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WELL-POSEDNESS AND LONG-TIME BEHAVIOUR FOR THE NON-ISOTHERMAL CAHN-HILLIARD EQUATION WITH MEMORY

JAN PRÜSS, VICENTE VERGARA, AND RICO ZACHER

Abstract. In this paper we study a temperature dependent phase field model with memory. The case where both, the equation for the temperature and that for the order parameter is of fractional time order is covered. Under physically reasonable conditions on the nonlinearities we prove global well-posedness in the L_p setting and show that each solution converges to a steady state as time goes to infinity.

1. Introduction

This paper is concerned with a temperature dependent phase field model with memory. To describe the model, let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\Gamma = \partial\Omega$ of class C^4, and $J = [0,T]$, $T > 0$. Let ψ denote the phase field (the order parameter), ϑ the temperature field, j the phase flux, q the heat flux, e the internal energy, and μ the chemical potential. Phase function and internal energy are assumed to be conserved quantities, the according conservation laws read as

$$\partial_t e + \operatorname{div} q = 0, \quad \partial_t \psi + \operatorname{div} j = 0, \quad \text{in } J \times \Omega.$$

In the classical theory the heat flux q resp. the phase flux j are simply proportional to the gradient of the local temperature field ϑ resp. to the gradient of the chemical potential μ. Here we consider a more general model that takes into account memory effects in both the process of heat conduction and phase changes. Following Coleman and Gurtin [14] we assume that the heat flux is given by

$$q(t,x) = -a_{01}\nabla\vartheta(t,x) - \int_{-\infty}^{t} a_1(t-s)\nabla\vartheta(s,x)\,ds, \quad t \in J,\, x \in \Omega,$$

where a_{01} is a non-negative constant and $a_1 \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is a positive and nondecreasing kernel. Analogously, see Binder, Frisch, and Jäckle [6], we assume that the phase flux is given by

$$j(t,x) = -a_{02}\nabla\mu(t,x) - \int_{-\infty}^{t} a_2(t-s)\nabla\mu(s,x)\,ds, \quad t \in J,\, x \in \Omega,$$

where a_{02} is a nonnegative constant and a_2 enjoys the same properties as the kernel a_1. The kinetic equations then become

$$\begin{aligned} &\partial_t e - a_{01}\Delta\vartheta - \int_{-\infty}^{t} a_1(t-s)\Delta\vartheta(s,\cdot)ds = 0, && \text{in } J \times \Omega; \\ &\partial_t \psi - a_{02}\Delta\mu - \int_{-\infty}^{t} a_2(t-s)\Delta\mu(s,\cdot)ds = 0, && \text{in } J \times \Omega. \end{aligned} \tag{1.1}$$

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To ensure that ψ and e are conserved quantities we assume Neumann boundary conditions for μ and ϑ, i.e.

$$\partial_\nu \vartheta = \partial_\nu \mu = 0, \text{ on } J \times \Gamma. \tag{1.2}$$

Consider the free energy functional

$$\mathcal{F}(\psi, \vartheta) := \int_\Omega \left(\frac{1}{2}|\nabla \psi|^2 + \phi(\psi) - \lambda(\psi)\vartheta - \frac{1}{2}\vartheta^2 \right) dx, \tag{1.3}$$

where ϕ, the physical potential, and λ are given functions. Typically, they are of the form $\lambda(\psi) = \lambda_0(\psi - \psi_c)$ and $\phi(\psi) = \frac{\phi_0}{4}(\psi^2 - 1)^2$, where $\lambda_0, \phi_0 > 0$ and ψ_c are constants; in this case λ_0 is called latent heat and ϕ a double-well potential.

The chemical potential μ and the internal energy e are defined by means of the functional derivatives of the free energy via the relations

$$\mu = \frac{\partial \mathcal{F}}{\partial \psi}(\psi, \vartheta) = -\Delta\psi + \phi'(\psi) - \lambda'(\psi)\vartheta$$

and

$$e = -\frac{\partial \mathcal{F}}{\partial \vartheta}(\psi, \vartheta) = \lambda(\psi) + \vartheta.$$

The second line in (1.1) yields an equation of fourth order, therefore in order to obtain well-posedness a second boundary condition for ψ is needed. Usually one imposes the Neumann boundary condition $\partial_\nu \psi = 0$, which we will also employ in this paper.

For simplicity we assume trivial history up to time $t = 0$ (i.e. $\vartheta(t, \cdot) \equiv 0$ and $\psi(t, \cdot) \equiv 0$ for $t < 0$) and that the system is then exposed to a sudden change of temperature and of the order parameter.

Denoting by $f * g$ the convolution on $\mathbb{R}_+$, that is $(f * g)(t) = \int_0^t f(t-s)g(s)ds$, $t \geq 0$, the system under consideration then reads as follows

$$\partial_t(\vartheta + \lambda(\psi)) = a_{01}\Delta\vartheta + a_1 * \Delta\vartheta \quad \text{in } J \times \Omega, \tag{1.4}$$

$$\mu = -\Delta\psi + \phi'(\psi) - \lambda'(\psi)\vartheta \quad \text{in } J \times \Omega, \tag{1.5}$$

$$\partial_t \psi = a_{02}\Delta\mu + a_2 * \Delta\mu \quad \text{in } J \times \Omega, \tag{1.6}$$

$$\partial_\nu \mu = \partial_\nu \psi = \partial_\nu \vartheta = 0 \quad \text{on } J \times \Gamma, \tag{1.7}$$

$$\psi|_{t=0} = \psi_0, \ \vartheta|_{t=0} = \vartheta_0 \qquad \text{in } \Omega. \tag{1.8}$$

A typical example for the kernels a_i we have in mind is given by

$$a_i(t) = \frac{t^{\alpha_i - 1}}{\Gamma(\alpha_i)} e^{-\gamma_i t}, \ t > 0, \ i = 1, 2, \tag{1.9}$$

with $\alpha_i \in (0, 1)$ and $\gamma_i > 0$, $i = 1, 2$, and where $\Gamma(\cdot)$ means the Gamma function. If in addition $a_{01} = a_{02} = 0$ then equations (1.4) and (1.6) are of *fractional time order*. For this case no theory seems to be available.

There is a vast literature on the system (1.4)–(1.8) and variants of it (e.g. with dynamic boundary conditions) in the isothermal case without memory. Then the problem reduces to the well-known Cahn-Hilliard equation, see e.g. [7, 16, 23, 21, 10], and the references given therein. For the nonisothermal Cahn-Hilliard equation with a dynamic boundary condition for the order parameter global well-posedness and convergence to steady state was obtained in [22]. As to phase-field systems with memory, in [3] the system (1.4)–(1.8) is studied with $a_{0i} > 0$, $i = 1, 2$, $a_2 = 0$, ϕ a double-well potential, and with $\lambda(\psi) = \lambda_0(\psi - \psi_c)$; in particular it is shown that global bounded solutions converge to an equilibrium. Global well-posedness in a weak setting for a nonconserved phase-field model with a memory term in both the equation for ϑ and that for ψ has been established in [24]. In [28] global well-posedness in the L_p setting was obtained for (1.4)–(1.8) in the case $a_{01} \geq 0$, $a_{02} = 0$, nontrivial kernels a_1, a_2, and $\lambda(\psi) = \lambda_0(\psi - \psi_c)$, see also the second author's thesis [27], which also contains a similar result in the nonconserved case (see [29]).

Concerning convergence to steady state of solutions of (1.4)–(1.8) nothing seems to be known in the situation where $a_{01} = a_{02} = 0$, in particular in the time fractional case. It is the purpose of the present paper to close this gap. Under physically reasonable conditions on the functions λ, ϕ and on the kernels a_i we will show that the problem

$$
\begin{aligned}
\partial_t(\vartheta + \lambda(\psi)) &= a_1 * \Delta\vartheta \text{ in } J \times \Omega,\\
\mu &= -\Delta\psi + \phi'(\psi) - \lambda'(\psi)\vartheta \text{ in } J \times \Omega,\\
\partial_t\psi &= a_2 * \Delta\mu \text{ in } J \times \Omega,\\
\partial_\nu\mu = \partial_\nu\psi &= \partial_\nu\vartheta = 0 \text{ on } J \times \Gamma,\\
\vartheta|_{t=0} = \vartheta_0,\ \psi|_{t=0} &= \psi_0, \text{ in } \Omega,
\end{aligned}
\tag{1.10}
$$

is globally well-posed in the L_p setting, and that the solutions converge to steady states of the problem in energy norm as $t \to \infty$.

Our proof of the local well-posedness uses the contraction mapping principle and optimal L_p regularity results for the linearized problem. The latter rely on techniques developed in [20] and [34]. Global well-posedness is then established by means of suitable a priori estimates, which are derived using energy estimates and the Gagliardo-Nirenberg inequality. The proof of the convergence result is based on appropriate new Lyapunov functionals, compactness properties, and on the Łojasiewicz-Simon inequality. We point out that, in general, construction of a Lyapunov functional for problems with memory is a highly nontrivial task, cf. the remarks in [18, Chapter 14]. Here we make use of ideas and results recently obtained by Vergara and Zacher [30], the key ingredient being Theorem 3.6 below. The relative compactness of the orbit in the natural energy space is shown by means of an iteration argument of Nash-Moser type. Finally, the Łojasiewicz-Simon inequality is the key tool to prove convergence to steady state. We remark that this method has been used by many authors to obtain similar convergence results for several types of evolution equations, see e.g. [2, 1, 19, 3, 8, 17, 32, 9, 22].

The paper is organized as follows. In Section 2 we describe all assumptions and formulate our main result, Theorem 2.2. Section 3 recalls some basic definitions and auxiliary results concerning sectorial operators and evolutionary integral equations, and collects important properties of the operator $Bu = \frac{d}{dt}(k * u)$ with nonnegative, nonincreasing kernel k, including the crucial result from [30]. Section 4 and Section 5 are devoted respectively to well-posedness and long-time behaviour.

2. Assumptions and Main Result

Concerning the nonlinearities we will suppose the subsequent growth conditions.

(H1) $\phi \in C^{4-}(\mathbb{R})$, and there are constants $C > 0$, $\beta < 3$ such that

$$|\phi'''(s)| \le C(1 + |s|)^\beta,\ s \in \mathbb{R}.$$

(H2) There are constants $c_0 \in \mathbb{R}$, $c_1 < \lambda_1$ such that

$$\phi(s) \ge -\frac{c_1}{2}s^2 - c_0,\ s \in \mathbb{R},$$

where $\lambda_1 > 0$ denotes the first nontrivial eigenvalue of the nonnegative Neumann-Laplacian on Ω.

(H3) $\lambda \in C^{4-}(\mathbb{R})$, and $\lambda', \lambda'', \lambda''' \in L_\infty(\mathbb{R})$.

Observe that the double-well potential $\phi(s) = \frac{\phi_0}{4}(s^2 - 1)^2$, and $\lambda(s) = \lambda_0(s - s_0)$, $s \in \mathbb{R}$, $\phi_0, \lambda_0 > 0$, satisfy (H1)–(H3), respectively.

To formulate the assumptions on the kernels, let $\hat{f}$ denote the Laplace transform of the function f. We recall that a kernel $a \in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth is *k-regular* ($k \in \mathbb{N}$), if there is a constant $M > 0$ such that $|z^j \hat{a}^{(j)}(z)| \le M|\hat{a}(z)|$ for all $\operatorname{Re} z > 0, 0 \le j \le k$, cf. [20, Definition 3.3]. Further, a kernel $a \in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth satisfying $\hat{a}(z) \ne 0$, $\operatorname{Re} z > 0$, is called *θ-sectorial* ($\theta > 0$) if $|\arg \hat{a}(z)| \le \theta$ for all $\operatorname{Re} z > 0$, cf. [20, Definition 3.2]. Following [34] we say that a kernel $a \in L_{1,loc}(\mathbb{R}_+)$ of subexponential growth belongs to the class $\mathcal{K}^1(\alpha,\theta)$ with $\alpha > 0, \theta \ge 0$, if a is 1-regular, θ-sectorial, and a satisfies

$$\limsup_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^{\alpha} < \infty,\ \liminf_{\mu\to\infty} |\,\hat{a}(\mu)|\,\mu^{\alpha} > 0,\ \liminf_{\mu\to 0} |\,\hat{a}(\mu)| > 0.$$

As to the kernels a_1, a_2 we assume that

(A1) $a_i \in \mathcal{K}^1(\alpha_i,\theta_i)$ for some $\alpha_i \in (0,1)$ and $\theta_i \in [0,\pi/2)$.

This is the basic assumption for local well-posedness. To establish global well-posedness and convergence to equilibrium as $t \to \infty$ we need the following additional assumptions.

(A2) There are nonnegative nonincreasing kernels $k_i \in L_{1,loc}(\mathbb{R}_+)$ such that

$$(k_i * a_i)(t) = 1,\ t > 0,\ i = 1, 2.$$

(A3) There are constants $\gamma_i > 0$ and strictly positive kernels $b_i \in L_1(\mathbb{R}_+)$ such that

$$b_i(t) + \gamma_i(1 * b_i)(t) = k_i(t),\ t > 0,\ i = 1, 2. \tag{2.1}$$

(A4) a_1, a_2 are 2-regular, $\lim_{z\to 0} \hat{a}_j(z)$ exists and is not zero, and $\left(\frac{1}{\hat{a}_j(i\cdot)}\right)' \in L_1(-1,1)$, $j = 1, 2$.

Remark 2.1. (i) (A2) implies that a_1, a_2 are completely positive, in particular a_1, a_2 are nonnegative, see e.g. [12, 13, 20].

(ii) The kernels b_i are nonincreasing. This follows from positivity of b_i and the assumption that the kernels k_i are nonincreasing (see (A2)). Note that

$$b_i(t) = k_i(t) - \gamma_i(e^{-\gamma_i\cdot} * k_i)(t), \quad t > 0,\ i = 1, 2.$$

(iii) If we weaken the assumption (A3) by replacing 'strictly positive' with 'nonnegative', then by decreasing γ_i, we obtain again a decomposition of the form (2.1) with strictly positive b_i, see [30, Remark 3.1].

(iv) Note that (A2) and (A3) entail that $k_i(t) \ge \lim_{t\to\infty} k_i(t) = \gamma_i |b_i|_{L_1(\mathbb{R}_+)} =: k_i^\infty > 0$.

(v) Condition (A2) is satisfied if and only if $[z\hat{a}_i(z)]^{-1}$ and $-\hat{a}_i'(z)/\hat{a}_i^2(z)$ are completely monotonic, while (A2) and (A3) hold if and only if $[(z+\gamma_i)\hat{a}_i(z)]^{-1}$ and $-\hat{a}_i'(z)/\hat{a}_i^2(z)$ are completely monotonic, see [20, Chapter 4].

(vi) Observe that the functions $a_i(t)$, $ta_i(t)$, and $tb_i(t)$ belong to $L_1(\mathbb{R}_+)$, $i = 1, 2$. This can be seen by looking at their Laplace transforms.

(vii) Observe that (A2) and (A3) imply the second and third condition in (A4).

(viii) Observe that the kernels in (1.9) satisfy conditions (A1)–(A4), here the kernels b_i in condition (A3) are given by $b_i(t) = \frac{t^{-\alpha_i}}{\Gamma(1-\alpha_i)} e^{-\gamma_i t}$, $t > 0$, $i = 1, 2$.

For an interval $J \subset \mathbb{R}$, a Banach space Y, $s > 0$ and $1 < p < \infty$, by $H_p^s(J;Y)$ we mean the vector-valued Bessel potential space, and by $B_{pp}^s(J;Y)$ the vector-valued Sobolev-Slobodeckij space of Y-valued functions on J, see e.g. [5, 25].

We are now in position to state our main result. Observe that the system (1.10) for ϑ and ψ is equivalent to the subsequent system (2.2) for the functions e and ψ.

$$\begin{aligned}
\partial_t e &= a_1 * \Delta e - a_1 * \Delta\big(\lambda(\psi)\big) \ \text{ in } J \times \Omega,\\
\mu &= -\Delta\psi + \phi'(\psi) - \lambda'(\psi)\big(e - \lambda(\psi)\big) \ \text{ in } J \times \Omega,\\
\partial_t \psi &= a_2 * \Delta\mu \ \text{ in } J \times \Omega,\\
\partial_\nu \mu &= \partial_\nu \psi = \partial_\nu e = 0 \ \text{ on } J \times \Gamma,\\
e|_{t=0} &= e_0,\ \psi|_{t=0} = \psi_0 \ \text{ in } \Omega.
\end{aligned} \tag{2.2}$$

Theorem 2.2. Part I (Global Well-Posedness): *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary Γ of class C^4. Suppose that the assumptions (H1)–(H3) and (A1) are satisfied. Let $p \in [2,\infty)$ be such that $p - 1 \notin \{\frac{4}{3(1+\alpha_2)}, \frac{2}{1+\alpha_1}, \frac{4}{1+\alpha_2}\}$ and $\alpha_i \neq 1/p$, $i = 1, 2$. Suppose further that the initial data are subject to the following conditions.*

(i) $e_0 \in B_{pp}^{2(1-\frac{1}{p(1+\alpha_1)})}(\Omega)$, $\psi_0 \in B_{pp}^{4(1-\frac{1}{p(1+\alpha_2)})}(\Omega)$; *(regularity)*
(ii) $\partial_\nu e_0 = 0$ *if* $p > 1 + \frac{2}{1+\alpha_1}$, $\partial_\nu \psi_0 = 0$ *if* $p > 1 + \frac{4}{3(1+\alpha_2)}$,
$\partial_\nu \Delta\psi_0 = 0$ *if* $p > 1 + \frac{4}{1+\alpha_2}$. *(compatibility conditions)*

Then for every $T > 0$ *there exists a unique solution* $(e, \psi) \in Z_1 \times Z_2$ *of the system (2.2) with*

$$Z_i = H_p^{1+\alpha_i}([0,T]; L_p(\Omega)) \cap L_p([0,T]; H_p^{2i}(\Omega)), \quad i = 1, 2.$$

The temperature $\vartheta = e - \lambda(\psi)$ *belongs to the space* Z_ϑ *given by*

$$Z_\vartheta = H_p^{1+\min\{\alpha_1,\alpha_2\}}([0,T]; L_p(\Omega)) \cap L_p([0,T]; H_p^2(\Omega)).$$

Part II (Long-Time Behaviour): *Suppose in addition that the functions* ϕ *and* λ *are real analytic, and that the assumptions (A2)–(A4) hold. Assume further that the initial data additionally satisfy* $\vartheta_0 \in H_2^2(\Omega)$ *and* $\psi_0 \in H_2^4(\Omega)$ *with* $\partial_\nu \vartheta_0 = \partial_\nu \psi_0 = \partial_\nu \Delta\psi_0 = 0$ *on* Γ.

Then the solution (ϑ, ψ) *of (1.10) is globally bounded, that is* $\vartheta, \psi \in L_\infty(\mathbb{R}_+ \times \Omega)$. *Moreover,* (ϑ, ψ) *converges in* $L_2(\Omega) \times H_2^1(\Omega)$ *as time goes to infinity to a stationary solution* $(\vartheta_\infty, \psi_\infty)$ *of (1.10), that is*

$$\begin{aligned} -\Delta\psi_\infty + \phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty &= \mu_\infty, \quad x \in \Omega, \\ \partial_\nu \psi_\infty &= 0, \quad x \in \Gamma. \end{aligned}$$

Here ϑ_∞ *and* μ_∞ *are constants,* $\psi_\infty \in H_2^2(\Omega)$, *and*

$$\mu_\infty = \frac{1}{|\Omega|} \int_\Omega \big(\phi'(\psi_\infty(x)) - \lambda'(\psi_\infty(x))\vartheta_\infty\big)\, dx.$$

Furthermore, the internal energy e *converges in* $L_2(\Omega)$ *as* $t \to \infty$ *to* $e_\infty = \vartheta_\infty + \lambda(\psi_\infty)$.

Remark 2.3. Theorem 2.2 is still valid when the assumption $\lambda' \in L_\infty(\mathbb{R})$ is dropped in (H3). Thus e.g. the function $\lambda(s) = s^2 + c$, $s \in \mathbb{R}$, which is sometimes used in the literature, would be admissible. For the L_∞ bounds and the relative compactness of the orbit (see Section 5.1) it suffices to have $\lambda'' \in L_\infty(\mathbb{R})$. Once these results are established, the remaining part of the proof of Theorem 2.2 is the same. This has been already observed in [31, Section 3.5] in the case without memory.

3. Preliminaries

3.1. **Sectorial Operators.** Let Y be a complex Banach space and A be a closed linear operator in Y. Then A is called *pseudo-sectorial* if $(-\infty, 0)$ is contained in the resolvent set of A and the estimate $|t(t+A)^{-1}|_{\mathcal{B}(Y)} \le M$, $t > 0$, holds for some constant $M > 0$. If in addition the null space $N(A) = \{0\}$, and the domain $D(A)$ as well as the range $R(A)$ of A are dense in Y then A is called *sectorial*. The class of sectorial operators in Y is denoted by $\mathcal{S}(Y)$. We recall that in case Y is reflexive and A pseudo-sectorial, the space Y decomposes as $Y = N(A) \oplus \overline{R(A)}$. Thus in such a situation A is sectorial on $\overline{R(A)}$. Putting $\Sigma_\theta = \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| < \theta\}$, the *spectral angle* $\phi_A \in [0, \pi)$ of $A \in \mathcal{S}(Y)$ is defined by $\phi_A = \inf\{\phi : \rho(-A) \supset \Sigma_{\pi-\phi}, \sup\{\lambda \in \Sigma_{\pi-\phi} : |\lambda(\lambda+A)^{-1}| < \infty\}\}$. We remark that for sectorial operators the Dunford functional calculus is available; in particular the complex powers A^z, $z \in \mathbb{C}$ are well-defined closed linear operators in X; see e.g. [15].

Let D_A denote the domain of A equipped with the graph norm, $1 \le p < \infty$, and let $\gamma \in (0,1)$. For $A \in \mathcal{S}(Y)$ the real interpolation space $(Y, D_A)_{\gamma,p}$ coincides with the space $D_A(\gamma, p)$ defined by $D_A(\gamma,p) := \{y \in Y : [y]_{D_A(\gamma,p)} < \infty\}$, with the seminorm $[y]_{D_A(\gamma,p)} = (\int_0^\infty (t^\gamma |A(t+A)^{-1}y|_Y)^p \frac{dt}{t})^{1/p}$, see e.g. [11, Proposition 3].

An operator $A \in \mathcal{S}(Y)$ is said to admit *bounded imaginary powers*, if the imaginary powers A^{is} form a bounded C_0-group on Y. The type θ_A of this group is called the *power angle* of A. We denote the class of operators with bounded imaginary powers by $\mathcal{BIP}(Y)$. For more details on sectorial operators and the class $\mathcal{BIP}(Y)$ we refer to [4] and [15].

We recall further that a Banach space Y belongs to the class $\mathcal{HT}$, if the Hilbert transform is bounded on $L_2(\mathbb{R}; Y)$.

3.2. **Abstract Volterra Equations.** Let Y be a Banach space, A a closed linear, in general unbounded operator in Y with dense domain $D(A)$, and let $a \in L^1_{loc}(\mathbb{R}_+)$ be a scalar kernel. We consider the Volterra equation

$$u(t) + (a * Au)(t) = f(t),\ t \in J. \tag{3.1}$$

Definition 3.1. *A family $\{S(t)\}_{t\geq 0} \subset \mathcal{B}(Y)$ of bounded linear operators in Y is called a resolvent for* (3.1) *if the following conditions are satisfied.*

- **(S1)** *$S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;*
- **(S2)** *$S(t)$ commutes with A, which means that $S(t)D(A) \subset D(A)$ and $AS(t)y = S(t)Ay$ for all $y \in D(A)$ and $t \geq 0$;*
- **(S3)** *the resolvent equation holds*

$$S(t)y + \int_0^t a(t-s)AS(s)ds = y, \ \text{ for all } y \in D(A),\ t \geq 0.$$

We remark that if $A \in \mathcal{S}(Y)$ with spectral angle $\phi_A < \pi$ and if the kernel a is 1-regular and θ-sectorial with $\theta < \pi$ such that the condition of parabolicity $\theta + \phi_A < \pi$ holds, then there is a resolvent operator $S \in C((0,+\infty);\mathcal{B}(Y))$ for (3.1), which is also uniformly bounded in $\mathbb{R}_+$, see [20, Proposition 3.1 and Theorem 3.1].

The following result concerns the property of maximal regularity of type L_p for equation (3.1).

Theorem 3.2. *Let $Y \in \mathcal{HT}$, $p \in (1,\infty)$, and $J = [0,T]$. Suppose that $A \in \mathcal{BIP}(Y)$, and $a \in \mathcal{K}^1(\alpha,\theta_a)$ with $\alpha \in (0,2)\setminus\{1/p, 1+1/p\}$. Assume the* parabolicity condition $\theta_a + \theta_A < \pi$. *Then (3.1) has a unique solution in $Z = H^\alpha_p(J;Y) \cap L_p(J;D_A)$ if and only if the function f satisfies the subsequent conditions:*

- (i) *$f \in H^\alpha_p(J;Y)$;*
- (ii) *$f(0) \in D_A(1-\frac{1}{p\alpha},p)$, if $\alpha > 1/p$;*
- (iii) *$\dot{f}(0) \in D_A(1-\frac{1}{\alpha}-\frac{1}{p\alpha},p)$, if $\alpha > 1+1/p$.*

Theorem 3.2 even holds true in the more general case where A is an $\mathcal{R}$-sectorial operator with $\mathcal{R}$-angle $\phi^R_A < \pi - \theta_a$, see [34, Theorem 3.4]. Note that $A \in \mathcal{BIP}(X)$ implies that A is an $\mathcal{R}$-sectorial operator with $\phi^R_A \leq \theta_A$ provided that the underlying Banach space X belongs to the class $\mathcal{HT}$, cf. [15].

Remark 3.3. (i) In the situation of Theorem 3.2 there exists a unique operator $\mathcal{B} \in \mathcal{BIP}(L_p(J;Y))$ with power angle $\theta_{\mathcal{B}} \leq \theta_a$ such that $\mathcal{B}$ is invertible satisfying $\mathcal{B}^{-1}v = a * v$ for all $v \in L_p(J;Y)$. Moreover $D(\mathcal{B}) = {}_0H^\alpha_p(J;Y)$ where the subscript 0 means that the function and its derivative vanish at $t = 0$ whenever these traces exist. See [34, Corollary 2.1].

(ii) In the special case $f = a * g$ with $g \in L_p(J;Y)$ the solution u of (3.1) belongs to the space ${}_0Z = {}_0H^\alpha_p(J;Y) \cap L_p(J;D_A)$. Denoting by $\mathcal{A}$ the natural extension of A to $L_p(J;Y)$, the solution is given by $u = (\mathcal{B}+\mathcal{A})^{-1}g$, where the sum $\mathcal{B}+\mathcal{A}$ is defined by $(\mathcal{B}+\mathcal{A})v = \mathcal{B}v + \mathcal{A}v$, $v \in D(\mathcal{B}+\mathcal{A}) = D(\mathcal{B}) \cap D(\mathcal{A}) = {}_0Z$. This follows by the Dore-Venni theorem, see [20, Section 8] or [34].

The next result concerns the integrability of the resolvent $S(\cdot)$ for the Volterra equation

$$u(t) + (1 * a * Au)(t) = f(t), \ \ t > 0. \tag{3.2}$$

Theorem 3.4. *Let Y be a Banach space, $A \in \mathcal{S}(Y)$ be an invertible operator with spectral angle ϕ_A. Assume that the kernel a in* (3.2) *belongs to $L_1(\mathbb{R}_+)$ and that it satisfies the following assumptions:*

- (i) *a is 2-regular and θ_a-sectorial such that $\phi_A + \theta_a < \pi/2$ holds;*
- (ii) *$\lim_{\lambda\to 0}\hat{a}(\lambda) \neq 0$ and $\left(\frac{1}{\hat{a}(i\cdot)}\right)' \in L_1(-1,1)$.*

Then there exists a uniform integrable resolvent family S for equation (3.2)*, i.e. $S \in L_1(\mathbb{R}_+;\mathcal{B}(Y))$. Moreover, for each $\kappa \in [0,1)$, $A^\kappa a * S \in L_1(\mathbb{R}_+;\mathcal{B}(Y))$.*

Theorem 3.4 follows by combining the proofs of Theorem 10.1 and 10.2 in [20] and by using Lemma 10.2 from the same monograph. For more details we also refer the reader to [27, Lemma 6.13].

3.3. **Useful Inequalities.** The following results are used in the construction of Lyapunov functionals. The first is a simple lemma, which can be found in [30, Lemma 2.1].

Lemma 3.5. *Let H be a real Hilbert space and $T>0$. Suppose that $l \in L_{1,loc}(\mathbb{R}_+)$ is nonnegative. Then for any $v \in L_2([0,T];H)$ there holds*

$$|(l*v)(t)|_H^2 \le (l*|v|_H^2)(t)\,(1*l)(t), \quad a.a.\ t\in(0,T).$$

The next result is the key tool in the construction of Lyapunov functionals for the system (1.10). It is due to Vergara and Zacher [30]. By $\langle\cdot,\cdot\rangle_H$ we denote the inner product in the Hilbert space H.

Theorem 3.6. *Let H be a real Hilbert space and $T>0$. Let $k\in L_{1,loc}(\mathbb{R}_+)$ be a nonnegative and nonincreasing kernel. Assume that there is a nonnegative kernel $a\in L_{1,loc}(\mathbb{R}_+)$ such that $k*a=1$ in $(0,\infty)$. Let $v\in L_2([0,T];H)$ and suppose that $k*v\in {}_0H_2^1([0,T];H)$ as well as $k*|v|_H^2\in {}_0H_1^1([0,T])$. Then*

$$2\left\langle \frac{d}{dt}(k*v)(t), v(t)\right\rangle_H \ge \frac{d}{dt}(k*|v|_H^2)(t)+k(t)|v(t)|_H^2, \quad \textit{for a.a. } t\in(0,T).$$

For a large class of kernels the second regularity assumption on v in Theorem 3.6 follows already from the first one, as the following result (cf. [30, Proposition 2.1]) and Remark 3.3(i) show.

Proposition 3.7. *Let Y be a Banach space of class $\mathcal{HT}$ and $T>0$. Let $a\in\mathcal{K}^1(\alpha,\theta)$ with $\alpha\in(0,1)$ and $\theta<\pi$. Assume there exists a kernel $k\in L_{1,loc}(\mathbb{R}_+)$ such that $a*k=1$ on $(0,\infty)$. Suppose $v\in {}_0H_2^\alpha([0,T];Y)$. Then*

$$k*v\in {}_0H_2^1([0,T];Y) \text{ and } k*|v|_Y^2\in {}_0H_1^1([0,T]).$$

4. Well-Posedness

Let $2\le p<\infty$, $J=[0,T]$, $T>0$, and let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary $\Gamma=\partial\Omega$ of class C^4. Set $X=L_p(\Omega)$, $Y=\{v\in X: \int_\Omega v(x)\,dx=0\}$ and define the operator A in Y by $Av:=-\Delta v$, $v\in D(A):=\{u\in H_p^2(\Omega)\cap Y: \partial_\nu u|_\Gamma=0\}$. It is well known that A and thus also A^2 belongs to the class $\mathcal{BIP}(Y)$ with power angle $\theta_A=0$ and $\theta_{A^2}=0$, respectively. Let $Pv=v-\bar v$ with $\bar v=\frac{1}{|\Omega|}\int_\Omega v(x)\,dx$, that is, P is the canonical projection in X onto Y. Let further $\mathcal{A}^i$ denote the natural extension of A^i to $L_p(J;Y)$, $i=1,2$. Then $\mathcal{A}^i$ belongs to $\mathcal{BIP}(L_p(J;Y))$ with power angle $\theta_{\mathcal{A}^i}=0$, $i=1,2$.

In what follows we will always assume that $\overline{\psi_0}=\overline{e_0}=\overline{\vartheta_0}+\overline{\lambda(\psi_0)}=0$. This is not a restriction of generality. In fact, $\bar e$ and $\overline{\psi}$ are conserved quantities, hence we may introduce the new variables $\tilde\vartheta:=\vartheta-\overline{\vartheta_0}$, $\tilde\psi:=\psi-\overline{\psi_0}$, and replace the functions λ and ϕ by $\tilde\lambda(s)=\lambda(s+\overline{\psi_0})-\overline{\lambda(\psi_0)}$ and $\tilde\phi(s)=\phi(s+\overline{\psi_0})-\tilde\lambda(s)\overline{\vartheta_0}$, respectively. Then $\tilde e:=\tilde\vartheta+\tilde\lambda(\tilde\psi)=e-\overline{e_0}$, and so the problem is reduced to the previous case.

System (2.2) can then be written in abstract form as follows.

$$\partial_t e(t)+a_1*Ae(t)=a_1*AP\lambda(\psi)(t), \qquad t\in J, \tag{4.1}$$

$$\partial_t\psi(t)+a_2*A^2\psi(t)=-a_2*AP\big(\phi'(\psi)-\lambda'(\psi)[e-\lambda(\psi)]\big)(t),\quad t\in J, \tag{4.2}$$

$$e(0)=e_0, \tag{4.3}$$

$$\psi(0)=\psi_0. \tag{4.4}$$

We are looking for a solution (e,ψ) of this system such that $(e,\psi)\in\mathcal{Z}_1\times\mathcal{Z}_2$ where

$$\mathcal{Z}_i=H_p^{1+\alpha_i}(J;Y)\cap L_p(J;D(A^i)),\quad i=1,2.$$

4.1. **Local well-posedness.** Let $p \in [2,\infty)$ be as in the first part of Theorem 2.2. Suppose $\phi, \lambda \in C^{4-}(\mathbb{R})$ and that the condition (A1) is satisfied. Observe first that assuming ψ to be known, the function e can be determined by solving the *linear* problem (4.1),(4.3). In fact, $e = u_1 + z_1$, where u_1 and z_1 solve

$$u_1 + 1 * a_1 * Au_1 = e_0, \quad t \in J, \tag{4.5}$$

and

$$z_1 + 1 * a_1 * Az_1 = 1 * a_1 * AP\lambda(\psi), \quad t \in J, \tag{4.6}$$

respectively. Since $a_1 \in \mathcal{K}^1(\alpha_1, \theta_1)$ with $\alpha_1 \in [0,1)$ and $\theta_1 \in [0, \pi/2)$, we have $1 * a_1 \in \mathcal{K}^1(1 + \alpha_1, \theta_1 + \pi/2)$, see e.g. [33, Lemma 2.6.2] and [20]. By the assumptions on e_0 in the first part of Theorem 2.2, we further have $e_0 \in D_A(1 - \frac{1}{p(1+\alpha_1)})$. Thus Theorem 3.2 yields a unique solution $u_1 \in \mathcal{Z}_1$ of (4.5).

As to z_1, suppose that

$$AP\lambda(\psi) \in L_p(J;Y). \tag{4.7}$$

Let $\mathcal{B}_1 \in \mathcal{S}(L_p(J;Y))$ be the operator $(1 * a_1 *)^{-1}$ from Remark 3.3(i). Then by Theorem 3.2 and Remark 3.3(ii), (4.6) admits a unique solution $z_1 \in {}_0\mathcal{Z}_1$ and we have the representation

$$e = u_1 + z_1 = u_1 + (\mathcal{B}_1 + \mathcal{A})^{-1} AP\lambda(\psi). \tag{4.8}$$

Using (4.8) we will now set up a fixed point problem for the function ψ in an appropriate space. To this purpose let $u_2 \in \mathcal{Z}_2$ denote the solution of

$$u_2 + 1 * a_2 * A^2 u_2 = \psi_0, \quad t \in J.$$

In fact, we have $\psi_0 \in D_{A^2}(1 - \frac{1}{p(1+\alpha_2)})$ and so once again Theorem 3.2 yields the unique solvability in the desired regularity class. We may then write

$$e = u_1 + (\mathcal{B}_1 + \mathcal{A})^{-1} AP\lambda(u_2) + F(\psi), \tag{4.9}$$

where we have set

$$F(\psi) = (\mathcal{B}_1 + \mathcal{A})^{-1} AP\big(\lambda(\psi) - \lambda(u_2)\big).$$

Inserting (4.9) into the equation for ψ and integrating gives

$$\psi + 1 * a_2 * A^2 \psi = \psi_0 - 1 * a_2 * G(\psi), \tag{4.10}$$

with

$$G(\psi) := AP\Big[\phi'(\psi) - \lambda'(\psi)\big(u_1 + (\mathcal{B}_1 + \mathcal{A})^{-1} AP\lambda(u_2) + F(\psi) - \lambda(\psi)\big)\Big].$$

By definition of u_2, equation (4.10) is equivalent to

$$\psi = u_2 - (\mathcal{B}_2 + \mathcal{A}^2)^{-1} G(\psi), \tag{4.11}$$

provided that $G(\psi) \in L_p(J;Y)$. Setting $z_2 = \psi - u_2$ we obtain the following fixed point problem for z_2:

$$z_2 = -(\mathcal{B}_2 + \mathcal{A}^2)^{-1} G(z_2 + u_2) =: \mathcal{T}(z_2). \tag{4.12}$$

The map $\mathcal{T} : {}_0\mathcal{Z}_2 \to {}_0\mathcal{Z}_2$ is well defined, whenever $z_2 \in {}_0\mathcal{Z}_2$ entails $G(z_2 + u_2) \in L_p(J;Y)$. To see the latter, note first that $p \geq 2$ implies the condition $p > 3/4 + 1/(1+\alpha_2)$, and thus the embedding

$$Z_2 \hookrightarrow C(J; B_{pp}^{4-4/p(1+\alpha_2)}(\Omega)) \hookrightarrow C(J \times \overline{\Omega}), \tag{4.13}$$

holds true. We will next prove the following property.

Lemma 4.1. *Under the above assumptions there holds the implication*

$$\psi \in \mathcal{Z}_2 \subset Z_2 \Rightarrow \lambda(\psi) \in Z_2. \tag{4.14}$$

Proof. Suppose $\psi \in \mathcal{Z}_2$. From (4.13) and (H3) it follows that $\lambda^{(i)}(\psi) \in L_\infty(J\times\Omega)$ for all $i = 0,\dots,4$. Hence, to verify that all components of $\nabla^4\big(\lambda(\psi)\big)$ belong to $L_p(J; L_p(\Omega))$ it suffices to show that all products of the form $\partial^{\varsigma_1}\psi\,\partial^{\varsigma_2}\psi\,\partial^{\varsigma_3}\psi\,\partial^{\varsigma_4}\psi$ with multiindices $\varsigma_j \in (\mathbb{N}\cup\{0\})^3$ satisfying $\sum_{j=1}^4 |\varsigma_j| = 4$ do so. If $|\varsigma_j| = 4$ for some $j \in \{1,\dots,4\}$, then the desired property is evidently satisfied, by boundedness of ψ and Hölder's inequality. By the mixed derivative theorem and Sobolev embedding, we have for each multiindex ς with $|\varsigma| \in \{1,2,3\}$

$$\begin{aligned}\partial^\varsigma\psi &\in H_p^{(1+\alpha_2)(1-|\varsigma|/4)}(J; L_p(\Omega)) \cap L_p(J; H_p^{4-|\varsigma|}(\Omega))\\ &\hookrightarrow H_p^{(1-|\varsigma|/4)/p}(J; H_p^{(4-|\varsigma|)(1-1/p)}(\Omega)) \hookrightarrow L_{4p/|\varsigma|}(J\times\Omega),\end{aligned}$$

where for the last embedding we employ the inequality

$$(4-|\varsigma|)(1-\frac{1}{p}) - \frac{3}{p} \geq -\frac{3|\varsigma|}{4p},$$

which is equivalent to

$$16p + 7|\varsigma| \geq 4p|\varsigma| + 28. \tag{4.15}$$

It is easy to check that (4.15) is satisfied for all $p \geq 2$ and each $|\varsigma| \in \{1,2,3\}$. Using Hölder's inequality, we conclude that $\partial^{\varsigma_1}\psi\,\partial^{\varsigma_2}\psi\,\partial^{\varsigma_3}\psi\,\partial^{\varsigma_4}\psi \in L_p(J; L_p(\Omega))$ whenever $\sum_{j=1}^4 |\varsigma_j| = 4$.

It remains to show that $\lambda'(\psi)\partial_t\psi \in H_p^{\alpha_2}(J; L_p(\Omega))$. From the characterization of the Bessel potential spaces via differences (see [26]) and Hölder's inequality, it follows that

$$|uv|_{H_p^{\alpha_2}(L_p)} \leq C\left(|u|_{H^{\alpha_2}_{\sigma_1' p}(L_{r_1' p})}|v|_{L_{\sigma_1 p}(L_{r_1 p})} + |u|_{L_{\sigma_2' p}(L_{r_2' p})}|v|_{H^{\alpha_2}_{\sigma_2 p}(L_{r_2 p})}\right), \tag{4.16}$$

where $1/\sigma_i + 1/\sigma_i' = 1/r_i + 1/r_i' = 1$, $i = 1,2$. We want to apply (4.16) to $u = \lambda'(\psi)$ and $v = \partial_t\psi$. Since $\lambda'(\psi) \in L_\infty(J\times\Omega)$ and $\partial_t\psi \in H_p^{\alpha_2}(J; L_p(\Omega))$, the second summand on the right-hand side of (4.16) is finite when taking $r_2 = \sigma_2 = 1$. The first summand is more involved. By the mixed derivative theorem,

$$\partial_t\psi \in H_p^{\alpha_2}(J; L_p(\Omega)) \cap L_p(J; H_p^{4(1-\frac{1}{1+\alpha_2})}(\Omega)) \hookrightarrow H_p^{\alpha_2\theta}(J; H_p^{\frac{4\alpha_2}{1+\alpha_2}(1-\theta)}(\Omega)), \quad \theta\in[0,1].$$

Therefore $\partial_t\psi \in L_{\sigma_1 p}(J; L_{r_1 p}(\Omega))$ provided that for some $\theta \in [0,1]$

$$\alpha_2\theta - \frac{1}{p} \geq -\frac{1}{\sigma_1 p}, \quad \text{and} \quad \frac{4\alpha_2}{1+\alpha_2}(1-\theta) - \frac{3}{p} \geq -\frac{3}{r_1 p}. \tag{4.17}$$

Next observe that $\lambda'(\psi) \in H_p^1(J; L_p(\Omega)) \cap L_p(J; H_p^3(\Omega))$. Hence

$$\lambda'(\psi) \in H_p^{\tilde\theta}(J; H_p^{3(1-\tilde\theta)}(\Omega)) \hookrightarrow H^{\alpha_2}_{\sigma_1' p}(J; L_{r_1' p}(\Omega)),$$

provided that for some $\tilde\theta \in [\alpha_2, 1]$ there holds

$$\tilde\theta - \frac{1}{p} \geq \alpha_2 - \frac{1}{\sigma_1' p}, \quad \text{and} \quad 1-\tilde\theta - \frac{1}{p} \geq -\frac{1}{r_1' p}. \tag{4.18}$$

Adding the first conditions of (4.17) and (4.18), and adding the second conditions in (4.17) and (4.18) leads to

$$\tilde\theta - \frac{1}{p} \geq \alpha_2(1-\theta) \geq \frac{3(1+\alpha_2)}{4}(\tilde\theta + \frac{1}{p} - 1), \tag{4.19}$$

that is we need $\tilde\theta \geq 1/p$ and

$$\tilde\theta(1-3\alpha_2) \geq \frac{7+3\alpha_2}{p} - 3(1+\alpha_2) =: \eta. \tag{4.20}$$

For $\alpha_2 = 1/3$ (4.20) is satisfied for any $\tilde\theta \in [0,1]$ and $p \geq 2$. In the case $\alpha_2 \in (0, 1/3)$ there exists $\tilde\theta \in [\eta/(1-3\alpha_2), 1]$, thanks to $p \geq 2$. In the case $\alpha_2 \geq 1/3$ there exists $\tilde\theta \in [\min\{\alpha_2, 1/p\}, \eta/(1-3\alpha_2)]$ at least if $\alpha_2 \leq 1/2$. Assuming the latter we thus find $\tilde\theta \in [\min\{\alpha_2, 1/p\}, 1]$ that satisfies (4.20). It is then not difficult to see that there are $\theta \in [0,1]$ and $r_1, \sigma_1 \in [1,\infty]$ such that all conditions in (4.17) and (4.18) are fulfilled. So we are done in the case $\alpha_2 \le 1/2$.

Suppose now that $\alpha_2 \ge 1/2$. Replacing the function λ by its derivative λ' it follows from what we have just proved that $\lambda'(\psi) \in H_p^{3/2}(J; L_p(\Omega)) \cap L_p(J; H_p^3(\Omega))$. Therefore

$$\lambda'(\psi) \in H_p^{\frac{3\tilde\theta}{2}}(J; H_p^{3(1-\tilde\theta)}(\Omega)) \hookrightarrow H^{\alpha_2}_{\sigma_1' p}(J; L_{r_1' p}(\Omega)),$$

provided that $\tilde\theta \ge 2\alpha_2/3$ and

$$\frac{3\tilde\theta}{2} - \frac{1}{p} \ge \alpha_2 - \frac{1}{\sigma_1' p}, \quad \text{and} \quad 1 - \tilde\theta - \frac{1}{p} \ge -\frac{1}{r_1' p}. \tag{4.21}$$

Proceeding as above with (4.21) instead of (4.18) leads to the condition

$$\tilde\theta \ge \frac{\eta}{3(1-\alpha_2)}. \tag{4.22}$$

The assumption $p \ge 2$ ensures that $1 \ge \eta/[3(1-\alpha_2)]$, hence there is $\tilde\theta \in [2\min\{\alpha_2, 1/p\}/3, 1]$ such that (4.22) holds. As above it is now not difficult to see that there are $\theta \in [0,1]$ and $r_1, \sigma_1 \in [1,\infty]$ such that all conditions in (4.17) and (4.21) are satisfied. This shows that $\lambda'(\psi)\partial_t\psi \in H_p^{\alpha_2}(J; L_p(\Omega))$, thereby completing the proof of the lemma. □

Lemma 4.1 shows that $\lambda(z_2 + u_2) - \lambda(u_2) \in {}_0Z_2$ for all $z_2 \in {}_0\mathcal{Z}_2$. By the mixed derivative theorem ${}_0Z_2 \hookrightarrow {}_0H_p^{(1+\alpha_2)/2}(J; H_p^2(\Omega))$, and thus $AP\big(\lambda(z_2+u_2) - \lambda(u_2)\big) \in {}_0H_p^{(1+\alpha_2)/2}(J; L_p(\Omega))$. By boundedness of $\mathcal{A}(\mathcal{B}_1 + \mathcal{A})^{-1}P$ in ${}_0H_p^{(1+\alpha_2)/2}(J; L_p(\Omega))$ we then infer that

$$APF(z_2 + u_2) \in {}_0H_p^{(1+\alpha_2)/2}(J; L_p(\Omega)) \quad \text{for all } z_2 \in {}_0\mathcal{Z}_2. \tag{4.23}$$

Decomposing $G(\psi)$ according to $G(\psi) = G_1(\psi) + G_2(\psi)$ with

$$\begin{aligned} G_1(\psi) &= AP\Big[\phi'(\psi) - \lambda'(\psi)\big(F(\psi) - \lambda(\psi)\big)\Big], \\ G_2(\psi) &= -AP\Big[\lambda'(\psi)\big(u_1 + (\mathcal{B}_1 + \mathcal{A})^{-1}AP\lambda(u_2)\big)\Big], \end{aligned}$$

it is then not difficult to see that $G_2(z_2 + u_2)$ is just in $L_p(J; Y)$, whereas $G_1(z_2 + u_2)$ belongs to some space $H_p^\epsilon(J; Y)$ with $\epsilon > 0$.

Let now $B_R^T(0) = \{z \in {}_0\mathcal{Z}_2 : |z|_{\mathcal{Z}_2} \le R\}$ and $B_R^T(u_2) = u_2 + B_R^T(0)$, where $R > 0$ is a fixed number. It can be shown that for sufficiently small $T > 0$ one has $\mathcal{T}B_R^T \subset B_R^T$ and that $\mathcal{T}$ is a strict contraction in B_R^T. To see this let us indicate the dependence on T by writing $\mathcal{Z}_i^T$ instead of $\mathcal{Z}_i$, $i = 1, 2$.

Lemma 4.2. *Let $\alpha_2 \in (0,1)$, $p \ge 2$, and $\phi, \lambda \in C^{4-}(\mathbb{R})$. Then there exists a nonincreasing function $c(T)$ with $c(T) \to 0$ as $T \to 0+$ such that for all $w, z \in B_R^T(0)$,*

$$|\mathcal{T}w - \mathcal{T}z|_{\mathcal{Z}_2^T} \le c(T)|w - z|_{\mathcal{Z}_2^T}, \tag{4.24}$$

$$|\mathcal{T}w|_{\mathcal{Z}_2^T} \le c(T)[|w|_{\mathcal{Z}_2^T} + |u_2|_{\mathcal{Z}_2^T}]. \tag{4.25}$$

For the proof of this result note that, it suffices to check that the inequalities

$$\begin{aligned} |\Delta\phi'(u) - \Delta\phi'(v)|_{L_p([0,T]\times\Omega)} &\le c(T)|u - v|_{\mathcal{Z}_2^T}, \\ |\Delta(\lambda'(u)F(u)) - \Delta(\lambda'(v)F(v))|_{L_p([0,T]\times\Omega)} &\le c(T)|u - v|_{\mathcal{Z}_2^T}, \\ |\Delta(\lambda'(u)f_1) - \Delta(\lambda'(v)f_1)|_{L_p([0,T]\times\Omega)} &\le c(T)|u - v|_{\mathcal{Z}_2^T}, \end{aligned}$$

are satisfied for all $u, v \in B_R^T(u_2)$, where $f_1 \in \mathcal{Z}_1$ is defined by $f_1 = u_1 + (\mathcal{B}_1 + \mathcal{A})^{-1}AP\lambda(u_2)$, and $c(T)$ behaves as in the statement of Lemma 4.2. These inequalities can be obtained by similar arguments as in [22, Proposition 3.2]. The regularity property (4.23) is basic to derive the second estimate.

Lemma 4.2 and the contraction mapping principle imply that equation (4.12) possesses a unique fixed point $z_2 \in {}_0\mathcal{Z}_2^T$, provided T is chosen sufficiently small. It is then clear that (e,ψ) defined by $\psi = u_2 + z_2$ and (4.8) belongs to the space $Z_1^T \times Z_2^T$ and that it is the unique solution of the system (4.1)–(4.4).

Summarizing, we have proved the following result.

Theorem 4.3. *Let $p \in [2,\infty)$, $\phi, \lambda \in C^{4-}(\mathbb{R})$, and suppose that (A1) holds. Assume that $\alpha_i \neq 1/p$, $i = 1,2$ and that $p-1 \notin \{\frac{4}{3(1+\alpha_2)}, \frac{2}{1+\alpha_1}, \frac{4}{1+\alpha_2}\}$. Suppose further that the initial data e_0 and ψ_0 satisfy the regularity and compatibility conditions (i) and (ii) in the first part of Theorem 2.2.*

Then there exists $T > 0$, such that the system (4.1)–(4.4) *admits a unique solution $(e,\psi) \in Z_1^T \times Z_2^T$. The temperature $\vartheta = e - \lambda(\psi)$ belongs to the space Z_ϑ^T given by*

$$Z_\vartheta^T = H_p^{1+\min\{\alpha_1,\alpha_2\}}([0,T]; L_p(\Omega)) \cap L_p([0,T]; H_p^2(\Omega)).$$

4.2. **Global well-posedness.** The local solution obtained in Theorem 4.3 can be continued to some larger interval $[0, T+\delta]$. In fact, define

$$\mathcal{M}_{z_2}^{T+\delta} := \{w \in {}_0\mathcal{Z}_2^{T+\delta} : w|_{[0,T]} = z_2\},$$

where, as above, $z_2 = \psi - u_2$. The set $\mathcal{M}_{z_2}^{T+\delta}$ is not empty and becomes a complete metric space when endowed with the metric induced by the norm of $\mathcal{Z}_2^{T+\delta}$. Using the estimates from Lemma 4.2 and the contraction mapping principle, one can show that $v = \mathcal{T}(v)$ has a unique fixed point in $\mathcal{M}_{z_2}^{T+\delta}$ provided $\delta > 0$ is selected sufficiently small. This in turn then yields a solution (e,ψ) of (4.1)–(4.4) on $[0,T+\delta]$ with $(e,\psi) \in Z_1^{T+\delta} \times Z_2^{T+\delta}$.

Repeating this argument we obtain an interval of maximal existence $[0, t_{max})$, i.e. $t_{max} \in (0,\infty]$ is the supremum of all $t_1 > 0$ such that for all $T \in (0,t_1)$ the system (4.1)-(4.4) has a unique solution $(e,\psi) \in Z_1^T \times Z_2^T$.

Suppose now in addition that ϕ, λ satisfy the growth conditions from (H1)–(H3). We want to show that in this situation we have global existence for (4.1)–(4.4), that is $t_{max} = \infty$.

To derive suitable a priori estimates, we first remark that the operator A defined at the beginning of Section 4 is selfadjoint and positive definite in case $p = 2$, the spectrum $\sigma(A)$ is independent of $p \in (1,\infty)$, it consists of semisimple eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$. In particular, the Poincaré-Wirtinger inequality is valid:

$$|\nabla u|_2^2 = (Au|u) \geq \lambda_1 |u|_2^2, \quad u \in D(A). \tag{4.26}$$

Here $|\cdot|_2$ and $(\cdot|\cdot)$ stand for the norm and the inner-product of $L_2(\Omega)$, respectively.

Let $T \in (0, t_{max})$, and suppose that $(e,\psi) \in Z_1^T \times Z_2^T$ is the solution of (4.1)-(4.4), which is equivalent to (1.10) with $\vartheta = e - \lambda(\psi)$. Multiply the first equation of (1.10) by ϑ and the third equation of that system by μ. Adding the resulting relations and integrating by parts yields the energy identity

$$\frac{1}{2}\frac{d}{dt}\Big(|\vartheta|_2^2 + |\nabla\psi|_2^2 + 2\int_\Omega \phi(\psi)\,dx\Big) + (a_1 * \nabla\vartheta|\nabla\vartheta) + (a_2 * \nabla\mu|\nabla\mu) = 0. \tag{4.27}$$

Set

$$E_0(\vartheta,\psi) = \frac{1}{2}|\vartheta|_2^2 + \frac{1}{2}|\nabla\psi|_2^2 + \int_\Omega \phi(\psi)\,dx, \quad t \in [0,T].$$

Since the kernels a_i, $i = 1,2$, are of positive type (see e.g. [18] or [20]), relation (4.27) implies

$$E_0(\vartheta(t),\psi(t)) \leq E_0(\vartheta_0,\psi_0), \quad t \in [0,T]. \tag{4.28}$$

On the other hand, we may use (H2) and inequality (4.26) to estimate

$$\int_\Omega \phi(\psi)\,dx \geq -\frac{c_1}{2}|\psi|_2^2 - c_0|\Omega| \geq -\frac{c_1}{2\lambda_1}|\nabla\psi|_2^2 - c_0|\Omega|,$$

which together with (4.28) gives

$$\frac{\lambda_1 - c_1}{2\lambda_1}\,|\nabla\psi|_2^2 + \frac{1}{2}\,|\vartheta|_2^2 \le E_0(\vartheta_0,\psi_0) + c_0|\Omega|, \quad t \in [0,T].$$

This estimate and (4.26) yield a uniform bound

$$|\vartheta|_{L_\infty([0,T];L_2(\Omega))} + |\psi|_{L_\infty([0,T];H_2^1(\Omega))} \le C(\vartheta_0,\psi_0,|\Omega|) =: C. \tag{4.29}$$

The following lemma is basic to obtain global existence.

Lemma 4.4. *Suppose $p \ge 2$ and let $\psi \in \mathcal{Z}_2^T$ be the second component of the solution of* (4.1)-(4.4) *on* $[0,T]$. *Let $G(\psi)$ be as in equation (4.10). Then there exist constants $C_1, m > 0$ and $\varrho \in (0,1)$, independent of $T > 0$, such that*

$$|G(\psi)|_{L_p([0,T];L_p(\Omega))} \le C_1\left(1 + |\psi|^{\varrho}_{Z_2^T}\,|\psi|^m_{L_\infty([0,T];H_2^1(\Omega))}\right).$$

The proof of this lemma relies on the Gagliardo-Nirenberg inequality and is basically the same as in [22, Lemma 4.1], see also [28].

We will now show that $|\psi|_{Z_2^T}$ stays bounded as $T \nearrow t_{max}$. Note that

$$|(\mathcal{B}_2 + \mathcal{A}^2)^{-1}|_{\mathcal{B}(L_p([0,T];Y),Z_2^T)} \le M$$

where $M > 0$ does not depend on $T > 0$. Using this, the decomposition (4.11), the estimate (4.29), and Lemma 4.4, we have

$$\begin{aligned}
|\psi|_{Z_2^T} &\le |u_2|_{Z_2^T} + |(\mathcal{B}_2 + \mathcal{A}^2)^{-1}G(\psi)|_{Z_2^T} \\
&\le |u_2|_{Z_2^T} + M|G(\psi)|_{L_p([0,T];L_p(\Omega))} \\
&\le |u_2|_{Z_2^T} + MC_1\left(1 + |\psi|^{\varrho}_{Z_2^T}|\psi|^m_{L_\infty([0,T];H_2^1(\Omega))}\right) \\
&\le |u_2|_{Z_2^T} + MC_1\left(1 + C|\psi|^{\varrho}_{Z_2^T}\right),
\end{aligned}$$

which implies

$$|\psi|_{Z_2^T} \le \tilde{C}(1 + |u_2|_{Z_2^T}),$$

with some constant $\tilde{C} > 0$ not depending on T. Hence we have global existence. This completes the proof of the first part of Theorem 2.2.

Remark 4.5. Global existence and the uniform estimate (4.29) imply that the solution (ϑ,ψ) of (1.10) belongs to $L_\infty(\mathbb{R}_+;L_2(\Omega)) \times L_\infty(\mathbb{R}_+;H_2^1(\Omega))$.

5. Long-Time Behaviour

In this section we study the long-time behaviour of the solution of the system (1.10). Recall that we assume without loss of generality $\overline{\psi_0} = \overline{e_0} = \overline{\vartheta_0} + \overline{\lambda(\psi_0)} = 0$.

5.1. L_∞-bounds and relative compactness of the orbit.

We show first that the solution (ϑ,ψ) of (1.10) is globally bounded, that is $(\vartheta,\psi) \in L_\infty(\mathbb{R}_+ \times \Omega)^2$, and that its orbit is relatively compact in the natural energy space W defined by

$$W = \{(\vartheta,\psi) \in L_2(\Omega) \times H_2^1(\Omega) : \bar{\psi} = 0\}.$$

The proof relies on a bootstrap (Nash-Moser iteration) argument.

Let A_2 denote the operator A in the space $\{v \in L_2(\Omega) : \int_\Omega v(x)\,dx = 0\}$ defined as in Section 4.

Theorem 5.1. *Let the assumptions of Theorem 2.2 Part I be satisfied. Suppose in addition that condition (A4) holds and that $(\vartheta_0,\psi_0) \in D(A_2) \times D(A_2^2)$. Then the solution (ϑ,ψ) of (1.10) is globally bounded and the orbit $\{(\vartheta(t),\psi(t)) : t \ge 0\}$ is relatively compact in the energy space W.*

Proof. Recall that system (1.10) is equivalent to the system (4.1)–(4.4) for the functions e and ψ.

Let $\{S_i(t)\}_{t\geq 0} \subset \mathcal{B}(Y)$, $i=1,2$, be the resolvent family for the Volterra equation

$$z + 1 * a_i * A^i z = 1 * f + z_0, \quad t \geq 0. \tag{5.1}$$

Evidently, (5.1) is equivalent to

$$\dot{z} + a_i * A^i z = f,\ t \geq 0, \quad z(0) = z_0,$$

and the variation of parameters formula yields

$$z(t) = S_i(t) z_0 + \int_0^t S_i(t-\tau) f(\tau)\, d\tau, \quad t \geq 0.$$

Hence we can rewrite (4.1)–(4.4) as the system of integral equations

$$e(t) = S_1(t)e_0 + (A^{1/2} a_1 * S_1 * A^{1/2} P\lambda(\psi))(t),\ t > 0, \tag{5.2}$$

$$\psi(t) = S_2(t)\psi_0 - (A a_2 * S_2 * P[\phi'(\psi) - \lambda'(\psi)(e - \lambda(\psi))])(t),\ t > 0. \tag{5.3}$$

Observe that in view of $(\vartheta_0, \psi_0) \in D(A_2) \times D(A_2^2)$ we have $S_1(\cdot)e_0 \in L_\infty(\mathbb{R}_+; H_2^2(\Omega))$ as well as $S_2(\cdot)\psi_0 \in L_\infty(\mathbb{R}_+; H_2^4(\Omega))$. Note that here and in the subsequent lines we use the same notation for the resolvent of (5.1) and the Neumann-Laplacian, respectively, for all $p \geq 2$.

We know already (cf. Remark 4.5) that $\psi \in L_\infty(\mathbb{R}_+; H_2^1(\Omega))$ and $\vartheta \in L_\infty(\mathbb{R}_+; L_2(\Omega))$. This is the starting point for the following iteration argument of Nash-Moser type. Set $p_0 = 6$ and $r_0 = 2$. Suppose we know already that

$$\psi \in L_\infty(\mathbb{R}_+; H^1_{r_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{p_n}(\Omega)), \text{ with } \frac{1}{p_n} = \frac{1}{r_n} - \frac{1}{3}.$$

By (H3), we then have $\lambda'(\psi) \in L_\infty(\mathbb{R}_+ \times \Omega)$ and $\lambda(\psi) \in L_\infty(\mathbb{R}_+; H^1_{r_n}(\Omega))$, i.e. $A^{1/2}P\lambda(\psi) \in L_\infty(\mathbb{R}_+; L_{r_n}(\Omega))$, where we use the fact that $D(A^{1/2}) = H^1_p(\Omega) \cap Y$. Appealing to Theorem 3.4 we have $A^{1-\delta/2} a_1 * S_1 \in L_1(\mathbb{R}_+; \mathcal{B}(L_{r_n}(\Omega)))$, for each $\delta \in (0,1)$. Thus (5.2) implies that

$$e \in L_\infty(\mathbb{R}_+; H^{1-\delta}_{r_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{s_n}(\Omega)), \text{ with } \frac{1}{s_n} = \frac{1}{p_n} + \frac{\delta}{3}.$$

Since $\lambda(\psi) \in L_\infty(\mathbb{R}_+; H^1_{r_n}(\Omega))$ we also obtain that $\vartheta = e - \lambda(\psi) \in L_\infty(\mathbb{R}_+; H^{1-\delta}_{r_n}(\Omega))$. Turning to the equation for ψ, we have $\phi'(\psi) \in L_\infty(\mathbb{R}_+; L_{p_n/(\beta+2)}(\Omega))$, by assumption (H1). Therefore

$$\phi'(\psi) - \lambda'(\psi)\vartheta \in L_\infty(\mathbb{R}_+; L_{q_n}(\Omega)), \text{ with } \frac{1}{q_n} = \frac{\beta+2}{p_n} + \frac{\delta}{3}.$$

Since $(A^2)^{(1-\delta/2)} a_2 * S_2 \in L_1(\mathbb{R}_+; \mathcal{B}(L_{q_n}(\Omega)))$, by Theorem 3.4, it follows from (5.3) that

$$\psi \in L_\infty(\mathbb{R}_+; H^{2-\delta}_{q_n}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; H^1_{r_{n+1}}(\Omega)) \hookrightarrow L_\infty(\mathbb{R}_+; L_{p_{n+1}}(\Omega)),$$

where

$$\frac{1}{r_{n+1}} - \frac{1}{3} = \frac{1}{p_{n+1}} = \frac{\beta+2}{p_n} - 2\frac{1-\delta}{3}.$$

Inductively this yields

$$\frac{1}{p_n} = (\beta+2)^n \left[\frac{1}{p_0} - \frac{2-\delta}{3(\beta+1)} \right] + \frac{2-\delta}{3(\beta+1)}.$$

Since by assumption $\beta < 3$, we may choose $0 < \delta < (3-\beta)/2$ to get the bracket negative. Then the iteration ends after finitely many steps. As a consequence we obtain

$$\psi \in L_\infty(\mathbb{R}_+ \times \Omega).$$

It is then also clear that $\vartheta \in L_\infty(\mathbb{R}_+ \times \Omega)$. Moreover, since $H^{1-\delta}_{r_n}(\Omega)$ and $H^{2-\delta}_{q_n}(\Omega)$ are compactly embedded in $L_2(\Omega)$ and $H^1_2(\Omega)$, respectively, it follows that $\{(\vartheta(t), \psi(t)) : t \geq 0\}$ is relatively compact in the energy space W. □

Remark 5.2. Observe that the above proof also yields a bound for ψ in the space $L_\infty(\mathbb{R}_+; H^1_3(\Omega))$.

5.2. **Lyapunov functional and properties of the ω-limit set.** We define an energy functional E on W by means of

$$E(\vartheta,\psi)=\int_\Omega\left(\frac{1}{2}|\nabla\psi|^2+\phi(\psi)+\frac{1}{2}\left(\vartheta+\overline{\lambda(\psi)}\right)^2\right)dx+\frac{|\Omega|}{2}\,(\overline{\lambda(\psi)})^2.$$

Note that the critical points of E in W with constraint $\overline{\vartheta+\lambda(\psi)}=0$ are precisely the solutions of

$$\begin{aligned} -\Delta\psi_\infty+\phi'(\psi_\infty)-\lambda'(\psi_\infty)\vartheta_\infty &= \mu_\infty, \quad x\in\Omega,\\ \partial_\nu\psi_\infty &= 0, \qquad x\in\Gamma, \end{aligned}\tag{5.4}$$

where ϑ_∞ and μ_∞ are constants, $\psi_\infty\in H_2^2(\Omega)$, and

$$\overline{\psi_\infty}=\overline{\vartheta_\infty+\lambda(\psi_\infty)}=0,\quad \mu_\infty=\overline{\phi'(\psi_\infty)-\lambda'(\psi_\infty)\vartheta_\infty},\tag{5.5}$$

cf. [10, Proposition 6.4] and [22, Proposition 5.2]). Furthermore, the system (5.4), (5.5) is the stationary problem to (1.10); recall that we assume $\overline{\psi_0}=\overline{\vartheta_0+\lambda(\psi_0)}=0$.

Moreover, the Fréchet-derivative of E is given by

$$\begin{aligned}\langle E'(\vartheta,\psi)|(k,h)\rangle_{W^*\times W}=&\int_\Omega\Big[\nabla\psi\cdot\nabla h+\phi'(\psi)h+\overline{\lambda(\psi)}\lambda'(\psi)h+(\vartheta+\overline{\lambda(\psi)})k\Big]\,dx\\ &+\overline{(\vartheta+\lambda(\psi))}(\lambda'(\psi)|h),\quad k\in L_2(\Omega),\ h\in\hat H_2^1(\Omega),\end{aligned}\tag{5.6}$$

where we have set $\hat H_2^1(\Omega)=\{v\in H_2^1(\Omega):\int_\Omega v(x)\,dx=0\}$. Assuming $\psi\in D(A_2)$ we thus obtain with $\overline{\vartheta+\lambda(\psi)}=0$

$$\frac{1}{c}\left(|A_2^{-1/2}PM(\psi)|_2^2+|P\vartheta|_2^2\right)^{1/2}\le|E'(\vartheta,\psi)|_{W^*}\le c\left(|A_2^{-1/2}PM(\psi)|_2^2+|P\vartheta|_2^2\right)^{1/2},\tag{5.7}$$

where c is a positive constant and

$$M(\psi)=A_2\psi+\phi'(\psi)+\overline{\lambda(\psi)}\lambda'(\psi),$$

see also [10, 22]. In the remaining part of the paper we will suppress the subscript 2 in the notation for the Neumann-Laplacian on $PL_2(\Omega)$, that is $A:=A_2$.

Using the Neumann boundary conditions for ϑ and μ as well as the property $\int_\Omega(\vartheta+\overline{\lambda(\psi)})=0$, it follows from the energy relation (4.27) that

$$\frac{d}{dt}E(\vartheta,\psi)+\left(a_1*A^{1/2}P\vartheta|A^{1/2}P\vartheta\right)+\left(a_2*A^{1/2}P\mu|A^{1/2}P\mu\right)=0.\tag{5.8}$$

Hence integrating over $[0,t]$, we obtain

$$E(\vartheta,\psi)(t)+\int_0^t\Big[\left(a_1*A^{1/2}P\vartheta|A^{1/2}P\vartheta\right)(s)+\left(a_2*A^{1/2}P\mu|A^{1/2}P\mu\right)(s)\Big]\,ds=E(\vartheta_0,\psi_0).$$

Since the kernels a_i are of positive type, this implies

$$E(\vartheta,\psi)(t)\le E(\vartheta_0,\psi_0).$$

By assumption (H2) and the Poincaré-Wirtinger inequality (4.26), $E(\vartheta,\psi)$ is also bounded from below on $\mathbb{R}_+$, cf. Section 4.2.

Unfortunately, the energy functional E is not a Lyapunov functional since it is not decreasing. To construct a proper Lyapunov functional we will use the ideas from [30]. For this approach we need the assumptions (A2) and (A3) on the kernels a_i.

Define the functions v_1 and v_2 by

$$\begin{aligned} v_1&=a_1*A^{1/2}P\vartheta=-A^{-1/2}\partial_t(\vartheta+\lambda(\psi))=-A^{-1/2}\partial_t e,\\ v_2&=a_2*A^{1/2}P\mu=-A^{-1/2}\partial_t\psi.\end{aligned}\tag{5.9}$$

Thanks to (A2) we can then rewrite the second respectively third term on the left-hand side of (5.8) as follows.

$$\begin{aligned} \big(a_1 * A^{1/2}P\vartheta|A^{1/2}P\vartheta\big) &= (\partial_t(k_1 * v_1)|v_1)\,, \\ \big(a_2 * A^{1/2}P\mu|A^{1/2}P\mu\big) &= (\partial_t(k_2 * v_2)|v_2)\,. \end{aligned} \tag{5.10}$$

By (A1), v_i $(i = 1, 2)$ satisfies the assumptions of Proposition 3.7 with $Y = L_2(\Omega)$ and $a = a_i$. Thus we may apply Theorem 3.6, which yields

$$2(\partial_t(k_i * v_i)(t)|v_i(t)) \geq \partial_t(k_i * |v_i|_2^2)(t) + k_i(t)|v_i(t)|_2^2, \quad \text{a.a. } t > 0. \tag{5.11}$$

Furthermore, by (A3) we have

$$\partial_t(k_i * |v_i|_2^2)(t) = \partial_t(b_i * |v_i|_2^2)(t) + \gamma_i(b_i * |v_i|_2^2)(t). \tag{5.12}$$

Combining (5.8),(5.10), and (5.12) shows that the functional H_0 defined by

$$H_0(\vartheta, \psi)(t) := E(\vartheta, \psi)(t) + \frac{1}{2}\big(b_1 * |v_1|_2^2\big)(t) + \frac{1}{2}\big(b_2 * |v_2|_2^2\big)(t), \tag{5.13}$$

is a Lyapunov functional. Indeed, $H_0 \in H^1_{1,loc}(\mathbb{R}_+)$ and for a.a. $t > 0$ there holds

$$\frac{d}{dt}H_0(\vartheta, \psi)(t) \leq -\frac{1}{2}\Big(k_1(t)|v_1(t)|_2^2 + k_2(t)|v_2(t)|_2^2 + \gamma_1\big(b_1 * |v_1|_2^2\big)(t) + \gamma_2\big(b_2 * |v_2|_2^2\big)(t)\Big). \tag{5.14}$$

Since $E(\vartheta, \psi)$ is bounded from below on $\mathbb{R}_+$, $H_0(\vartheta, \psi)$ evidently enjoys the same property. Recall that $k_i(t) \geq \lim_{t\to\infty} k_i(t) = \gamma_i|b_i|_{L_1(\mathbb{R}_+)} =: k_i^\infty > 0$, $i = 1, 2$, cf. Remark 2.1(iv). Hence, (5.14) implies that

$$v_i \in L_2(\mathbb{R}_+ \times \Omega),\ b_i * |v_i|_2^2 \in L_1(\mathbb{R}_+),\ i = 1, 2. \tag{5.15}$$

For the next result we recall that the ω-limit set of the global solution (ϑ, ψ) of (1.10) in the energy space W is defined by

$$\omega(\vartheta, \psi) = \{(\vartheta_\infty, \psi_\infty) \in W : \text{ there exists } (t_n) \nearrow \infty \text{ s.t. } (\vartheta(t_n), \psi(t_n)) \to (\vartheta_\infty, \psi_\infty) \text{ in } W\}.$$

By Theorem 5.1, $\omega(\vartheta, \psi) \neq \emptyset$ and the orbit $\{(\vartheta(t), \psi(t)) : t \geq 0\}$ is relatively compact in W. It follows from well-known results that the ω-limit set is compact and connected.

Proposition 5.3. *Let the assumptions of Theorem 5.1 be satisfied. Suppose in addition that (A2) and (A3) hold. Let (ϑ, ψ) be the global solution of (1.10). Then*

(i) $A^{-1/2}\partial_t\psi, A^{-1/2}\partial_t e \in L_2(\mathbb{R}_+; L_2(\Omega))$;
(ii) *The functional E is constant on $\omega(\vartheta, \psi)$ and $\lim_{t\to\infty} E(\vartheta(t), \psi(t))$ exists;*
(iii) $b_i * |v_i|_2^2 \in C_0(\mathbb{R}_+)$, $i \in \{1, 2\}$;
(iv) *For every $(\vartheta_\infty, \psi_\infty) \in \omega(\vartheta, \psi)$ there holds that ϑ_∞ is constant, $\psi_\infty \in D(A)$, and $(\vartheta_\infty, \psi_\infty)$ solves the stationary problem (5.4) with $\mu_\infty = \overline{\phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty}$;*
(v) *For every $(\vartheta_\infty, \psi_\infty) \in \omega(\vartheta, \psi)$ one has $E'(\vartheta_\infty, \psi_\infty) = 0$.*

Proof. The first claim follows directly from (5.15) and the definition of v_i in (5.9). Next, let $(\vartheta_\infty, \psi_\infty) \in \omega(\vartheta, \psi)$ and let $(t_n) \nearrow \infty$ such that $(\vartheta(t_n), \psi(t_n)) \to (\vartheta_\infty, \psi_\infty)$ in W as $n \to \infty$. Then

$$e(t_n) = \vartheta(t_n) + \lambda(\psi(t_n)) \to \vartheta_\infty + \lambda(\psi_\infty) =: e_\infty \quad \text{in } L_2(\Omega). \tag{5.16}$$

Since $A^{-1/2}\partial_t\psi \in L_2(\mathbb{R}_+; L_2(\Omega))$, we have $A^{-1/2}\psi(t_n + s) \to A^{-1/2}\psi_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$. By the relative compactness of $\psi(\mathbb{R}_+)$ in $\hat{H}^1_2(\Omega)$, this also yields $\psi(t_n + s) \to \psi_\infty$ in $\hat{H}^1_2(\Omega)$ for all $s \in [0, 1]$. An analogous argument shows that (5.16), the second part of (i), and the relative compactness of $e(\mathbb{R}_+)$ in $L_2(\Omega)$ imply that $e(t_n + s) \to e_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$. Since $\vartheta = e - \lambda(\psi)$, it follows then that $\vartheta(t_n + s) \to \vartheta_\infty$ in $L_2(\Omega)$ for all $s \in [0, 1]$. By continuity, we also have $E(\vartheta(t_n+s), \psi(t_n+s)) \to E(\vartheta_\infty, \psi_\infty)$ for all $s \in [0, 1]$, and thus, by the dominated convergence theorem,

$$E(\vartheta_\infty, \psi_\infty) = \lim_{n\to\infty} \int_0^1 E(\vartheta(t_n + s), \psi(t_n + s))\, ds. \tag{5.17}$$

Next, integrate (5.13) over $[t_n, t_n + 1]$ and send $n \to \infty$; using (5.17) and the second part of (5.15) this yields

$$H_0^\infty := \lim_{t\to\infty} H_0(\vartheta(t), \psi(t)) = \lim_{n\to\infty} \int_0^1 H_0(\vartheta(t_n + s), \psi(t_n + s))\, ds = E(\vartheta_\infty, \psi_\infty).$$

Since $(\vartheta_\infty, \psi_\infty)$ was chosen arbitrarily in $\omega(\vartheta, \psi)$, this shows that E is constant on $\omega(\vartheta, \psi)$. Moreover, by relative compactness of the orbit of (ϑ, ψ) in W, we obtain $\lim_{t\to\infty} E(\vartheta(t), \psi(t)) = H_0^\infty$. Thus (ii) is proved. Claim (iii) follows from $\lim_{t\to\infty} H_0(\vartheta(t), \psi(t)) = \lim_{t\to\infty} E(\vartheta(t), \psi(t)) = H_0^\infty$ and the positivity of the kernels b_i, $i = 1, 2$.

To establish (iv), let $(\vartheta_\infty, \psi_\infty) \in \omega(\vartheta, \psi)$ and let $(t_n) \nearrow \infty$ such that $(\vartheta(t_n), \psi(t_n)) \to (\vartheta_\infty, \psi_\infty)$ in W as $n \to \infty$. We know already that this implies $(\vartheta(t_n + s), \psi(t_n + s)) \to (\vartheta_\infty, \psi_\infty)$ in W for all $s \in [0, 1]$. By (A2), (5.9), and the dominated convergence theorem, we thus have in $L_2(\Omega)$

$$\begin{aligned} P\vartheta_\infty &= \lim_{n\to\infty} \int_0^1 P\vartheta(t_n + s)\, ds \\ &= \lim_{n\to\infty} \int_{t_n}^{t_n+1} \frac{d}{dt}\,(k_1 * a_1 * P\vartheta)(s)\, ds \\ &= \lim_{n\to\infty} A^{-1/2}\Big((k_1 * v_1)(t_n + 1) - (k_1 * v_1)(t_n)\Big). \end{aligned}$$

Denote the term in brackets by f_n. Using (A3), (5.15), property (iii), and Lemma 3.5 we have

$$\begin{aligned} |f_n|_2^2 &\le 3|(b_1 * v_1)(t_n + 1)|_2^2 + 3|(b_1 * v_1)(t_n)|_2^2 + 3\gamma_1^2 \int_{t_n}^{t_n+1} |b_1 * v_1|_2^2(s)\, ds \\ &\le 3|b_1|_{L_1(\mathbb{R}_+)}\Big(b_1 * |v_1|_2^2)(t_n + 1) + b_1 * |v_1|_2^2)(t_n) + \gamma_1^2 \int_{t_n}^{t_n+1} \big(b_1 * |v_1|_2^2\big)(s)\, ds\Big) \to 0 \end{aligned}$$

as n tends to ∞. Hence $P\vartheta_\infty = 0$, i.e. ϑ_∞ is constant.

Using (H1), (H3), $\vartheta_\infty = \mathit{const}$, and the global boundedness of ψ we deduce from the convergence $(\vartheta(t_n + s), \psi(t_n + s)) \to (\vartheta_\infty, \psi_\infty)$ in W for all $s \in [0, 1]$ that

$$A^{-1/2}P\mu(t_n + s) \to A^{1/2}\psi_\infty + A^{-1/2}P\big(\phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty\big) =: m_\infty$$

in $L_2(\Omega)$ for all $s \in [0, 1]$. Similarly as above we then have in $L_2(\Omega)$

$$\begin{aligned} m_\infty &= \lim_{n\to\infty} \int_0^1 A^{-1/2}P\mu(t_n + s)\, ds \\ &= \lim_{n\to\infty} \int_{t_n}^{t_n+1} A^{-1/2}\,\frac{d}{dt}\,(k_2 * a_2 * P\mu)(s)\, ds \\ &= \lim_{n\to\infty} A^{-1}\Big((k_2 * v_2)(t_n + 1) - (k_2 * v_2)(t_n)\Big). \end{aligned}$$

We may now argue as above, employing now the global estimates for v_2 and $b_2 * |v_2|_2^2$, to see that the term in brackets tends to 0 as $n \to \infty$. Hence $m_\infty = 0$, which in turn implies $\psi_\infty \in D(A)$ and (5.4) with $\mu_\infty := \overline{\phi'(\psi_\infty) - \lambda'(\psi_\infty)\vartheta_\infty}$. Thus (iv) is proved.

Assertion (v) is a consequence of (iv) by the remarks following the definition of the functional E at the beginning of this subsection. □

5.3. **Łojasiewicz-Simon Inequality and Modified Lyapunov Functional.** The following result is a key ingredient in the proof of convergence to equilibrium.

Proposition 5.4. *Suppose that the assumptions (H1)–(H3) are satisfied and assume in addition that ϕ and λ are real analytic on $\mathbb{R}$. Then for any critical point $(\vartheta_\infty, \psi_\infty)$ of the functional E in the energy space W there are constants $c > 0$, $\sigma > 0$ and $\theta \in (0, 1/2]$ such that*

$$|E(\vartheta, \psi) - E(\vartheta_\infty, \psi_\infty)|^{1-\theta} \le c|E'(\vartheta, \psi)|_{W^*}, \tag{5.18}$$

whenever $|(\vartheta, \psi) - (\vartheta_\infty, \psi_\infty)|_W \le \sigma$.

This has been proved in [22, Proposition 5.3], see also [10, Proposition 6.6].

Let (ϑ, ψ) be the global solution of (1.10). Then we know already that the ω-limit set $\omega(\vartheta, \psi)$ is compact in W. Thus we can cover it by a union of finitely many balls with centers in $\omega(\vartheta, \psi)$. Since E is constant on $\omega(\vartheta, \psi)$ (cf. Proposition 5.3(ii)), say $E = E_\infty$, there exist uniform constants $c > 0$, $\theta \in (0, 1/2]$, and an open set $\mathcal{O} \supset \omega(\vartheta, \psi)$ such that

$$|E(\tilde{\vartheta}, \tilde{\psi}) - E_\infty|^{1-\theta} \le c|E'(\tilde{\vartheta}, \tilde{\psi})|_{W^*}, \quad \text{for all } (\tilde{\vartheta}, \tilde{\psi}) \in U. \tag{5.19}$$

We will next modify H_0 defined in (5.13) in order to produce a new Lyapunov functional which is suitable for the approach via Łojasiewicz-Simon inequality.

Multiplying the identity

$$P\vartheta = -A^{-1}\partial_t[k_1 * \partial_t(\vartheta + \lambda(\psi))] = A^{-1/2}\partial_t(k_1 * v_1)$$

by $P\vartheta$ and integrating over Ω gives

$$\begin{aligned}|P\vartheta|_2^2 &= (\partial_t(k_1 * v_1)|A^{-1/2}P\vartheta)\\ &= (\partial_t(b_1 * v_1)|A^{-1/2}P\vartheta) + \gamma_1(b_1 * v_1|A^{-1/2}P\vartheta)\\ &= \partial_t(b_1 * v_1|A^{-1/2}P\vartheta) - (b_1 * v_1|A^{-1/2}\partial_t P\vartheta) + \gamma_1(b_1 * v_1|A^{-1/2}P\vartheta).\end{aligned}$$

Hence

$$\begin{aligned}-\partial_t(b_1 * v_1|A^{-1/2}P\vartheta) = &- |P\vartheta|_2^2 + \gamma_1(b_1 * v_1|A^{-1/2}P\vartheta)\\ &+ (b_1 * v_1|v_1) + (b_1 * v_1|A^{-1/2}\partial_t P\lambda(\psi)).\end{aligned} \tag{5.20}$$

The last term in (5.20) can be rewritten as follows.

$$\begin{aligned}(b_1 * v_1|A^{-1/2}\partial_t P\lambda(\psi)) &= -(b_1 * v_1|A^{-1/2}P[\lambda'(\psi)A^{1/2}v_2])\\ &= -(\lambda'(\psi)A^{-1/2}(b_1 * v_1)|A^{1/2}v_2)\\ &= (\lambda'(\psi)A^{-1/2}(b_1 * v_1)|\Delta A^{-1/2}v_2)\\ &= -\big([\lambda''(\psi)\nabla\psi]A^{-1/2}(b_1 * v_1) + \lambda'(\psi)\nabla A^{-1/2}(b_1 * v_1)|\nabla A^{-1/2}v_2\big)_{L_2(\Omega)^3}.\end{aligned} \tag{5.21}$$

Here we used the property that $\partial_\nu A^{-1/2}v_2 = \partial_\nu a_2 * P\mu = 0$ on $\mathbb{R}_+ \times \Gamma$. Note that $\nabla A^{-1/2}P \in \mathcal{B}(L_2(\Omega); L_2(\Omega)^3)$. Using the L_∞-bounds for ψ (cf. Theorem 5.1 and Remark 5.2), Hölder's and Young's inequality as well as assumption (A3) and Lemma 3.5 we conclude from (5.20) and (5.21) that

$$-\partial_t(b_1 * v_1|A^{-1/2}P\vartheta) \le -\frac{1}{2}|P\vartheta|_2^2 + C_1\big(|v_1|_2^2 + |v_2|_2^2 + b_1 * |v_1|_2^2\big), \tag{5.22}$$

for some constant $C_1 > 0$.

Furthermore we have

$$\begin{aligned}-\partial_t(A^{-1}PM(\psi)|A^{-1/2}b_2 * v_2) = &(v_2 - A^{-3/2}P[(\phi''(\psi) + \overline{\lambda(\psi)}\lambda''(\psi))\partial_t\psi]|b_2 * v_2)\\ &+ (A^{-3/2}P\lambda'(\psi)|b_2 * v_2)|\Omega|^{-1}(A^{1/2}P\lambda'(\psi)|v_2)\\ &+ \gamma_2(A^{-3/2}PM(\psi)|b_2 * v_2)\\ &+ (A^{-1}PM(\psi)|A^{-1}\partial_t(k_2 * \partial_t\psi)).\end{aligned} \tag{5.23}$$

By the equation for ψ and due to $\overline{\lambda(\psi) + \vartheta} = 0$, there holds

$$\begin{aligned}A^{-1}\partial_t(k_2 * \partial_t\psi) &= -P\mu\\ &= P(-A\psi - \phi'(\psi) + \lambda'(\psi)\vartheta)\\ &= -PM(\psi) + P(\lambda'(\psi)P\vartheta).\end{aligned} \tag{5.24}$$

Inserting (5.24) into (5.23) yields

$$
\begin{aligned}
-\partial_t(A^{-1}PM(\psi)|A^{-1/2}b_2*v_2) =& (v_2 - A^{-3/2}P[(\phi''(\psi)+\overline{\lambda(\psi)}\lambda''(\psi))\partial_t\psi]|b_2*v_2)\\
&+(A^{-3/2}P\lambda'(\psi)|b_2*v_2)|\Omega|^{-1}(A^{1/2}P\lambda'(\psi)|v_2)\\
&+\gamma_2(A^{-3/2}PM(\psi)|b_2*v_2)\\
&-|A^{-1/2}PM(\psi)|_2^2+(A^{-1}PM(\psi)|P(\lambda'(\psi)P\vartheta)).
\end{aligned}\tag{5.25}
$$

The term $(A^{-3/2}P[(\phi''(\psi)+\overline{\lambda(\psi)}\lambda''(\psi))\partial_t\psi]|b_2*v_2)$ in (5.25) can be rewritten analogously to (5.21). Employing the L_∞-bounds for ψ (cf. Theorem 5.1 and Remark 5.2), we may then proceed similarly as above to obtain

$$
-\partial_t(A^{-1}PM(\psi)|A^{-1/2}b_2*v_2) \le -\frac{1}{2}|A^{-1/2}PM(\psi)|_2^2 + C_2\big(|v_2|_2^2+b_2*|v_2|_2^2+|P\vartheta|_2^2\big), \tag{5.26}
$$

with some constant $C_2>0$.

Define now the function $H_1:\mathbb{R}_+\to\mathbb{R}$ by

$$
H_1(t)=H_0(t)-\delta_1(b_1*v_1|A^{-1/2}P\vartheta)-\delta_2(A^{-1}PM(\psi)|A^{-1/2}b_2*v_2),
$$

where δ_1 and δ_2 are positive constants. Choosing first δ_1 small and then δ_2 even smaller we obtain from the estimates (5.14), (5.22), (5.26), and (5.7) that for a.a. $t>0$

$$
\begin{aligned}
-\frac{d}{dt}H_1 \ge& C_3\Big(|v_1|_2^2+|v_2|_2^2+b_1*|v_1|_2^2+b_2*|v_2|_2^2+|P\vartheta|_2^2+|A^{-1/2}PM(\psi)|_2^2\Big)\\
\ge& C_4\Big(|v_1|_2^2+|v_2|_2^2+b_1*|v_1|_2^2+b_2*|v_2|_2^2+|E'(\vartheta,\psi)|_{W^*}^2\Big),
\end{aligned}\tag{5.27}
$$

where $C_3,C_4>0$ are constants. Thus H_1 is decreasing on $\mathbb{R}_+$. By Lemma 3.5 and Proposition 5.3(iii),

$$
(|b_i*v_i|_2^2)(t)\le |b_i|_{L_1(\mathbb{R}_+)}(b_i*|v_i|_2^2)(t)\to 0\quad \text{as } t\to\infty,\ i=1,2.
$$

From this, the bounds for (ϑ,ψ), and from Proposition 5.3(ii) we then infer that

$$
\lim_{t\to\infty}H_1(t)=\lim_{t\to\infty}H_0(t)=\lim_{t\to\infty}E(\vartheta,\psi)(t)=E_\infty.
$$

Define the function H on $\mathbb{R}_+$ by $H(t)=H_1(t)-E_\infty$. Then H is locally absolutely continuous, nonnegative, and decreasing, and we have $\lim_{t\to\infty}H(t)=0$. If $H(t_0)=0$ for some $t_0\ge 0$, then $H(t)=0$ for all $t\ge t_0$, and hence $v_1=v_2=0$ on $[t_0,\infty)\times\Omega$, in view of (5.27); but this implies that $(\vartheta(t),\psi(t))$ is constant in time for $t\ge t_0$. So we may assume that $H(t)>0$ for all $t\ge 0$.

From the definition of H_1 and (5.7) we deduce by means of Young's inequality and Lemma 3.5 that for all $t\ge 0$

$$
\begin{aligned}
H(t)^{1-\theta}\le& C_5\Big(|E(\vartheta(t),\psi(t))-E_\infty|^{1-\theta}+(b_1*|v_1|_2^2)(t)^{\frac{2(1-\theta)}{2}}+(b_2*|v_2|_2^2)(t)^{\frac{2(1-\theta)}{2}}\\
&+|E'(\vartheta(t),\psi(t))|_{W^*}+(b_1*|v_1|_2^2)(t)^{\frac{1-\theta}{2\theta}}+(b_2*|v_2|_2^2)(t)^{\frac{1-\theta}{2\theta}}\Big),
\end{aligned}
$$

for some constant $C_5>0$. Observe that $\theta\in(0,1/2]$ entails $2(1-\theta)\ge 1$ and $(1-\theta)/\theta\ge 1$. Recall that $(b_i*|v_i|_2^2)(t)\to 0$ as $t\to\infty$ for $i=1,2$ (cf. Proposition 5.3(iii)). Further, since $\lim_{t\to\infty}\mathrm{dist}((\vartheta(t),\psi(t)),\omega(\vartheta,\psi))=0$, there exists a $t_1\ge 0$ such that $(\vartheta(t),\psi(t))\in\mathcal{O}$ for all $t\ge t_1$. Hence using the Łojasiewicz-Simon inequality (5.19) it follows that for all sufficiently large t, say $t\ge t_2\ge t_1$, there holds

$$
H(t)^{1-\theta}\le C_6\Big(|E'(\vartheta(t),\psi(t))|_{W^*}+b_1*|v_1|_2^2(t)^{\frac{1}{2}}+b_2*|v_2|_2^2(t)^{\frac{1}{2}}\Big), \tag{5.28}
$$

for some constant $C_6 > 0$. From (5.27) and (5.28) we then obtain for a.a. $t > t_2$

$$\begin{aligned}
-\frac{d}{dt}\left(H(t)^\theta\right) &= -\,\theta H(t)^{\theta-1}\,\frac{d}{dt}H_1(t)\\
&\geq C_7\,\frac{|v_1(t)|_2^2 + |v_2(t)|_2^2 + (b_1 * |v_1|_2^2)(t) + (b_2 * |v_2|_2^2)(t) + |E'(\vartheta(t),\psi(t))|_{W^*}^2}{(b_1 * |v_1|_2^2)(t)^{1/2} + (b_2 * |v_2|_2^2)(t)^{1/2} + |E'(\vartheta(t),\psi(t))|_{W^*}}\\
&\geq C_8\Big(|v_1(t)|_2 + |v_2(t)|_2 + (b_1 * |v_1|_2^2)(t)^{1/2} + (b_2 * |v_2|_2^2)(t)^{1/2} + |E'(\vartheta(t),\psi(t))|_{W^*}\Big)\\
&\geq C_8\Big(|A^{-1/2}\partial_t\psi|_2 + |A^{-1/2}\partial_t e|_2\Big),
\end{aligned}$$

with some positive constants C_7 and C_8. This shows that

$$A^{-1/2}\partial_t\psi,\ A^{-1/2}\partial_t(\vartheta + \lambda(\psi)) \in L_1(\mathbb{R}_+; L_2(\Omega)). \tag{5.29}$$

Hence the limits $\lim_{t\to\infty} A^{-1/2}\psi(t)$ and $\lim_{t\to\infty} A^{-1/2}e(t)$ exist in $L_2(\Omega)$. By relative compactness of $\psi(\mathbb{R}_+)$ in $\hat{H}_2^1(\Omega)$ and $e(\mathbb{R}_+)$ in $L_2(\Omega)$, respectively, this implies that $\lim_{t\to\infty}\psi(t) =: \psi_\infty$ and $\lim_{t\to\infty} e(t) =: e_\infty$ exist in $\hat{H}_2^1(\Omega)$ and $L_2(\Omega)$, respectively. Hence, as $t \to \infty$,

$$\vartheta(t) = e(t) - \lambda(\psi(t)) \to e_\infty - \lambda(\psi_\infty) =: \vartheta_\infty \quad \text{in } L_2(\Omega),$$

by the assumptions on λ. This shows the convergence of $(\vartheta(t), \psi(t))$ in W as $t \to \infty$. The limit $(\vartheta_\infty, \psi_\infty)$ is a solution of the stationary problem, by Proposition 5.3(iv). The proof of Theorem 2.2 is complete.

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