 \leq \nu < \frac{2(\gamma-1)}{\alpha}$ and in case $G \neq \mathbb{R}^N_+$ let $\nu \in [0, \frac{2(\gamma-1)}{\alpha}) \cap [0,4)$. Then the following hold if $b \in U^0_{\nu,\gamma}$, given by (5.4).*

- *(i) The Dirichlet problem (5.1), i.e. $\mathcal{D} = I$, admits a unique solution u in the regularity class $Z_{\nu+1}$ given by (5.3). If, in addition, $\nu \leq 3$ and Hypothesis (ψ_0) is valid, then membership of b to the class $U_{\nu,\gamma}$ is necessary and sufficient.*
- *(ii) The Neumann problem (5.1), i.e. $\mathcal{D} = \partial_\nu$, admits a unique solution u in the regularity class $Z_{\nu+3}$ given by (5.3). If, in addition, $\nu \leq 1$ and Hypothesis (ψ_0) is valid, then membership of b to the class $U_{\nu,\gamma}$ is necessary and sufficient.*

Note that the additional assumption $\nu < 4$, when $G \neq \mathbb{R}^N_+$, is only restrictive in the case of subdiffusion, if $\alpha < \frac{\gamma-1}{2}$. However, in view of maximal L_2-regularity it is not obstructive at all. If one seeks for strong solutions of problem (5.1) with Dirichlet boundary condition, Theorem 5.2 shows that one necessarily has to assume that $\gamma > \frac{3\alpha+2}{2}$. On the other hand, in view of the Neumann problem the above theorem yields the existence of strong solutions, only if $\gamma > \frac{\alpha+2}{2}$. The following corollaries concern a result on mixed regularity classes with either full spatial regularity or full temporal regularity. In the Dirichlet case, i.e. $\mathcal{D} = I$, this results reads as

Corollary 5.3. *The Dirichlet problem (5.1) admits a unique solution u and*

(i) $u \in {}_0W_2^{\vartheta\alpha}(J; W_2^2(G; L_2(\Omega)))$, $\qquad \vartheta \geq 0$,
(ii) $u \in {}_0W_2^{\alpha}(J; W_2^{2\vartheta}(G; L_2(\Omega)))$, $\qquad \vartheta \in [0, 1]$,

provided that

(a) $\vartheta < \frac{\gamma-1}{2\alpha} - \frac{3}{4}$*;*
(b) $\vartheta < \frac{1}{4}$*, in case* $G \neq \mathbb{R}_+^N$*;*
(c) $b \in U^0_{4\vartheta+3,\gamma}$*.*

Proof. Set $\nu = 4\vartheta + 3$, then clearly $3 \leq \nu < \frac{2(\gamma-1)}{\alpha}$ and, in addition, $\nu < 4$ if $G \neq \mathbb{R}_+^N$, thus Theorem 5.2 yields $u \in Z_{\nu+1}$. By the mixed derivative theorem we obtain

$$Z_{\nu+1} \hookrightarrow {}_0W_2^{\frac{\alpha(\nu+1)}{4}\theta}\left(J; W_2^{\frac{\nu+1}{2}(1-\theta)}(G; L_2(\Omega))\right), \qquad \theta \in [0, 1]$$

and the choice $\theta = \frac{\nu-3}{\nu+1}$ proves assertion (i), while $\theta = \frac{4}{\nu+1}$ gives (ii). □

In case of a boundary condition of Neumann type, i.e. $\mathcal{D} = \partial_\nu$, we deduce

Corollary 5.4. *The Neumann problem (5.1) admits a unique solution u and*

(i) $u \in {}_0W_2^{\vartheta\alpha}(J; W_2^2(G; L_2(\Omega)))$, $\qquad \vartheta \geq 0$,
(ii) $u \in {}_0W_2^{\alpha}(J; W_2^{2\vartheta}(G; L_2(\Omega)))$, $\qquad \vartheta \in [0, 1]$,

provided that

(a) $\vartheta < \frac{\gamma-1}{2\alpha} - \frac{1}{4}$*;*
(b) $\vartheta < \frac{3}{4}$*, in case* $G \neq \mathbb{R}_+^N$*;*
(c) $b \in U^0_{4\vartheta+1,\gamma}$*.*

Proof. Repeat the arguments of the proof of Corollary 5.3 with $\nu = 4\vartheta + 1$. □

Here we discuss mainly the $L_2(\Omega)$-valued case. The subsequent proposition – which is easy to prove but never the less useful – covers, in the presence of Theorem 5.2, results in the pathwise sense.

Proposition 5.5. *Let u belong to the space Z_δ given by (5.3) and $s \geq 0$ a real number.*

(i) If $\delta > \frac{2}{\alpha}(2s + 1)$*, then* $u \in L_2(\Omega; C^s(J; L_2(G)))$*.*

(ii) *If* $2s+N<\delta\leq 4$*, then* $u\in L_2(\Omega;L_2(J;C^s(G)))$.
(iii) *If* $\delta>4$ *and* $s<\frac{4-N}{2}$*, then* $u\in L_2(\Omega;L_2(J;C^s(G)))$.
(iv) *If* $\delta>\max\left\{4;\frac{8(2s+N+1)}{\alpha(3-2s-N)}\right\}$ *and* $s<\frac{3-N}{2}$*, then* $u\in L_2(\Omega;C^s(J\times G))$.

Proof. Fubini's Theorem yields

$$Z_\delta=L_2(\Omega;{}_0W_2^{\frac{\alpha\delta}{4}}(J;L_2(G)))\cap L_2(\Omega;L_2(J;W_2^{\min\{\frac{\delta}{2};2\}}(G)))$$

and it is due to Sobolev imbedding that ${}_0W_2^\theta(K)\hookrightarrow C^s(K)$ if $s<\theta-\frac{\dim K}{2}$. Then a simple computation confirms (i), (ii) and (iii). Turning to (iv) we allude to

$$Z_\delta\hookrightarrow L_2\left(\Omega;W_2^{\frac{2\alpha\delta}{\alpha\delta+8}}(J\times G)\right)\qquad\text{if}\qquad\delta>4,$$

which is due to the mixed derivative theorem and the claim follows via Sobolev imbedding. □

It is worthwhile to mention that in view of Proposition 5.5, there exists a feasible δ for (ii) and (iii) only if $N\leq 3$, and for (iv) only if $N\leq 2$.

5.2. **Proof of the main results.**

5.2.1. *Weak solutions.* We call a function $u\in X$ weak solution of the Dirichlet problem (5.1), i.e. $\mathcal{D}=I$, if it satisfies the integral equation

$$-\int_G\int_J(\partial_t^2\phi_D)(g_{2-\alpha}*u)\mathrm{d}t\mathrm{d}x+\int_G\int_J(\Delta\phi_D)u\mathrm{d}t\mathrm{d}x=\int_{\partial G}\int_J(\partial_\nu\phi_D)\psi\mathrm{d}t\mathrm{d}x\quad(5.6)$$

for all test functions ϕ_D in the class

$$\{\phi_D\in W_2^2(G;L_2(J)):\phi_D\mid_{\partial G}=0\}\cap\{\phi_D\in W_2^2(J;L_2(G)):\phi_D(T)=\partial_t\phi_D(T)=0\}.\quad(5.7)$$

Similarly, we call a function $u\in V$ weak solution of the Neumann problem (5.1), i.e. $\mathcal{D}=\partial_\nu$, if is satisfies the integral equation

$$-\int_G\int_J(\partial_t^2\phi_N)(g_{2-\alpha}*u)\mathrm{d}t\mathrm{d}x+\int_G\int_J(\Delta\phi_N)u\mathrm{d}t\mathrm{d}x=\int_{\partial G}\int_J\phi_N\psi\mathrm{d}t\mathrm{d}x\qquad(5.8)$$

for all test functions ϕ_N in the class

$$\{\phi_N\in W_2^2(G;L_2(J)):\partial_\nu\phi_N\mid_{\partial G}=0\}\cap\{\phi_N\in W_2^2(J;L_2(G)):\phi_N(T)=\partial_t\phi_N(T)=0\}.\quad(5.9)$$

Equations (5.6) resp. (5.8) can be obtained by multiplying problem (5.1) with ϕ_D resp. ϕ_N and integrating over J and G. Note that by construction every strong solution is also a weak solution. The converse is not true in general. Observe that the classes (5.7) and (5.9) are nontrivial and dense in V, since they contain the C^∞-functions with compact support in $(0,T)\times G$.

In the half-space setting, that is if $G=\mathbb{R}^N_+$, one achieves a more explicit representation of a weak solution of problem (5.1). To this purpose we define the operator

$$F:=(\partial_t^\alpha-\Delta_{x'})^{1/2}\qquad(5.10)$$

acting on the basic space Y with domain

$$D(F) = {}_0W_2^{\frac{\alpha}{2}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^1(\mathbb{R}^{N-1}; L_2(\Omega))). \tag{5.11}$$

Since the operator ∂_t^α is sectorial of angle $\frac{\alpha\pi}{2}$ and, moreover, commutes with the negative Laplacian $-\Delta_{x'}$ it is due to the Kalton-Weis-Theorem [5, Theorem 6.3], that the operator F is sectorial of angle $\frac{\alpha\pi}{4}$, hence is the negative generator of an analytic C_0-semigroup, provided $0 < \alpha < 2$.

Let $\Lambda : \{\partial_\nu, I\} \to \{0, 1\}$ be the function which indicates the Neumann problem; precisely $\Lambda_{\mathcal{D}} := \Lambda(\mathcal{D}) = 1$ if and only if $\mathcal{D} = \partial_\nu$. We are now in the position to rewrite problem (5.1) in coordinates according to (5.5) as the ordinary differential equation

$$\begin{cases} -\partial_y^2 u(y) + F^2 u(y) = 0, & y > 0, \\ (1 - \Lambda_{\mathcal{D}})u(0) - \Lambda_{\mathcal{D}}\partial_y u(0) = \psi. \end{cases} \tag{5.12}$$

The deterministic case (cf. [6, Section 3]) gives raise to call a function u a (weak) solution of (5.12), if it satisfies

$$u(y) = e^{-Fy}F^{-\Lambda_{\mathcal{D}}}\psi, \quad t > 0, \tag{5.13}$$

where as usual $F^0 := I$. Here e^{tA} denotes the analytic C_0-semigroup generated by the operator A. In particular, this formula depicts the well-posedness of problem (5.1) in the sense of Hadamard, i.e. the problem admits a unique solution which depends continuously on the data, in some reasonable topology.

In order to show, that a weak solution of the form (5.13) satisfies the representation formula (5.6) resp. (5.8), we make use of an approximation argument. We exemplify this argument for the case of a boundary condition of Dirichlet type. To this end let ψ_n belong to

$$D(F^{\frac{3}{2}}) = {}_0W_2^{\frac{3\alpha}{4}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^{\frac{3}{2}}(\mathbb{R}^{N-1}; L_2(\Omega)))$$

for all $n \in \mathbb{N}$ so that $\psi_n \to \psi \in Y$ as n tends to infinity. Theorem 5.2 yields that the function $u_n(y) = e^{-Fy}\psi_n$ affiliates to the class Z_4, hence is a strong solution of the Dirichlet problem

$$\begin{cases} \partial_t^\alpha u_n(t,x) - \Delta u_n(t,x) = 0, & t \in J, \quad x \in \mathbb{R}^N_+, \\ u_n(t,x) = \psi_n(t,x), & t \in J, \quad x \in \mathbb{R}^{N-1}, \\ u_n(0,x) = 0, & x \in \mathbb{R}^N_+ \end{cases}$$

for every $n \in \mathbb{N}$. It is due to the C_0-property of the semigroup e^{-Fy} and Theorem 5.2 that $u_n \to u \in Z_1$ as $n \to \infty$ and in particular by maximal regularity (the functions u_n are strong solutions for all $n \in \mathbb{N}$) and representation (5.6) we have the validity of the integral equation

$$-\int_{\mathbb{R}^N_+}\int_J (\partial_t^2 \phi_D)(g_{2-\alpha} * u_n)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}^N_+}\int_J (\Delta\phi_D)u_n \mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^{N-1}}\int_J (\partial_\nu \phi_D)\psi_n \mathrm{d}t\mathrm{d}x$$

for all $n \in \mathbb{N}$. Passing n to the limit we see that in the half-space setting, a weak solution of the form (5.13) satisfies equation (5.6). In this sense formulae (5.13) and (5.6) resp. (5.8) are connected.

We are now in the position to proof the main result in the half-space setting.

5.2.2. *Proof of Theorem 5.2: Half-space setting.* Let $G = \mathbb{R}^N_+$, given by (5.5). The unique existence of a solution u of problem (5.1) is clear by (5.13). By Theorems 4.1 and 4.4 we have the implication

$$b \in U^0_{\nu,\gamma} \quad \Longrightarrow \quad \psi \in D(F^{\frac{\nu}{2}}),$$

where the operator F is given by (5.10) and

$$D(F^\theta) = {}_0W_2^{\frac{\alpha\theta}{2}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^\theta(\mathbb{R}^{N-1}; L_2(\Omega))), \tag{5.14}$$

for $\theta \geq 0$. If, in addition, Hypothesis (ψ_0) is valid, then Theorems 4.1 and 4.4 enforce

$$b \in U_{\nu,\gamma} \quad \Longleftrightarrow \quad \psi \in D(F^{\frac{\nu}{2}}).$$

Assertion (i) is proven, if we can show that $\psi \in D(F^{\frac{\nu}{2}})$ is equivalent to $u \in Z_{\nu+1}$. To this end we denote by $z \in \widetilde{V} := L_2(\mathbb{R} \times \mathbb{R}^N_+ \times \Omega)$ the solution of the problem

$$\begin{cases} -\partial_y^2 z(y) + \widetilde{F}^2 z(y) = 0, & y > 0, \\ \qquad\qquad\qquad z(0) = \Psi, \end{cases}$$

where the process Ψ belongs to $\widetilde{Y} := L_2(\mathbb{R} \times \mathbb{R}^{N-1} \times \Omega)$ so that $\Psi\mid_{t\in J} = \psi$ holds and we define

$$\widetilde{F} := \sqrt{\partial_t^\alpha - \Delta_{x'} + I}$$

with domain

$$D(\widetilde{F}) = {}_0W_2^{\frac{\alpha}{2}}(\mathbb{R}; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(\mathbb{R}; W_2^1(\mathbb{R}^{N-1}; L_2(\Omega))).$$

Recall that by (5.13) z is of the form $z(y) := e^{-\widetilde{F}y}\Psi$ with $y \geq 0$.

In what follows $\mathcal{F}$ means the Fourier transform with respect to time t and tangential variable x'. Let $m = m(\lambda, \xi) = \sqrt{\lambda^\alpha + |\xi|^2 + 1}$ with $\lambda = i\rho$, $\rho \in \mathbb{R}$, $\xi \in \mathbb{R}^{N-1}$, denote the Fourier symbol of $\widetilde{F}(t, x')$. Suppressing the argument $\omega \in \Omega$, Plancherel's Theorem yields

$$\begin{aligned} \|\widetilde{F}^{\frac{\nu+1}{2}} z\|^2_{\widetilde{V}} = \|\mathcal{F}\{\widetilde{F}^{\frac{\nu+1}{2}} z\}\|^2_{\widetilde{V}} &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |m^{\frac{1}{2}} \mathcal{F}\{\widetilde{F}^{\frac{\nu}{2}} z(y)\}(\lambda,\xi)|^2 \mathrm{d}\xi \mathrm{d}\rho \mathrm{d}y \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \int_0^\infty |m| e^{-2\operatorname{Re} my} |\mathcal{F}\{\widetilde{F}^{\frac{\nu}{2}}\Psi\}(\lambda,\xi)|^2 \mathrm{d}y \mathrm{d}\xi \mathrm{d}\rho \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|m|}{2\operatorname{Re} m} |\mathcal{F}\{\widetilde{F}^{\frac{\nu}{2}}\Psi\}(\lambda,\xi)|^2 \mathrm{d}\xi \mathrm{d}\rho. \end{aligned}$$

Observe now, that due to $\alpha \in (0, 2)$ the symbol m takes values in an open sector of the complex plane, symmetric with respect to the positive real half axis $\mathbb{R}_+$, with vertex 0 and opening angle $\vartheta < \pi$. This captures the existence of constants $c_1, c_2 > 0$, such that

$$c_1|m| \leq \operatorname{Re} m \leq c_2|m|$$

holds. Therefrom we obtain for $\nu \in [0, \frac{2(\gamma-1)}{\alpha}) \cap [0, 3]$

$$\|\widetilde{F}^{\frac{\nu}{2}}\Psi\|^2_{\widetilde{Y}} \leq c \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|m|}{2\operatorname{Re} m} |\mathcal{F}\{\widetilde{F}^{\frac{\nu}{2}}\Psi\}(\lambda,\xi)|^2 \mathrm{d}\xi \mathrm{d}\rho = \|\widetilde{F}^{\frac{\nu+1}{2}} z\|^2_{\widetilde{V}}$$

which is the key to necessity. Turning to sufficiency we deduce for $\nu \in [0, \frac{2(\gamma-1)}{\alpha})$

$$\|\widetilde{F}^{\frac{\nu+1}{2}} z\|^2_{\widetilde{V}} \leq c \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |\mathcal{F}\{\widetilde{F}^{\frac{\nu}{2}}\Psi\}(\lambda,\xi)|^2 \mathrm{d}\xi \mathrm{d}\rho = c\|\widetilde{F}^{\frac{\nu}{2}}\Psi\|^2_{\widetilde{Y}},$$

hence $\bar{z} := z\mid_{t\in J} \in Z_{\nu+1}$, where $\bar{z} = z\mid_{t\in J}$ denotes the restriction of z to J. Observe now, that $u = \bar{z} + w$, where $w \in V$ is the solution of the problem

$$\begin{cases} -\partial_y^2 w(y) + F^2 w(y) = \bar{z}, & y > 0, \\ w(0) = 0, & \end{cases}$$

with F given by (5.10). It is due to [10, Theorem 3.1] that

$$w \in {}_0W_2^{\alpha + \frac{\alpha(\nu+1)}{4}}(J; L_2(G; L_2(\Omega))) \cap L_2\left(J; {}_0W_2^2(G; L_2(\Omega))\right) = Z_{\nu+5},$$

which in turn yields $u \in Z_{\nu+1}$ and assertion (i) is proven.

Turning to (ii) let us denote by v the solution of the Dirichlet problem (5.1). Recall that by (5.13) it is $u = F^{-1}v$, and so in particular we have

$$v \in Z_{\nu+1} \quad \Longleftrightarrow \quad u \in Z_{\nu+3}$$

for all $0 \le \nu < \frac{2(\gamma-1)}{\alpha}$. Employing (i) completes the proof.

5.2.3. *Proof of Theorem 5.2: Setting for domains.* The results when G is a domain, follow essentially from the half-space case by means of localization, perturbation and coordinate transforms.

References

1. H. Amann, *Linear and quasilinear parabolic problems. Vol. I*, Monographs in Mathematics, vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995, Abstract linear theory.
2. U. Frisch, *Turbulence*, Cambridge University Press, Cambridge, 1995, The legacy of A. N. Kolmogorov.
3. J. A. Gubner, *Theorems and fallacies in the theory of long-range-dependent processes*, IEEE Trans. Inform. Theory **51** (2005), no. 3, 1234–1239.
4. G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, Reprint of the 1952 edition.
5. N. J. Kalton and L. Weis, *The H^∞-calculus and sums of closed operators*, Math. Ann. **321** (2001), no. 2, 319–345.
6. J. Prüss, *Maximal regularity for abstract parabolic problems with inhomogeneous boundary data in L_p-spaces*, Proceedings of EQUADIFF, 10 (Prague, 2001), vol. 127, 2002, pp. 311–327.
7. Th. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications, vol. 3, Walter de Gruyter & Co., Berlin, 1996.
8. H. Triebel, *Theory of function spaces*, Monographs in Mathematics, vol. 78, Birkhäuser Verlag, Basel, 1983.
9. A. M. Yaglom, *Correlation theory of stationary and related random functions. Vol. I*, Springer Series in Statistics, Springer-Verlag, New York, 1987, Basic results.
10. R. Zacher, *Maximal regularity of type L_p for abstract parabolic Volterra equations*, J. Evol. Equ. **5** (2005), no. 1, 79–103.

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