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AN INTEGRATION CALCULUS FOR STOCHASTIC PROCESSES WITH STATIONARY INCREMENTS AND SPECTRAL DENSITY WITH APPLICATIONS TO PARABOLIC VOLTERRA EQUATIONS

STEFAN SPERLICH

Abstract. Aim of this paper is the introduction to a rigorous stochastic integration theory with respect to stochastic processes with stationary increments and spectral density. Once provided we will conclude with applications to parabolic Volterra equations.

1. Introduction

So far, stochastic integration calculi where tailor-made for a precious few random motions. This is by no means satisfactory and embarrasses the progress in many applied fields. Aim of this paper is to remedy this obstruction and it is structured as follows. In Section 2 we give meaning to some mathematical notations and provide briefly some fundamentals of the theory of parabolic problems. Then, in Section 3 we introduce the announced stochastic integration theory in the real-vealued and also in the vector-valued case. We conclude with Section 4, where we first present the main results in Subsection 4.1 which will be proven in Subsection 4.2.

2. Foundations

In what follows let X and Y be Banach spaces and $\mathcal{H}$ be a separable Hilbert space. $J \subset [0,\infty)$ will usually mean a bounded or unbounded time interval. We endeavor to denote the norm in X with $\|\cdot\|_X$, but from time to time we may write $\|\cdot\|$ or $|\cdot|_X$ if it is conductive to brevity. An inner product will be denoted by $(\cdot|\cdot)$ and if there is any risk of confusion we will add a lower index to designate the affiliation to a certain inner product space.

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural, real and complex numbers, respectively, and let further $\mathbb{R}_+ = [0,\infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The symbol

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$\mathcal{B}(X;Y)$ means the space of all bounded linear operators from X to Y and we write $\mathcal{B}(X) = \mathcal{B}(X;X)$ for short.

If A is an operator in X, $\mathcal{D}(A)$ and $\mathcal{R}(A)$ stand for domain and range of A, respectively, while $\rho(A)$, $\sigma(A)$ designate the resolvent set and the spectrum of A.

As usual we employ the star $*$ for the convolution of functions defined on $\mathbb{R}$

$$(f * g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}, \tag{2.1}$$

and

$$(f * g)(t) = \int_{0}^{t} f(t-\tau)g(\tau)\mathrm{d}\tau, \qquad t \geq 0, \tag{2.2}$$

for f, g supported on the half-ray $\mathbb{R}_+$. Observe that (2.1) and (2.2) are equivalent for functions which vanish on $(-\infty, 0)$; therefore there will be no danger of confusion.

For $u \in L_{1,\mathrm{loc}}(\mathbb{R}_+; X)$ of exponential growth, i.e. $\int_0^\infty e^{-\omega t}|u(t)|\mathrm{d}t < \infty$ with some $\omega \in \mathbb{R}$, the Laplace transform of u is defined by

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t}u(t)\mathrm{d}t, \quad \operatorname{Re}\lambda \geq \omega.$$

For $f \in L_1(\mathbb{R}; X)$, the Fourier transform of f is the function $\mathcal{F}f : \mathbb{R} \to X$ defined by

$$(\mathcal{F}f)(\xi) = \int_{\mathbb{R}} e^{-i\xi t}f(t)\mathrm{d}t.$$

Throughout this paper we will denote by χ_M the characteristic function of the set M, that is $\chi_M(x) = 1$ if $x \in M$ and $\chi_M(x) = 0$ otherwise.

2.1. **Spaces of nuclear and Hilbert-Schmidt operators.** In what follows let $\mathcal{H}$ a separable Hilbert space. The symbols $\mathscr{L}_1(\mathcal{H})$ and $\mathscr{L}_2(\mathcal{H})$ denote the spaces of nuclear operators and Hilbert-Schmidt operators on $\mathcal{H}$, respectively. Thereby a bounded operator T on $\mathcal{H}$ is called nuclear (that is $T \in \mathscr{L}_1(\mathcal{H})$) if there are sequences $(x_n^*) \subset \mathcal{H}^*$ and $(y_n) \subset \mathcal{H}$ with $\sum_{n=1}^\infty \|x_n^*\|\|y_n\| < \infty$ so that

$$Tx = \sum_{n=1}^\infty x_n^*(x)y_n$$

holds for all $x \in \mathcal{H}$. On the other hand a bounded operator T on $\mathcal{H}$ is said to be a Hilbert-Schmidt operator (meaning $T \in \mathscr{L}_2(\mathcal{H})$), if there is an orthonormal basis $(e_n) \subset \mathcal{H}$, so that

$$\sum_{n=1}^\infty \|Te_n\|^2 < \infty.$$

If this is true for one orthonormal basis, it is true for any other orthonormal basis of $\mathcal{H}$. We have

$$\mathscr{L}_1(\mathcal{H}) \hookrightarrow \mathscr{L}_2(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}).$$

In case the operator $T : \mathcal{H} \to \mathcal{H}$ is self-adjoint with eigenvalues $\lambda = (\lambda_n)_{n\in\mathbb{N}}$, the norms in these spaces can be written as

$$\|T\|_{\mathscr{L}_1(\mathcal{H})} = \|\lambda\|_{\ell_1},$$
$$\|T\|_{\mathscr{L}_2(\mathcal{H})} = \|\lambda\|_{\ell_2}.$$

For nuclear operators T on $\mathcal{H}$ one can define the trace of T by means of

$$\operatorname{Tr}[T] = \sum_{n=1}^{\infty} (Tg_n \mid g_n)_{\mathcal{H}},$$

where $(g_n)_{n\in\mathbb{N}}$ is an arbitrary orthonormal basis in $\mathcal{H}$. Due to this property nuclear operators are also called operators of trace class. One can show, that $|\operatorname{Tr}[T]| \le \|T\|_{\mathscr{L}_1(\mathcal{H})}$ holds for every $T \in \mathscr{L}_1(\mathcal{H})$ and, moreover, that $\operatorname{Tr}[T] = \|T\|_{\mathscr{L}_1(\mathcal{H})}$ if T is positive semi-definite.

2.2. **Homogeneous Bessel potential spaces.** Within this paper we make use of the notion of the (left-sided) fractional differintegral of order $\alpha \in (-2, 2)$ of a test-function ϕ by $\partial^\alpha \phi$ being defined as

$$(\partial^\alpha \phi)(t) := \frac{\mathrm{d}^2}{(\mathrm{d}t)^2} \int_{-\infty}^{t} g_{2-\alpha}(t-\tau)\phi(\tau)\mathrm{d}\tau, \qquad t \in \mathbb{R}, \tag{2.3}$$

where

$$g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}, \qquad t \ge 0, \qquad \kappa > 0 \tag{2.4}$$

denotes the Riemann-Liouville kernel. Note that g_κ is of subexponential growth, i.e.

$$\int_0^\infty e^{-\omega t} |g_\kappa(t)| \mathrm{d}t < \infty$$

for arbitrary small $\omega > 0$. This means that that Laplace transform $\widehat{g}_\kappa$ of g_κ is well-defined, and we have

$$(\widehat{\partial^\alpha \phi})(\lambda) = \lambda^\alpha \widehat{\phi}(\lambda), \qquad \operatorname{Re} \lambda > 0$$

for all test-functions ϕ supported on $(0, \infty)$.

For an open subset $D \subset \mathbb{R}$, $\mathrm{H}_p^m(D; X)$ with $m \in \mathbb{N}$ denotes the classical Sobolev space, that is the space of all functions $f : D \to X$ having distributional derivatives $\partial^\alpha f \in L_p(D; X)$ of order $0 \le \alpha \le m$. For $1 \le p < \infty$ the norm in $\mathrm{H}_p^m(D; X)$ is given by

$$\|f\|_{\mathrm{H}_p^m(D;X)} := \left[\sum_{\alpha \le m} \|\partial^\alpha f\|_p^p \right]^{\frac{1}{p}}.$$

Further, for $0 < s < 1$, we define the Bessel potential spaces $\mathrm{H}_p^{sm}(D; X)$, by means of complex interpolation via

$$\mathrm{H}_p^{sm}(D; X) := \left[L_p(D; X); \mathrm{H}_p^m(D; X) \right]_s.$$

Let now $f \in \mathcal{S}^*(\mathbb{R})$, that is a tempered distribution, and X be a Hilbert or UMD space. Then we have the norm representation

$$\|f\|_{\mathrm{H}_2^s(\mathbb{R};X)} = \|(1+|\cdot|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L_2(\mathbb{R};X)}, \qquad s>0. \tag{2.5}$$

If $U \subset \mathbb{R}$ is a subset of $\mathbb{R}$, then $\mathrm{H}_2^s(U;X)$ denotes the restriction of the functions $f \in \mathrm{H}_2^s(\mathbb{R};X)$ to the subset U. Then, by $\dot{\mathrm{H}}_2^s(\mathbb{R})$, we mean the homogenous Bessel potential space of order $s>0$, defined as

$$\dot{\mathrm{H}}_2^s(\mathbb{R}) := \left\{ f \in \mathcal{S}^*(\mathbb{R}) : \||\cdot|^s\mathcal{F}f\|_{L_2(\mathbb{R})} < \infty \right\}.$$

By means of the fractional derivatives (2.3) and Plancherel's Theorem we obtain the identity

$$\int_{\mathbb{R}} |(\mathcal{F}f)(\xi)|^2|\xi|^{2s}\mathrm{d}\xi = \int_{\mathbb{R}} |\partial^s f(t)|^2\,\mathrm{d}t,$$

so that we have alternatively

$$\|f\|_{\dot{\mathrm{H}}_2^s(\mathbb{R})} = \|\partial^s f\|_{L_2(\mathbb{R})}, \qquad 0<s<2. \tag{2.6}$$

Observe, that (2.3), (2.5) and (2.6) allow us to define the (homogenous) Bessel potential spaces also for negative orders $s \in (-2,0)$.

2.3. **Evolutionary integral equations.** The notion of parabolic problems used in this study is widely taken from the monograph of Prüss [6].

Let $\mathcal{H}$ be a separable Hilbert space, A a closed linear, but in general unbounded operator in $\mathcal{H}$ with dense domain $\mathcal{D}(A)$, and let $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth. Then it is readily seen that the Laplace transform $\widehat{a}(\lambda)$ of a exists for $\operatorname{Re}\lambda > 0$. We consider the Volterra equation

$$u(t) + (a*Au)(t) = f(t), \qquad t \geq 0, \tag{2.7}$$

where $f:\mathbb{R}_+ \to \mathcal{H}$ is a given function, strongly measurable and locally integrable, at least.

In the sequel we denote by $\mathcal{H}_A$ the domain of A equipped with the graph norm $|x|_A := |x| + |Ax|$. $\mathcal{H}_A$ is a Banach space since A is closed, and it is continuously embedded into $\mathcal{H}$. The following notions of solutions of (2.7) are natural. Again we let $J \subset \mathbb{R}_+$.

Definition 2.1 (Strong and mild solutions)**.** *A function $u \in C(J;\mathcal{H})$ is called*

(a) strong solution of (2.7) on J if $u \in C(J;\mathcal{H}_A)$ and (2.7) holds on J;
(b) mild solution of (2.7) on J if $a*u \in C(J;\mathcal{H}_A)$ and $u(t) = f(t) - A(a*u)(t)$ on J.

Obviously, every strong solution of (2.7) is a mild one. The converse is not true, in general.

Definition 2.2 (Parabolicity). *Problem (2.7) is called parabolic, if*

(i) $\widehat{a}(\lambda) \neq 0$ *and* $\frac{1}{\widehat{a}(\lambda)} \in \rho(A)$ *for all* $\operatorname{Re}\lambda > 0$*;*
(ii) there is a constant $M \geq 1$ *such that*

$$\left|\frac{1}{\lambda}(I + \widehat{a}(\lambda)A)^{-1}\right| \leq \frac{M}{|\lambda|} \quad \textit{for all } \operatorname{Re}\lambda > 0.$$

The notion of sectorial kernels is given by

Definition 2.3 (Sectoriality). *Let* $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ *be of subexponential growth and suppose* $\widehat{a}(\lambda) \neq 0$ *for all* $\operatorname{Re}\lambda > 0$*.* a *is called sectorial with angle* $\theta > 0$ *(or merely* θ*-sectorial) if*

$$|\arg \widehat{a}(\lambda)| \leq \theta \quad \textit{for all } \operatorname{Re}\lambda > 0. \tag{2.8}$$

Here, $\arg \widehat{a}(\lambda)$ is defined as the imaginary part of a fixed branch of $\log \widehat{a}(\lambda)$, and θ in (2.8) is allowed to be greater than π. In case a is sectorial, we always choose that branch of $\log \widehat{a}(\lambda)$ which yields the smallest angle θ; in particular, if $\widehat{a}(\lambda)$ is real for real λ we choose the principal branch. In the following, we denote by $\Sigma(\omega, \theta)$ the open sector in the complex plane with vertex $\omega \in \mathbb{R}$ and opening angle 2θ which is symmetric with respect to the real positive axis. A standard situation leading to parabolic equations is described in

Proposition 2.4 ([6, Proposition 3.1]). *Let* $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ *be* θ*-sectorial for some* $\theta < \pi$*, suppose* A *is closed linear densely defined, such that* $\rho(A) \supset \Sigma(0, \theta)$*, and*

$$|(\mu + A)^{-1}| \leq \frac{M}{|\mu|} \quad \textit{for all } \mu \in \Sigma(0, \theta).$$

Then (2.7) is parabolic.

Definition 2.5 (k-regular kernels). *Let* $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ *be of subexponential growth and* $k \in \mathbb{N}$*.* a *is called* k*-regular if there is a constant* $c > 0$ *such that*

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \leq c|\widehat{a}(\lambda)|, \quad \textit{for all } \operatorname{Re}\lambda > 0,\ 0 \leq n \leq k.$$

It is not difficult to see that convolutions of k-regular kernels are again k-regular. Furthermore, k-regularity is preserved by integration and differentiation, while sums and differences of k-regular kernels need not be k-regular. However, if $a(t)$ and $b(t)$ are k-regular and

$$|\arg \widehat{a}(\lambda) - \arg \widehat{b}(\lambda)| \leq \theta < \pi, \quad \operatorname{Re}\lambda > 0$$

then $a(t) + b(t)$ is k-regular as well. In general, nonnegative, nonincreasing kernels are not 1-regular, but if the kernel is also convex, then it is 1-regular (cf. [6, Section I.3]). We call a kernel $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ 1-monotone if $a(t)$ is nonnegative and nonincreasing; for $k \geq 2$ we define

Definition 2.6 (k-monotone kernels)**.** *Let $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ and $k \geq 2$. $a(t)$ is called k-monotone if $a \in C^{k-2}(0,\infty)$, $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0$, $0 \leq n \leq k-2$, and $(-1)^{k-2} a^{(k-2)}(t)$ is nonincreasing and convex.*

Proposition 2.7 ([6, Proposition 3.3])**.** *Let $k \geq 1$ and suppose $a \in L_{1,\mathrm{loc}}$ is $(k+1)$-monotone. Then $a(t)$ is k-regular and of positive type, i.e. $\frac{\pi}{2}$-sectorial.*

If A is sectorial with angle ϕ_A (for a detailed survey we refer to Denk et al. [4, Section 1]), and a is ϕ_a-sectorial, then (2.7) is parabolic provided that $\phi_A + \phi_a < \pi$, cf. [7, Proposition 3.1]. An important property of parabolic Volterra equations is the fact that they admit bounded resolvents whenever the kernel a is 1-regular, see [7, Theorem 3.1]. By a resolvent for (2.7) we mean a family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in $\mathcal{H}$ which satisfy the following conditions:

- (S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
- (S2) $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$, $t \geq 0$;
- (S3) $S(t)x + A(a * Sx)(t) = x$, for all $x \in \mathcal{H}$, $t \geq 0$.

(S3) is called resolvent equation. One can show that (2.7) admits at most one resolvent, and if it exists, then (2.7) has a unique mild solution u represented by the variation of parameters formula

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(t-\tau) f(\tau) \mathrm{d}\tau, \qquad t \geq 0, \tag{2.9}$$

at least for such f for which (2.9) is meaningful. If (2.7) admits an analytic resolvent $S(t)$ (cf. [6, Section I.1 and I.2]) which is bounded on some sector $\Sigma(0,\theta)$, then (2.7) is parabolic; the converse is not true in general.

2.4. **Stochastic processes with stationary increments.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let J be an interval of $\mathbb{R}$. An arbitrary family $\{X(t)\}_{t\in J}$

defined on Ω, such that $X(t) : \Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable for each $t \in J$ is called a stochastic process and we set $X(t,\omega) = X(t)(\omega)$ for all $t \in J$ and $\omega \in \Omega$. The functions $X(\cdot,\omega)$ are called trajectories of X. For the reader's convenience we recall some basic definitions of regularity for a process X on J.

(a) X is mean-square continuous at $t_0 \in J$, if $\lim_{t\to t_0} \mathbb{E}[|X(t) - X(t_0)|^2] = 0$.
(b) X is mean-square continuous on J, if it is mean-square continuous at every point of J.
(c) X is continuous (with probability 1), if its trajectories $X(\cdot,\omega)$ are continuous almost surely.
(d) X is Hölder-continuous of order α (with probability 1), if its trajectories $X(\cdot,\omega)$ are Hölder-continuous of order α almost surely.

In what follows we denote by

$$D_3(t;u,v) := \mathbb{E}[(X(u) - X(t))(X(v) - X(t))]$$

the structure function of the real-valued process X.

Definition 2.8 (Processes with stationary increments)**.** *We call the random process $X = \{X(t)\}_{t\in\mathbb{R}}$ a process with stationary increments if*

(i) the mean value of its increments depends only on the length $t-s$ of the interval $[s,t]$, i.e.

$$\mathbb{E}[X(t) - X(s)] = \mathbb{E}[X(t-s) - X(0)];$$

(ii) for $u,v,t \in \mathbb{R}$ the structure function $D_3(t;u,v)$ of the process X depends only on the differences $u-t$ and $v-t$, i.e.

$$D_3(t;u,v) = D_3(0;u-t,v-t) =: D_2(u-t,v-t).$$

At this point we should be more careful and say that the processes under consideration have stationary increments in the wider sense. However this refinement is unnecessary in this paper where more special processes with strictly stationary increments will not be considered at all. Definition 2.8 particularly yields that a real-valued process X with stationary increments, which in addition satisfies $\mathbb{E}[X(0)] = 0$ is characterized by a function (the mean of the increments) of one variable

$$\mathbb{E}[X(t+\tau) - X(t)] =: m(\tau) \tag{2.10}$$

and by a function $D(\cdot)$ of one variable

$$\mathbb{E}|X(t+\tau) - X(t)|^2 =: D(\tau). \tag{2.11}$$

The function $D_2(\cdot,\cdot)$ can then be obtained via the identity

$$D_2(\tau_1,\tau_2) = \frac{1}{2}[D(\tau_1) + D(\tau_2) - D(|\tau_1 - \tau_2|)]. \tag{2.12}$$

Definition 2.9 (Centered processes). *A process $X := \{X(t)\}_{t\in J}$ is called centered, if $\mathbb{E}[X(t)] = 0$ holds for all $t \in J$.*

Remark 2.10. Observe, that if the process X is centered, then

$$D(\tau) = \mathrm{Var}[X(\tau)] \quad \text{and} \quad D_2(\tau_1,\tau_2) = \mathrm{Cov}[X(\tau_1), X(\tau_2)]. \tag{2.13}$$

Looking for a general form of the function $D(\tau) = D_2(\tau,\tau)$ we follow Yaglom [11, Chapter 4] and note that a process with stationary increments admits a spectral density if there is a function $\phi : (0,\infty) \to \mathbb{R}_+$ so that

$$D(\tau) = 4\int_0^\infty (1 - \cos\lambda\tau)\phi(\lambda)\mathrm{d}\lambda, \tag{2.14}$$

and moreover

$$\int_0^{\lambda_0} \lambda^2\phi(\lambda)\mathrm{d}\lambda + \int_{\lambda_0}^\infty \phi(\lambda)\mathrm{d}\lambda < \infty.$$

As a matter of fact, ϕ is even as soon as it exists.

3. Stochastic Integration

In what follows let X be a process with stationary increments, so that X has the spectral density ϕ and $X(t) \in L_2(\Omega)$ for every $t \in \mathbb{R}$, if not indicated otherwise. Furthermore we presume, that $X(0) = 0$ a.s. and $\phi(\lambda) > 0$ almost everywhere.

Typical examples of those processes are

(a) centered Lévy processes, whose spectral density is of the form $\phi(\lambda) = c|\lambda|^{-2}$ with a constant $c > 0$.
(b) fractional Brownian motions with Hurst parameter $H \in (0,1)$, whose spectral density is of the form $\phi(\lambda) = c|\lambda|^{-2H-1}$ with a constant $c > 0$.
(c) fractional Riesz-Bessel motions $\mathcal{RB}^\alpha_\beta$ (see [1]), whose spectral density is of the form $\phi(\lambda) = |\lambda|^{-2\alpha}(1+\lambda^2)^{-\beta}$.

3.1. **The real-valued case.**

We study the stochastic integral

$$\mathscr{I}_X(f) := \int_{\mathbb{R}} f(\tau)\mathrm{d}X(\tau), \tag{3.1}$$

where the integrant $f : \mathbb{R} \to \mathbb{R}$ is supposed to be deterministic. If f is a step function given by

$$f(t) = \sum_{i=-n}^{n} f_i \chi_{[t_i, t_{i+1})}(t), \tag{3.2}$$

where $t_0 = 0$, we define (3.1) to be

$$\mathscr{I}_X(f) = \sum_{i=-n}^{n} f_i [X(t_{i+1}) - X(t_i)].$$

Obviously, $\mathscr{I}_X(af + bg) = a\mathscr{I}_X(f) + b\mathscr{I}_X(g)$ for any $a, b \in \mathbb{R}$ and step functions f and g. Our aim is to construct a preferably large class of deterministic integrands f so that $\mathscr{I}_X(f)$ is a well-defined random variable with finite second moment.

Remark 3.1. It is readily seen from the construction of $\mathscr{I}_X(f)$ that

(1) if X is centered, then so is $\mathscr{I}_X(f)$;
(2) if X is Gaussian distributed, then so is $\mathscr{I}_X(f)$.

For the sake of completeness we recall, in the proposition below, how to construct classes of integrands $\mathcal{C}$ for integrals of the form (3.1). This is a generalized version of [5, Proposition 2.1] in this sense, that it is formulated not exclusively for integrals with respect to a fractional Brownian motion, but with respect to a process X with stationary increments featuring the spectral density ϕ. However, except of some notational details, the proof can be completely adopted from [5].

Proposition 3.2. *Suppose that $\mathcal{C}$ is a set of deterministic functions defined on $\mathbb{R}$ such that*

(a) $\mathcal{C}$ is an inner product space with an inner product $(f \mid g)_{\mathcal{C}}$, for $f, g \in \mathcal{C}$,
(b) $\mathcal{E} \subset \mathcal{C}$ and $(f \mid g)_{\mathcal{C}} = (\mathscr{I}_X(f) \mid \mathscr{I}_X(g))_{L_2(\Omega)}$, for $f, g \in \mathcal{E}$,
(c) the set $\mathcal{E}$ is dense in $\mathcal{C}$.

Then there is an isometry between the space $\mathcal{C}$ and a linear subspace of

$$\mathrm{Sp}(X) := \{Y \in L_2(\Omega) : \|\mathscr{I}_X(f_n) - Y\|_{L_2(\Omega)} \to 0, \text{ for some } (f_n) \subset \mathcal{E}\}.$$

which is an extension of the map $f \mapsto \mathscr{I}_X(f)$, for $f \in \mathcal{E}$.

Let us now introduce the weighted homogeneous Bessel potential space $\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})$ with weight-function $\lambda^2\phi(\lambda)$ and the inner product

$$(f \mid g)_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})} := \int_{\mathbb{R}} \mathcal{F}f(\lambda)\overline{\mathcal{F}g(\lambda)}\lambda^2\phi(\lambda)\mathrm{d}\lambda.$$

Observe that the inner product $(\cdot \mid \cdot)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})}$ is well-defined and real-valued if the function ϕ is even and almost everywhere positive at least. Observe further, that the class of step functions lies dense in $\dot{\mathrm{H}}_2^\phi(\mathbb{R})$, provided that ϕ is a spectral density (this can be easily seen by modifying [5, Lemma 5.1] appropriately.

Lemma 3.3. *Let $X = \{X(t)\}_{t\in\mathbb{R}} \subset L_2(\Omega)$ be a process with stationary increments having the spectral density ϕ. Then we have for $t, s \in \mathbb{R}$ and any $h \geq 0$*

$$(X(t+h) - X(t) \mid X(s+h) - X(s))_{L_2(\Omega)} = \left(\chi_{[t,t+h)} \mid \chi_{[s,s+h)}\right)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})}.$$

Proof. In view of identities (2.12) and (2.14) we obtain

$$\begin{aligned}
&(X(t+h) - X(t) \mid X(s+h) - X(s))_{L_2(\Omega)} \\
&\qquad = D_2(t+h, s+h) + D_2(t,s) - D_2(t+h,s) - D_2(t, s+h) \\
&\qquad = \frac{1}{2}\left[D(|t-s+h|) - 2D(|t-s|) + D(|t-s-h|)\right] \\
&\qquad = 2\int_0^\infty \{2\cos[(t-s)\lambda] - \cos[(t-s-h)\lambda] - \cos[(t-s+h)\lambda]\}\,\phi(\lambda)\mathrm{d}\lambda \\
&\qquad = \mathrm{Re}\left[\int_{\mathbb{R}} \left\{2e^{-i(s-t)\lambda} - e^{-i(s-t+h)\lambda} - e^{-i(s-t-h)\lambda}\right\}\phi(\lambda)\mathrm{d}\lambda\right] \\
&\qquad = \mathrm{Re}\left[\int_{\mathbb{R}} \frac{e^{-i\lambda s} - e^{-i\lambda(s+h)}}{i\lambda} \cdot \overline{\left(\frac{e^{-i\lambda t} - e^{-i\lambda(t+h)}}{i\lambda}\right)} \cdot \lambda^2\phi(\lambda)\mathrm{d}\lambda\right] \\
&\qquad = \mathrm{Re}\left[\int_{\mathbb{R}} \mathcal{F}\chi_{[s,s+h)}(\lambda)\overline{\mathcal{F}\chi_{[t,t+h)}(\lambda)}\lambda^2\phi(\lambda)\mathrm{d}\lambda\right] \\
&\qquad = \left(\chi_{[s,s+h)} \mid \chi_{[t,t+h)}\right)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})}
\end{aligned}$$

since as already mentioned the inner product in $\dot{\mathrm{H}}_2^\phi(\mathbb{R})$ is real-valued as soon as ϕ is even. □

We are now in the position to formulate the main result for the stochastic integration with respect to random processes with stationary increments and spectral density. Note that the subsequent theorem also allocates an isometry of Itô-type.

Theorem 3.4. *Let $X = \{X(t)\}_{t\in\mathbb{R}} \subset L_2(\Omega)$ be a process with stationary increments having the spectral density ϕ. Then for $f, g \in \dot{\mathrm{H}}_2^\phi(\mathbb{R})$ it is*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}} f(\tau)\mathrm{d}X(\tau)\right)\left(\int_{\mathbb{R}} g(\tau)\mathrm{d}X(\tau)\right)\right] = (f \mid g)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})}.$$

In particular, for integrands $f \in \dot{\mathrm{H}}_2^\phi(\mathbb{R})$ the integral $\mathscr{I}_X(f)$ given by (3.1) is a well-defined random variable with

$$\mathbb{E}[\mathscr{I}_X(f)]^2 = \|f\|^2_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})}.$$

Proof. Proposition 3.2 yields that it suffices to prove the claim for step functions. For this purpose let $f, g \in \mathcal{E} \subset \dot{\mathrm{H}}_2^\phi(\mathbb{R})$, that is f, g is of the form (3.2). With the aid of Lemma 3.3 we verify

$$\begin{aligned}
&(\mathscr{I}_X(f) \mid \mathscr{I}_X(g))_{L_2(\Omega)} \\
&\quad= \left(\sum_{j=-n}^{n} f_j [X(t_j+1) - X(t_j)] \mid \sum_{k=-n}^{n} g_k [X(t_k+1) - X(t_k)] \right)_{L_2(\Omega)} \\
&\quad= \sum_{j=-n}^{n} \sum_{k=-n}^{n} f_j g_k \left(X(t_j+1) - X(t_j) \mid X(t_k+1) - X(t_k) \right)_{L_2(\Omega)} \\
&\quad= \sum_{j=-n}^{n} \sum_{k=-n}^{n} f_j g_k \left(\chi_{[t_j, t_j+1)} \mid \chi_{[t_k, t_k+1)} \right)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})} \\
&\quad= \left(\sum_{j=-n}^{n} f_j \chi_{[t_j, t_j+1)} \mid \sum_{k=-n}^{n} g_k \chi_{[t_k, t_k+1)} \right)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})} \\
&\quad= (f \mid g)_{\dot{\mathrm{H}}_2^\phi(\mathbb{R})},
\end{aligned}$$

which completes the proof. □

We now turn our attention to a vector-valued process $\mathcal{X}$ which is in some sense generated by a mutually independent sequence of processes $(X_n)_{n\in\mathbb{N}}$ having a unique spectral density.

3.2. **The vector-valued case.** Let $\mathcal{H}$ be a separable Hilbert space. We want to deduce properties of functions $R : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ such that the integral

$$\int_{\mathbb{R}} R(t) \mathrm{d}(Q^{1/2}\mathcal{X})(t). \tag{3.3}$$

is well defined. The process $Q^{1/2}\mathcal{X}$ is supposed to satisfy

Hypothesis (X). *The operator Q belongs to $\mathscr{L}_1(\mathcal{H})$ is self-adjoint, positive definite and is diagonal with respect to the orthonormal basis $(e_n)_{n\in\mathbb{N}}$ of $\mathcal{H}$, i.e. $Qe_n = \nu_n e_n$ and $\nu_n > 0$ for all $n \in \mathbb{N}$. $\mathcal{X} := \{\mathcal{X}(t)\}_{t\in\mathbb{R}}$ is of the form*

$$(\mathcal{X}(t)|x) = \sum_{n=0}^{\infty} X_n(t)(e_n|x), \quad t \in \mathbb{R}, \quad x \in \mathcal{H}, \tag{3.4}$$

where X_n are mutually independent, real-valued and centered processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover, for every $n \in \mathbb{N}$ the process X_n features the unique spectral density ϕ.

As we will see, $\mathcal{X}$ (as in (3.4)) is not a well defined $\mathcal{H}$-valued random variable. However, due to $\mathcal{X}(t) : \Omega \to \mathcal{H}_{Q^{-1/2}}$, where $\mathcal{H}_{Q^{-1/2}}$ is the completion of $\mathcal{H}$ with respect to the norm $|x|^2_{Q^{-1/2}} := |Q^{-1/2}x|_{\mathcal{H}}$, $x \in \mathcal{H}$, the process $Q^{1/2}\mathcal{X}$ converges in $L_2(\Omega;\mathcal{H})$. Note that $Q^{1/2}$ is well-defined and belongs to $\mathscr{L}_2(\mathcal{H})$.

Let us deduce the distributional properties of the process $Q^{1/2}\mathcal{X}$, where the covariance operator Q and the process $\mathcal{X}$ satisfy Hypothesis (X). It is obvious that $Q^{1/2}\mathcal{X}$ is again a process, so it remains to calculate the mean value and the covariance.

$$\mathbb{E}\left(Q^{1/2}\mathcal{X}(t) \mid x\right) = \mathbb{E}\left(\sum_{n=1}^{\infty} \sqrt{\nu_n} X_n(t) e_n \mid x\right) = \sum_{n=1}^{\infty} \sqrt{\nu_n}\mathbb{E}(X_n(t))(e_n \mid x) = 0,$$

for every $x \in \mathcal{H}$ and $t \in \mathbb{R}$. Regarding the covariance we have for all $s, t \in \mathbb{R}$

$$\begin{aligned}\left(\mathrm{Cov}[Q^{1/2}\mathcal{X}(t), Q^{1/2}\mathcal{X}(s)]e_m \mid e_n\right) &= \mathbb{E}\left[\left(Q^{1/2}\mathcal{X}(t) \mid e_m\right)\left(Q^{1/2}\mathcal{X}(s) \mid e_n\right)\right] \\ &= \delta_{mn}\sqrt{\nu_m\nu_n}\mathbb{E}\left[X_m(t)X_n(s)\right].\end{aligned}$$

Thus it is meaningful to define a vector-valued process of a certain spectral type as

Definition 3.5 (Vector-valued processes of spectral type ϕ)**.** *Let Q and $\mathcal{X}$ be subject to Hypothesis (X), then we call the process $Q^{1/2}\mathcal{X} = \{Q^{1/2}\mathcal{X}(t)\}_{t\in\mathbb{R}}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ a $\mathcal{H}$-valued process of spectral type ϕ.*

By the definition of a stochastic integral it is

$$\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) := \sum_{n=1}^{\infty}\int_{\mathbb{R}} R(t)Q^{1/2}e_n \mathrm{d}X_n(t).$$

Our next goal is to calculate the covariance operator. For every $x, y \in \mathcal{H}$ it is

$$\begin{aligned}&\mathbb{E}\left[\left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid x\right)_{\mathcal{H}}\left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid y\right)_{\mathcal{H}}\right] \\ &\quad= \mathbb{E}\left[\sum_{k=1}^{\infty}\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}}\mathrm{d}X_k(t)\sum_{l=1}^{\infty}\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}}\mathrm{d}X_l(t)\right] \\ &\quad= \mathbb{E}\left[\sum_{k=l}\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}}\mathrm{d}X_k(t)\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}}\mathrm{d}X_l(t)\right] \\ &\qquad + 2\mathbb{E}\left[\sum_{k<l}\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}}\mathrm{d}X_k(t)\int_{\mathbb{R}}\left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}}\mathrm{d}X_l(t)\right]\end{aligned}$$

and with the aid of the independence of X_k and X_l for $k \neq l$ we proceed in view of Theorem 3.4 with

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid x\right)_{\mathcal{H}} \left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid y\right)_{\mathcal{H}}\right] \\
&\quad = \sum_{n=1}^{\infty} \mathbb{E}\left[\left(\int_{\mathbb{R}} \left(R(t)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \mathrm{d}X_n(t)\right) \left(\int_{\mathbb{R}} \left(R(t)Q^{1/2}e_n \mid y\right)_{\mathcal{H}} \mathrm{d}X_n(t)\right)\right] \\
&\quad = \sum_{n=1}^{\infty} \left(\left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \mid \left(R(\cdot)Q^{1/2}e_n \mid y\right)_{\mathcal{H}}\right)_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}.
\end{aligned}$$

Now we may choose $x = y$ to obtain the variance

$$\mathbb{E}\left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid x\right)_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} \left\| \left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \right\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2$$

for all $x \in \mathcal{H}$. Hence, letting $(h_n)_{n\in\mathbb{N}}$ an arbitrary orthonormal basis in $\mathcal{H}$, then Parseval's equation yields

$$\begin{aligned}
\mathbb{E}\left|\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t)\right|_{\mathcal{H}}^2 &= \sum_{k=1}^{\infty} \mathbb{E}\left(\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) \mid h_k\right)_{\mathcal{H}}^2 \\
&= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left\| \left(R(\cdot)Q^{1/2}e_n \mid h_k\right)_{\mathcal{H}} \right\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2 .
\end{aligned}$$

Note, that $R \in \dot{\mathrm{H}}_2^{\phi}(\mathbb{R};\mathcal{B}(\mathcal{H}))$ implies $RQ^{1/2} \in \dot{\mathrm{H}}_2^{\phi}(\mathbb{R};\mathscr{L}_2(\mathcal{H}))$ as well as $\left(RQ^{1/2}e_n \mid x\right)_{\mathcal{H}} \in \dot{\mathrm{H}}_2^{\phi}(\mathbb{R})$ for all $x \in \mathcal{H}$ and for all $n \in \mathbb{N}$. Resuming, we deduced the following identity.

Theorem 3.6. *Let $Q^{1/2}\mathcal{X}$ be an $\mathcal{H}$-valued process of spectral type ϕ. Let further $R : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ and $(h_n)_{n\in\mathbb{N}}$ an orthonormal basis in $\mathcal{H}$. Then the identity*

$$\mathbb{E}\left|\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t)\right|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \left\| \left(R(\cdot)Q^{1/2}e_n \mid h_k\right)_{\mathcal{H}} \right\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2$$

holds and the left hand-side is independent from the choice of the basis $(h_n)_{n\in\mathbb{N}}$. In particular the stochastic integral (3.3) is well defined, if $R \in \dot{\mathrm{H}}_2^{\phi}(\mathbb{R};\mathcal{B}(\mathcal{H}))$.

Suppose now, that there is a second orthonormal system $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ and scalar functions $r(\cdot,\cdot) : \mathbb{R} \times \mathbb{C} \to \mathbb{R}$, so that the operator $R(t)$ decomposes into

$$R(t)x = \sum_{n=1}^{\infty} r(t,\mu_n)(x \mid g_n)g_n, \quad t \geq 0, \quad x \in \mathcal{H}.$$

Note that the operator valued function $R : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ particularly admits this property if $R(t)$ is self-adjoint and the resolvent set $\rho(R(t))$ is compact for all

$t \in \mathbb{R}$. We have

$$\begin{aligned}\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t) &= \sum_{n=1}^{\infty}\int_{\mathbb{R}} R(t)Q^{1/2}e_n \mathrm{d}X_n(t)\\ &= \sum_{n=1}^{\infty}\sqrt{\nu_n}\int_{\mathbb{R}} R(t)e_n\mathrm{d}X_n(t)\\ &= \sum_{n=1}^{\infty}\sqrt{\nu_n}\int_{\mathbb{R}}\sum_{k=1}^{\infty}(e_n \mid g_k)r(t,\mu_k)g_k\mathrm{d}X_n(t)\\ &= \sum_{n=1}^{\infty}\sqrt{\nu_n}\int_{\mathbb{R}}\sum_{k,l}(e_n \mid g_k)r(t,\mu_k)(g_k \mid e_l)e_l\mathrm{d}X_n(t)\\ &= \sum_{n,k,l}\sqrt{\nu_n}(e_n \mid g_k)(g_k \mid e_l)\int_{\mathbb{R}} r(t,\mu_k)\mathrm{d}X_n(t)e_l\end{aligned}$$

and Theorem 3.4 together with the elementary identity $(x \mid y) = \sum_j (x \mid e_j)(y \mid e_j)$ for $x, y \in \mathcal{H}$, yields

$$\begin{aligned}&\mathbb{E}\left|\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t)\right|_{\mathcal{H}}^2\\ &\quad= \sum_{n,k,l,m}\nu_n(e_n \mid g_k)(g_k \mid e_l)(e_n \mid g_m)(g_m \mid e_l)\left(r(\cdot,\mu_k) \mid r(\cdot,\mu_m)\right)_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}\\ &\quad= \sum_{n,k,m}\nu_n(e_n \mid g_k)(e_n \mid g_m)\left(r(\cdot,\mu_k) \mid r(\cdot,\mu_m)\right)_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}\left[\sum_l (g_k \mid e_l)(g_m \mid e_l)\right]\\ &\quad= \sum_{n,k}\nu_n(e_n \mid g_k)^2\left(r(\cdot,\mu_k) \mid r(\cdot,\mu_k)\right)_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}\\ &\quad= \sum_k \|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}\left[\sum_n (Q^{1/2}e_n \mid g_k)^2\right]\\ &\quad= \sum_k \|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}\left[\sum_n (e_n \mid Q^{1/2}g_k)^2\right]\end{aligned}$$

and we may employ Parseval's equation $|x|^2_{\mathcal{H}} = \sum_j (x, e_j)^2$ for $x \in \mathcal{H}$ to conclude

$$\mathbb{E}\left|\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t)\right|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty}|Q^{1/2}g_k|^2_{\mathcal{H}}\|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}.$$

Let us introduce a notation of a shifted, time inverted and truncated function f supported on $\mathbb{R}$ in virtue of

$$f^{\langle t\rangle}(\tau) := f(t-\tau)\chi_{(-\infty,t]}(\tau) = \begin{cases} f(t-\tau) & : -\infty < \tau \le t;\\ 0 & : \tau > t.\end{cases} \tag{3.5}$$

For brevity we write $r_n(\tau) := r(\tau, \mu_n)$. The latter observations lead us to

$$\mathbb{E}\left|\int_{\mathbb{R}}(R^{\langle t\rangle}(\tau) - R^{\langle s\rangle}(\tau))\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2 = \sum_{k=1}^{\infty} |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2.$$

Observe that due to

$$\begin{aligned}(\mathcal{F}f^{\langle t\rangle})(\xi) &= \int_{\mathbb{R}} f(t-\tau)\chi_{(-\infty,t]}(\tau)e^{-i\xi\tau}\mathrm{d}\tau \\ &= \int_{\mathbb{R}} f(-s)\chi_{(-\infty,0]}(s)e^{-i\xi(s+t)}\mathrm{d}s = e^{-i\xi t}(\mathcal{F}f^{\langle 0\rangle})(\xi)\end{aligned}$$

we find for all $t \in \mathbb{R}$

$$\|f^{\langle t\rangle}\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R})} = \|f^{\langle 0\rangle}\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R})} = \|f\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R}_+)}.$$

Next, deduce that if there is a number $\gamma \in (1,3)$ such that $\sup_{\lambda\in\mathbb{R}} |\lambda|^{\gamma}\phi(\lambda) < \infty$ we have

$$\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}) \hookrightarrow \dot{\mathrm{H}}_2^{\phi}(\mathbb{R}),$$

and therefrom we obtain by designating with $c > 0$ a generic constant

$$\begin{aligned}\|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2 &\le c\|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R})}^2 \\ &= c\left\|\partial^{\frac{2-\gamma}{2}}(r_k^{\langle t\rangle} - r_k^{\langle s\rangle})\right\|_{L_2(\mathbb{R})}^2 \\ &= c\int_{\mathbb{R}}\left|(\partial^{\frac{2-\gamma}{2}}r_k^{\langle t\rangle})(\tau) - (\partial^{\frac{2-\gamma}{2}}r_k^{\langle s\rangle})(\tau)\right|^2\mathrm{d}\tau \\ &= c\int_{\mathbb{R}}\left|(\partial^{\frac{2-\gamma}{2}}r_k^{\langle 0\rangle})(\tau-t) - (\partial^{\frac{2-\gamma}{2}}r_k^{\langle 0\rangle})(\tau-s)\right|^2\mathrm{d}\tau \\ &= c\left\|(\partial^{\frac{2-\gamma}{2}}r_k^{\langle 0\rangle})(s-t+\cdot) - (\partial^{\frac{2-\gamma}{2}}r_k^{\langle 0\rangle})(\cdot)\right\|_{L_2(\mathbb{R})}^2 \\ &\le c\left\|\partial^{\frac{2-\gamma}{2}}r_k^{\langle 0\rangle}\right\|_{B_{2,\infty}^{\theta}(\mathbb{R})}^2 |t-s|^{2\theta},\end{aligned}$$

where $0 \le \theta < \frac{\gamma-1}{2}$ and $B_{2,\infty}^{\theta}(\mathbb{R})$ denotes a Besov space with the equivalent norm

$$\|f\|_{B_{2,\infty}^{\theta}(\mathbb{R})} = \left[\|f\|_{L_2(\mathbb{R})}^2 + \sup_{h\neq 0}\int_{\mathbb{R}}\frac{|f(y+h)-f(y)|^2}{|h|^{2\theta}}\mathrm{d}y\right]^{1/2}$$

and cf. [10, Section 2.3.2] to verify

$$\mathrm{H}_2^{\theta}(\mathbb{R}) \hookrightarrow B_{2,\infty}^{\theta}(\mathbb{R}).$$

Finally, with the apparent relation

$$\|f\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R})} + \|f\|_{\dot{\mathrm{H}}_2^{\theta+\sigma}(\mathbb{R})} = \|\partial^{\sigma}f\|_{\mathrm{H}_2^{\theta}(\mathbb{R})}, \qquad -\tfrac{1}{2} < \sigma < \tfrac{1}{2},$$

we derived the estimate

$$\|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|_{\dot{\mathrm{H}}_2^{\phi}(\mathbb{R})}^2 \le \left[\|r_k\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{2-\gamma}{2}}(\mathbb{R}_+)}\right]^2 |t-s|^{2\theta}.$$

Let us hypothesize

Hypothesis (X_ϕ). *$Q^{1/2}\mathcal{X}$ is a $\mathcal{H}$-valued process of spectral type ϕ and there is a number $\gamma \in (1,3)$ such that $\sup_{\lambda\in\mathbb{R}} |\lambda|^\gamma \phi(\lambda) < \infty$.*

We have then observed

Theorem 3.7. *Let $Q^{1/2}\mathcal{X}$ be subject to Hypothesis (X_ϕ). Let further $R : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ and $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, such that $R(t)$ decomposes into*

$$R(t)x = \sum_{n=1}^{\infty} r(t,\mu_n)(x \mid g_n)g_n, \quad t \in \mathbb{R}, \qquad x \in \mathcal{H}.$$

Then there is a constant $c > 0$ such that

$$\mathbb{E}\left|\int_{\mathbb{R}} R(t)\mathrm{d}(Q^{1/2}\mathcal{X})(t)\right|_{\mathcal{H}}^2 \le c\sum_{k=1}^{\infty} |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R})}.$$

By means of the notation (3.5)

$$\mathbb{E}\left|\int_{\mathbb{R}} R^{\langle t\rangle}(\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2 \le c\sum_{k=1}^{\infty} |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)}$$

holds true for all $t \in \mathbb{R}$. Moreover, for $\theta \in [0, \frac{\gamma-1}{2})$ and $s, t \in \mathbb{R}$ it is

$$\begin{aligned}\mathbb{E}&\left|\int_{\mathbb{R}} (R^{\langle t\rangle}(\tau) - R^{\langle s\rangle}(\tau))\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2 \\ &\qquad \le c\sum_{k=1}^{\infty} |Q^{1/2}g_k|_{\mathcal{H}}^2 \left[\|r_k\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{2-\gamma}{2}}(\mathbb{R}_+)}\right]^2 |t-s|^{2\theta}.\end{aligned}$$

Concerning integrals evaluated on an interval $[0, t_0]$, where $t_0 > 0$, we deduce the following corollary.

Corollary 3.8. *Let $Q^{1/2}\mathcal{X}$ be subject to Hypothesis (X_ϕ). Let further $R : \mathbb{R} \to \mathcal{B}(\mathcal{H})$ and $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, such that $R(t)$ decomposes into*

$$R(t)x = \sum_{n=1}^{\infty} r(t,\mu_n)(x \mid g_n)g_n, \quad t \in \mathbb{R}, \qquad x \in \mathcal{H}$$

and let $0 \le s \le t_0$. Then there is a constant $c > 0$ such that

$$\mathbb{E}\left|\int_0^{t_0} R(\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2 \le c\sum_{k=1}^{\infty} |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}([0,t_0])}.$$

Moreover, we have

$$\mathbb{E}\left|\int_0^{t_0} R(t_0-\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2 \le c\sum_{k=1}^{\infty}|Q^{1/2}g_k|_{\mathcal{H}}^2\|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}([0,t_0])}$$

and by means of notation (3.5)

$$\mathbb{E}\left|\int_0^{t_0}(R^{\langle t_0\rangle}(\tau)-R^{\langle s\rangle}(\tau))\mathrm{d}(Q^{1/2}\mathcal{X})(\tau)\right|_{\mathcal{H}}^2$$
$$\le c\sum_{k=1}^{\infty}|Q^{1/2}g_k|_{\mathcal{H}}^2\left[\|r_k\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}([0,t_0])}+\|r_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{2-\gamma}{2}}([0,t_0])}\right]^2|t_0-s|^{2\theta}.$$

Proof. The claim follows by employing Theorem 3.7 to the operators $R\chi_{[0,t_0]}$, $R^{\langle t_0\rangle}\chi_{[0,t_0]}$ and $(R^{\langle t_0\rangle}-R^{\langle s\rangle})\chi_{[0,t_0]}$ respectively. □

4. Parabolic Volterra equations

Throughout this chapter $\mathcal{H}$ is a separable Hilbert space and $Q^{1/2}\mathcal{X}$ is subject to Hypothesis (X_ϕ).

4.1. **Main results.** Let A be a closed linear densely defined operator in $\mathcal{H}$, and $b\in L_1(\mathbb{R}_+)$ a scalar kernel. Let us consider the problem

$$u(t)+\int_0^t b(t-\tau)Au(\tau)\mathrm{d}\tau = Q^{1/2}\mathcal{X}(t),\qquad t\ge 0 \tag{4.1}$$

in the Hilbert space $\mathcal{H}$. In particular we recall that Hypothesis (X_ϕ) forces the existence of a sequence $(\nu_n)_{n\in\mathbb{N}}\in\ell_1(\mathbb{R}_+)$ and an orthonormal basis $(e_n)_{n\in\mathbb{N}}\subset\mathcal{H}$, such that $Qe_n=\nu_n e_n$ for every $n\in\mathbb{N}$.

Because problem (4.1) is motivated from applications of linear viscoelastic material behavior, we consider the operator $-A$ to be an elliptic differential operator like the Laplacian, the elasticity operator, or the Stokes operator, together with appropriate boundary conditions (e.g. Prüss [6, Section I.5]). We formulate abstractly

Hypothesis (A). *A is an unbounded, self-adjoint, positive definite operator in $\mathcal{H}$ with compact resolvent. Consequently, the eigenvalues μ_n of A form a strictly positive, nondecreasing sequence with $\lim_{n\to\infty}\mu_n=\infty$, the corresponding eigenvectors $(a_n)_{n\in\mathbb{N}}\subset\mathcal{H}$ form an orthonormal basis of $\mathcal{H}$.*

Observe, that Hypothesis (A) implies the sectoriality of the operator A with angle $\phi_A=0$ (cf. [4, Section 1]). This observation allows us to define complex powers A^z for arbitrary $z\in\mathbb{C}$; cf. [6, Section 8.1].

The kernel b is supposed to be the antiderivative of a 3-monotone scalar function (see Definition 2.6); more precisely b is subject to

Hypothesis (b): *The kernel b is of the form*

$$b(t) = b_0 + \int_0^t b_1(\tau)\mathrm{d}\tau, \quad t > 0, \tag{4.2}$$

where $b_0 \geq 0$ and $b_1(t)$ is 3-monotone with $\lim_{t\to\infty} b_1(t) = 0$; in addition,

$$\lim_{t\downarrow 0} \frac{\frac{1}{t}\int_0^t \tau b_1(\tau)\mathrm{d}\tau}{b_0 + \int_0^t -\tau \dot{b}_1(\tau)\mathrm{d}\tau} < \infty. \tag{4.3}$$

In case (A) and (b) are valid, problem (4.1) is well-posed and parabolic; for kernels subject to (4.2), condition (4.3) is in fact equivalent to parabolicity. Typical examples of kernels arising from the theory of linear viscoelasticity (cf. [6, Section I.5]), which satisfy Hypothesis (b) are the material functions of Newtonian fluids ($b_0 > 0$, $b_1 \equiv 0$), Maxwell fluids ($b_0 = 0$, $b_1(t) = \sigma \exp\{-\frac{\sigma t}{\nu}\}$) and of power type materials ($b_0 = 0$, $b_1(t) = g_\alpha(t)$, $\alpha \in (1,2)$). Define

$$\rho := \frac{2}{\pi} \sup\left\{ |\arg \widehat{b}(\lambda)| : \mathrm{Re}\,\lambda > 0 \right\},$$

then we obtain the subsequent existence and regularity results for the mild solution, in the sense of Definition 2.1, of (4.1).

Theorem 4.1. *Let Hypotheses (A), (b) and (X_ϕ) are valid.*

(i) If $QA^{\frac{1-\gamma}{\rho}} \in \mathscr{L}_1(\mathcal{H})$, then the mild solution u of (4.1) exists and is mean-square continuous on $\mathbb{R}_+$. Moreover, the trajectories of u are continuous on the half-line $\mathbb{R}_+$ almost sure.

(ii) If in addition, there is $\theta \in (0, \frac{\gamma-1}{2})$ such that $QA^{\frac{1-\gamma}{\rho}+\frac{2\theta}{\rho}} \in \mathscr{L}_1(\mathcal{H})$, then the trajectories of u are locally Hölder-continuous of any order strictly less then θ almost sure.

Regarding existence and regularity it is easily seen from Theorem 4.1 and Theorem 4.3 (see below) that the results are independent from the choice of the eigensystems of the operators A and Q.

Remark 4.2. The case $b \equiv \text{const}$ merely corresponds to the stochastic differential equation

$$\begin{cases} \dot{u} + Au = Q^{1/2}\dot{\mathcal{X}}, & t > 0, \\ u(0) = 0. \end{cases}$$

It is then obvious that Theorem 4.1 applies with $\rho = 1$. Moreover, the notions of strong and mild solutions in the sense of Definition 2.1 are equivalent in all the cases where $b \equiv \text{const}$.

Let us take up a different viewpoint to Volterra equations with fractional noise. We consider the problems

$$u(t) + \int_0^t g_\alpha(t-\tau)Au(\tau)\mathrm{d}\tau = \int_0^t g_\beta(t-\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau), \qquad t \geq 0 \tag{4.4}$$

in the Hilbert space $\mathcal{H}$, where the operator A is subject to Hypothesis (A) and g_κ denotes the Riemann-Liouville kernel; see (2.4).

In case (A) is valid and $0 < \alpha < 2$, problem (4.4) is well-posed and parabolic.

Theorem 4.3. *Assume Hypotheses (A) and (X_ϕ) are valid and let $\alpha \in (0, 2)$, $\beta > 0$, $\theta \in [0, 1]$, such that $\beta \in (\frac{3-\gamma}{2} + \theta, \frac{3-\gamma}{2} + \theta + \alpha)$.*

- *(i) If $QA^{\frac{3-2\beta-\gamma}{\alpha}} \in \mathscr{L}_1(\mathcal{H})$ then the mild solution u of (4.4) exists and is mean-square continuous on $\mathbb{R}_+$. Moreover, the trajectories of u are almost surely continuous on $\mathbb{R}_+$.*
- *(ii) If $QA^{\frac{3-2\beta-\gamma}{\alpha}+\frac{2\theta}{\alpha}} \in \mathscr{L}_1(\mathcal{H})$ then the trajectories of u are locally Hölder-continuous of any order strictly less then θ almost sure.*

On the first view Theorem 4.1 seems to be a variant of Theorem 4.3 for the case $\beta = 1$ and more general kernels. However, Hypothesis (b) is too stringent as to countenance standard kernels g_α with $\alpha < 1$. Similar results for the special cases where $\mathcal{X}$ is modeled to be a vector-valued Wiener process or a vector-valued fractional Brownian motion were obtained by Clément et al. [3], Bonaccorsi [2], Sp. & Wilke [9] and Sp. [8]. However, all those cases are completely covered by our approach.

Remark 4.4. Note that

(1) if $\mathcal{X}$ is chosen to be a vector-valued centered Lévy process, then the above results hold with $\gamma = 2$.
(2) if $\mathcal{X} = B^H$ is chosen to be a vector-valued fractional Brownian motion with Hurst parameter $0 < H < 1$, the above results hold with $\gamma = 2H + 1$.
(3) if $\mathcal{X} = \mathcal{RB}^\eta_\sigma$ is chosen to be a vector-valued fractional Riesz-Bessel motion with parameters $\frac{1}{2} < \eta < \frac{3}{2}$ and $\sigma > \max\{0, 1-\eta\}$, then the above results hold true for every $\gamma \in [2\eta, 2(\sigma + \eta)] \cap [2\eta, 3)$.

4.2. **Proof of the main results.** Before proving the main results we gather some elementary results needed later on. The first is taken from [8, Lemma 4.1] and reads as

Lemma 4.5. *Let $\mu > 0$, the function b satisfying Hypothesis (b) with $\rho \in [1,2)$ and denote by $s : \mathbb{R}_+ \to \mathbb{R}$ the solution of the problem*

$$s(t) + \mu(b * s)(t) = 1, \qquad t \geq 0$$

Then the following are true.

(i) $|s(t)| \leq 1$ *for all* $t \geq 0$*;*
(ii) $\|\dot{s}\|_{L_1(\mathbb{R}_+)} \leq c$*;*
(iii) $\|s\|_{L_1(\mathbb{R}_+)} \leq c\mu^{-\frac{1}{\rho}}$*;*
(iv) $\|s\|_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-\sigma}(\mathbb{R}_+)} \leq c\mu^{\frac{\theta-\sigma}{\rho}}$ *for* $\sigma \in (0,1)$, $\theta \in [0,\sigma)$ *and* $\mu \geq 1$,

where $c > 0$ denotes a constant which is independent of μ.

Lemma 4.6. *Let $\mu > 0$, $\alpha \in (0,2)$, $\beta > 0$, $\sigma \in (0,1)$, $\theta \in [0,1]$ and denote by $r : \mathbb{R}_+ \to \mathbb{R}$ the solution of the problem*

$$r(t) + \mu(g_\alpha * r)(t) = g_\beta(t), \qquad t \geq 0,$$

where g_κ means the Riemann-Liouville kernel of fractional integration; see (2.4). Then there is a constant $c > 0$ so that

$$\|r\|_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-\sigma}(\mathbb{R}_+)} \leq c\mu^{\frac{2(1-\beta+\theta-\sigma)}{\alpha}},$$

whenever $\beta \in (1-\sigma+\theta, 1-\sigma+\theta+\alpha)$.

Proof. By the Paley-Wiener theorem, a function f belongs to $L_2(\mathbb{R})$ if and only if $\widehat{f} \in \mathscr{H}_2(\mathbb{C}_+)$, the Hardy space of exponent 2, and the theorem also yields

$$\|f\|^2_{L_2(\mathbb{R})} = \frac{1}{2\pi}\|\widehat{f}\|^2_{\mathscr{H}(\mathbb{C}_+)} = \frac{1}{2\pi}\int_{\mathbb{R}} \left|\widehat{f}(i\rho)\right|^2 \mathrm{d}\rho = \frac{1}{2\pi}\int_{\mathbb{R}} |(\mathcal{F}f)(\rho)|^2 \,\mathrm{d}\rho.$$

Extending the function r trivially by zero for $t < 0$, we have by means of identity (2.6)

$$\|r\|^2_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-\sigma}(\mathbb{R}_+)} = \|\partial^{\theta+\frac{1}{2}-\sigma} r\|^2_{L_2(\mathbb{R})} = \int_{\mathbb{R}} |\mathcal{F}r(\rho)|^2 \, |\rho|^{2\theta+1-2\sigma}\mathrm{d}\rho = \int_{\mathbb{R}} |\widehat{r}(i\rho)|^2 \, |\rho|^{2\theta+1-2\sigma}\mathrm{d}\rho.$$

Observe now, that due to (**??**) we obtain a representation of r in terms of its Laplace transform, that is

$$\widehat{r}(\lambda) = \frac{\widehat{g}_\beta(\lambda)}{1+\mu\widehat{g}_\alpha(\lambda)} = \frac{\lambda^\alpha}{\lambda^\beta(\lambda^\alpha+\mu)}, \qquad \operatorname{Re}\lambda \geq 0, \quad \lambda \neq 0.$$

Hence, we can proceed with

$$
\begin{aligned}
\|r\|^2_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-\sigma}(\mathbb{R}_+)} &= \int_{\mathbb{R}} |\widehat{r}(i\rho)|^2\, |\rho|^{2\theta+1-2\sigma}\mathrm{d}\rho \\
&= \int_{\mathbb{R}} \left[\frac{|\rho|^{\alpha}}{|\rho|^{\beta}(|\rho|^{\alpha}+\mu)}\right]^2 |\rho|^{2\theta+1-2\sigma}\mathrm{d}\rho \\
&= 2\int_0^{\infty} \left[\frac{\rho^{\alpha}}{\rho^{\beta}(\rho^{\alpha}+\mu)}\right]^2 \rho^{2\theta+1-2\sigma}\mathrm{d}\rho \\
&= 2\mu^{\frac{2(1-\beta-\sigma)}{\alpha}} \int_0^{\infty} \left[\frac{\tau^{\theta+\alpha-\beta-\sigma+\frac{1}{2}}}{1+\tau^{\alpha}}\right]^2 \mathrm{d}\tau
\end{aligned}
$$

and the last integral is finite if and only if $1-\sigma+\theta<\beta<1-\sigma+\theta+\alpha$. $\square$

4.2.1. *Proof of Theorem 4.1.* It is due to [6, Section I.1], that if (A) and (b) are valid, problem (4.1) admits a resolvent $S(t)$, so that $S\in L_1(\mathbb{R}_+;\mathcal{B}(\mathcal{H}))$, $S(t)$ is strongly continuous, is uniformly bounded by 1 and $\lim_{t\to\infty}|S(t)|_{\mathcal{B}(\mathcal{H})}=0$. Consequently, the unique mild solution u of problem (4.1) exists and is given by the variation of parameters formula (2.9)

$$
u(t)=\int_0^t S(t-\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau), \qquad t\geq 0.
$$

By means of the spectral decomposition of the operator A, the resolvent family decomposes into

$$
S(t)x=\sum_{k=1}^{\infty} s_k(t)(x|a_n)a_n, \qquad t\geq 0, \qquad x\in\mathcal{H},
$$

where the scalar functions $s_n(t):=s(t,\mu_n)$ are the solutions of the scalar problems

$$
s_n(t)+\mu_n\int_0^t b(t-\tau)s_n(\tau)\mathrm{d}\tau=1, \qquad t\geq 0.
$$

Observe now, that due to Hypothesis (A) there is a positive integer $N_\mu\in\mathbb{N}$, so that $\mu_n\geq 1$ for all $n>N_\mu$. Then Corollary 3.8 yields the existence of a constant

$c > 0$ (in the sequel generic) so that for $t \geq 0$ in view of Lemma 4.5 (iv) it is

$$\begin{aligned}
\mathbb{E}\,|u(t)|^2_{\mathcal{H}} &\leq c \sum_{k=1}^{\infty} |Q^{1/2} a_k|^2_{\mathcal{H}} \|s_k\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}([0,t])} \\
&\leq c \sum_{k=1}^{\infty} |Q^{1/2} a_k|^2_{\mathcal{H}} \|s_k\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} \\
&\leq c \left[\sum_{k=1}^{N_\mu} |Q^{1/2} a_k|^2_{\mathcal{H}} \|s_k\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} + \sum_{k=N_\mu+1}^{\infty} |Q^{1/2} a_k|^2_{\mathcal{H}} \mu_k^{\frac{1-\gamma}{\rho}} \right] \\
&= c \left[C(N_\mu) + \sum_{k=N_\mu+1}^{\infty} (Q a_k | a_k)_{\mathcal{H}} \mu_k^{\frac{1-\gamma}{\rho}} \right] \\
&= c \left[C(N_\mu) + \sum_{k=N_\mu+1}^{\infty} (Q a_k | A^{\frac{1-\gamma}{\rho}} a_k)_{\mathcal{H}} \right] \\
&\leq c \left[C(N_\mu) + \left\| Q A^{\frac{1-\gamma}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})} \right]
\end{aligned}$$

holds true. Then for $s, t \geq 0$ and $0 < \theta < \frac{\gamma-1}{2}$ the estimate

$$\begin{aligned}
\mathbb{E}\,|u(t) - u(s)|^2_{\mathcal{H}} &\leq c \sum_{k=1}^{\infty} |Q^{1/2} a_k|^2_{\mathcal{H}} \left[\|s_k\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} + \|s_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{2-\gamma}{2}}(\mathbb{R}_+)} \right]^2 |t-s|^{2\theta} \\
&\leq c \left[C(N_\mu) + \sum_{k=N_\mu+1}^{\infty} (Q a_k | a_k)_{\mathcal{H}} \mu_k^{\frac{2\theta-\gamma+1}{\rho}} \right] |t-s|^{2\theta} \\
&\leq c \left[C(N_\mu) + \left\| Q A^{\frac{2\theta-\gamma+1}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})} \right] |t-s|^{2\theta}
\end{aligned}$$

yields the mean-square continuity of u. Lastly, we may employ the Kahane-Khinchine inequality to obtain for all $2 < p < \infty$

$$\mathbb{E}|u(t) - u(s)|^p_{\mathcal{H}} \leq c_p \left(\mathbb{E}|u(t) - u(s)|^2_{\mathcal{H}} \right)^{\frac{p}{2}} \leq c_p \left[C(N_\mu) + \left\| Q A^{\frac{2\theta-\gamma+1}{\rho}} \right\| \right]^{\frac{p}{2}}_{\mathscr{L}_1(\mathcal{H})} |t-s|^{p\theta}$$

and the Kolmogorov-Čentsov-Theorem yields the claimed Hölderianity for every $\Theta \in (0, \theta - \frac{1}{p})$ for all $p \in (2, \infty)$.

4.2.2. *Proof of Theorem 4.3.* For $\alpha \in (0, 2)$ and $\beta > 0$ problem (4.4) admits a resolvent $R(t)$ which decomposes into

$$R(t)x = \sum_{k=1}^{\infty} r_k(t)(x|a_n)a_n, \qquad t \geq 0, \qquad x \in \mathcal{H},$$

where the scalar fundamental solutions $r_n(t) := r(t, \mu_n)$, $n \in \mathbb{N}$, of (4.4) can be expressed in terms of its Laplace transform

$$\widehat{r}_n(\lambda) = \frac{\widehat{g}_\beta(\lambda)}{1 + \mu_n \widehat{g}_\alpha(\lambda)} = \frac{\lambda^\alpha}{\lambda^\beta(\lambda^\alpha + \mu_n)}, \qquad \operatorname{Re}\lambda \geq 0, \quad \lambda \neq 0, \quad \mu_n > 0.$$

Thus the mild solution u of problem (4.4) exists and the variation of parameters formula (2.9) yields

$$u(t) = \int_0^t R(t-\tau)\mathrm{d}(Q^{1/2}\mathcal{X})(\tau), \qquad t \geq 0.$$

Then Corollary 3.8 ensures the existence of a constant $c > 0$ (in the sequel generic), so that in view of Lemma 4.6 we deduce for $t \geq 0$

$$\begin{aligned}\mathbb{E}\,|u(t)|^2_{\mathcal{H}} &\leq c\sum_{k=1}^{\infty}|Q^{1/2}a_k|^2_{\mathcal{H}}\|r_k\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}([0,t])} \leq c\sum_{k=1}^{\infty}|Q^{1/2}a_k|^2_{\mathcal{H}}\|r_k\|^2_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)}\\ &\leq c\sum_{k=1}^{\infty}|Q^{1/2}a_k|^2_{\mathcal{H}}\mu_k^{\frac{3-2\beta-\gamma}{\alpha}} = c\sum_{k=1}^{\infty}(Qa_k|a_k)_{\mathcal{H}}\mu_k^{\frac{3-2\beta-\gamma}{\alpha}}\\ &= c\sum_{k=1}^{\infty}(Qa_k|A^{\frac{3-2\beta-\gamma}{\alpha}}a_k)_{\mathcal{H}} = c\left\|QA^{\frac{3-2\beta-\gamma}{\alpha}}\right\|_{\mathscr{L}_1(\mathcal{H})}.\end{aligned}$$

In the same manner

$$\begin{aligned}\mathbb{E}\,|u(t)-u(s)|^2_{\mathcal{H}} &\leq c\sum_{k=1}^{\infty}|Q^{1/2}a_k|^2_{\mathcal{H}}\left[\|r_k\|_{\dot{\mathrm{H}}_2^{\frac{2-\gamma}{2}}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{2-\gamma}{2}}(\mathbb{R}_+)}\right]^2|t-s|^{2\theta}\\ &\leq c\sum_{k=1}^{\infty}(Qa_k|a_k)_{\mathcal{H}}\mu_k^{\frac{3-2\beta+2\theta-\gamma}{\alpha}}|t-s|^{2\theta}\\ &\leq c\left\|QA^{\frac{3-2\beta+2\theta-\gamma}{\alpha}}\right\|_{\mathscr{L}_1(\mathcal{H})}|t-s|^{2\theta}\end{aligned}$$

holds for $s,t \in \mathbb{R}_+$ and $\theta \in [0,1]$ and yields the mean-square continuity of u.

Again we may employ the Kahane-Khinchine inequality to obtain for all $2 < p < \infty$

$$\mathbb{E}|u(t)-u(s)|^p_{\mathcal{H}} \leq c_p\left(\mathbb{E}|u(t)-u(s)|^2_{\mathcal{H}}\right)^{\frac{p}{2}} \leq c_p\left\|QA^{\frac{3-2\beta+2\theta-\gamma}{\alpha}}\right\|^{\frac{p}{2}}_{\mathscr{L}_1(\mathcal{H})}|t-s|^{p\theta}$$

and the Kolmogorov-Čentsov-Theorem yields the claimed Hölderianity for every $\Theta \in (0, \theta - \frac{1}{p})$ for all $p \in (2,\infty)$. This completes the proof.

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