Martin–Luther–Universität
Halle–Wittenberg
Institut für Mathematik

L^{∞}–Estimates for Nonlinear Elliptic Neumann Boundary Value Problems

Patrick Winkert

Report No. 3 (2009)

Editors:
Professors of the Institute for Mathematics, Martin-Luther-University Halle-Wittenberg.

Electronic version: see `http://www2.mathematik.uni-halle.de/institut/reports/`

$L^\infty-$Estimates for Nonlinear Elliptic Neumann Boundary Value Problems

Patrick Winkert

Report No. 3 (2009)

Patrick Winkert
Martin-Luther-Universität Halle-Wittenberg
Naturwissenschaftliche Fakultät III
Institut für Mathematik
Theodor-Lieser-Str. 5
D-06120 Halle/Saale, Germany
Email: patrick.winkert@mathematik.uni-halle.de

L^∞–ESTIMATES FOR NONLINEAR ELLIPTIC NEUMANN BOUNDARY VALUE PROBLEMS

PATRICK WINKERT

Martin-Luther-University Halle-Wittenberg, Department of Mathematics, 06099 Halle, Germany

Abstract. In this paper we prove the L^∞-boundedness of solutions of the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned}$$

where A is a second order quasilinear differential operator and $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ and $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions satisfying some growth conditions. Our main result is given in Theorem 3.1 and is based on the Moser iteration technique along with the real interpolation theory. In addition, the solutions of the elliptic equation above belong to $C^{1,\alpha}(\overline{\Omega})$, if g is locally Hölder continuous.

1. Introduction

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} Au &= f(x,u,\nabla u) && \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} &= g(x,u) && \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\frac{\partial u}{\partial \nu}$ denotes the conormal derivative of u. Here, A is a second-order quasilinear differential operator in divergence form of Leray-Lions type given by

$$Au(x) = -\sum_{i=1}^{N}\frac{\partial}{\partial x_i}a_i(x,u(x),\nabla u(x)), \tag{1.2}$$

and $f:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}$ as well as $g:\partial\Omega\times\mathbb{R}\to\mathbb{R}$ are some Carathéodory functions. For $u\in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\gamma: W^{1,p}(\Omega)\to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation γu and write for short u, respectively, $g(x,u):=g(x,\gamma u)$. The main goal of this paper is to prove a priori estimates for solutions of the nonlinear elliptic equation in (1.1). For this purpose, we use some important tools

E-mail address: patrick.winkert@mathematik.uni-halle.de.

2000 *Mathematics Subject Classification.* 35B45, 35J25, 35J60.

Key words and phrases. A priori estimates, Neumann boundary values, Nonlinear elliptic equations.

like the Moser iteration technique and the real interpolation theory. By an a priori estimate, we mean an estimate of the form

$$\|u\|_{L^\infty(\Omega)} \leq C,$$

for all possible solutions of (1.1) with some constant C independent of u.
Many papers about a priori estimates for the Dirichlet problem were published in the last years. In [15] the author considers the weighted $p-$Laplacian eigenvalue problem given by

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{p-2}\nabla u) &= \lambda b(x)|u|^{p-2}u \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where $a, b : \Omega \to \mathbb{R}$ are measurable functions. He proved in Lemma 3.5 that any positive eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of problem (1.3) belongs to $L^\infty(\Omega)$. A related work of problem (1.3) in all of $\mathbb{R}^N$ can be found in [16], where the L^∞-boundedness for every positive solution has been demonstrated by using the Moser iteration. A priori estimates for the $p-$Laplacian equations with Dirichlet boundary values were also developed in [14], [26],[36],[39] and concerning the Laplace operator ($p = 2$) we refer to [5],[17],[19],[25]. A priori bounds for solutions of the Dirichlet problem for elliptic equations with discontinuous coefficients in unbounded domains were studied in [9]. Existence and a priori bounds for the semilinear elliptic problem of the form

$$\begin{aligned} -\Delta w &= w^Q + \mu \qquad &&\text{in } \Omega, \\ w &= \lambda &&\text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

were investigated in [7], where $Q > 0$, and μ and λ are nonnegative Radon measures in Ω and $\partial\Omega$, with $\int_\Omega \rho d\mu < +\infty$, where ρ is the distance to $\partial\Omega$. Moreover, the authors extended their results to the elliptic system

$$\begin{aligned} -\Delta u = v^p + \mu,\ -\Delta v &= u^q + \eta \qquad &&\text{in } \Omega, \\ u = \lambda,\ v &= \kappa &&\text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

with $p, q > 0$ and the same assumptions on η and κ. To obtain a priori estimates for elliptic systems of Laplace type we also refer to the publications in [11] and [28].
In [6] the authors study the uniformly elliptic Liouville-type problem

$$\begin{aligned} -\nabla(A\nabla u) &= \mu \frac{Ke^u}{\int_\Omega Ke^u dx} \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where Ω is a bounded smooth domain in $\mathbb{R}^2$, $A(x)$ is a positive-definite matrix and $K > 0$. In particular, they showed a priori bounds for solutions of problem (1.6).
New a priori estimates of the solutions to the second-order elliptic interface problem

$$\begin{aligned} -\nabla(\beta(x)\nabla u(x)) &= f(u) \qquad &&\text{in } \Omega, \\ u &= 0 &&\text{on } \partial\Omega, \end{aligned}$$

were studied from Huang and Zou in [22], where Ω is a convex polyhedral domain in $\mathbb{R}^3$ and $\beta(x)$ is positive and piecewise constant in Ω. To complete the references, we point out the papers in [2, 4, 20, 21, 30, 32].

Concerning a priori bounds for elliptic equations with zero Neumann conditions we refer to the results in [35] and [38], where they consider problems of the form

$$\begin{aligned} -\Delta u + \lambda u = f(u),\ u > 0 \qquad & \text{in } \Omega,\\ \frac{\partial u}{\partial \nu} = 0 \qquad & \text{on } \partial\Omega, \end{aligned}$$

in a bounded convex domain $\Omega \subset \mathbb{R}^3$ with smooth boundary and $\lambda > 0$. Motreanu et al. have applied the Moser iteration, too, in [27, Proof of Proposition 2.5] to prove L^∞-boundedness for solutions of the Neumann problem

$$\begin{aligned} -\operatorname{div}\vartheta_\varepsilon(z,\nabla v_\varepsilon) = f_0(z,v_\varepsilon) + \lambda_\varepsilon f_0(z,x_0) - \lambda_\varepsilon |v_\varepsilon - x_0|^{p-2}(v_\varepsilon - x_0) \qquad & \text{in } Z,\\ \frac{\partial u}{\partial \nu} = 0 \qquad & \text{on } \partial Z, \end{aligned}$$

where

$$\vartheta_\varepsilon(z,\xi) = |\xi|^{p-2}\xi + \lambda_\varepsilon |\nabla x_0|^{p-2}\nabla x_0 + \lambda_\varepsilon|\xi - \nabla x_0|^{p-2}(\xi - \nabla x_0),$$

with $Z \subset \mathbb{R}^N$ is a bounded domain with a C^2-boundary $\partial Z, 0 < \lambda_\varepsilon \leq 1, \varepsilon \in (0,1]$, $x_0 \in L^\infty(\Omega)$ fixed and with a Carathéodory function $f_0 : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$ satisfying some growth condition. We also refer to [1] and [10].

The novelty of our paper is the demonstration of a priori estimates for nonlinear elliptic equations with nonlinear nonzero Neumann boundary values of the form (1.1), where the Carathéodory functions f and g depend on $u, \nabla u$ and u, respectively, satisfying a natural growth condition. Additionally, we extend our results and show that every solution of (1.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ in case g is locally Hölder continuous. This fact follows directly from the L^∞-boundedness by applying the results of Liebermann in [24].

2. Notations and Hypotheses

Let $\frac{1}{p} + \frac{1}{q} = 1$. We suppose the following conditions for the operator A and the nonlinearities $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$.

(A1) Each $a_i(x,s,\xi)$ satisfies Carathéodory conditions, i.e., is measurable in $x \in \Omega$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s,ξ) for a.e. $x \in \Omega$. Furthermore, a constant $c_0 > 0$ and a function $k_0 \in L^q(\Omega)$ exist so that

$$|a_i(x,s,\xi)| \leq k_0(x) + c_0(|s|^{p-1} + |\xi|^{p-1}), \tag{2.1}$$

for a.e. $x \in \Omega$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $|\xi|$ denotes the Euclidian norm of the vector ξ.

(A2) The coefficients a_i satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^N (a_i(x,s,\xi) - a_i(x,s,\xi'))(\xi_i - \xi_i') > 0, \tag{2.2}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi, \xi' \in \mathbb{R}^N$ with $\xi \neq \xi'$.

(A3) A constant $c_1 > 0$ and a function $k_1 \in L^\infty(\Omega)$ exists such that

$$\sum_{i=1}^N a_i(x,s,\xi)\xi_i \geq c_1|\xi|^p - k_1(x), \tag{2.3}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$, and for all $\xi \in \mathbb{R}^N$.

(F1) $x \mapsto f(x,s,\xi)$ is measurable in Ω for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$.
(F2) $(s,\xi) \mapsto f(x,s,\xi)$ is continuous in $\mathbb{R} \times \mathbb{R}^N$ for almost all $x \in \Omega$.
(F3) There exists a constant $c_2 > 0$ such that

$$|f(x,s,\xi)| \le c_2(1 + |s|^{p-1} + |\xi|^{p-1}), \tag{2.4}$$

for a.e. $x \in \Omega$, for all $s \in \mathbb{R}$ and for all $\xi \in \mathbb{R}^N$.
(G1) $x \mapsto g(x,s)$ is measurable in $\partial\Omega$ for all $s \in \mathbb{R}$.
(G2) $s \mapsto g(x,s)$ is continuous in $\mathbb{R}$ for almost all $x \in \partial\Omega$.
(G3) There exists a constant $c_3 > 0$ such that

$$|g(x,s)| \le c_3(1 + |s|^{p-1}), \tag{2.5}$$

for a.e. $x \in \partial\Omega$ and for all $s \in \mathbb{R}$.

Condition (A1) implies that $A : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded continuous and along with (A2) it holds that A is pseudomonotone. Due to (A1) the operator A generates a mapping from $W^{1,p}(\Omega)$ into its dual space defined by

$$\langle Au, \varphi \rangle = \int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u) \frac{\partial \varphi}{\partial x_i} dx, \tag{2.6}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and $(W^{1,p}(\Omega))^*$. Assumption (A3) is a coercivity type condition and the assumptions (F3) and (G3) ensure that the corresponding Nemytskij operators $F : L^p(\Omega) \to L^q(\Omega)$ and $G : L^p(\partial\Omega) \to L^q(\partial\Omega)$ defined by

$$Fu(x) = f(x,u(x),\nabla u(x)), \qquad Gu(x) = g(x,u(x)), \tag{2.7}$$

are bounded and continuous (see e.g. [37]). The definition of a solution of problem (1.1) in the weak sense is defined as follows.

Definition 2.1. A function $u \in W^{1,p}(\Omega)$ is said to be a weak solution of (1.1) if the following holds

$$\int_\Omega \sum_{i=1}^N a_i(x,u,\nabla u) \frac{\partial \varphi}{\partial x_i} dx = \int_\Omega f(x,u,\nabla u)\varphi dx + \int_{\partial\Omega} g(x,u)\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega).$$

3. Main Results

Theorem 3.1. *Let the conditions (A1)-(A3),(F1)-(F3) and (G1)-(G3) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in L^\infty(\Omega)$.*

Proof. To prove the $L^\infty-$regularity of u, we will use the Moser iteration technique (see e.g. [15],[16],[17], [18], [23]). It suffices to consider the proof in case $1 \le p \le N$, otherwise we would be done. First we are going to show that $u^+ = \max\{u, 0\}$ belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{u^+(x), M\}$. Letting $K(t) = t$ if $t \le M$ and $K(t) = M$ if $t > M$, it follows by [23, Theorem B.3] that $K \circ u^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. For real $k \ge 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$ and $\varphi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$. Notice that $u(x) \le 0$ implies directly $v_M(x) = 0$. Testing the weak formulation in Definition

2.1 with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned}(kp+1)\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+) v_M^{kp}\frac{\partial v_M}{\partial x_i}dx \\ = \int_\Omega f(x,u^+,\nabla u^+) v_M^{kp+1}dx + \int_{\partial\Omega} g(x,u^+) v_M^{kp+1} d\sigma.\end{aligned} \tag{3.1}$$

By applying condition (A3) and the Hölder inequality, the left-hand-side of (3.1) can be estimated to obtain

$$\begin{aligned}&(kp+1)\int_\Omega \sum_{i=1}^N a_i(x,u^+,\nabla u^+) v_M^{kp}\frac{\partial v_M}{\partial x_i}dx \\ &= (kp+1)\int_\Omega \sum_{i=1}^N a_i(x,v_M,\nabla v_M)\frac{\partial v_M}{\partial x_i} v_M^{kp}dx \\ &\geq (kp+1)\int_\Omega (c_1|\nabla v_M|^p - k_1) v_M^{kp}dx \\ &\geq c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx - e_1(kp+1)\int_\Omega (u^+)^{kp}dx \\ &\geq c_1\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx - e_1(kp+1)|\Omega|^{\frac{1}{k+1}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}.\end{aligned} \tag{3.2}$$

The assumption (F3) along with the Hölder inequality and Young's inequality implies

$$\begin{aligned}&\int_\Omega f(x,u^+,\nabla u^+) v_M^{kp+1}dx \\ &\leq c_2\int_\Omega (1+|u^+|^{p-1}+|\nabla u^+|^{p-1}) v_M^{kp+1}dx \\ &\leq c_2|\Omega|^{\frac{p-1}{(k+1)p}}\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + c_2\int_\Omega (u^+)^{(k+1)p}dx \\ &\quad + c_2\int_\Omega \delta|\nabla u^+|^{(p-1)q} v_M^{k(p-1)q}dx + c_2\int_\Omega C(\delta) v_M^{(k+1)p}dx \\ &\leq e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx \\ &\quad + \frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx.\end{aligned} \tag{3.3}$$

The same arguments for the boundary integral provide

$$\begin{aligned}\int_{\partial\Omega} g(x,u^+) v_M^{kp+1}d\sigma &\leq c_3\int_{\partial\Omega}(1+|u^+|^{p-1}) v_M^{kp+1}d\sigma \\ &\leq e_3\left(\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}} + e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma.\end{aligned} \tag{3.4}$$

Applying the estimates (3.2)–(3.4) to (3.1) one gets

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p dx\\
&\leq e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\
&\quad+\frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\
&\quad+e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}.
\end{aligned}\tag{3.5}$$

We have $\lim_{M\to\infty} v_M(x) = u^+(x)$ for a.e. $x\in\Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx\\
&\leq e_2\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + (1+C(\delta))c_2\int_\Omega (u^+)^{(k+1)p}dx\\
&\quad+\frac{c_2\delta}{(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx + e_3\left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\\
&\quad+e_4\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_5(kp+1)\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}.
\end{aligned}\tag{3.6}$$

We have either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}}\leq 1 \quad\text{or}\quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}}\leq \int_\Omega (u^+)^{(k+1)p}dx,$$

respectively, either

$$\left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\leq 1 \quad\text{or}\quad \left(\int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}\leq \int_{\partial\Omega} (u^+)^{(k+1)p}d\sigma,$$

and finally, either

$$\left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}\leq 1 \quad\text{or}\quad \left(\int_\Omega (u^+)^{(k+1)p}dx\right)^{\frac{kp}{(k+1)p}}\leq \int_\Omega (u^+)^{(k+1)p}dx.$$

Using the calculation above to (3.6), we obtain

$$\begin{aligned}
&\left[\frac{kp+1}{(k+1)^p}-\frac{c_2\delta}{(k+1)^p}\right]\int_\Omega |\nabla (u^+)^{k+1}|^p dx\\
&\leq (C(\delta)c_2+e_6(kp+1))\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_\Omega (u^+)^{(k+1)p}d\sigma + e_8,
\end{aligned}\tag{3.7}$$

where the choice $\delta=\frac{kp+1}{2c_2}$ results in

$$\begin{aligned}
&\frac{kp+1}{2(k+1)^p}\int_\Omega |\nabla (u^+)^{k+1}|^p dx\\
&\leq e_9(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8.
\end{aligned}\tag{3.8}$$

It should be pointed out that

$$C(\delta) = (\delta p)^{-\frac{q}{p}} \cdot \frac{1}{q} = \left(\frac{2c_2}{p}\right)^{\frac{q}{p}} \cdot \left(\frac{1}{kp+1}\right)^{\frac{q}{p}} \cdot \frac{1}{q} \leq e_{10}$$

with a positive constant e_{10} where we have set $e_9 = e_{10}c_2 + e_6$. Adding on both sides of (3.8) the positive integral $\frac{kp+1}{2(k+1)^p}\int_\Omega (u^+)^{(k+1)p}dx$ yields

$$\begin{aligned} &\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla(u^+)^{k+1}|^p dx + \int_\Omega (u^+)^{(k+1)p}dx\right] \\ &\leq e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8, \end{aligned} \tag{3.9}$$

due to the fact that $\frac{kp+1}{2(k+1)^p} < kp+1$ for all $k \geq 0$. Next we want to estimate the boundary integral by an integral in the domain Ω. To this end, we need the following continuous embeddings

$$\begin{aligned} &T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \\ &T_2 : B^{s-\frac{1}{p}}_{pp}(\partial\Omega) = F^{s-\frac{1}{p}}_{pp}(\partial\Omega) \to F^0_{p2}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p}, \end{aligned}$$

where Ω is a bounded C^∞-domain (see [29, Page 75 and Page 82], [33, 2.3.1 and 2.3.2] and [34, 3.3.1]). Let $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \leq \iota < 1$. Then the embeddings are also valid if $\partial\Omega \in C^{m,1}$ ([31]). In [12, Satz 9.40] a similar proof is given for $p = 2$, however, it can be extend to $p \in (1,\infty)$ by using the Fourier transformation in $L^p(\Omega)$ ([13]).
Here B^s_{pq} and F^s_{pq} denote the Besov and Lizorkin-Triebel spaces, respectively, which are equal in case $p = q$ with $1 < p < \infty$ and $-\infty < s < \infty$. We set $s = \frac{1}{p} + \widetilde{\varepsilon}$, where $\widetilde{\varepsilon} > 0$ is arbitrarily fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$. Thus the embeddings are valid for a Lipschitz boundary $\partial\Omega$. This yields

$$T_3 : B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega) \to L^p(\partial\Omega). \tag{3.10}$$

The real interpolation theory implies

$$\left(F^0_{p2}(\Omega), F^1_{p2}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \left(L^p(\Omega), W^{1,p}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega),$$

(for more details see [3],[33],[34]) which ensures the norm estimate

$$\|v\|_{B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)} \leq e_{12}\|v\|^{\frac{1}{p}+\widetilde{\varepsilon}}_{W^{1,p}(\Omega)}\|v\|^{1-\frac{1}{p}-\widetilde{\varepsilon}}_{L^p(\Omega)}, \ \forall v \in W^{1,p}(\Omega), \tag{3.11}$$

with a positive constant e_{12} only depending on the boundary $\partial\Omega$. Using (3.10), (3.11) and Young's inequality yields

$$\begin{aligned}
&\int_{\partial\Omega}((u^+)^{k+1})^p d\sigma\\
&= \|(u^+)^{k+1}\|^p_{L^p(\partial\Omega)}\\
&\le e_{13}^p \|(u^+)^{k+1}\|^p_{B_{pp}^{\frac{1}{p}+\tilde{\varepsilon}}(\Omega)}\\
&\le e_{13}^p e_{12}^p \|(u^+)^{k+1}\|^{(\frac{1}{p}+\tilde{\varepsilon})p}_{W^{1,p}(\Omega)} \|(u^+)^{k+1}\|^{(1-\frac{1}{p}-\tilde{\varepsilon})p}_{L^p(\Omega)}\\
&\le e_{13}^p e_{12}^p \left(\delta' \|(u^+)^{k+1}\|^{(1+\tilde{\varepsilon}p)\tilde{q}}_{W^{1,p}(\Omega)} + C(\delta')\|(u^+)^{k+1}\|^{(p-1-\tilde{\varepsilon}p)\tilde{q}'}_{L^p(\Omega)}\right)\\
&= e_{13}^p e_{12}^p \left(\delta' \|(u^+)^{k+1}\|^{p}_{W^{1,p}(\Omega)} + C(\delta')\|(u^+)^{k+1}\|^{p}_{L^p(\Omega)}\right),
\end{aligned} \tag{3.12}$$

where $\tilde{q} = \frac{p}{1+\tilde{\varepsilon}p}$ and $\tilde{q}' = \frac{p}{p-1-\tilde{\varepsilon}p}$ satisfy $\frac{1}{\tilde{q}} + \frac{1}{\tilde{q}'} = 1$ and δ' is a free parameter specified later. Note that the positive constant $C(\delta')$ depends on δ'. Applying (3.12) to (3.9) shows

$$\begin{aligned}
&\frac{kp+1}{2(k+1)^p}\left[\int_\Omega |\nabla(u^+)^{k+1}|^p dx + \int_\Omega (u^+)^{(k+1)p}dx\right]\\
&\le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_7\int_{\partial\Omega}(u^+)^{(k+1)p}d\sigma + e_8\\
&\le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_{14}\delta'\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)}\\
&\quad + e_{14}C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8,
\end{aligned}$$

where $e_{14} = e_7 e_{13}^p e_{12}^p$ is a positive constant. We take $\delta' = \frac{kp+1}{e_{14}4(k+1)^p}$ to get

$$\begin{aligned}
&\left(\frac{kp+1}{2(k+1)^p} - e_{14}\frac{kp+1}{e_{14}4(k+1)^p}\right)\left[\int_\Omega |\nabla(u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right]\\
&\le e_{11}(kp+1)\int_\Omega (u^+)^{(k+1)p}dx + e_{14}C(\delta')\|(u^+)^{k+1}\|^p_{L^p(\Omega)} + e_8\\
&\le e_{15}(kp+1+C(\delta'))\int_\Omega (u^+)^{(k+1)p}dx + e_8,
\end{aligned} \tag{3.13}$$

where it holds

$$\begin{aligned}
kp+1+C(\delta') &= kp+1+\left(\frac{4e_{14}}{p}\right)^{\frac{q}{p}}\cdot\left(\frac{(k+1)^p}{kp+1}\right)^{\frac{q}{p}}\cdot\frac{1}{q}\\
&\le e_{16}\left(kp+1+\left(\frac{(k+1)^{\frac{p}{p-1}}}{(kp+1)^{\frac{1}{p-1}}}\right)\right)\\
&\le e_{17}(kp+1)^{\frac{p}{p-1}}.
\end{aligned}$$

Using the calculations above to (3.13) provides

$$\begin{aligned}
&\frac{kp+1}{4(k+1)^p}\left[\int_\Omega |\nabla(u^+)^{k+1}|^p dx + \int_\Omega |(u^+)^{k+1}|^p dx\right]\\
&\le e_{18}(kp+1)^{\frac{p}{p-1}}\left[\int_\Omega (u^+)^{(k+1)p}dx + 1\right],
\end{aligned}$$

equivalently

$$\|(u^+)^{k+1}\|^p_{W^{1,p}(\Omega)} \le (kp+1)^{\frac{1}{p-1}}(k+1)^p e_{19}\left[\int_\Omega (u^+)^{(k+1)p}dx + 1\right].$$

By Sobolev's embedding theorem a positive constant e_{20} exists such that

$$\|(u^+)^{k+1}\|_{L^{p^*}(\Omega)} \le e_{20}\|(u^+)^{k+1}\|_{W^{1,p}(\Omega)}, \tag{3.14}$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We get

$$\begin{aligned}
&\|u^+\|_{L^{(k+1)p^*}(\Omega)}\\
&= \|(u^+)^{k+1}\|^{\frac{1}{k+1}}_{L^{p^*}(\Omega)}\\
&\le e_{20}^{\frac{1}{k+1}}\|(u^+)^{k+1}\|^{\frac{1}{k+1}}_{W^{1,p}(\Omega)}\\
&\le e_{20}^{\frac{1}{k+1}}\Big((kp+1)^{\frac{1}{(p-1)p}}(k+1)\Big)^{\frac{1}{k+1}} e_{19}^{\frac{1}{(k+1)p}}\left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}.
\end{aligned}$$

Since $\Big((kp+1)^{\frac{1}{(p-1)p}}(k+1)\Big)^{\frac{1}{\sqrt{k+1}}} \ge 1$ and $\lim_{k\to\infty}\Big((kp+1)^{\frac{1}{(p-1)p}}(k+1)\Big)^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{21} > 1$ such that $\Big((kp+1)^{\frac{1}{(p-1)p}}(k+1)\Big)^{\frac{1}{k+1}} \le e_{21}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{20}^{\frac{1}{k+1}} e_{21}^{\frac{1}{\sqrt{k+1}}} e_{19}^{\frac{1}{(k+1)p}}\left[\int_\Omega (u^+)^{(k+1)p})dx + 1\right]^{\frac{1}{(k+1)p}}. \tag{3.15}$$

Now, we will use the bootstrap arguments similarly as in the proof of [18, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le c(k)$$

for any finite number $k > 0$ which shows that $u^+ \in L^r(\Omega)$ for any $r \in (1,\infty)$. To prove the uniform estimate with respect to k we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$\int_\Omega (u^+)^{(k_n+1)p}dx \le 1,$$

we have immediately

$$\|u^+\|_{L^\infty(\Omega)} \le 1,$$

(cf. the proof of [18, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$\int_\Omega (u^+)^{(k+1)p}dx > 1$$

for any $k \ge k_0$. Then we conclude from (3.15)

$$\|u^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{20}^{\frac{1}{k+1}} e_{21}^{\frac{1}{\sqrt{k+1}}} e_{22}^{\frac{1}{(k+1)p}}\|u^+\|_{L^{(k+1)p}}, \text{ for any } k \ge k_0, \tag{3.16}$$

where $e_{22} = 2e_{19}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$\|u^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{20}^{\frac{1}{k_1+1}} e_{21}^{\frac{1}{\sqrt{k_1+1}}} e_{22}^{\frac{1}{(k_1+1)p}}\|u^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{3.17}$$

Next, we can choose k_2 in (3.16) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$\begin{aligned} \|u^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{20}^{\frac{1}{k_2+1}} e_{21}^{\frac{1}{\sqrt{k_2+1}}} e_{22}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_2+1)p}(\Omega)} \\ &= e_{20}^{\frac{1}{k_2+1}} e_{21}^{\frac{1}{\sqrt{k_2+1}}} e_{22}^{\frac{1}{(k_2+1)p}} \|u^+\|_{L^{(k_1+1)p^*}(\Omega)}. \end{aligned} \tag{3.18}$$

By induction we obtain

$$\begin{aligned} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\le e_{20}^{\frac{1}{k_n+1}} e_{21}^{\frac{1}{\sqrt{k_n+1}}} e_{22}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{20}^{\frac{1}{k_n+1}} e_{21}^{\frac{1}{\sqrt{k_n+1}}} e_{22}^{\frac{1}{(k_n+1)p}} \|u^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \end{aligned} \tag{3.19}$$

where the sequence (k_n) is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = \left(\frac{p^*}{p}\right)^n$. Thus

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} = e_{20}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{21}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{22}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)}, \tag{3.20}$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{23} > 0$ such that

$$\|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \le e_{23} \|u^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{3.21}$$

Let us assume that $u^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $u^+(x) \ge e_{23}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} &\ge \left(\int_A |u^+(x)|^{(k_n+1)p^*}\right)^{\frac{1}{(k_n+1)p^*}} \\ &\ge (e_{23}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta)|A|^{\frac{1}{(k_n+1)p^*}}. \end{aligned}$$

Passing to the limes inferior in the inequality above yields

$$\liminf_{n\to\infty} \|u^+\|_{L^{(k_n+1)p^*}(\Omega)} \ge e_{23}\|u^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (3.21) and hence, $u^+ \in L^\infty(\Omega)$. In a similar way one shows that $u^- = \max\{-u, 0\} \in L^\infty(\Omega)$. This proves $u = u^+ - u^- \in L^\infty(\Omega)$. □

Let us now suppose an additional condition to the function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

(G4) g is locally Hölder continuous in $\partial\Omega \times \mathbb{R}$, that is,

$$|g(x_1, s_1) - g(x_2, s_2)| \le L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0, 1]$.

Theorem 3.2. *Let the conditions (A1)-(A3),(F1)-(F3) and (G1)-(G4) be satisfied. Let $u \in W^{1,p}(\Omega)$ be a solution of (1.1). Then $u \in C^{1,\alpha}(\overline{\Omega})$.*

Proof. Theorem 3.1 implies $u \in L^\infty(\Omega)$. Moreover, we see at once that the assumptions (0.3a)–(0.3d) and (0.6) in [24] are satisfied which yields in view of [24, Theorem 2] the assertion. □

Example 3.3. *Let $A = -\Delta_p, 1 < p < \infty$, be the p–Laplacian which is defined by*

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{where} \quad \nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_N). \tag{3.22}$$

The coefficients $a_i, i = 1, \dots, N$ are given by

$$a_i(x, s, \xi) = |\xi|^{p-2}\xi_i.$$

Thus, hypothesis (A1) is satisfied with $k_0 = 0$ and $c_0 = 1$. Hypothesis (A2) is a consequence of the inequalities from the vector-valued function $\xi \mapsto |\xi|^{p-2}\xi$ (see [8, Page 37]*) and (A3) is satisfied with $c_1 = 1$ and $k_1 = 0$. Our equations in (1.1) gets the form*

$$\begin{aligned} -\Delta_p u &= f(x, u, \nabla u) \qquad && \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= g(x, u) && \text{on } \partial\Omega, \end{aligned} \tag{3.23}$$

where $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of u with respect to $\partial\Omega$. Theorem 3.1 and Theorem 3.2 ensure under the assumptions (F1)–(F3) and (G1)–(G4) that every solution u of (3.23) satisfies $u \in L^\infty(\Omega)$ and $u \in C^{1,\alpha}(\overline{\Omega})$.

References

[1] A. Alberico and A. Cianchi. Borderline sharp estimates for solutions to Neumann problems. *Ann. Acad. Sci. Fenn. Math.*, 32(1):27–53, 2007.

[2] A. Alberico and A. Cianchi. Optimal summability of solutions to nonlinear elliptic problems. *Nonlinear Anal.*, 67(6):1775–1790, 2007.

[3] H. Amann. *Linear and quasilinear parabolic problems. Vol. I*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1995. Abstract linear theory.

[4] A. A. Arkhipova. New a priori estimates for q-nonlinear elliptic systems with strong nonlinearities in the gradient, $1 < q < 2$. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 310(Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 35 [34]):19–48, 226, 2004.

[5] D. Bartolucci. A priori estimates for an elliptic equation with exponential nonlinearity. *Asymptot. Anal.*, 35(3-4):325–347, 2003.

[6] D. Bartolucci and L. Orsina. Uniformly elliptic Liouville type equations: concentration compactness and a priori estimates. *Commun. Pure Appl. Anal.*, 4(3):499–522, 2005.

[7] M. F. Bidaut-Véron and C. Yarur. Semilinear elliptic equations and systems with measure data: existence and a priori estimates. *Adv. Differential Equations*, 7(3):257–296, 2002.

[8] S. Carl, V. K. Le, and D. Motreanu. *Nonsmooth variational problems and their inequalities.* Springer Monographs in Mathematics. Springer, New York, 2007. Comparison principles and applications.

[9] L. Caso, P. Cavaliere, and M. Transirico. A priori bounds for elliptic equations. *Ricerche Mat.*, 51(2):381–396 (2003), 2002.

[10] A. Cianchi and V. G. Maz'ya. Neumann problems and isocapacitary inequalities. *J. Math. Pures Appl. (9)*, 89(1):71–105, 2008.

[11] D. G. de Figueiredo and B. Sirakov. Liouville type theorems, monotonicity results and a priori bounds for positive solutions of elliptic systems. *Math. Ann.*, 333(2):231–260, 2005.

[12] M. Dobrowolski. *Applied functional analysis. Functional analysis, Sobolev spaces and elliptic differential equations. (Angewandte Funktionalanalysis. Funktionalanalysis, Sobolev-Räume und elliptische Differentialgleichungen.).* Berlin: Springer. xii, 266 p. EUR 29.95; sFr. 51.00, 2006.

[13] M. Dobrowolski. private communication, 2008.

[14] W. Dong. A priori estimates and existence of positive solutions for a quasilinear elliptic equation. *J. London Math. Soc. (2)*, 72(3):645–662, 2005.

[15] P. Drábek. The least eigenvalues of nonhomogeneous degenerated quasilinear eigenvalue problems. *Math. Bohem.*, 120(2):169–195, 1995.

[16] P. Drábek. Nonlinear eigenvalue problem for p-Laplacian in $\mathbf{R}^N$. *Math. Nachr.*, 173:131–139, 1995.

[17] P. Drábek and J. Hernández. Existence and uniqueness of positive solutions for some quasilinear elliptic problems. *Nonlinear Anal.*, 44(2, Ser. A: Theory Methods):189–204, 2001.

[18] P. Drábek, A. Kufner, and F. Nicolosi. *Quasilinear elliptic equations with degenerations and singularities*, volume 5 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1997.

[19] Y. Du and S. Li. Nonlinear Liouville theorems and a priori estimates for indefinite superlinear elliptic equations. *Adv. Differential Equations*, 10(8):841–860, 2005.

[20] N. Grenon, F. Murat, and A. Porretta. Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms. *C. R. Math. Acad. Sci. Paris*, 342(1):23–28, 2006.

[21] C. P. Gupta. A priori estimates and solvability of a non-resonant generalized multi-point boundary value problem of mixed Dirichlet-Neumann-Dirichlet type involving a p-Laplacian type operator. *Appl. Math.*, 52(5):417–430, 2007.

[22] J. Huang and J. Zou. Some new a priori estimates for second-order elliptic and parabolic interface problems. *J. Differential Equations*, 184(2):570–586, 2002.

[23] A. Lê. Eigenvalue problems for the p-Laplacian. *Nonlinear Anal.*, 64(5):1057–1099, 2006.

[24] G. M. Lieberman. Boundary regularity for solutions of degenerate elliptic equations. *Nonlinear Anal.*, 12(11):1203–1219, 1988.

[25] Linfeng Mei. A priori bounds for an indefinite superlinear elliptic equation with exponential growth. *J. Math. Anal. Appl.*, 339(2):1294–1304, 2008.

[26] M. Montenegro and F. O. V. de Paiva. A-priori bounds and positive solutions to a class of quasilinear elliptic equations. In *Contributions to nonlinear analysis*, volume 66 of *Progr. Nonlinear Differential Equations Appl.*, pages 407–413. Birkhäuser, Basel, 2006.

[27] D. Motreanu, V. V. Motreanu, and N. S. Papageorgiou. Nonlinear neumann problems near resonance. *Indiana Univ. Math. J.*, 2008 (in press).

[28] P. Quittner and Ph. Souplet. A priori estimates and existence for elliptic systems via bootstrap in weighted Lebesgue spaces. *Arch. Ration. Mech. Anal.*, 174(1):49–81, 2004.

[29] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of *de Gruyter Series in Nonlinear Analysis and Applications*. Walter de Gruyter & Co., Berlin, 1996.

[30] V. Shevchik. An a priori estimate for a model semi-elliptic differential operator. *J. Math. Anal. Appl.*, 268(2):385–399, 2002.

[31] W. Sickel. private communication, 2008.

[32] B. Sirakov. Existence results and a priori bounds for higher order elliptic equations and systems. *J. Math. Pures Appl. (9)*, 89(2):114–133, 2008.

[33] H. Triebel. *Interpolation theory, function spaces, differential operators*. VEB Deutscher Verlag der Wissenschaften, Berlin, 1978.

[34] H. Triebel. *Theory of function spaces*, volume 38 of *Mathematik und ihre Anwendungen in Physik und Technik [Mathematics and its Applications in Physics and Technology]*. Akademische Verlagsgesellschaft Geest & Portig K.-G., Leipzig, 1983.

[35] J. Wei and X. Xu. Uniqueness and a priori estimates for some nonlinear elliptic Neumann equations in $\mathbb{R}^3$. *Pacific J. Math.*, 221(1):159–165, 2005.

[36] Z. Yang and H. Yang. A priori estimates for a quasilinear elliptic partial differential equations non-positone problems. *Nonlinear Anal.*, 43(2, Ser. A: Theory Methods):173–181, 2001.

[37] E. Zeidler. *Nonlinear functional analysis and its applications. II/B*. Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.

[38] M. Zhu. Uniqueness results through a priori estimates. I. A three-dimensional Neumann problem. *J. Differential Equations*, 154(2):284–317, 1999.

[39] H. H. Zou. A priori estimates and existence for quasi-linear elliptic equations. *Calc. Var. Partial Differential Equations*, 33(4):417–437, 2008.

Reports of the Institutes 2009

01-09. Jan Prüss, Vicente Vergara, Rico Zacher, *Well-Posedness and Long-Time Behaviour for the Non-Isothermal Cahn-Hilliard Equation with Memory*

02-09. Siegfried Carl, Patrick Winkert, *General comparison principle for variational-hemivariational inequalities*

Reports are available via WWW: http://www2.mathematik.uni-halle.de/institut/reports/