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CONSTANT-SIGN AND SIGN-CHANGING SOLUTIONS FOR NONLINEAR ELLIPTIC EQUATIONS WITH NEUMANN BOUNDARY VALUES

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Abstract. In this paper we study the existence of multiple solutions to the equation

$$-\Delta_p u = f(x,u) - |u|^{p-2}u$$

with the nonlinear boundary condition

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u + g(x,u).$$

We establish the existence of a smallest positive solution, a greatest negative solution, and a nontrivial sign-changing solution when the parameter λ is greater than the second eigenvalue of the Steklov eigenvalue problem. Our approach is based on truncation techniques and comparison principles for nonlinear elliptic differential inequalities. In particular, we make use of variational and topological tools, such as critical point theory, Mountain-Pass Theorem, Second Deformation Lemma and variational characterization of the second eigenvalue of the Steklov eigenvalue problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the quasilinear elliptic equation

$$\begin{aligned} -\Delta_p u &= f(x,u) - |u|^{p-2}u \qquad &&\text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u + g(x,u), &&\text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $-\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the negative $p-$Laplacian, $\frac{\partial u}{\partial \nu}$ means the outer normal derivative of u with respect to $\partial\Omega$, λ is a real parameter and the nonlinearities $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are some Carathéodory functions. For $u \in W^{1,p}(\Omega)$ defined on the boundary $\partial\Omega$, we make use of the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ which is well known to be compact. For easy readability we will drop the notation $\tau(u)$ and write for short u.

Neumann boundary value problems in the form (1.1) arise in different areas of pure

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and applied mathematics, for example in the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [28],[59]), in the study of optimal constants for the Sobolev trace embedding (see [18],[32],[33],[31]) or at non-Newtonian fluids, flow through porus media, nonlinear elasticity, reaction diffusion problems, glaciology and so on (see [4],[6],[5],[19]).

Our main goal is to provide the existence of multiple solutions of (1.1) meaning that for all values $\lambda > \lambda_2$, where λ_2 denotes the second eigenvalue of $(-\Delta_p, W^{1,p}(\Omega))$ known as the Steklov eigenvalue problem (see, e.g., [35, 49, 56]) given by

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u \qquad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

there exist at least three nontrivial solutions. More precisely, we obtain two constant-sign solutions and one sign-changing solution of problem (1.1). This is the main result of the present paper and it is formulated in the Theorems 4.3 and 6.3, respectively. In our consideration, the nonlinearities f and g only need to be Carathéodory functions which are bounded on bounded sets whereby their growth does not need to be necessarily polynomial. We only require some growth properties at zero and infinity given by

$$\lim_{s\to 0}\frac{f(x,s)}{|s|^{p-2}s} = \lim_{s\to 0}\frac{g(x,s)}{|s|^{p-2}s} = 0, \qquad \lim_{|s|\to\infty}\frac{f(x,s)}{|s|^{p-2}s} = \lim_{|s|\to\infty}\frac{g(x,s)}{|s|^{p-2}s} = -\infty$$

and we suppose the existence of $\delta_f > 0$ such that $f(x,s)/|s|^{p-2}s \geq 0$ for all $0 < |s| \leq \delta_f$.

In the last years many papers about the existence of the Neumann problems like the form (1.1) were developed (see, e.g., [3, 17, 30, 34, 48, 68]). Martínez et al [48] proved the existence of weak solutions of the Neumann boundary problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}u + f(x,u) \qquad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u - h(x,u), \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where the perturbations $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $h : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are bounded Carathéodory functions satisfying an integral condition of Landesmann-Lazer type. Their main result is given in [48, Theorem 1.2] which yields the existence of a weak solution of (1.3) with $\lambda = \lambda_1$, where λ_1 is the first eigenvalue of the Steklov eigenvalue problem (see (1.2)). Moreover, they supposed in their main theorem the boundedness of $f(x,t)$ and $h(x,t)$ by functions $\overline{f} \in L^q(\Omega)$ and $\overline{h} \in L^q(\partial\Omega)$ for all $(x,t) \in \Omega \times \mathbb{R}$ and $(x,t) \in \partial\Omega \times \mathbb{R}$, respectively. A similar work of (1.1) can be found in [31]. There the authors get as well three nontrivial solutions for the nonlinear boundary value problem

$$\begin{aligned} -\Delta_p u + |u|^{p-2}u &= f(x,u) \qquad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= g(x,u), \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.4}$$

where they assume among others that the Carathéodory functions f and g are also continuously differentiable in the second argument. The proof is based on the Lusternik-Schnirelmann method for non-compact manifolds. If the Neumann boundary values are defined by a function $f : \mathbb{R} \to \mathbb{R}$ meaning the problem

$$\begin{aligned} \Delta_p u &= |u|^{p-2}u \qquad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= f(u), \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.5}$$

we refer to the results of J. Fernández Bonder and J.D. Rossi in [34]. They consider various cases where f has subcritical growth, critical growth and supercritical growth, respectively. In the first two cases the existence of infinitely many solutions under some conditions to the exponents of the growth were demonstrated.
Another result to obtain multiple solutions with nonlinear boundary conditions can be found in the paper of J.H. Zhao and P.-H. Zhao [68]. They study the equation

$$\begin{aligned} -\Delta_p u + \lambda(x)|u|^{p-2}u &= f(x,u) \qquad \text{in } \Omega \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \eta|u|^{p-2}u, \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where $\lambda(x) \in L^\infty(\Omega)$ satisfying ess $\inf_{x\in\overline{\Omega}} \lambda(x) > 0$ and η is a real parameter. They prove the existence of infinitely many solutions when f is superlinear and subcritical with respect to u by using the fountain theorem and the dual fountain theorem, respectively. In case that f has the form $f(x,u) = |u|^{p^*-2}u + |u|^{r-2}u$ they get at least one nontrivial solution when $p < r < p^*$ and infinitely many solutions when $1 < r < p$ by using the Mountain-Pass Theorem and the "concentration-compactness principle", respectively. A similar result of the same authors is also developed in [67]. The existence of multiple solutions and sign-changing solutions for zero Neumann boundary values have been proven in [44, 54, 55, 65] and [68], respectively. Analog results for the Dirichlet problem have been recently obtained in [10, 11, 12, 13, 26, 50, 51]. An interesting problem about the existence of multiple solutions for both, the Dirichlet problem and the Neumann problem, can be found in [15]. The authors study the existence of multiple solutions to the abstract equation $J_p u = N_f u$, where J_p is the duality mapping on a real reflexive and smooth Banach space X, corresponding to the gauge function $\varphi(t) = t^{p-1}, 1 < p < \infty$ and $N_f : L^q(\Omega) \to L^{q'}(\Omega), 1/q + 1/q' = 1$, is the Nemytskij operator generated by a function $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$.
The novelty of our paper is the fact that we do not need differentiability, polynomial growth or some integral conditions on the mappings f and g. In order to prove our main results we make use of variational and topological tools, e.g. critical point theory, Mountain-Pass Theorem, Second Deformation Lemma and variational characterization of the second eigenvalue of the Steklov eigenvalue problem. This paper is motivated by recent publications of S. Carl and D. Motreanu in [12] and [11], respectively. In [12] the authors consider the Dirichlet problem $-\Delta_p u = \lambda|u|^{p-2}u + g(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, and show the existence of at least three nontrivial solutions for all values $\lambda > \lambda_2$, where λ_2 denotes the second eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$. Therein, the main theorem about the existence of a sign-changing solution is also based on the Mountain-Pass Theorem and the Second Deformation Lemma. These results have been extended by themselves to the equation $-\Delta_p u = a(u^+)^{p-1} - b(u^-)^{p-1} + g(x,u)$ in Ω, $u = 0$ on $\partial\Omega$, where $u^+ = \max\{u, 0\}$ and $u^- = \max\{-u, 0\}$ denote the positive and negative part of u, respectively. Carl et al have shown that at least three nontrivial solutions exist provided the value (a, b) is above the first nontrivial curve $\mathcal{C}$ of the Fŭcik spectrum constructed by Cuesta et al in [16].

The rest of the paper is organized as follows. In Section 2 and Section 3, we recall some preliminaries and formulate our notations and hypothesis, respectively. In Section 4, we will show the existence of specific sub- and supersolutions of problem (1.1), then we will prove that every solution between these pairs of sub- and supersolutions belongs to $\operatorname{int}(C^1(\overline{\Omega})_+)$ and finally we will provide the existence of extremal constant-sign solutions. A variational characterization of these extremal solutions is given in Section 5 and our main result about the existence of a nontrivial sign-changing solution is proven in the last section by using the Mountain-Pass Theorem.

2. Preliminaries

Let us consider some nonlinear boundary value problems with Neumann conditions involving the p-Laplacian. In [47] the authors study the Steklov problem

$$\begin{aligned} -\Delta_p u &= -|u|^{p-2}u \qquad \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &= \lambda |u|^{p-2}u \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.1}$$

The trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is linear bounded (and even compact), thus a best constant λ_1 exists such that

$$\lambda_1^{1/p}\|u\|_{L^p(\partial\Omega)} \leq \|u\|_{W^{1,p}(\Omega)}.$$

The best Sobolev trace constant λ_1 can be characterized as

$$\lambda_1 = \inf_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega [|\nabla u|^p + |u|^p] dx \text{ such that } \int_{\partial\Omega} |u|^p d\sigma = 1 \right\},$$

and λ_1 is the first eigenvalue of (2.1). Martínez et al. showed that the first eigenvalue $\lambda_1 > 0$ is isolated and simple. The corresponding eigenfunction φ_1 is strictly positive in $\overline{\Omega}$ and belongs to $L^\infty(\Omega)$ (cf. [43, Lemma 5.6 and Theorem 4.3]). Applying the results of Lieberman in [45, Theorem 2] implies $\varphi_1 \in C^{1,\alpha}(\overline{\Omega})$. This fact along with $\varphi_1(x) > 0$ in $\overline{\Omega}$ yields $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$, where $\operatorname{int}(C^1(\overline{\Omega})_+)$ denotes the interior of the positive cone $C^1(\overline{\Omega})_+ = \{u \in C^1(\overline{\Omega}) : u(x) \geq 0, \forall x \in \Omega\}$ in the Banach space $C^1(\overline{\Omega})$, given by

$$\operatorname{int}(C^1(\overline{\Omega})_+) = \left\{u \in C^1(\overline{\Omega}) : u(x) > 0, \forall x \in \overline{\Omega}\right\}.$$

The study of Neumann eigenvalue problems with or without weights are also considered in [17, 27, 41, 43, 60]. Analog to the results for the Dirichlet eigenvalue problem (see [16]), there also exists a variational characterization of the second eigenvalue of (2.1) meaning that λ_2 can be represented as follows

$$\lambda_2 = \inf_{g \in \Gamma} \max_{u \in g([-1,1])} \int_\Omega \Big(|\nabla u|^p + |u|^p\Big) dx, \tag{2.2}$$

where

$$\Gamma = \{g \in C([-1,1], S) \mid g(-1) = -\varphi_1, g(1) = \varphi_1\}, \tag{2.3}$$

and

$$S = \left\{u \in W^{1,p}(\Omega) : \int_{\partial\Omega} |u|^p d\sigma = 1\right\}. \tag{2.4}$$

The proof of this result can be found in [49]. In our considerations we make use of the following strong maximum principle proven by Vázguez in [63].

Theorem 2.1 (Vázguez's strong maximum principle). *Let $u \in C^1(\Omega)$ such that*

(1) $\Delta_p u \in L^2_{\mathrm{loc}}(\Omega)$,

(2) $u \geq 0$ a.e. in Ω and $u \not\equiv 0$ in Ω,

(3) $\Delta_p u \leq \beta(u)$ a.e. in Ω with $\beta : [0, \infty) \to \mathbb{R}$ continuous, nondecreasing, $\beta(0) = 0$ and either
  (i) $\beta(s) = 0$ for some $s > 0$ or,
  (ii) $\beta(s) > 0$ for all $s > 0$ with $\int_0^1 (\beta(s)s)^{-1/p} ds = +\infty$.

Then it holds

$$u(x) > 0 \text{ a.e. in } \Omega.$$

Moreover, if $u \in C^1(\Omega \cup x_0)$ for an $x_0 \in \partial\Omega$ satisfying an interior sphere condition and $u(x_0) = 0$, then

$$\frac{\partial u}{\partial \nu}(x_0) < 0,$$

where ν is the outer normal derivative of u at $x_0 \in \partial\Omega$.

We recall that a point $x_0 \in \partial\Omega$ satisfies the interior sphere condition if there exists an open ball $B = B_R(x_1) \subset \Omega$ such that $\overline{B} \cap \partial\Omega = \{x_0\}$. Then one can choose a unit vector

$$\nu = (x_0 - x_1)/|x_0 - x_1|,$$

and ν is a normal to ∂B at x_0 pointing outside. A sufficient condition to satisfy the interior sphere condition is a C^2-boundary. Now we consider solutions of the Neumann boundary value problem

$$\begin{aligned} -\Delta_p u &= -\varsigma |u|^{p-2} u + 1 && \text{in } \Omega, \\ |\nabla u|^{p-2} \frac{\partial u}{\partial \nu} &= 1 && \text{on } \partial\Omega, \end{aligned} \tag{2.5}$$

where $\varsigma > 1$ is a constant. Let $B : L^p(\Omega) \to L^q(\Omega)$ be the Nemytskij operator defined by $Bu(x) := \varsigma |u(x)|^{p-2} u(x)$. It is well known that $B : L^p(\Omega) \to L^q(\Omega)$ is bounded and continuous. We set $\widehat{B} := i^* \circ B \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$, where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ is the adjoint operator of the compact embedding $i : W^{1,p}(\Omega) \to L^p(\Omega)$. The operator $\widehat{B}$ is bounded, continuous, completely continuous and thus, also pseudomonotone. We denote by $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ the trace operator and with $\tau^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ its adjoint operator. The weak formulation of (2.5) is given by

$$u \in W^{1,p}(\Omega) : \ \langle -\Delta_p u + \widehat{B} u - i^*(1) - \tau^*(1), \varphi \rangle = 0, \ \forall \varphi \in W^{1,p}(\Omega), \tag{2.6}$$

meaning

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \varphi dx + \varsigma \int_\Omega |u|^{p-2} u \varphi dx - \int_\Omega \varphi dx - \int_{\partial\Omega} \varphi d\sigma = 0, \ \forall \varphi \in W^{1,p}(\Omega),$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and its dual space $(W^{1,p}(\Omega))^*$. The negative p-Laplacian $-\Delta_p$ is pseudomonotone and therefore, the sum $-\Delta_p + \widehat{B}$ is pseudomonotone. The coercivity of $-\Delta_p + \widehat{B}$ follows directly and thus, using classical existence results implies the existence of a solution of problem

(2.5). Let e_1, e_2 be solutions of (2.5) satisfying $e_1 \neq e_2$. Subtracting the corresponding weak formulation of (2.5) with respect to e_1, e_2 and taking $\varphi = e_1 - e_2$ yields

$$\begin{aligned}&\int_\Omega [|\nabla e_1|^{p-2}\nabla e_1 - |\nabla e_2|^{p-2}\nabla e_2]\nabla(e_1 - e_2)dx\\&+\varsigma\int_\Omega [|e_1|^{p-2}e_1 - |e_2|^{p-2}e_2](e_1 - e_2)dx = 0.\end{aligned}$$

As the left-hand-side is strictly positive for $e_1 \neq e_2$, we obtain a contradiction and thus, $e_1 = e_2$. Let e be the unique solution of (2.5) in the weak sense. Choosing the test function $\varphi = e^- = \max\{-e, 0\} \in W^{1,p}(\Omega)$ results in

$$-\int_{\{x\in\Omega:e(x)<0\}} |\nabla e|^p dx - \varsigma\int_{\{x\in\Omega:e(x)<0\}} |e|^p dx = \int_\Omega e^- dx + \int_{\partial\Omega} e^- d\sigma \geq 0,$$

which proves that e is nonnegative. Notice that e is not identically zero. Applying the Moser Iteration (cf. [25],[43] or see the proof of Proposition 5.3) yields $e \in L^\infty(\Omega)$ and thus, the regularity results of Lieberman (see [45, Theorem 2]) ensure $e \in C^{1,\alpha}(\overline{\Omega})$. From (2.5) we conclude

$$\Delta_p e = \varsigma|e|^{p-2}e - 1 \leq \varsigma e^{p-1} \text{ a.e. in } \Omega.$$

Setting $\beta(s) = \varsigma s^{p-1}$ for $s > 0$ allows us to apply Vázguez's strong maximum principle stated in Theorem 2.1 which is possible since $\int_{0^+} \frac{1}{(s\beta(s))^{1/p}} ds = +\infty$. This shows that $e(x) > 0$ for a.a. $x \in \Omega$. If there exists $x_0 \in \partial\Omega$ such that $e(x_0) = 0$, we obtain by applying again Vázguez's strong maximum principle that $\frac{\partial e}{\partial\nu}(x_0) < 0$, which is a contradiction since $|\nabla e|^{p-2}\frac{\partial e}{\partial\nu}(x_0) = 1$. Hence, $e(x) > 0$ in $\overline{\Omega}$ and therefore, we get $e \in \operatorname{int}(C^1(\overline{\Omega})_+)$.

The following theorem is an important theorem to prove the existence of minimum points of weakly coercive functionals (cf. [66, Theorem 25.D]).

Theorem 2.2 (Main Theorem on Weakly Coercive Functionals). *Suppose that the functional $f : M \subseteq X \to \mathbb{R}$ has the following three properties:*

(1) M is a nonempty closed convex set in the reflexive Banach space X.
(2) f is weakly sequentially lower semicontinuous on M.
(3) f is weakly coercive.

Then f has a minimum on M.

A criterion for the weak sequential lower semicontinuity of C^1-functionals can be read as follows. For more details we refer to Zeidler [66, Proposition 25.21].

Proposition 2.3. *Let $f : M \subseteq X \to \mathbb{R}$ be a C^1-functional on the open convex set M of the real Banach space X, and let f' be pseudomonotone and bounded. Then, f is weakly sequentially lower semicontinuous on M.*

A significant tool in the proof for the existence of a nontrivial sign-changing solution is the following Mountain Pass Theorem (see [57]). First, we give the definition of the Palais-Smale-Condition.

Definition 2.4 (Palais-Smale-Condition). Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. The functional I is said to satisfy the Palais-Smale-Condition if for each sequence $(u_n) \subset E$ which fulfills

(1) $I(u_n)$ is bounded,
(2) $I'(u_n) \to 0$ as $n \to \infty$,

there exists a strong convergent subsequence of (u_n).

Theorem 2.5 (Mountain-Pass Theorem). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfying the Palais-Smale-Condition. Suppose*

(I_1) there are constants $\rho, \alpha > 0$ and an $e_1 \in E$ such that $I_{\partial B_\rho(e_1)} \geq \alpha$, and
(I_2) there is an $e_2 \in E \setminus \overline{B_\rho(e_1)}$ such that $I(e_2) \leq I(e_1) < \alpha$.

Then I possesses a critical value c corresponding to a critical point u_0 such that $I(u_0) = c \geq \alpha$. Moreover c can be characterized as

$$c = \inf_{g \in \Gamma} \max_{u \in g([-1,1])} I(u), \tag{2.7}$$

where

$$\Gamma = \{g \in C([-1,1], E) \mid g(-1) = e_1, g(1) = e_2\}.$$

3. Notations and hypotheses

We impose the following conditions on the nonlinearities f and g in problem (1.1). The mappings $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions (that is, measurable in the first argument and continuous in the second argument) such that

(f1) $\displaystyle\lim_{s \to 0} \frac{f(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$.
(f2) $\displaystyle\lim_{|s| \to \infty} \frac{f(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \Omega$.
(f3) f is bounded on bounded sets.
(f4) There exists $\delta_f > 0$ such that $\dfrac{f(x,s)}{|s|^{p-2}s} \geq 0$ for all $0 < |s| \leq \delta_f$.
(g1) $\displaystyle\lim_{s \to 0} \frac{g(x,s)}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \partial\Omega$.
(g2) $\displaystyle\lim_{|s| \to \infty} \frac{g(x,s)}{|s|^{p-2}s} = -\infty$, uniformly with respect to a.a. $x \in \partial\Omega$.
(g3) g is bounded on bounded sets.
(g4) g is locally Hölder continuous in $\partial\Omega \times \mathbb{R}$, that is,

$$|g(x_1, s_1) - g(x_2, s_2)| \leq L\Big[|x_1 - x_2|^\alpha + |s_1 - s_2|^\alpha\Big],$$

for all pairs $(x_1, s_1), (x_2, s_2)$ in $\partial\Omega \times [-M_0, M_0]$, where M_0 is a positive constant and $\alpha \in (0, 1]$.

Note that the function $s \mapsto |s|^{p-2}s$ is locally Hölder continuous in $\mathbb{R}$. This implies in view of (g4) that the mapping $\Phi : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by $\Phi(x,s) := \lambda|s|^{p-2}s + g(x,s)$ is locally Hölder continuous in $\partial\Omega \times \mathbb{R}$. Recall that we write $g(x, u(x)) := g(x, \tau(u(x)))$ for $u \in W^{1,p}(\Omega)$, where $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ stands for the trace operator. With a view to the conditions (f1) and (g1), we see at once that $f(x,0) = g(x,0) = 0$ and thus, $u = 0$ is a trivial solution of problem (1.1).

Corollary 3.1. *Let (f1),(f3) and (g1),(g3) be satisfied. Then, for each $a > 0$ there exist constants $b_1, b_2 > 0$ such that*

$$\begin{aligned} |f(x,s)| &\le b_1|s|^{p-1}, \ \forall s : 0 \le |s| \le a, \\ |g(x,s)| &\le b_2|s|^{p-1}, \ \forall s : 0 \le |s| \le a. \end{aligned} \tag{3.1}$$

Proof. The assumption (f1) implies that for each $c_1 > 0$ there exists $\delta > 0$ such that

$$|f(x,s)| \le c_1|s|^{p-1}, \ \forall s : 0 \le |s| \le \delta. \tag{3.2}$$

Due to condition (f3), there exists a constant $c_2 > 0$ such that for a given $a > 0$ holds

$$|f(x,s)| \le c_2, \ \forall s : 0 \le |s| \le a. \tag{3.3}$$

If $\delta > a$, then inequality (3.2), in particular, implies

$$|f(x,s)| \le b_1|s|^{p-1}, \ \forall s : 0 \le |s| \le a,$$

where $b_1 := c_1$. Let us assume $\delta < a$. From (3.3) we obtain

$$|f(x,s)| \le \frac{c_2}{\delta^{p-1}}|s|^{p-1}, \ \forall s : \delta \le |s| \le a, \tag{3.4}$$

and thus, combining (3.2) and (3.4) yields

$$|f(x,s)| \le (c_1 + \frac{c_2}{\delta^{p-1}})|s|^{p-1}, \ \forall s : 0 \le |s| \le a,$$

where the setting $b_1 := c_1 + \frac{c_2}{\delta^{p-1}}$ proves (3.1). In the same way, one shows the assertion for g. $\square$

Example 3.2. *Consider the functions $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ defined by*

$$f(x,s) = \begin{cases} |s|^{p-2}s(1 + (s+1)e^{-s}) & \text{if } s \le -1 \\ \operatorname{sgn}(s)\dfrac{|s|^p}{2}(|(s-1)\cos(s+1)| + s + 1) & \text{if } -1 \le s \le 1 \\ s^{p-1}e^{1-s} - |x|(s-1)s^{p-1}e^s & \text{if } s \ge 1, \end{cases}$$

and

$$g(x,s) = \begin{cases} |s|^{p-2}s(s + 1 + e^{s+1}) & \text{if } s \le -1 \\ |s|^{p-1}se^{(s^2+1)\sqrt{|x|}} & \text{if } -1 \le s \le 1 \\ s^{p-1}(\cos(1-s) + (1-s)e^s) & \text{if } s \ge 1. \end{cases}$$

One verifies that all assumptions (f1)-(f4) and (g1)-(g4) are satisfied.

The definition of a solution of problem (1.1) in the weak sense is defined as follows.

Definition 3.3. A function $u \in W^{1,p}(\Omega)$ is called a solution of (1.1) if the following holds:

$$\begin{aligned}&\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi dx\\&= \int_\Omega (f(x,u) - |u|^{p-2}u)\varphi dx + \int_{\partial\Omega} (\lambda |u|^{p-2}u + g(x,u))\varphi d\sigma,\ \forall \varphi \in W^{1,p}(\Omega).\end{aligned}$$

Next, we recall the notations of sub- and supersolutions of problem (1.1).

Definition 3.4. A function $\underline{u} \in W^{1,p}(\Omega)$ is called a subsolution of (1.1) if the following holds:

$$\begin{aligned}&\int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \nabla \varphi dx\\&\leq \int_\Omega (f(x,\underline{u}) - |\underline{u}|^{p-2}\underline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\underline{u}|^{p-2}\underline{u} + g(x,\underline{u}))\varphi d\sigma,\ \forall \varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

Definition 3.5. A function $\overline{u} \in W^{1,p}(\Omega)$ is called a supersolution of (1.1) if the following holds:

$$\begin{aligned}&\int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \nabla \varphi dx\\&\geq \int_\Omega (f(x,\overline{u}) - |\overline{u}|^{p-2}\overline{u})\varphi dx + \int_{\partial\Omega} (\lambda |\overline{u}|^{p-2}\overline{u} + g(x,\overline{u}))\varphi d\sigma,\ \forall \varphi \in W^{1,p}(\Omega)_+.\end{aligned}$$

Here, $W^{1,p}(\Omega)_+ := \{\varphi \in W^{1,p}(\Omega) : \varphi \geq 0\}$ stands for all nonnegative functions of $W^{1,p}(\Omega)$. Recall that if $u \in W^{1,p}(\Omega)$ satisfies $v \leq u \leq w$, where v, w are some functions in $W^{1,p}(\Omega)$, then it holds $\tau(v) \leq \tau(u) \leq \tau(w)$, where $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ denotes the trace operator.

4. Extremal constant-sign solutions

We start by generating two ordered pairs of sub- and supersolutions of problem (1.1) having constant signs. Here and in the following we denote by $\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ the first eigenfunction of the Steklov eigenvalue problem (2.1) corresponding to the first eigenvalue λ_1.

Lemma 4.1. *Assume (f1)–(f4), (g1)–(g4) and $\lambda > \lambda_1$ and let e be the unique solution of problem (2.5). Then there exists a constant $\vartheta > 0$ such that ϑe and $-\vartheta e$ are supersolution and subsolution, respectively, of problem (1.1). In addition, $\varepsilon\varphi_1$ is a subsolution and $-\varepsilon\varphi_1$ is a supersolution of problem (1.1) provided the number $\varepsilon > 0$ is sufficiently small.*

Proof. Let $\underline{u} = \varepsilon\varphi_1$, where ε is a positive constant specified later. In view of the Steklov eigenvalue problem (2.1) it holds

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\&= -\int_\Omega (\varepsilon\varphi_1)^{p-1}\varphi dx + \int_{\partial\Omega} \lambda_1(\varepsilon\varphi_1)^{p-1}\varphi d\sigma, \forall \varphi \in W^{1,p}(\Omega).\end{aligned} \tag{4.1}$$

We are going to prove that Definition 3.4 is satisfied for $\underline{u} = \varepsilon\varphi_1$ meaning that the inequality

$$\begin{aligned}&\int_\Omega |\nabla(\varepsilon\varphi_1)|^{p-2}\nabla(\varepsilon\varphi_1)\nabla\varphi dx\\&\leq \int_\Omega (f(x,\varepsilon\varphi_1) - (\varepsilon\varphi_1)^{p-1})\varphi dx + \int_{\partial\Omega}(\lambda(\varepsilon\varphi_1)^{p-1} + g(x,\varepsilon\varphi_1))\varphi d\sigma,\end{aligned} \tag{4.2}$$

is valid for all $\varphi \in W^{1,p}(\Omega)_+$. Therefore, (4.2) is fulfilled provided the following holds true

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx + \int_{\partial\Omega}((\lambda_1-\lambda)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma \leq 0,\ \forall\varphi\in W^{1,p}(\Omega)_+.$$

Condition (f4) implies for $\varepsilon \in (0, \delta_f/\|\varphi_1\|_\infty]$

$$\int_\Omega -f(x,\varepsilon\varphi_1)\varphi dx = \int_\Omega -\frac{f(x,\varepsilon\varphi_1)}{(\varepsilon\varphi_1)^{p-1}}(\varepsilon\varphi_1)^{p-1}\varphi dx \leq 0,$$

where $\|\cdot\|_\infty$ stands for the supremum norm. Due to assumption (g1) there exists a number $\delta_\lambda > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} < \lambda - \lambda_1 \text{ for a.a. } x\in\partial\Omega \text{ and all } 0<|s|\leq\delta_\lambda.$$

If $\varepsilon \in \left(0, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}\right]$, we get

$$\begin{aligned}\int_{\partial\Omega}((\lambda_1-\lambda)(\varepsilon\varphi_1)^{p-1} - g(x,\varepsilon\varphi_1))\varphi d\sigma &\leq \int_{\partial\Omega}\left(\lambda_1 - \lambda + \frac{|g(x,\varepsilon\varphi)|}{(\varepsilon\varphi_1)^{p-1}}\right)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma\\&< \int_{\partial\Omega}(\lambda_1-\lambda+\lambda-\lambda_1)(\varepsilon\varphi_1)^{p-1}\varphi d\sigma\\&= 0.\end{aligned}$$

Choosing $0 < \varepsilon \leq \min\{\delta_f/\|\varphi_1\|_\infty, \delta_\lambda/\|\varphi_1\|_\infty\}$ proves that $\underline{u} = \varepsilon\varphi_1$ is a positive subsolution. In a similar way one proves that $\overline{u} = -\varepsilon\varphi_1$ is a negative supersolution. Let $\overline{u} = \vartheta e$, where ϑ is a positive constant specified later. From the auxiliary problem (2.5) we conclude

$$\begin{aligned}&\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx\\&= -\varsigma\int_\Omega(\vartheta e)^{p-1}\varphi dx + \int_\Omega \vartheta^{p-1}\varphi dx + \int_{\partial\Omega}\vartheta^{p-1}\varphi d\sigma, \forall\varphi\in W^{1,p}(\Omega).\end{aligned} \tag{4.3}$$

In order to fulfill the assertion of the lemma, we have to show the validity of Definition 3.5 for $\overline{u} = \vartheta e$ meaning that for all $\varphi \in W^{1,p}(\Omega)_+$ holds

$$\begin{aligned}&\int_\Omega |\nabla(\vartheta e)|^{p-2}\nabla(\vartheta e)\nabla\varphi dx\\&\geq \int_\Omega(f(x,\vartheta e) - (\vartheta e)^{p-1})\varphi dx + \int_{\partial\Omega}(\lambda(\vartheta e)^{p-1} + g(x,\vartheta e))\varphi d\sigma.\end{aligned} \tag{4.4}$$

With a view to (4.3) we see at once that inequality (4.4) is satisfied if the following holds

$$\begin{aligned}&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x,\vartheta e))\varphi dx\\&\quad + \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x,\vartheta e))\varphi d\sigma \geq 0,\end{aligned} \tag{4.5}$$

where $\widetilde{c} = \varsigma - 1$ with $\widetilde{c} > 0$. By (f2) there exists $s_\varsigma > 0$ such that

$$\frac{f(x,s)}{s^{p-1}} < -\widetilde{c}, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\varsigma,$$

and by (f3) we have

$$|-f(x,s) - \widetilde{c}s^{p-1}| \leq |f(x,s)| + \widetilde{c}s^{p-1} \leq c_\varsigma, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\varsigma].$$

Thus, we get

$$f(x,s) \leq -\widetilde{c}s^{p-1} + c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \geq 0. \tag{4.6}$$

Applying (4.6) to the first integral in (4.5) yields

$$\begin{aligned}&\int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} - f(x,\vartheta e))\varphi dx\\&\geq \int_\Omega (\vartheta^{p-1} - \widetilde{c}(\vartheta e)^{p-1} + \widetilde{c}(\vartheta e)^{p-1} - c_\varsigma)\varphi dx\\&= \int_\Omega (\vartheta^{p-1} - c_\varsigma)\varphi dx,\end{aligned}$$

which shows that for $\vartheta \geq c_\varsigma^{\frac{1}{p-1}}$ the integral is nonnegative. Due to hypothesis (g2) there is $s_\lambda > 0$ such that

$$\frac{g(x,s)}{s^{p-1}} < -\lambda, \quad \text{for a.a. } x \in \Omega \text{ and all } s > s_\lambda.$$

Assumption (g3) ensures the existence of a constant $c_\lambda > 0$ such that

$$|-g(x,s) - \lambda s^{p-1}| \leq |g(x,s)| + \lambda s^{p-1} \leq c_\lambda, \quad \text{for a.a. } x \in \Omega \text{ and all } s \in [0, s_\lambda].$$

We obtain

$$g(x,s) \leq -\lambda s^{p-1} + c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \geq 0. \tag{4.7}$$

Using (4.7) to the second integral in (4.5) provides

$$\begin{aligned}&\int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} - g(x,\vartheta e))\varphi dx\\&\geq \int_{\partial\Omega} (\vartheta^{p-1} - \lambda(\vartheta e)^{p-1} + \lambda(\vartheta e)^{p-1} - c_\lambda)\varphi dx\\&\geq \int_{\partial\Omega} (\vartheta^{p-1} - c_\lambda)\varphi dx.\end{aligned}$$

Choosing $\vartheta := \max\left\{c_\varsigma^{\frac{1}{p-1}}, c_\lambda^{\frac{1}{p-1}}\right\}$ proves that both integrals in (4.5) are nonnegative and thus, $\overline{u} = \vartheta e$ is a positive supersolution of problem (1.1). In order to prove that $\underline{u} = -\vartheta e$ is a negative subsolution we make use of the following estimates

$$\begin{aligned} f(x,s) &\geq -\widetilde{c}s^{p-1} - c_\varsigma, \text{ for a.a. } x \in \Omega \text{ and all } s \leq 0, \\ g(x,s) &\geq -\lambda s^{p-1} - c_\lambda, \text{ for a.a. } x \in \partial\Omega \text{ and all } s \leq 0. \end{aligned} \tag{4.8}$$

which can be derivated as stated above. With the aid of (4.8) one verifies that $\underline{u} = -\vartheta e$ is a negative subsolution of problem (1.1). □

According to Lemma 4.1 we obtain a positive pair $[\varepsilon\varphi_1, \vartheta e]$ and a negative pair $[-\vartheta e, -\varepsilon\varphi_1]$ of sub- and supersolutions of problem (1.1) assumed $\varepsilon > 0$ is sufficiently small.
The next lemma will prove the $C^{1,\alpha}$ regularity of solutions of problem (1.1) lying in the order interval $[0, \vartheta e]$ and $[-\vartheta e, 0]$, respectively. Note that $\underline{u} = \overline{u} = 0$ is both, a subsolution and a supersolution due to the assumptions (f1) and (g1). In the following proof we make use of the regularity results of Lieberman (see [45]) and Vázguez in [63]. To obtain regularity results, in particular for elliptic Neumann problems, we refer also to the papers of Tolksdorf in [59] and DiBenedetto in [20].

Lemma 4.2. *Let the conditions (f1)–(f4) and (g1)–(g4) be satisfied and let $\lambda > \lambda_1$. If $u \in [0, \vartheta e]$ (respectively, $u \in [-\vartheta e, 0]$) is a solution of problem (1.1) satisfying $u \not\equiv 0$ in Ω, then it holds $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (respectively, $u \in -\operatorname{int}(C^1(\overline{\Omega})_+)$).*

Proof. Let u be a solution of (1.1) such that $0 \leq u \leq \vartheta e$. Then it follows $u \in L^\infty(\Omega)$ and thus, $u \in C^{1,\alpha}(\overline{\Omega})$ by Lieberman [45, Theorem 2] (see also Fan [29]). The conditions (f1),(f3),(g1) and (g3) (cf. Corollary 3.1) imply the existence of constants $c_f, c_g > 0$ such that

$$\begin{aligned} |f(x,s)| &\leq c_f s^{p-1} \ \text{ for a.a. } x \in \Omega \text{ and all } 0 \leq s \leq \vartheta \|e\|_\infty, \\ |g(x,s)| &\leq c_g s^{p-1} \ \text{ for a.a. } x \in \partial\Omega \text{ and all } 0 \leq s \leq \vartheta \|e\|_\infty. \end{aligned} \tag{4.9}$$

Applying the first line in (4.9) along with (1.1) yields $\Delta_p u \leq \widetilde{c} u^{p-1}$ a.e. in Ω, where $\widetilde{c}$ is a positive constant. This allows us to apply Vázguez's strong maximum principle (see [63, Theorem 5]). We take $\beta(s) = \widetilde{c}s^{p-1}$ for all $s > 0$ which is possible because $\int_{0^+} \frac{1}{(s\beta(s))^{\frac{1}{p}}} ds = +\infty$. We get $u > 0$ in Ω. Let us assume there exists $x_0 \in \partial\Omega$ such that $u(x_0) = 0$. By applying again the maximum principle we obtain $\frac{\partial u}{\partial \nu}(x_0) < 0$. But taking into account $g(x_0, u(x_0)) = g(x_0, 0) = 0$ along with the Neumann condition in (1.1) yields $\frac{\partial u}{\partial \nu}(x_0) = 0$, which is a contradiction. Thus, $u > 0$ in $\overline{\Omega}$ which proves $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. The proof in case $u \in [-\vartheta e, 0]$ can be shown in an analogous manner. □

The result of the existence of extremal constant-sign solutions is read as follows.

Theorem 4.3. *Assume (f1)–(f4) and (g1)–(g4). Then for every $\lambda > \lambda_1$ there exists a smallest positive solution $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[0, \vartheta e]$ and a greatest negative solution $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$ in the order interval $[-\vartheta e, 0]$ with $\vartheta > 0$ stated in Lemma 4.1.*

Proof. We fix $\lambda > \lambda_1$. On the basis of Lemma 4.1, there exists an ordered pair of a positive supersolution $\overline{u} = \vartheta e \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and a positive subsolution $\underline{u} = \varepsilon\varphi_1 \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) assuming $\varepsilon > 0$ is sufficiently small such that $\varepsilon\varphi_1 \leq \vartheta e$. The method of sub- and supersolution (see [9]) with respect to the order interval $[\varepsilon\varphi_1, \vartheta e]$ implies the existence of a smallest positive solution $u_\varepsilon = u_\varepsilon(\lambda)$ of problem (1.1) satisfying $\varepsilon\varphi_1 \le u_\varepsilon \le \vartheta e$ which ensures $u_\varepsilon \in \operatorname{int}(C^1(\overline{\Omega})_+)$ (see Lemma 4.2). Hence, for every positive integer n sufficiently large there exists a smallest solution $u_n \in \operatorname{int}(C^1(\overline{\Omega})_+)$ of problem (1.1) in the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$ and therefore, we have

$$u_n \downarrow u_+ \text{ for almost all } x \in \Omega, \tag{4.10}$$

where $u_+ : \Omega \to \mathbb{R}$ is some function satisfying $0 \le u_+ \le \vartheta e$. We are going to show that u_+ is a solution of problem (1.1). Since u_n belongs to the order interval $[\frac{1}{n}\varphi_1, \vartheta e]$, it follows that u_n is bounded in $L^p(\Omega)$. Moreover, we obtain the boundedness of u_n in $L^p(\partial\Omega)$ because $\tau(u_n) \le \tau(\vartheta e)$. As u_n solves (1.1) in the weak sense, one has by setting $\varphi = u_n$ in Definition 3.3

$$\begin{aligned} \|\nabla u_n\|^p_{L^p(\Omega)} &\le \int_\Omega |f(x,u_n)|u_n dx + \|u_n\|^p_{L^p(\Omega)} + \lambda\|u_n\|^p_{L^p(\partial\Omega)} + \int_\Omega |g(x,u_n)|u_n d\sigma \\ &\le \|u_n\|^p_{L^p(\Omega)} + a_1\|u_n\|_{L^p(\Omega)} + \lambda\|u_n\|^p_{L^p(\partial\Omega)} + a_2\|u_n\|_{L^p(\partial\Omega)} \\ &\le a_3, \end{aligned}$$

where $a_i, i = 1,\dots,3$ are some positive constants independent of n. Thus, u_n is bounded in $W^{1,p}(\Omega)$. The reflexivity of $W^{1,p}(\Omega), 1 < p < \infty$, ensures the existence of a weak convergent subsequence of u_n. Due to the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$, the monotony of u_n and the compactness of the trace operator τ, we get for the entire sequence u_n

$$\begin{aligned} &u_n \rightharpoonup u_+ \quad \text{in } W^{1,p}(\Omega), \\ &u_n \to u_+ \quad \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega, \\ &u_n \to u_+ \quad \text{in } L^p(\partial\Omega). \end{aligned} \tag{4.11}$$

Due to the fact that u_n solves problem (1.1), one has for all $\varphi \in W^{1,p}(\Omega)$

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi dx \\ &= \int_\Omega (f(x,u_n) - u_n^{p-1})\varphi dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))\varphi d\sigma. \end{aligned} \tag{4.12}$$

The choice $\varphi = u_n - u_+ \in W^{1,p}(\Omega)$ is admissible in equation (4.12) which implies

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+) dx \\ &= \int_\Omega (f(x,u_n) - u_n^{p-1})(u_n - u_+) dx + \int_{\partial\Omega} (\lambda u_n^{p-1} + g(x,u_n))(u_n - u_+) d\sigma. \end{aligned} \tag{4.13}$$

Applying (4.11) and the conditions (f3), (g3) result in

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u_+) dx \le 0, \tag{4.14}$$

which ensures by the S_+−property of $-\Delta_p$ on $W^{1,p}(\Omega)$ combined with (4.11)

$$u_n \to u_+ \text{ in } W^{1,p}(\Omega). \tag{4.15}$$

Taking account of the uniform boundedness of the sequence (u_n) in combination with the strong convergence in (4.15) and the assumptions (f3) and (g3) allows us to pass to the limit in (4.12) which proves that u_+ is a solution of problem (1.1). As u_+ is a solution of (1.1) belonging to $[0, \vartheta e]$, we can use Lemma 4.2 provided

$u_+ \not\equiv 0$. We argue by contradiction and assume that $u_+ \equiv 0$ which in view of (4.10) results in

$$u_n(x) \downarrow 0 \quad \text{for all } x \in \Omega. \tag{4.16}$$

We set

$$\widetilde{u}_n = \frac{u_n}{\|u_n\|_{W^{1,p}(\Omega)}} \quad \text{for all } n. \tag{4.17}$$

Obviously, the sequence $(\widetilde{u}_n)$ is bounded in $W^{1,p}(\Omega)$ which implies the existence of a weakly convergent subsequence of $\widetilde{u}_n$, not relabeled, such that

$$\begin{aligned} &\widetilde{u}_n \rightharpoonup \widetilde{u} \quad \text{in } W^{1,p}(\Omega),\\ &\widetilde{u}_n \to \widetilde{u} \quad \text{in } L^p(\Omega) \text{ and for a.a. } x \in \Omega,\\ &\widetilde{u}_n \to \widetilde{u} \quad \text{in } L^p(\partial\Omega), \end{aligned} \tag{4.18}$$

where $\widetilde{u} : \Omega \to \mathbb{R}$ is some function belonging to $W^{1,p}(\Omega)$. Moreover, we may suppose there are functions $z_1 \in L^p(\Omega)_+, z_2 \in L^p(\partial\Omega)_+$ such that

$$\begin{aligned} |\widetilde{u}_n(x)| &\le z_1(x) \quad \text{for a.a. all } x \in \Omega,\\ |\widetilde{u}_n(x)| &\le z_2(x) \quad \text{for a.a. all } x \in \partial\Omega. \end{aligned} \tag{4.19}$$

By means of (4.12), we get for $\widetilde{u}_n$ the following variational equation

$$\begin{aligned} \int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla \varphi dx = \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1} \right) \varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1} \varphi d\sigma\\ + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} \varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \end{aligned} \tag{4.20}$$

Choosing $\varphi = \widetilde{u}_n - \widetilde{u} \in W^{1,p}(\Omega)$ in the last equality, we obtain

$$\begin{aligned} &\int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla (\widetilde{u}_n - \widetilde{u}) dx\\ &= \int_\Omega \left(\frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} - \widetilde{u}_n^{p-1} \right) (\widetilde{u}_n - \widetilde{u}) dx + \int_{\partial\Omega} \lambda \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma\\ &\quad + \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma. \end{aligned} \tag{4.21}$$

Using (4.9) along with (4.19) implies

$$\frac{|f(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \le c_f z_1(x)^{p-1} (z_1(x) + |\widetilde{u}(x)|), \tag{4.22}$$

respectively,

$$\frac{|g(x,u_n(x))|}{u_n^{p-1}(x)} \widetilde{u}_n^{p-1}(x) |\widetilde{u}_n(x) - \widetilde{u}(x)| \le c_g z_2(x)^{p-1} (z_2(x) + |\widetilde{u}(x)|). \tag{4.23}$$

The right-hand-sides of (4.22) and (4.23) are in $L^1(\Omega)$ and $L^1(\partial\Omega)$, respectively, which allows us to apply Lebesgue's dominated convergence theorem. This fact and the convergence properties in (4.18) show

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \frac{f(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) dx = 0,\\ &\lim_{n\to\infty} \int_{\partial\Omega} \frac{g(x,u_n)}{u_n^{p-1}} \widetilde{u}_n^{p-1} (\widetilde{u}_n - \widetilde{u}) d\sigma = 0. \end{aligned} \tag{4.24}$$

From (4.18), (4.21), (4.24) we conclude

$$\limsup_{n\to\infty} \int_\Omega |\nabla \widetilde{u}_n|^{p-2} \nabla \widetilde{u}_n \nabla (\widetilde{u}_n - u_n) dx = 0. \tag{4.25}$$

Taking into account the S_+−property of $-\Delta_p$ with respect to $W^{1,p}(\Omega)$, we have

$$\widetilde{u}_n \to \widetilde{u} \quad \text{in } W^{1,p}(\Omega). \tag{4.26}$$

Notice that $\|\widetilde{u}\|_{W^{1,p}(\Omega)} = 1$. The statements in (4.16), (4.26) and (4.20) yield along with the conditions (f1),(g1)

$$\int_\Omega |\nabla \widetilde{u}|^{p-2} \nabla \widetilde{u} \nabla \varphi dx = -\int_\Omega \widetilde{u}^{p-1} \varphi dx + \int_{\partial\Omega} \lambda \widetilde{u}^{p-1} \varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega). \tag{4.27}$$

Due to $\widetilde{u} \not\equiv 0$, the equation (4.27) is the Steklov eigenvalue problem in (2.1), where $\widetilde{u} \geq 0$ is the eigenfunction corresponding to the eigenvalue $\lambda > \lambda_1$. The fact that $\widetilde{u} \geq 0$ is nonnegative in $\overline{\Omega}$ yields a contradiction to the results of Martínez et al. in [47, Lemma 2.4] because $\widetilde{u}$ must change sign on $\partial\Omega$. Thus, $u_+ \not\equiv 0$ and we obtain by applying Lemma 4.2 that $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$.
Now we need to show that u_+ is the least positive solution of (1.1) within $[0, \vartheta e]$. Let $u \in W^{1,p}(\Omega)$ be a positive solution of (1.1) lying in the order interval $[0, \vartheta e]$. Lemma 4.2 implies $u \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Then there exists an integer n sufficiently large such that $u \in [\frac{1}{n}\varphi_1, \vartheta e]$. On the basis that u_n is the least solution of (1.1) in $[\frac{1}{n}\varphi_1, \vartheta e]$ it holds $u_n \leq u$. This yields by passing to the limit $u_+ \leq u$. Hence, u_+ must be the least positive solution of (1.1). In similar way one proves the existence of the greatest negative solution of (1.1) within $[-\vartheta e, 0]$. This completes the proof of the theorem. □

5. Variational characterization of extremal solutions

Theorem 4.3 implies the existence of extremal positive and negative solutions of (1.1) for all $\lambda > \lambda_1$ denoted by $u_+ = u_+(\lambda) \in \operatorname{int}(C^1(\overline{\Omega})_+)$ and $u_- = u_-(\lambda) \in -\operatorname{int}(C^1(\overline{\Omega})_+)$, respectively. Now, we introduce truncation functions $T_+, T_-, T_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ and $T_+^{\partial\Omega}, T_-^{\partial\Omega}, T_0^{\partial\Omega} : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ as follows.

$$T_+(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}, \quad T_+^{\partial\Omega}(x,s) = \begin{cases} 0 & \text{if } s \leq 0 \\ s & \text{if } 0 < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}$$

$$T_-(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \geq 0 \end{cases}, \quad T_-^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < 0 \\ 0 & \text{if } s \geq 0 \end{cases}$$

$$T_0(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases},$$

$$T_0^{\partial\Omega}(x,s) = \begin{cases} u_-(x) & \text{if } s \leq u_-(x) \\ s & \text{if } u_-(x) < s < u_+(x) \\ u_+(x) & \text{if } s \geq u_+(x) \end{cases}$$

For $u \in W^{1,p}(\Omega)$ the truncation operators on $\partial\Omega$ apply to the corresponding traces $\tau(u)$. We just write for simplification $T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u), T_+^{\partial\Omega}(x,u)$ without τ. Furthermore, the truncation operators are continuous and uniformly bounded on $\Omega \times \mathbb{R}$ (respectively, on $\partial\Omega \times \mathbb{R}$) and they are Lipschitz continuous with respect to the second argument (see, e.g. [40]). By means of these truncations, we define the following associated functionals given by

$$\begin{aligned} E_+(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_+(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda T_+^{\partial\Omega}(x,s)^{p-1} + g(x, T_+^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.1}$$

$$\begin{aligned} E_-(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_-(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_-^{\partial\Omega}(x,s)|^{p-2} T_-^{\partial\Omega}(x,s) + g(x, T_-^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.2}$$

$$\begin{aligned} E_0(u) =& \frac{1}{p}[\|\nabla u\|_{L^p(\Omega)}^p + \|u\|_{L^p(\Omega)}^p] - \int_\Omega \int_0^{u(x)} f(x, T_0(x,s)dsdx \\ & - \int_{\partial\Omega} \int_0^{u(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2} T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] dsd\sigma, \end{aligned} \tag{5.3}$$

which are well-defined and belong to $C^1(W^{1,p}(\Omega))$.

Lemma 5.1. *The functionals $E_+, E_-, E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ are coercive and weakly sequentially lower semicontinuous.*

Proof. First, we introduce the Nemytskij operators $F, F^\Omega : L^p(\Omega) \to L^q(\Omega)$ and $G, F^{\partial\Omega} : L^p(\partial\Omega) \to L^q(\partial\Omega)$ by

$$\begin{aligned} Fu(x) &= f(x, T_+(x, u(x))), & F^\Omega u(x) &= |u(x)|^{p-2}u(x), \\ Gu(x) &= g(x, T_+^{\partial\Omega}(x, u(x))), & F^{\partial\Omega} u(x) &= \lambda |T_+^{\partial\Omega}(x, u(x))|^{p-2} T_+^{\partial\Omega}(x, u(x)). \end{aligned}$$

It is clear that $E_+ \in C^1(W^{1,p}(\Omega))$. The embedding $i : W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ are compact. We set

$$\begin{aligned} \widehat{F} &:= i^* \circ F \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{F}^\Omega &:= i^* \circ F^\Omega \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{G} &:= \tau^* \circ G \circ \tau : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{F}^{\partial\Omega} &:= \tau^* \circ F^{\partial\Omega} \circ \tau : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \end{aligned}$$

where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\tau^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ denote the adjoint operators. With a view to (5.1) we obtain

$$\langle E_+'(u), \varphi \rangle = \langle -\Delta_p u, \varphi \rangle + \langle \widehat{F}^\Omega u, \varphi \rangle - \langle \widehat{F} u, \varphi \rangle - \langle \widehat{F}^{\partial\Omega} u + \widehat{G} u, \varphi \rangle, \tag{5.4}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $W^{1,p}(\Omega)$ and its dual space $(W^{1,p}(\Omega))^*$. The operators $\widehat{F}, \widehat{F}^\Omega, \widehat{F}^{\partial\Omega}$ and $\widehat{G}$ are bounded, completely continuous and hence also pseudomonotone. Since the sum of pseudomonotone operators is also pseudomonotone, we obtain that $E'_+ : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is pseudomonotone. Note that the negative p-Laplacian $-\Delta_p : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ is bounded and pseudomonotone for $1 < p < \infty$. Using Proposition 2.3 shows that E_+ is weakly sequentially lower semicontinuous. Applying the assumptions in (f3),(g3), the boundedness of the truncation operators and the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$, we obtain for a positive constant c

$$\frac{E_+(u)}{\|u\|_{W^{1,p}(\Omega)}} \geq \frac{\frac{1}{p}\|u\|^p_{W^{1,p}(\Omega)} - c\|u\|_{W^{1,p}(\Omega)}}{\|u\|_{W^{1,p}(\Omega)}} \to \infty \text{ as } \|u\|_{W^{1,p}(\Omega)} \to \infty, \tag{5.5}$$

which proves the coercivity. In the same manner, one shows this lemma for E_- and E_0, respectively. □

Lemma 5.2. *Let u_+ and u_- be the extremal constant-sign solutions of (1.1). Then the following holds:*

(i) A critical point $v \in W^{1,p}(\Omega)$ of E_+ is a (nonnegative) solution of (1.1) satisfying $0 \leq v \leq u_+$.
(ii) A critical point $v \in W^{1,p}(\Omega)$ of E_- is a (nonpositive) solution of (1.1) satisfying $u_- \leq v \leq 0$.
(iii) A critical point $v \in W^{1,p}(\Omega)$ of E_0 is a solution of (1.1) satisfying $u_- \leq v \leq u_+$.

Proof. Let v be a critical point of E_+, that is, it holds $E'_+(v) = 0$. In view of (5.1) we obtain

$$\begin{aligned}&\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi dx \\ &= \int_\Omega [f(x, T_+(x, v)) - |v|^{p-2}v]\varphi dx \\ &\quad + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x, v)^{p-1} + g(x, T_+^{\partial\Omega}(x, v))]\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega).\end{aligned} \tag{5.6}$$

Since u_+ is a positive solution of (1.1) we have by using Definition 3.3

$$\begin{aligned}\int_\Omega |\nabla u_+|^{p-2}\nabla u_+ \nabla\varphi dx &= \int_\Omega [f(x, u_+) - u_+^{p-1}]\varphi dx \\ &\quad + \int_{\partial\Omega} [\lambda u_+^{p-1} + g(x, u_+)]\varphi d\sigma, \ \forall \varphi \in W^{1,p}(\Omega).\end{aligned} \tag{5.7}$$

Choosing $\varphi = (v - u_+)^+ \in W^{1,p}(\Omega)$ in (5.7) and (5.6) and subtracting (5.7) from (5.6) results in

$$\begin{aligned}&\int_\Omega [|\nabla v|^{p-2}\nabla v - |\nabla u_+|^{p-2}\nabla u_+]\nabla(v - u_+)^+ dx + \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v - u_+)^+ dx \\ &= \int_\Omega [f(x, T_+(x, v)) - f(x, u_+)](v - u_+)^+ dx \\ &\quad + \int_{\partial\Omega} [\lambda T_+^{\partial\Omega}(x, v)^{p-1} - \lambda u_+^{p-1} + g(x, T_+^{\partial\Omega}(x, v)) - g(x, u_+)](v - u_+)^+ d\sigma \\ &= 0.\end{aligned}$$

As the first term on the left-hand-side of the last equality is nonnegative, we obtain

$$0 = \int_\Omega [|v|^{p-2}v - u_+^{p-1}](v-u_+)^+ dx, \tag{5.8}$$

which implies $(v-u_+)^+ = 0$ and thus, $v \leq u_+$. Taking $\varphi = v^- = \max(-v, 0)$ in (5.6) yields

$$\int_{\{x: v(x)<0\}} |\nabla v|^p dx + \int_{\{x: v(x)<0\}} |v|^p \varphi dx = 0,$$

consequently, it holds $\|v^-\|^p_{W^{1,p}(\Omega)} = 0$ and equivalently $v^- = 0$, that is, $v \geq 0$. By the definition of the truncation operator we see at once that $T_+(x,v) = v, T_+^{\partial\Omega}(x,v) = v$ and therefore, v is a solution of (1.1) satisfying $0 \leq v \leq u_+$. The statements in (ii) and (iii) can be shown in a similar way. $\square$

The next result matches $C^1(\overline{\Omega})$ and $W^{1,p}(\Omega)$-local minimizers for a large class of C^1- functionals. We will show that every local C^1-minimizer of E_0 is a local $W^{1,p}(\Omega)$-minimizer of E_0. This result first proven for the Dirichlet problem by Brezis and Nirenberg [8] when $p = 2$ and was extended by García Azorero et al in [37] for $p \neq 2$ (see also [39] when $p > 2$). For the zero Neumann problem we refer to the recent results of Motreanu et al. in [52] for $1 < p < \infty$. In case of nonsmooth functionals the authors in [53] and [7] proved the same result for the Dirichlet problem and the zero Neumann problem when $p \geq 2$. We give the proof for the nonlinear nonzero Neumann problem for any $1 < p < \infty$.

Proposition 5.3. *If $z_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})-$minimizer of E_0 meaning that there exists $r_1 > 0$ such that*

$$E_0(z_0) \leq E_0(z_0 + h) \quad \textit{for all } h \in C^1(\overline{\Omega}) \textit{ with } \|h\|_{C^1(\overline{\Omega})} \leq r_1,$$

then z_0 is a local minimizer of E_0 in $W^{1,p}(\Omega)$ meaning that there exists $r_2 > 0$ such that

$$E_0(z_0) \leq E_0(z_0 + h) \quad \textit{for all } h \in W^{1,p}(\Omega) \textit{ with } \|h\|_{W^{1,p}(\Omega)} \leq r_2.$$

Proof. Let $h \in C^1(\overline{\Omega})$. If $\beta > 0$ is small, we have

$$0 \leq \frac{E_0(z_0 + \beta h) - E_0(z_0)}{\beta},$$

meaning that the directional derivative of E_0 at z_0 in direction h satisfies

$$0 \leq E_0(z_0; h) \quad \text{for all } h \in C^1(\overline{\Omega}).$$

We recall that $h \mapsto E_0(z_0; h)$ is continuous on $W^{1,p}(\Omega)$ and the density of $C^1(\overline{\Omega})$ in $W^{1,p}(\Omega)$ results in

$$0 \leq E_0(z_0; h) \quad \text{for all } h \in W^{1,p}(\Omega).$$

Therefore, setting $-h$ instead of h, we get

$$0 = E_0'(z_0),$$

which yields

$$
\begin{aligned}
0 = &\int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla\varphi dx - \int_\Omega (f(x,z_0) - |z_0|^{p-2}z_0)\varphi dx \\
&- \int_{\partial\Omega} \lambda |z_0|^{p-2} z_0 \varphi d\sigma - \int_{\partial\Omega} g(x,z_0)\varphi d\sigma,\ \forall \varphi \in W^{1,p}(\Omega).
\end{aligned}
\tag{5.9}
$$

By means of Lemma 5.2, we obtain $u_- \le z_0 \le u_+$ and thus, $z_0 \in L^\infty(\Omega)$. As before via the regularity results of Lieberman [45] and Vázguez [63], it follows that $z_0 \in \operatorname{int}(C^1(\overline{\Omega}))$ (cf. Lemma 4.2). Let us assume the proposition is not valid. The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is weakly sequentially lower semicontinuous (cf. Lemma 5.1) and the set $\overline{B}_\varepsilon = \{y \in W^{1,p}(\Omega) : \|y\|_{W^{1,p}(\Omega)} \le \varepsilon\}$ is weakly compact in $W^{1,p}(\Omega)$. Thus, for any $\varepsilon > 0$ we can find $y_\varepsilon \in \overline{B}_\varepsilon$ such that

$$
E_0(z_0 + y_\varepsilon) = \min\{E_0(z_0+y) : y \in \overline{B}_\varepsilon)\} < E_0(z_0). \tag{5.10}
$$

Obviously, y_ε is a solution of the following minimum-problem

$$
\begin{cases}
\min E_0(z_0+y) \\
y \in \overline{B}_\varepsilon, g_\varepsilon(y) := \frac{1}{p}(\|y\|^p_{W^{1,p}(\Omega)} - \varepsilon^p) \le 0.
\end{cases}
$$

Applying the Lagrange multiplier rule (see, e.g., [46] or [14]) yields the existence of a multiplier $\lambda_\varepsilon > 0$ such that

$$
E_0'(z_0 + y_\varepsilon) + \lambda_\varepsilon g_\varepsilon'(y_\varepsilon) = 0, \tag{5.11}
$$

which results in

$$
\begin{aligned}
&\int_\Omega |\nabla(z_0+y_\varepsilon)|^{p-2}\nabla(z_0+y_\varepsilon)\nabla\varphi dx \\
&- \int_\Omega (f(x, T_0(x, z_0+y_\varepsilon)) - |z_0+y_\varepsilon|^{p-2}(z_0+y_\varepsilon))\varphi dx \\
&- \int_{\partial\Omega} (\lambda |T_0^{\partial\Omega}(x, z_0+y_\varepsilon)|^{p-2} T_0^{\partial\Omega}(x, z_0+y_\varepsilon) + g(x, T_0^{\partial\Omega}(x, z_0+y_\varepsilon)))\varphi d\sigma \\
&+ \lambda_\varepsilon \int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla \varphi dx + \lambda_\varepsilon \int_\Omega |y_\varepsilon|^{p-2} y_\varepsilon \varphi dx = 0,
\end{aligned}
\tag{5.12}
$$

for all $\varphi \in W^{1,p}(\Omega)$. Notice that λ_ε cannot be zero since the constraints guarantee that y_ε belongs to $\overline{B}_\varepsilon$. Let $0 < \lambda_\varepsilon \le 1$ for all $\varepsilon \in (0,1]$. We multiply (5.9) by λ_ε, set $v_\varepsilon = z_0 + y_\varepsilon$ in (5.12) and add these new equations. One obtains

$$
\begin{aligned}
&\int_\Omega |\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon \nabla\varphi dx + \lambda_\varepsilon \int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla \varphi dx \\
&+ \lambda_\varepsilon \int_\Omega |\nabla(v_\varepsilon - z_0)|^{p-2}\nabla(v_\varepsilon - z_0)\nabla\varphi dx \\
&= \int_\Omega (\lambda_\varepsilon f(x,z_0) + f(x, T_0(x, v_\varepsilon)))\varphi dx \\
&\quad - \int_\Omega (\lambda_\varepsilon |z_0|^{p-2} z_0 + |v_\varepsilon|^{p-2}v_\varepsilon + \lambda_\varepsilon |v_\varepsilon - z_0|^{p-2}(v_\varepsilon - z_0))\varphi dx \\
&\quad + \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2} z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon)|^{p-2} T_0^{\partial\Omega}(x, v_\varepsilon))\varphi d\sigma \\
&\quad + \int_{\partial\Omega} (\lambda_\varepsilon g(x, z_0) + g(x, T_0^{\partial\Omega}(x, v_\varepsilon)))\varphi d\sigma.
\end{aligned}
\tag{5.13}
$$

Now, we introduce the maps $\mathcal{A}_\varepsilon : \Omega\times\mathbb{R}^N \to \mathbb{R}^N$, $\mathcal{B}_\varepsilon : \Omega\times\mathbb{R} \to \mathbb{R}$ and $\Phi_\varepsilon : \partial\Omega\times\mathbb{R} \to \mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{A}_\varepsilon(x,\xi) &= |\xi|^{p-2}\xi + \lambda_\varepsilon|H|^{p-2}H + \lambda_\varepsilon|\xi-H|^{p-2}(\xi-H)\\
-\mathcal{B}_\varepsilon(x,\psi) &= \lambda_\varepsilon f(x,z_0) + f(x,T_0(x,\psi))\\
&\quad - (\lambda_\varepsilon|z_0|^{p-2}z_0 + |\psi|^{p-2}\psi + \lambda_\varepsilon|\psi-z_0|^{p-2}(\psi-z_0))\\
\Phi_\varepsilon(x,\psi) &= \lambda(\lambda_\varepsilon|z_0|^{p-2}z_0 + |T_0^{\partial\Omega}(x,\psi)|^{p-2}T_0^{\partial\Omega}(x,\psi))\\
&\quad + \lambda_\varepsilon g(x,z_0) + g(x,T_0^{\partial\Omega}(x,\psi)),
\end{aligned}$$

where $H(x) = \nabla z_0(x)$ and $H \in (C^\alpha(\overline{\Omega}))^N$ for some $\alpha \in (0,1]$. Apparently, the operator $\mathcal{A}_\varepsilon(x,\xi)$ belongs to $C(\overline{\Omega}\times\mathbb{R}^N,\mathbb{R}^N)$. For $x \in \Omega$ we have

$$\begin{aligned}
&(\mathcal{A}_\varepsilon(x,\xi),\xi)_{\mathbb{R}^N}\\
&= \|\xi\|^p + \lambda_\varepsilon(|\xi-H|^{p-2}(\xi-H) - |-H|^{p-2}(-H), \xi - H - (-H))_{\mathbb{R}^N}\\
&\geq \|\xi\|^p \quad \text{for all } \xi\in\mathbb{R}^N,
\end{aligned} \tag{5.14}$$

where $(\cdot,\cdot)_{\mathbb{R}^N}$ stands for the inner product in $\mathbb{R}^N$. (5.14) shows that $\mathcal{A}_\varepsilon$ satisfies a strong ellipticity condition. Hence, the equation in (5.13) is the weak formulation of the elliptic Neumann problem

$$\begin{aligned}
-\operatorname{div}\mathcal{A}_\varepsilon(x,\nabla v_\varepsilon) + \mathcal{B}_\varepsilon(x,v_\varepsilon) &= 0 && \text{in } \Omega,\\
\frac{\partial v_\varepsilon}{\partial\nu} &= \Phi_\varepsilon(x,v_\varepsilon) && \text{on } \partial\Omega.
\end{aligned} \tag{5.15}$$

where $\frac{\partial v_\varepsilon}{\partial\nu}$ denotes the conormal derivative of v_ε. To prove the $L^\infty-$regularity of v_ε, we will use the Moser iteration technique (see e.g. [22],[23],[24], [25], [43]). It suffices to consider the proof in case $1 \leq p \leq N$, otherwise we would be done. First we are going to show that $v_\varepsilon^+ = \max\{v_\varepsilon,0\}$ belongs to $L^\infty(\Omega)$. For $M > 0$ we define $v_M(x) = \min\{v_\varepsilon^+(x), M\}$. Letting $K(t) = t$ if $t \leq M$ and $K(t) = M$ if $t > M$, it follows by [43, Theorem B.3] that $K\circ v_\varepsilon^+ = v_M \in W^{1,p}(\Omega)$ and hence $v_M \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. For real $k \geq 0$ we choose $\varphi = v_M^{kp+1}$, then $\nabla\varphi = (kp+1)v_M^{kp}\nabla v_M$ and $\varphi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$. Notice that $v_\varepsilon(x) \leq 0$ implies directly $v_M(x) = 0$. Testing (5.13) with $\varphi = v_M^{kp+1}$, one gets

$$\begin{aligned}
&(kp+1)\int_\Omega |\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+\nabla v_M v_M^{kp}dx + \int_\Omega |v_\varepsilon^+|^{p-2}v_\varepsilon^+ v_M^{kp+1}dx\\
&+ \lambda_\varepsilon(kp+1)\int_\Omega \Big[|\nabla(v_\varepsilon^+ - z_0)|^{p-2}\nabla(v_\varepsilon^+ - z_0) - |-\nabla z_0|^{p-2}(-\nabla z_0)\Big]\\
&\quad \times(\nabla v_M - \nabla z_0 - (-\nabla z_0))v_M^{kp}dx\\
&= \int_\Omega (\lambda_\varepsilon f(x,z_0) + f(x,T_0(x,v_\varepsilon^+)))v_M^{kp+1}dx\\
&\quad - \int_\Omega (\lambda_\varepsilon|z_0|^{p-2}z_0 + \lambda_\varepsilon|v_\varepsilon^+ - z_0|^{p-2}(v_\varepsilon^+ - z_0))v_M^{kp+1}dx\\
&\quad + \int_{\partial\Omega}\lambda(\lambda_\varepsilon|z_0|^{p-2}z_0 + |T_0^{\partial\Omega}(x,v_\varepsilon^+)|^{p-2}T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1}d\sigma\\
&\quad + \int_{\partial\Omega}(\lambda_\varepsilon g(x,z_0) + g(x,T_0^{\partial\Omega}(x,v_\varepsilon^+)))v_M^{kp+1}d\sigma.
\end{aligned} \tag{5.16}$$

Since $z_0 \in [u_-, u_+]$, $\tau(z_0) \in [\tau(u_-), \tau(u_+)]$, $T_0(x, v_\varepsilon) \in [u_-, u_+]$ and $T_0^{\partial\Omega}(x, v_\varepsilon) \in [\tau(u_-), \tau(u_+)]$ we get for the right-hand-side of (5.16) by using (f3) and (g3)

$$\begin{aligned}
&(1)\ \int_\Omega (\lambda_\varepsilon f(x,z_0) + f(x, T_0(x, v_\varepsilon^+)))v_M^{kp+1}dx \le e_1 \int_\Omega (v_\varepsilon^+)^{kp+1}dx \\
&(2)\ -\int_\Omega (\lambda_\varepsilon |z_0|^{p-2}z_0 + \lambda_\varepsilon |v_\varepsilon^+ - z_0|^{p-2}(v_\varepsilon^+ - z_0))v_M^{kp+1}dx \\
&\qquad \le e_2 \int_\Omega |v_\varepsilon^+|^{p-1}(v_\varepsilon^+)^{kp+1}dx + e_3 \int_\Omega |z_0|^{p-1}(v_\varepsilon^+)^{kp+1}dx \\
&\qquad \le \int_\Omega e_2 (v_\varepsilon^+)^{(k+1)p}dx + e_4 \int_\Omega (v_\varepsilon^+)^{kp+1}dx \\
&(3)\ \int_{\partial\Omega} \lambda(\lambda_\varepsilon |z_0|^{p-2}z_0 + |T_0^{\partial\Omega}(x, v_\varepsilon^+)|^{p-2}T_0^{\partial\Omega}(x, v_\varepsilon^+)))v_M^{kp+1}d\sigma \\
&\qquad \le e_5 \int_{\partial\Omega} (v_\varepsilon^+)^{kp+1}d\sigma \\
&(4)\ \int_{\partial\Omega} (\lambda_\varepsilon g(x,z_0) + g(x, T_0^{\partial\Omega}(x, v_\varepsilon^+)))v_M^{kp+1}d\sigma \\
&\qquad \le e_6 \int_{\partial\Omega} (v_\varepsilon^+)^{kp+1}d\sigma.
\end{aligned} \tag{5.17}$$

The left-hand-side of (5.16) can be estimated to obtain

$$\begin{aligned}
&(kp+1)\int_\Omega |\nabla v_\varepsilon^+|^{p-2}\nabla v_\varepsilon^+ \nabla v_M v_M^{kp}dx + \int_\Omega |v_\varepsilon^+|^{p-2}v_\varepsilon^+ v_M^{kp+1}dx \\
&+ \lambda_\varepsilon(kp+1)\int_\Omega \Big[|\nabla(v_\varepsilon^+ - z_0)|^{p-2}\nabla(v_\varepsilon^+ - z_0) - |-\nabla z_0|^{p-2}(-\nabla z_0)\Big] \\
&\quad \times (\nabla v_M - \nabla z_0 - (-\nabla z_0))v_M^{kp}dx \\
&\ge (kp+1)\int_\Omega |\nabla v_M|^p v_M^{kp}dx + \int_\Omega (v_\varepsilon^+)^{p-1}v_M^{kp+1}dx \\
&\ge \frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega (v_\varepsilon^+)^{p-1}v_M^{kp+1}dx\right].
\end{aligned} \tag{5.18}$$

Using the Hölder inequality we see at once

$$\int_\Omega 1 \cdot (v_\varepsilon^+)^{kp+1}dx \le |\Omega|^{\frac{p-1}{(k+1)p}} \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}}, \tag{5.19}$$

and analog for the boundary integral

$$\int_{\partial\Omega} 1 \cdot (v_\varepsilon^+)^{kp+1}d\sigma \le |\partial\Omega|^{\frac{p-1}{(k+1)p}} \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}. \tag{5.20}$$

Applying the estimates (5.17)–(5.20) to (5.16) one gets

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla v_M^{k+1}|^p dx + \int_\Omega (v_\varepsilon^+)^{p-1}v_M^{kp+1}dx\right] \\
&\le e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + e_7 \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx\right)^{\frac{kp+1}{(k+1)p}} + e_8 \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p}d\sigma\right)^{\frac{kp+1}{(k+1)p}}.
\end{aligned}$$

We have $\lim_{M\to\infty} v_M(x) = v_\varepsilon^+(x)$ for a.e. $x \in \Omega$ and can apply Fatou's Lemma which results in

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right]\\
&\leq e_2 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_7 \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}}\\
&\quad + e_8 \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}}.
\end{aligned} \tag{5.21}$$

We have either

$$\left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} \leq 1 \quad \text{or} \quad \left(\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx\right)^{\frac{kp+1}{(k+1)p}} \leq \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx,$$

respectively, either

$$\left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}} \leq 1 \quad \text{or} \quad \left(\int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma\right)^{\frac{kp+1}{(k+1)p}} \leq \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma.$$

Thus, we obtain from (5.21)

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla (v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right]\\
&\leq e_9 \int_\Omega (v_\varepsilon^+)^{(k+1)p} dx + e_{10} \int_{\partial\Omega} (v_\varepsilon^+)^{(k+1)p} d\sigma + e_{11}.
\end{aligned} \tag{5.22}$$

Next we want to estimate the boundary integral by an integral in the domain Ω. To make it, we need the following continuous embeddings

$$\begin{aligned}
&T_1 : B^s_{pp}(\Omega) \to B^{s-\frac{1}{p}}_{pp}(\partial\Omega), \quad \text{with } s > \frac{1}{p}\\
&T_2 : B^{s-\frac{1}{p}}_{pp}(\partial\Omega) \to B^0_{pp}(\partial\Omega) = L^p(\partial\Omega), \quad \text{with } s > \frac{1}{p},
\end{aligned}$$

where $s = m + \iota$ with $m \in \mathbb{N}_0$ and $0 \leq \iota < 1$ and $\partial\Omega \in C^{m,1}$ (see [58, Page 75 and Page 82], [64, Satz 8.7] or [21, Satz 9.40]). In [21] the proof is only given for $p = 2$, however, it can be extend to $p \in (1, \infty)$ by using the Fourier transformation in $L^p(\Omega)$.

Here $B^s_{pp}(\Omega)$ denotes the Sobolev-Slobodeckii space $W^{s,p}(\Omega)$ for $s \in \mathbb{R}$ which is equal to the usual Sobolev space $W^{s,p}(\Omega)$ for $s \in \mathbb{N}$. We set $s = \frac{1}{p} + \widetilde{\varepsilon}$ with $\widetilde{\varepsilon} > 0$ is arbitrary fixed such that $s = \frac{1}{p} + \widetilde{\varepsilon} < 1$ which only requires a Lipschitz boundary $\partial\Omega$ because $m = 0$. This yields the embedding

$$T_3 : B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega) \to L^p(\partial\Omega). \tag{5.23}$$

The pair $(B^0_{pp}(\Omega), B^1_{pp}(\Omega)) = (L^p(\Omega), W^{1,p}(\Omega))$ is an interpolation couple since there exist the embeddings $W^{1,p}(\Omega) \to L^p(\Omega)$ and $L^p(\Omega) \to L^p(\Omega)$ where $L^p(\Omega)$ is, in particular, a locally convex space. The real interpolation theory implies $\left(B^0_{pp}(\Omega), B^1_{pp}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = \left(L^p(\Omega), W^{1,p}(\Omega)\right)_{\frac{1}{p}+\widetilde{\varepsilon},p} = B^{\frac{1}{p}+\widetilde{\varepsilon}}_{pp}(\Omega)$ (for more details see

[2],[61],[62]) which ensures the norm estimate

$$\|v\|_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)} \leq e_{12}\|v\|_{W^{1,p}(\Omega)}^{\frac{1}{p}+\widetilde{\varepsilon}}\|v\|_{L^p(\Omega)}^{1-\frac{1}{p}-\widetilde{\varepsilon}}, \ \forall v \in B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega), \tag{5.24}$$

with a positive constant e_{12} only depending on the boundary $\partial\Omega$. Using (5.23), (5.24) and Young's inequality yields

$$\begin{aligned}
&\int_{\partial\Omega}((v_\varepsilon^+)^{k+1})^p d\sigma \\
&= \|(v_\varepsilon^+)^{k+1}\|_{L^p(\partial\Omega)}^p \\
&\leq e_{13}^p\|(v_\varepsilon^+)^{k+1}\|_{B_{pp}^{\frac{1}{p}+\widetilde{\varepsilon}}(\Omega)}^p \\
&\leq e_{13}^p e_{12}^p\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^{(\frac{1}{p}+\widetilde{\varepsilon})p}\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^{(1-\frac{1}{p}-\widetilde{\varepsilon})p} \\
&\leq e_{13}^p e_{12}^p(\delta\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^{(1+\widetilde{\varepsilon}p)\widetilde{q}} + C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^{(p-1-\widetilde{\varepsilon}p)\widetilde{q}'}) \\
&= e_{13}^p e_{12}^p(\delta\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^p + C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^p),
\end{aligned} \tag{5.25}$$

where $\widetilde{q} = \frac{p}{1+\widetilde{\varepsilon}p}$ and $\widetilde{q}' = \frac{p}{p-1-\widetilde{\varepsilon}p}$ are chosen such that $\frac{1}{\widetilde{q}} + \frac{1}{\widetilde{q}'} = 1$ and δ is a free parameter specified later. Note that the positive constant $C(\delta)$ depends only on δ. Applying (5.25) to (5.22) shows

$$\begin{aligned}
&\frac{kp+1}{(k+1)^p}\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p dx\right] \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p}dx + e_{10}\int_{\partial\Omega}(v_\varepsilon^+)^{(k+1)p}d\sigma + e_{11} \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p})dx + e_{14}\delta\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^p + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^p + e_{11},
\end{aligned}$$

where $e_{14} = e_{10}e_{13}^p e_{12}^p$ is a positive constant. We take $\delta = \frac{kp+1}{e_{14}2(k+1)^p}$ to get

$$\begin{aligned}
&\left(\frac{kp+1}{(k+1)^p} - e_{14}\frac{kp+1}{e_{14}2(k+1)^p}\right)\left[\int_\Omega |\nabla(v_\varepsilon^+)^{k+1}|^p dx + \int_\Omega |(v_\varepsilon^+)^{k+1}|^p)dx\right] \\
&\leq e_9\int_\Omega (v_\varepsilon^+)^{(k+1)p})dx + e_{14}C(\delta)\|(v_\varepsilon^+)^{k+1}\|_{L^p(\Omega)}^p + e_{11} \\
&\leq e_{15}\int_\Omega (v_\varepsilon^+)^{(k+1)p})dx + e_{11},
\end{aligned}$$

and hence,

$$\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^p \leq \frac{2(k+1)^p}{kp+1}\left[e_{15}\int_\Omega (v_\varepsilon^+)^{(k+1)p})dx + e_{11}\right].$$

By Sobolev's embedding theorem a positive constant e_{16} exists such that

$$\|(v_\varepsilon^+)^{k+1}\|_{L^{p^*}(\Omega)} \leq e_{16}\|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)} \tag{5.26}$$

where $p^* = \frac{Np}{N-p}$ if $1 < p < N$ and $p^* = 2p$ if $p = N$. We have

$$\begin{aligned}
&\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \\
&= \|(v_\varepsilon^+)^{k+1}\|_{L^{p^*}(\Omega)}^{\frac{1}{k+1}} \\
&\le e_{16}^{\frac{1}{k+1}} \|(v_\varepsilon^+)^{k+1}\|_{W^{1,p}(\Omega)}^{\frac{1}{k+1}} \\
&\le e_{16}^{\frac{1}{k+1}} \left(\frac{2^{\frac{1}{p}}(k+1)}{(kp+1)^{1/p}} \right)^{\frac{1}{k+1}} \left[e_{15} \int_\Omega (v_\varepsilon^+)^{(k+1)p}) dx + e_{11} \right]^{\frac{1}{(k+1)p}} \\
&\le e_{16}^{\frac{1}{k+1}} e_{17}^{\frac{1}{(k+1)p}} \left(\frac{(k+1)}{(kp+1)^{1/p}} \right)^{\frac{1}{k+1}} \left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}) dx + 1 \right]^{\frac{1}{(k+1)p}}.
\end{aligned}$$

where $e_{17} = 2\max\{e_{15}, e_{11}\}$.

Since $\left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}} \right)^{\frac{1}{\sqrt{k+1}}} \ge 1$ and $\lim_{k\to\infty} \left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}} \right)^{\frac{1}{\sqrt{k+1}}} = 1$, there exists a constant $e_{18} > 1$ such that $\left(\frac{(k+1)}{(kp+1)^{\frac{1}{p}}} \right)^{\frac{1}{k+1}} \le e_{18}^{\frac{1}{\sqrt{k+1}}}$. We obtain

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{16}^{\frac{1}{k+1}} e_{18}^{\frac{1}{\sqrt{k+1}}} e_{17}^{\frac{1}{(k+1)p}} \left[\int_\Omega (v_\varepsilon^+)^{(k+1)p}) dx + 1 \right]^{\frac{1}{(k+1)p}}. \tag{5.27}$$

Now, we will use the bootstrap arguments similarly as in the proof of [25, Lemma 3.2] starting with $(k_1+1)p = p^*$ to get

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \le c(k)$$

for any finite number $k > 0$ which shows that $v_\varepsilon^+ \in L^r(\Omega)$ for any $r \in (1, \infty)$. To prove the uniform estimate with respect to k we argue as follows. If there is a sequence $k_n \to \infty$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k_n+1)p} dx \le 1,$$

we have immediately

$$\|v_\varepsilon^+\|_{L^\infty(\Omega)} \le 1,$$

(cf. the proof of [25, Lemma 3.2]). In the opposite case there exists $k_0 > 0$ such that

$$\int_\Omega (v_\varepsilon^+)^{(k+1)p} dx > 1$$

for any $k \ge k_0$. Then we conclude from (5.27)

$$\|v_\varepsilon^+\|_{L^{(k+1)p^*}(\Omega)} \le e_{16}^{\frac{1}{k+1}} e_{18}^{\frac{1}{\sqrt{k+1}}} e_{19}^{\frac{1}{(k+1)p}} \|v_\varepsilon^+\|_{L^{(k+1)p}}, \text{ for any } k \ge k_0, \tag{5.28}$$

where $e_{19} = 2e_{17}$. Choosing $k := k_1$ such that $(k_1+1)p = (k_0+1)p^*$ yields

$$\|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)} \le e_{16}^{\frac{1}{k_1+1}} e_{18}^{\frac{1}{\sqrt{k_1+1}}} e_{19}^{\frac{1}{(k_1+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p}(\Omega)}. \tag{5.29}$$

Next, we can choose k_2 in (5.28) such that $(k_2+1)p = (k_1+1)p^*$ to get

$$\begin{aligned} \|v_\varepsilon^+\|_{L^{(k_2+1)p^*}(\Omega)} &\le e_{16}^{\frac{1}{k_2+1}} e_{18}^{\frac{1}{\sqrt{k_2+1}}} e_{19}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_2+1)p}(\Omega)} \\ &= e_{16}^{\frac{1}{k_2+1}} e_{18}^{\frac{1}{\sqrt{k_2+1}}} e_{19}^{\frac{1}{(k_2+1)p}} \|v_\varepsilon^+\|_{L^{(k_1+1)p^*}(\Omega)}. \end{aligned} \tag{5.30}$$

By induction we obtain

$$\begin{aligned} \|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} &\le e_{16}^{\frac{1}{k_n+1}} e_{18}^{\frac{1}{\sqrt{k_n+1}}} e_{19}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_n+1)p}(\Omega)} \\ &= e_{16}^{\frac{1}{k_n+1}} e_{18}^{\frac{1}{\sqrt{k_n+1}}} e_{19}^{\frac{1}{(k_n+1)p}} \|v_\varepsilon^+\|_{L^{(k_{n-1}+1)p^*}(\Omega)}, \end{aligned} \tag{5.31}$$

where the sequence (k_n) is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$ with $k_0 > 0$. One easily verifies that $k_n + 1 = \left(\frac{p^*}{p}\right)^n$. Thus

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} = e_{16}^{\sum_{i=1}^n \frac{1}{k_i+1}} e_{18}^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} e_{19}^{\sum_{i=1}^n \frac{1}{(k_i+1)p}} \|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)}, \tag{5.32}$$

with $r_n = (k_n+1)p^* \to \infty$ as $n \to \infty$. Since $\frac{1}{k_i+1} = (\frac{p}{p^*})^i$ and $\frac{p}{p^*} < 1$ there is a constant $e_{20} > 0$ such that

$$\|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \le e_{20} \|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} < \infty. \tag{5.33}$$

Let us assume that $v_\varepsilon^+ \notin L^\infty(\Omega)$. Then there exist $\eta > 0$ and a set A of positive measure in Ω such that $v_\varepsilon^+(x) \ge e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta$ for $x \in A$. It follows that

$$\begin{aligned} \|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} &\ge \left(\int_A |v_\varepsilon^+(x)|^{(k_n+1)p^*} \right)^{\frac{1}{(k_n+1)p^*}} \\ &\ge (e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta)|A|^{\frac{1}{(k_n+1)p^*}}. \end{aligned}$$

Passing to the limes inferior in the above inequality yields

$$\liminf_{n\to\infty} \|v_\varepsilon^+\|_{L^{(k_n+1)p^*}(\Omega)} \ge e_{20}\|v_\varepsilon^+\|_{L^{(k_0+1)p^*}(\Omega)} + \eta,$$

which is a contradiction to (5.33) and hence, $v_\varepsilon^+ \in L^\infty(\Omega)$. In a similar way one shows that $v_\varepsilon^- = \max\{-v_\varepsilon, 0\} \in L^\infty(\Omega)$. This proves $v_\varepsilon = v_\varepsilon^+ - v_\varepsilon^- \in L^\infty(\Omega)$.
In order to show some structure properties of $\mathcal{A}_\varepsilon$ note that its derivative has the form

$$\begin{aligned} D_\xi \mathcal{A}_\varepsilon(x,\xi) =& |\xi|^{p-2} I + (p-2)|\xi|^{p-4}\xi\xi^T \\ &+ \lambda_\varepsilon|\xi - H|^{p-2} I + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H)(\xi - H)^T, \end{aligned} \tag{5.34}$$

where I is the unit matrix and ξ^T stands for the transpose of ξ. Using (5.34) implies

$$\|D_\xi \mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N} \le a_1 + a_2|\xi|^{p-2}, \tag{5.35}$$

where a_1, a_2 are some positive constants. We also obtain

$$\begin{aligned} &(D_\xi \mathcal{A}_\varepsilon(x,\xi)y, y)_{\mathbb{R}^N} \\ &= |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi, y)^2_{\mathbb{R}^N} \\ &\quad + \lambda_\varepsilon|\xi - H|^{p-2}\|y\|^2_{\mathbb{R}^N} + \lambda_\varepsilon(p-2)|\xi - H|^{p-4}(\xi - H, y)^2_{\mathbb{R}^N} \\ &\ge \begin{cases} |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } p \ge 2 \\ (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } 1 < p < 2 \end{cases} \\ &\ge \min\{1, p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}. \end{aligned} \tag{5.36}$$

For the case $1 < p < 2$ in (5.36) we have used the estimate $|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi, y)^2_{\mathbb{R}^N} \geq (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}$. Because of (5.35) and (5.36), the operators $\mathcal{A}_\varepsilon, \mathcal{B}_\varepsilon$ and Φ_ε satisfy the assumptions (0.3a-d) and (0.6) of Lieberman in [45] and thus, Theorem 2 in [45] ensures the existence of $\alpha \in (0,1)$ and $M > 0$, both independent of $\varepsilon \in (0,1]$, such that

$$v_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \quad \text{and} \quad \|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \leq M, \quad \text{for all } \varepsilon \in (0,1]. \tag{5.37}$$

Due to $y_\varepsilon = v_\varepsilon - z_0$ and the fact that $v_\varepsilon, z_0 \in C^{1,\alpha}(\overline{\Omega})$, one realizes immediately that y_ε satisfies (5.37), too. Next, we assume $\lambda_\varepsilon > 1$ with $\varepsilon \in (0,1]$. Multiplying (5.9) by -1 and adding this new equation to (5.12) yields

$$\begin{aligned}
&\int_\Omega |\nabla(z_0 + y_\varepsilon)|^{p-2}\nabla(z_0 + y_\varepsilon)\nabla\varphi dx - \int_\Omega |\nabla z_0|^{p-2}\nabla z_0 \nabla\varphi dx \\
&+ \lambda_\varepsilon \int_\Omega |\nabla y_\varepsilon|^{p-2}\nabla y_\varepsilon \nabla\varphi dx \\
&= \int_\Omega (f(x, T_0(x, z_0 + y_\varepsilon)) - f(x, z_0))\varphi dx \\
&\quad + \int_\Omega (|z_0|^{p-2}z_0 - |z_0 + y_\varepsilon|^{p-2}(z_0 + y_\varepsilon) - \lambda_\varepsilon |y_\varepsilon|^{p-2}y_\varepsilon)\varphi dx \\
&\quad + \int_{\partial\Omega} \lambda(|T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)|^{p-2}T_0^{\partial\Omega}(x, z_0 + y_\varepsilon) - |z_0|^{p-2}z_0)\varphi d\sigma \\
&\quad + \int_{\partial\Omega} (g(x, T_0^{\partial\Omega}(x, z_0 + y_\varepsilon)) - g(x, z_0))\varphi d\sigma dx.
\end{aligned} \tag{5.38}$$

Defining again

$$\begin{aligned}
\mathcal{A}_\varepsilon(x, \xi) &= \frac{1}{\lambda_\varepsilon}(|H + \xi|^{p-2}(H + \xi) - |H|^{p-2}H) + |\xi|^{p-2}\xi \\
-\mathcal{B}_\varepsilon(x, \psi) &= f(x, T_0(x, z_0 + \psi)) - f(x, z_0) + |z_0|^{p-2}z_0 \\
&\quad - |z_0 + \psi|^{p-2}(z_0 + \psi) - \lambda_\varepsilon |\psi|^{p-2}y_\varepsilon \\
\Phi_\varepsilon(x, \psi) &= \lambda(|T_0^{\partial\Omega}(x, z_0 + \psi)|^{p-2}T_0^{\partial\Omega}(x, z_0 + \psi) - |z_0|^{p-2}z_0) \\
&\quad + g(x, T_0^{\partial\Omega}(x, z_0 + \psi)) - g(x, z_0),
\end{aligned} \tag{5.39}$$

and rewriting (5.38) yields the Neumann equation

$$\begin{aligned}
-\operatorname{div}\mathcal{A}_\varepsilon(x, \nabla y_\varepsilon) + \frac{1}{\lambda_\varepsilon}\mathcal{B}_\varepsilon(x, y_\varepsilon) &= 0 && \text{in } \Omega \\
\frac{\partial v_\varepsilon}{\partial \nu} &= \frac{1}{\lambda_\varepsilon}\Phi_\varepsilon(x, y_\varepsilon), && \text{on } \partial\Omega,
\end{aligned} \tag{5.40}$$

where $\frac{\partial v_\varepsilon}{\partial \nu}$ denotes the conormal derivative of v_ε. As above, we have the following estimate

$$\begin{aligned}
(\mathcal{A}_\varepsilon(x, \xi), \xi)_{\mathbb{R}^N} &= \frac{1}{\lambda_\varepsilon}(|H + \xi|^{p-2}(H + \xi) - |H|^{p-2}H, H + \xi - H)_{\mathbb{R}^N} + \|\xi\|^p \\
&\geq \|\xi\|^p \quad \text{for all } \xi \in \mathbb{R}^N,
\end{aligned} \tag{5.41}$$

and can write the derivative $D_\xi \mathcal{A}_\varepsilon(x,\xi)$ as

$$\begin{aligned} D_\xi \mathcal{A}_\varepsilon(x,\xi) =& \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}I + (p-2)|H+\xi|^{p-4}(H+\xi)(H+\xi)^T \\ & |\xi|^{p-2}I + (p-2)|\xi|^{p-4}\xi\xi^T. \end{aligned} \tag{5.42}$$

We have again the following estimate

$$\|D_\xi \mathcal{A}_\varepsilon(x,\xi)\|_{\mathbb{R}^N} \le a_1 + a_2|\xi|^{p-2}, \tag{5.43}$$

where a_1, a_2 are some positive constants. One also gets

$$\begin{aligned} &(D_\xi \mathcal{A}_\varepsilon(x,\xi)y,y)_{\mathbb{R}^N} \\ &= \frac{1}{\lambda_\varepsilon}(|H+\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|H+\xi|^{p-4}(H+\xi,y)^2_{\mathbb{R}^N}) \\ &\quad + |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} + (p-2)|\xi|^{p-4}(\xi,y)^2_{\mathbb{R}^N} \\ &\ge \begin{cases} |\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } p \ge 2 \\ (p-1)|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N} & \text{if } 1<p<2 \end{cases} \\ &\ge \min\{1,p-1\}|\xi|^{p-2}\|y\|^2_{\mathbb{R}^N}. \end{aligned} \tag{5.44}$$

As before, the nonlinear regularity theory implies the existence of $\alpha \in (0,1)$ and $M>0$, both independent of $\varepsilon \in (0,1)$, such that (5.37) holds for y_ε.
Let $\varepsilon \downarrow 0$. Using the compact embedding $C^{1,\beta}(\overline{\Omega}) \hookrightarrow C^1(\overline{\Omega})$ (cf. [42, p. 38] or [1, p. 11]), we may assume for a subsequence $y_\varepsilon \to \widetilde{y}$ in $C^1(\overline{\Omega})$. By construction we have $y_\varepsilon \to 0$ in $W^{1,p}(\Omega)$ and thus, it holds $\widetilde{y}=0$ which implies for a subsequence $\|y_\varepsilon\|_{C^1(\overline{\Omega})} \le r_1$. Hence, one has

$$E_0(z_0) \le E_0(z_0+y_\varepsilon),$$

which is a contradiction to (5.10). This completes the proof of the proposition. $\square$

Lemma 5.4. *Let $\lambda > \lambda_1$. Then the extremal positive solution u_+ (respectively, negative solution u_-) of (1.1) is the unique global minimizer of the functional E_+ (respectively, E_-). Moreover, u_+ and u_- are local minimizers of E_0.*

Proof. By Lemma 5.1 we know that $E_+ : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous. Therefore, by Theorem 2.2 there exists a global minimizer $v_+ \in W^{1,p}(\Omega)$ of E_+. Since v_+ is a critical point of E_+, Lemma 5.2 implies that v_+ is a nonnegative solution of (1.1) satisfying $0 \le v_+ \le u_+$. By (g1) we infer that

$$|g(x,s)| \le (\lambda-\lambda_1)s^{p-1}, \ \forall s : 0 < s \le \delta_\lambda. \tag{5.45}$$

Using (f4) and (5.45) and applying the Steklov eigenvalue problem in (2.1), we conclude for $\varepsilon < \min\left\{\frac{\delta_f}{\|\varphi_1\|_\infty}, \frac{\delta_\lambda}{\|\varphi_1\|_\infty}\right\}$

$$\begin{aligned} E_+(\varepsilon\varphi_1) &= -\int_\Omega \int_0^{\varepsilon\varphi_1(x)} f(x,s)dsdx + \frac{\lambda_1-\lambda}{p}\varepsilon^p\|\varphi_1\|^p_{L^p(\partial\Omega)} \\ &\quad - \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)} g(x,s)dsd\sigma \\ &< \frac{\lambda_1-\lambda}{p}\varepsilon^p\|\varphi_1\|_{L^p(\partial\Omega)} + \int_{\partial\Omega}\int_0^{\varepsilon\varphi_1(x)}(\lambda-\lambda_1)s^{p-1}dsd\sigma \\ &= 0. \end{aligned}$$

This shows $E_+(v_+) < 0$ and we obtain $v_+ \neq 0$. Applying Lemma 4.2 implies $v_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Since u_+ is the smallest positive solution of (1.1) in $[0, \vartheta e]$ and $0 \leq v_+ \leq u_+$, it holds $v_+ = u_+$. Thus, u_+ is the unique global minimizer of E_+. In the same way one verifies that u_- is the unique global minimizer of E_-. Now we want to show that u_+ and u_- are local minimizers of the functional E_0. As $u_+ \in \operatorname{int}(C^1(\overline{\Omega})_+)$ there exists a neighborhood V_{u_+} of u_+ in the space $C^1(\overline{\Omega})$ such that $V_{u_+} \subset C^1(\overline{\Omega})_+$. Hence $E_+ = E_0$ on V_{u_+} which ensures that u_+ is a local minimizer of E_0 on $C^1(\overline{\Omega})$. In view of Proposition 5.3, we obtain that u_+ is also a local minimizer of E_0 on the space $W^{1,p}(\Omega)$. By the same arguments as above we prove that u_- is a local minimizer of E_0. □

Lemma 5.5. *The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ has a global minimizer v_0 which is a nontrivial solution of (1.1) satisfying $u_- \leq v_0 \leq u_+$.*

Proof. The functional $E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ is coercive and weakly sequentially lower semicontinuous (see Lemma 5.1). Hence, a global minimizer v_0 of E_0 exists. Since v_0 is a critical point of E_0 we know by Lemma 5.2 that v_0 is a solution of (1.1) satisfying $u_- \leq v_0 \leq u_+$. Using $E_0(u_+) = E_+(u_+) < 0$ (cf. the proof of Lemma 5.4) shows that v_0 is nontrivial meaning $v_0 \neq 0$. □

6. Existence of sign-changing solutions

First, we are going to show that our functionals introduced in Section 5 satisfy the Palais-Smale condition. In order to prove this result, we will need a preliminary lemma which can be found in [48, Lemma 2.1-Lemma 2.3] in similar form.

Lemma 6.1. *Let $A, B, C : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*$ be given by*

$$\begin{aligned} \langle A(u), v\rangle &:= \int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_\Omega |u|^{p-2}uvdx \\ \langle B(u), v\rangle &:= \int_{\partial\Omega} \lambda |T_0^{\partial\Omega}(x,u)|^{p-2}T_0^{\partial\Omega}(x,u)vdx \\ \langle C(u), v\rangle &:= \int_\Omega f(x, T_0(x,u))vdx + \int_{\partial\Omega} g(x, T_0^{\partial\Omega}(x,u))vdx \end{aligned}$$

then A is continuous and continuously invertible and the operators B, C are continuous and compact.

Proof. According to Lemma 5.1 we introduce again the Nemytskij operators $F, F^{\Omega} : L^p(\Omega) \to L^q(\Omega)$ and $G, F^{\partial\Omega} : L^p(\partial\Omega) \to L^q(\partial\Omega)$ by

$$\begin{aligned} &Fu(x) = f(x, T_0(x, u(x))), \qquad F^{\Omega}u(x) = |u(x)|^{p-2}u(x), \\ &Gu(x) = g(x, T_0^{\partial\Omega}(x, u(x))), \qquad F^{\partial\Omega}u(x) = \lambda|T_0^{\partial\Omega}(x, u(x))|^{p-2}T_0^{\partial\Omega}(x, u(x)). \end{aligned}$$

The embedding $i : W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the trace operator $\tau : W^{1,p}(\Omega) \to L^p(\partial\Omega)$ are compact. We set

$$\begin{aligned} \widehat{F} &:= i^* \circ F \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{F}^{\Omega} &:= i^* \circ F^{\Omega} \circ i : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{G} &:= \tau^* \circ G \circ \tau : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \\ \widehat{F}^{\partial\Omega} &:= \tau^* \circ F^{\partial\Omega} \circ \tau : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))^*, \end{aligned}$$

where $i^* : L^q(\Omega) \to (W^{1,p}(\Omega))^*$ and $\tau^* : L^q(\partial\Omega) \to (W^{1,p}(\Omega))^*$ denote the adjoint operators. The operators $\widehat{F}, \widehat{F}^{\Omega}, \widehat{F}^{\partial\Omega}$ and $\widehat{G}$ are bounded, completely continuous and hence also compact. Thus, $B = \widehat{F}^{\partial\Omega}$ and $C = \widehat{F}^{\Omega} + \widehat{G}$ are bounded, continuous and compact. Since the negative $p-$Laplacian is bounded continuous for $1 < p < \infty$, we obtain that $A = -\Delta_p + \widehat{F}$ is bounded and continuous.
Finally, we have to show that A is continuously invertible. By Lemma 2.1 in [34] there exists for every fixed $\phi \in (W^{1,p}(\Omega))^*$ a unique solution of the equation

$$Au = -\Delta_p u + \widehat{F}^{\Omega} u = \phi, \tag{6.1}$$

which is a consequence of the Browder theorem (e.g. in [36]) since A is bounded, continuous, coercive and strictly monotone. This implies the surjectivity of A and since A is also injective, the mapping A^{-1} exists. To prove that A^{-1} is continuous, we make use of the following estimates

$$(|x|^{p-2}x - |y|^{p-2}y, x - y)_{\mathbb{R}^m} \geq \begin{cases} C(p)|x-y|^p & \text{if } p \geq 2 \\ C(p)\frac{|x-y|^2}{(|x|+|y|)^{2-p}} & \text{if } p \leq 2, \end{cases} \tag{6.2}$$

where $(\cdot,\cdot)_{\mathbb{R}^m}$ denotes the usual scalar product in $\mathbb{R}^m$. Let $\phi_1, \phi_2 \in (W^{1,p}(\Omega))^*$ be given and let $u_1 = A^{-1}\phi_1, u_2 = A^{-1}\phi_2$ be the corresponding solutions of (6.1). Testing the related weak formulation with $\varphi = u_1 - u_2$, subtracting the equations and using (6.2) for $p \geq 2$ yields

$$\begin{aligned} &\|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*}\|u_1 - u_2\|_{W^{1,p}(\Omega)} \\ &\geq \int_{\Omega} (\phi_1 - \phi_2)(u_1 - u_2)dx \\ &= \int_{\Omega} (|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2)(\nabla u_1 - \nabla u_2)dx \\ &\quad + \int_{\Omega} (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)dx \\ &\geq C(p)\int_{\Omega} (|\nabla u_1 - \nabla u_2|^p + |u_1 - u_2|^p)dx \\ &= C(p)\|u_1 - u_2\|^p_{W^{1,p}(\Omega)} \end{aligned}$$

Consequently,

$$\|A^{-1}(\phi_1) - A^{-1}(\phi_2)\|_{W^{1,p}(\Omega)} \leq C\|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*}^{\frac{1}{p-1}}.$$

Let us consider the case $p \leq 2$. We have

$$|\nabla(u_1 - u_2)|^p = \frac{|\nabla(u_1 - u_2)|^p}{(|\nabla u_1| + |\nabla u_2|)^{\frac{2-p}{2}p}}(|\nabla u_1| + |\nabla u_2|)^{\frac{2-p}{2}p}$$

$$|u_1 - u_2|^p = \frac{|u_1 - u_2|^p}{(|u_1| + |u_2|)^{\frac{2-p}{2}p}}(|u_1| + |u_2|)^{\frac{2-p}{2}p}$$

to obtain by applying the Hölder inequality

$$\int_\Omega |\nabla(u_1 - u_2)|^p dx \leq \left(\int_\Omega \frac{|\nabla(u_1 - u_2)|^2}{(|\nabla u_1| + |\nabla u_2|)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_1| + |\nabla u_2|)^p dx\right)^{\frac{2-p}{2}}$$

$$\int_\Omega |u_1 - u_2|^p dx \leq \left(\int_\Omega \frac{|u_1 - u_2|^2}{(|u_1| + |u_2|)^{2-p}} dx\right)^{\frac{p}{2}} \left(\int_\Omega (|u_1| + |u_2|)^p dx\right)^{\frac{2-p}{2}}.$$

From (6.2) and the estimates above we get

$$\frac{\|u_1 - u_2\|_{W^{1,p}(\Omega)}}{(\|u_1\|_{W^{1,p}(\Omega)} + \|u\|_{W^{1,p}(\Omega)})^{2-p}} \leq C\|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*}, \tag{6.3}$$

where C is a positive constant. The weak formulation of (6.1) implies for $u = u_i$ and $\varphi = u_i$

$$\|u_i\|_{W^{1,p}(\Omega)}^p \leq \|\phi_i\|_{(W^{1,p}(\Omega))^*}\|u_i\|_{W^{1,p}(\Omega)}, i = 1, 2,$$

and thus, (6.3) provides

$$\begin{aligned}
&\|A^{-1}(\phi_1) - A^{-1}(\phi_2)\|_{W^{1,p}(\Omega)} \\
&\leq C\left(\|\phi_1\|_{(W^{1,p}(\Omega))^*}^{\frac{1}{p-1}} + \|\phi_2\|_{(W^{1,p}(\Omega))^*}^{\frac{1}{p-1}}\right)^{2-p} \|\phi_1 - \phi_2\|_{(W^{1,p}(\Omega))^*},
\end{aligned}$$

which completes the proof. □

By means of this auxiliary lemma, we can prove the following.

Lemma 6.2. *The functionals $E_+, E_-, E_0 : W^{1,p}(\Omega) \to \mathbb{R}$ satisfy the Palais-Smale condition.*

Proof. We show this Lemma only for E_0. The proof for E_+, E_- is very similar. Let $(u_n) \subset W^{1,p}(\Omega)$ be a sequence such that $E_0(u_n)$ is bounded and $E_0'(u_n) \to 0$ as n tends to infinity. Since $|E_0(u_n)| \leq M$ for all n, we obtain by using Young's Inequality with Epsilon and the compact embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$

$$\begin{aligned}
M &\geq E_0(u_n) \\
&= \frac{1}{p}\left[\|\nabla u_n\|_{L^p(\Omega)}^p + \|u_n\|_{L^p(\Omega)}^p\right] - \int_\Omega \int_0^{u_n(x)} f(x, T_0(x,s)) ds dx \\
&\quad - \int_{\partial\Omega} \int_0^{u_n(x)} \left[\lambda |T_0^{\partial\Omega}(x,s)|^{p-2} T_0^{\partial\Omega}(x,s) + g(x, T_0^{\partial\Omega}(x,s))\right] ds d\sigma \\
&\geq (1/p - \varepsilon_1 - \varepsilon_2 - \varepsilon_3)||u_n\|_{W^{1,p}(\Omega)}^p - C.
\end{aligned}$$

Choosing $\varepsilon_i, i = 1, 2, 3$ sufficiently small yields the boundedness of u_n in $W^{1,p}(\Omega)$, and thus, we get $u_n \rightharpoonup u$ for a subsequence of u_n still denoted with u_n. We have

$$A(u_n) - \lambda B(u_n) - C(u_n) = E_0'(u_n) \to 0,$$

which implies the existence of a sequence $(\delta_n) \subset (W^{1,p}(\Omega))^*$ converging to zero such that

$$u_n = A^{-1}(\lambda B(u_n) + C(u_n) + \delta_n).$$

By Lemma 6.1 we know that B, C are compact and A^{-1} is continuous. Passing to the limit in the previous equality yields

$$u_n \to A^{-1}(\lambda B(u) + C(u)) =: u,$$

meaning that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. $\square$

Now, we can formulate our main result about the existence of a nontrivial solution of problem (1.1).

Theorem 6.3. *Under hypotheses (f1)–(f4), (g1)–(g4) and for every number $\lambda > \lambda_2$, problem (1.1) has a nontrivial sign-changing solution $u_0 \in C^1(\overline{\Omega})$.*

Proof. Lemma 5.2 implies that every critical point of E_0 is a solution of problem (1.1) in $[u_-, u_+]$. The coercivity and the weakly sequentially lower semicontinuity of E_0 ensure along with $\inf_{W^{1,p}(\Omega)} E_+(u) < 0$ (cf. the proof of Lemma 5.4) the existence of a global minimizer $v_0 \in W^{1,p}(\Omega)$ satisfying $v_0 \neq 0$. This means that v_0 is a nontrivial solution of (1.1) belonging to $[u_-, u_+]$. If $v_0 \neq u_-$ and $v_0 \neq u_+$, then $u_0 := v_0$ must be a sign-changing solution since u_- is the greatest negative solution and u_+ is the smallest positive solution of (1.1) which proves the theorem in this case. So, we still have to show that the theorem is also true in case that either $v_0 = u_-$ or $v_0 = u_+$. Without loss of generality we suppose $v_0 = u_+$. The function u_- can be assumed to be a strict local minimizer. Otherwise we would be done. Now, we can find a $\rho \in (0, \|u_+ - u_-\|_{W^{1,p}(\Omega)})$ such that

$$E_0(u_+) \leq E_0(u_-) < \inf\{E_0(u) : u \in \partial B_\rho(u_-)\}, \tag{6.4}$$

where $\partial B_\rho = \{u \in W^{1,p}(\Omega) : \|u - u_-\|_{W^{1,p}(\Omega)} = \rho\}$. Due to (6.4) along with the fact that E_+ satisfies the Palais-Smale condition (see Lemma 6.2) enables us to apply the Mountain-Pass Theorem to E_+ (cf. Theorem 2.5) which yields the existence of $u_0 \in W^{1,p}(\Omega)$ satisfying $E_0'(u_0) = 0$ and

$$\inf\{E_0(u) : u \in \partial B_\rho(u_-)\} \leq E_0(u_0) = \inf_{\gamma \in \Gamma} \max_{t \in [-1,1]} E_0(\gamma(t)), \tag{6.5}$$

where

$$\Gamma = \{\gamma \in C([-1, 1], W^{1,p}(\Omega)) : \gamma(-1) = u_-, \gamma(1) = u_+\}.$$

We see at once that (6.4) and (6.5) show $u_0 \neq u_-$ and $u_0 \neq u_+$, and therefore, u_0 is a sign-changing solution provided $u_0 \neq 0$. In order to prove $u_0 \neq 0$ we are going to show that $E_0(u_0) < 0$ which is satisfied if there exists a path $\widetilde{\gamma} \in \Gamma$ such that

$$E_0(\widetilde{\gamma}(t)) < 0, \ \forall t \in [-1, 1]. \tag{6.6}$$

Let $S = W^{1,p}(\Omega) \cap \partial B_1^{L^p(\partial\Omega)}$, where $\partial B_1^{L^p(\partial\Omega)} = \{u \in L^p(\partial\Omega) : \|u\|_{L^p(\partial\Omega)} = 1\}$, and $S_C = S \cap C^1(\overline{\Omega})$ be equipped with the topologies induced by $W^{1,p}(\Omega)$ and $C^1(\overline{\Omega})$, respectively. Furthermore, we set

$$\Gamma_0 = \{\gamma \in C([-1,1], S) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\},$$
$$\Gamma_{0,C} = \{\gamma \in C([-1,1], S_C) : \gamma(-1) = -\varphi_1, \gamma(1) = \varphi_1\}.$$

In view of assumption (g1) there exists a constant $\delta_2 > 0$ such that

$$\frac{|g(x,s)|}{|s|^{p-1}} \le \mu, \quad \text{for a.a. } x \in \partial\Omega \text{ and all } 0 < |s| \le \delta_2, \tag{6.7}$$

where $\mu \in (0, \lambda - \lambda_2)$. We select $\rho_0 \in (0, \lambda - \lambda_2 - \mu)$. Thanks to the results of Martínez and Rossi in [49] we have the following variational characterization of λ_2 given by (see (2.2)-(2.4) in Section 2)

$$\lambda_2 = \inf_{\gamma\in\Gamma_0} \max_{u\in\gamma([-1,1])} \int_\Omega \Big[|\nabla u|^p + |u|^p\Big] dx. \tag{6.8}$$

Since (6.8) there exists a $\gamma \in \Gamma_0$ such that

$$\max_{t\in[-1,1]} \|\gamma(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2 + \frac{\rho_0}{2}.$$

It is well known that S_C is dense in S. Let $\gamma \in \Gamma_0$ meaning $\gamma : [-1,1] \to S$ is continuous and let $t_0 \in [-1,1]$ fixed. The continuity of γ implies the existence of $\delta^1 > 0$ such that for $\varepsilon > 0$ holds

$$\|\gamma(t) - \gamma(t_0)\| \le \frac{\varepsilon}{3}, \ \forall t \in B(t_0, \delta^1),$$

where $B(t_0, \delta^1)$ stands for the open ball around t_0 with radius δ^1. Since S_C is dense in S, we find $\gamma_c \in \Gamma_{0,C}$ such that

$$\|\gamma_c(t_0) - \gamma(t_0)\| \le \frac{\varepsilon}{3}.$$

Applying again the continuity argument guarantees the existence of $\delta^2 > 0$ such that

$$\|\gamma_c(t_0) - \gamma(t)\| \le \frac{\varepsilon}{3}, \ \forall t \in B(t_0, \delta^2).$$

Let $\delta^3 := \min\{\delta^1, \delta^2\}$. Then we obtain

$$\begin{aligned}
&\|\gamma_c(t) - \gamma(t)\| \\
&\le \|\gamma_c(t) - \gamma_c(t_0)\| + \|\gamma_c(t_0) - \gamma(t_0)\| + \|\gamma(t_0) - \gamma(t)\| \\
&\le \varepsilon, \ \forall t \in B(t_0, \delta^3).
\end{aligned}$$

Hence, we have found an open cover of $[-1,1]$ such that

$$[-1,1] \subset \bigcup_{t_i\in[-1,1]} B(t_i, \delta(t_i)),$$

and due to the compactness of $[-1,1]$, there exists a finite open cover meaning

$$[-1,1] \subset \bigcup_{\substack{t_i\in[-1,1] \\ i=1,\dots,k}} B(t_i, \delta(t_i)),$$

which implies

$$\|\gamma_c(t)-\gamma(t)\| \le k\varepsilon =: \widetilde{\varepsilon},\ \forall t \bigcup_{\substack{t_i\in[-1,1]\\ i=1,\ldots,k}} B(t_i,\delta(t_i)) \supset [-1,1].$$

This proves the density of $\Gamma_{0,C}$ in Γ_0 and thus, for a fixed number r satisfying $0 < r \le (\lambda_2+\rho_0)^{\frac{1}{p}} - (\lambda_2+\frac{\rho_0}{2})^{\frac{1}{p}}$, there is a $\gamma_0 \in \Gamma_{0,C}$ such that

$$\max_{t\in[-1,1]} \|\gamma(t)-\gamma_0(t)\|^p_{W^{1,p}(\Omega)} < r.$$

This yields

$$\max_{t\in[-1,1]} \|\gamma_0(t)\|^p_{W^{1,p}(\Omega)} < \lambda_2+\rho_0.$$

Let $\delta := \min\{\delta_f,\delta_2\}$, where δ_f is the constant in condition (f4). Due to the boundedness of the set $\gamma_0([-1,1])(\overline{\Omega})$ in $\mathbb{R}$ ensures the existence of $\varepsilon_0>0$ such that

$$\varepsilon_0|u(x)| \le \delta \ \text{ for all } x\in\Omega \text{ and all } u\in\gamma_0([-1,1]). \tag{6.9}$$

Lemma 4.3 ensures that $u_+, -u_- \in \operatorname{int}(C^1(\overline{\Omega})_+)$. Thus, for every $u\in\gamma_0([-1,1])$ and any bounded neighborhood V_u of u in $C^1(\overline{\Omega})$ there exist positive numbers h_u and j_u satisfying

$$u_+ - \frac{1}{h}v \in \operatorname{int}(C^1(\overline{\Omega})_+) \ \text{ and } \ -u_- + \frac{1}{j}v \in \operatorname{int}(C^1(\overline{\Omega})_+), \tag{6.10}$$

if $h\ge h_u, j\ge j_u, v\in V_u$. By a compactness argument from (6.10) we conclude the existence of $\varepsilon_1>0$ such that

$$u_-(x) \le \widetilde{\varepsilon}u(x) \le u_+(x) \ \text{ for all } x\in\Omega, u\in\gamma_0([-1,1]) \text{ and } \forall\widetilde{\varepsilon}\in(0,\varepsilon_1). \tag{6.11}$$

Let $0<\varepsilon<\min\{\varepsilon_0,\varepsilon_1\}$. Now, we consider the continuous path $\varepsilon\gamma_0$ in $C^1(\overline{\Omega})$ joining $-\varepsilon\varphi_1$ and $\varepsilon\varphi_1$. We obtain by using hypothesis (f4)

$$-\int_\Omega\int_0^{\varepsilon\gamma_0(t)(x)} f(x,T_0(x,s))dsdx \le 0. \tag{6.12}$$

Applying (6.7), (6.9), (6.10), (6.11),(6.12) and the fact that $\gamma_0([-1,1])\subset\partial B_1^{L^p(\partial\Omega)}$ we have

$$\begin{aligned}
&E_0(\varepsilon\gamma_0(t))\\
&= \frac{\varepsilon^p}{p}[\|\nabla\gamma_0(t)\|^p_{L^p(\Omega)} + \|\gamma_0(t)\|^p_{L^p(\Omega)}] - \int_\Omega\int_0^{\varepsilon\gamma_0(t)(x)} f(x,T_0(x,s))dsdx\\
&\quad - \int_{\partial\Omega}\int_0^{\varepsilon\gamma_0(t)(x)} \left[\lambda|T_0^{\partial\Omega}(x,s)|^{p-2}T_0^{\partial\Omega}(x,s) + g(x,T_0^{\partial\Omega}(x,s))\right] dsd\sigma\\
&< \frac{\varepsilon^p}{p}(\lambda_2+\rho_0) - \frac{\varepsilon^p}{p}\lambda - \int_{\partial\Omega}\int_0^{\varepsilon\gamma_0(t)(x)} g(x,s)dsd\sigma\\
&< \frac{\varepsilon^p}{p}(\lambda_2+\rho_0-\lambda+\mu)\\
&< 0 \ \text{ for all } t\in[-1,1].
\end{aligned} \tag{6.13}$$

In the next step we are going to construct continuous paths γ_+, γ_- which join $\varepsilon\varphi_1$ and u_+, respectively, $-u_-$ and $-\varepsilon\varphi_1$. We denote

$$\begin{aligned} c_+ &= c_+(\lambda) = E_+(\varepsilon\varphi_1), \\ m_+ &= m_+(\lambda) = E_+(u_+), \\ E_+^{c_+} &= \{u \in W^{1,p}(\Omega) : E_+(u) \leq c_+\}. \end{aligned}$$

Since u_+ is a global minimizer of E_+, we see at once that $m_+ < c_+$. Using Lemma 5.2 yields the nonexistence of critical values in the interval $(m_+, c_+]$. Due to the coercivity of E_+ along with its property to satisfy the Palais-Smale condition (see Lemma 6.2), we can apply the second deformation lemma (see, e.g. [38, p. 366]) to E_+. This guarantees the existence of a continuous mapping $\eta \in C([0,1]\times E_+^{c_+}, E_+^{c_+})$ with the following properties:

(i) $\eta(0,u) = u$ for all $u \in E_+^{c_+}$
(ii) $\eta(1,u) = u_+$ for all $u \in E_+^{c_+}$
(iii) $E_+(\eta(t,u)) \leq E_+(u),\ \forall t \in [0,1]$ and $\forall u \in E_+^{c_+}$.

We introduce the path $\gamma_+ : [0,1] \to W^{1,p}(\Omega)$ given by $\gamma_+(t) = \eta(t,\varepsilon\varphi_1)^+ = \max\{\eta(t,\varepsilon\varphi_1), 0\}$ for all $t \in [0,1]$. Apparently, γ_+ is continuous in $W^{1,p}(\Omega)$ and joins $\varepsilon\varphi_1$ and u_+. Moreover, we have

$$E_0(\gamma_+(t)) = E_+(\gamma_+(t)) \leq E_+(\eta(t,\varepsilon\varphi_1)) \leq E_+(\varepsilon\varphi_1) < 0 \quad \text{for all } t \in [0,1]. \tag{6.14}$$

Analogously, we can apply the second deformation lemma to the functional E_- and obtain a continuous path $\gamma_- : [0,1] \to W^{1,p}(\Omega)$ between $-\varepsilon\varphi_1$ and u_- such that

$$E_0(\gamma_-(t)) < 0 \quad \text{for all } t \in [0,1]. \tag{6.15}$$

Putting the paths together, $\gamma_-, \varepsilon\gamma_0$ and γ_+ yield a continuous path $\widetilde{\gamma} \in \Gamma$ joining u_- and u_+. In view of (6.13), (6.14) and (6.15) it holds $u_0 \neq 0$. So, we have found a nontrivial sign-changing solution u_0 of problem (1.1) satisfying $u_- \leq u_0 \leq u_+$. This completes the proof. $\square$

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