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PARABOLIC PROBLEMS WITH FRACTIONAL BOUNDARY DISTURBANCE

STEFAN SPERLICH

ABSTRACT. This paper is devoted to the L_2-study of the disturbed parabolic problem of sub- and superdiffusion with stochastic boundary conditions of either Neumann or Dirichlet type on the half space $\mathbb{R}_+^N$ and on domains G with a somehow smooth boundary. All appearing disturbances may involve fractional Brownian motions. We determine the critical value for the smallest Hurst index, so that the unique solution affiliates to the maximal regularity class of type L_2 and construct the best possible solution space for Hurst indices smaller than the critical value.

1. INTRODUCTION AND NOTATION

In the last years parabolic problems with fractional noise in the interior have been studied by several authors (e.g. Duncan, Maslowski & Pasik-Duncan [6], Caithamer & Karczewska [3], Sperlich & Wilke [14], Sperlich [13]), such that a satisfactory theory has been established. Aim of this paper is to study a parabolic problem with a fractional stochastic perturbation on the boundary. We will consider both, half space problems and problems on bounded or unbounded domains with a somehow smooth boundary. Actually, for this situation results even for perturbations modeled as ordinary Wiener-processes, are rare. Two prototypes of applications to the problems we are discussing are given in Section 2.

Our plan is as follows. In *Section 2*, we introduce the problem under consideration. Then, in *Section 3*, we directly formulate the main results, which will be proven in *Section 4*.

We close this section with some notational remarks. Let X, Y be Banach spaces. By W_2^s we denote the Bessel-potential space (or Sobolev space of fractional order) defined by

$$W_2^s(\mathbb{R}^N; X) := \left\{ f \in \mathcal{S}' : \|\mathcal{F}^{-1}(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L_2(\mathbb{R}^N;X)} < \infty \right\},$$

where $s \geq 0$ is a real number, $\mathcal{S}'$ denotes the space of tempered distributions and $\mathcal{F}$ means the Fourier transform. In case $s > 0$ is an integer, this space is

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equivalent to the ordinary Sobolev space. As far as $s > 0$ is not an integer this space is isometrically isomorphic to the Slobodeckij space, since the exponent of integrability equals 2. In particular we have the equivalent norm representation

$$\|f\|_{W_2^s(\mathbb{R}^N;X)} \sim \|f\|_{W_2^{[s]}(\mathbb{R}^N;X)} + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|D^\alpha f(x) - D^\alpha f(y)|_X^2}{|x-y|^{1+2\{s\}}} \mathrm{d}x \mathrm{d}y \right)^{\frac{1}{2}},$$

where $s = [s] + \{s\}$ with $[s]$ integer and $0 < \{s\} < 1$. For a comprehensive account on Sobolev spaces of fractional order we refer to [12, Chapter 2]. Moreover, for $s \in [0,1)$ we denote by $C^s(X;Y)$ the space of all Hölder-continuous functions $f : X \to Y$ of order s. If $s \geq 1$ is not an integer, than $C^s(X;Y)$ means the space of all functions from X to Y, whose $[s]$-th derivative exists and belongs to $C^{\{s\}}(X;Y)$. Finally, if $s \geq 1$ is an integer, the space $C^s(X;Y)$ consists of all s-times continuously differentiable functions $f : X \to Y$. The symbol $\ell_2 := \ell_2(\mathbb{N};\mathbb{R})$ means the L_2 sequence space normed by

$$\|a\|_{\ell_2} = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{\frac{1}{2}}.$$

Further we will consider weighted L_2 and W_2^s spaces. They are weighted by the weight t^μ, $\mu \geq 0$, and are defined by

$$L_{2,\mu}(J;X) := \{f : J \to X : t^\mu f \in L_2(J;X)\},$$
$$W_{2,\mu}^s(J;X) := \{f : J \to X : t^\mu f \in W_2^s(J;X)\}.$$

It is easy to verify that $L_2(J;X) = L_{2,0}(J;X) \hookrightarrow L_{2,\mu_1}(J;X) \hookrightarrow L_{2,\mu_2}(J;X)$ holds if and only if $0 \leq \mu_1 \leq \mu_2$. With ${}_0W_{2,\mu}^s(J;X)$ we denote the space of all $W_{2,\mu}^s(J;X)$-functions whose trace at $t = 0$ is zero, if it exists. Throughout this paper we suppress the argument $\omega \in \Omega$, whenever there is no ambiguity.

2. The Problem

Let $\alpha \in (0,2)$, $G \subset \mathbb{R}^N$ to be a domain with C^2-boundary ∂G and $J = [0,T]$ a bounded time interval. We study the parabolic boundary problem of subdiffusion (if $\alpha < 1$), normal diffusion (if $\alpha = 1$), and superdiffusion (if $\alpha > 1$) with fractional stochastic disturbances on the boundary. This problem reads as

$$\begin{cases} \partial_t^\alpha u(t,x) - \Delta u(t,x) = 0, & t \in J, \quad x \in G, \\ \mathcal{D}u(t,x) = \psi(t,x), & t \in J, \quad x \in \partial G, \\ u(0,x) = 0, & x \in G, \end{cases} \tag{2.1}$$

in the basic space

$$X = L_2(J \times G \times \Omega),$$

where the boundary disturbance is modeled as

$$\psi(t,x,\omega) = \sum_{k=1}^{\infty} b_k(t,x) B_k^H(t,\omega). \tag{2.2}$$

Here $(B_i^H)_{i\in\mathbb{N}}$ are entirely independent scalar fractional Brownian motions (see Definition 4.1 below) with Hurst parameter $H \in (0,1)$ defined on the probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and the multipliers $b := (b_n)_{n\in\mathbb{N}}$ are subject to

Hypothesis (b). *The scalar functions $b_i \in L_2(J \times \partial G)$, $i \in \mathbb{N}$, are deterministic and nonnegative. The latter means $b_i(t,x) \geq 0$ for all $t \in J$, $x \in \partial G$, $i \in \mathbb{N}$.*

With

$$Y = L_2(J \times \partial G \times \Omega)$$

we denote the basic space for the boundary process ψ. The fractional derivative operator ∂_t^α is defined as

$$\partial_t^\alpha := \frac{\partial^2}{\partial t^2}(g_{2-\alpha} * \cdot), \qquad \alpha \in (0,2),$$

where $g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}$, $t \geq 0$, $\kappa > 0$, denotes the standard kernel of fractional integration. As the operator $\mathcal{D}$, we either choose $\mathcal{D} = \partial_\nu$ which links to the Neumann problem, or $\mathcal{D} = I$ to study the Dirichlet problem. As usual I denotes the identity mapping.

Such problems arise in the theory of normal and anomalous diffusion, where the boundary conditions prescribe a stochastic inflow in case $\mathcal{D} = \partial_\nu$, and a stochastic concentration on the boundary for $\mathcal{D} = I$, respectively. Another typical application of those problems is the heat conduction in materials with memory (e.g. polymeric fluids or solids).

We are seeking for conditions on the Hurst index H and the pointwise multiplier $b := (b_i)_{i\in\mathbb{N}}$, so that the solution u of (2.1) affiliates to the space Z_δ defined by

$$Z_\delta := {}_0W_2^{\frac{\alpha\delta}{4}}(J; L_2(G; L_2(\Omega))) \cap L_2\left(J; {}_0W_2^{\min\{\frac{\delta}{2};2\}}(G; L_2(\Omega))\right), \quad \delta \geq 0. \tag{2.3}$$

It will turn out, that these spaces are appropriate solution spaces. Note that the class Z_4 appears as the maximal regularity class of type L_2 associated to problem (2.1). Moreover, the spaces Z_δ, with $\delta \geq 4$ are tailored to capture results with a higher time regularity. Higher spacial regularity is not treated in this paper, since the resulting inevitable, purely technical, compatibility conditions cannot be motivated from the view of applications. For brevity we introduce the class $U_{\delta,H}$ for the pointwise multiplier $b := (b_i)_{i\in\mathbb{N}}$ as

$$U_{\delta,H} := {}_0W_{2,H}^{\frac{\alpha\delta}{4}}(J; L_2(\partial G; \ell_2)) \cap L_{2,H}\left(J; {}_0W_2^{\frac{\delta}{2}}(\partial G; \ell_2)\right), \quad \delta \geq 0. \tag{2.4}$$

3. Main results

In what follows let $H \in (0,1)$, $\alpha \in (0,2)$ and $G \subset \mathbb{R}^N$ be either the N dimensional half space, given by

$$\mathbb{R}^N_+ := \left\{ x := (x', y) \in \mathbb{R}^N : x' \in \mathbb{R}^{N-1}, y > 0 \right\}, \tag{3.1}$$

or a domain with compact boundary of class C^2, if not indicated otherwise.

Theorem 3.1. *Assume Hypothesis (**b**) holds. Let $0 \le \gamma < \frac{4}{\alpha}H$ and in case $G \neq \mathbb{R}^N_+$ let $\gamma \in [0, \frac{4}{\alpha}H) \cap [0,4)$. Then the following hold if $b \in U_{\gamma,H}$, given by (2.4).*

(a) The Dirichlet problem (2.1), i.e. $\mathcal{D} = I$, admits a unique solution u in the regularity class $Z_{\gamma+1}$ given by (2.3). If, in addition, $\gamma \le 3$ then membership of b to the class $U_{\gamma,H}$ is also necessary.

(b) The Neumann problem (2.1), i.e. $\mathcal{D} = \partial_\nu$, admits a unique solution u in the regularity class $Z_{\gamma+3}$ given by (2.3). If, in addition, $\gamma \le 1$ then membership of b to the class $U_{\gamma,H}$ is also necessary.

Note that the additional assumption $\gamma < 4$ is only restrictive in the case of subdiffusion, when $\alpha < H$. However, in view of maximal L_2-regularity it is not obstructive at all. If one seeks for strong solutions of problem (2.1) with Dirichlet boundary condition, Theorem 3.1 shows that one necessarily has to assume that $H > \frac{3\alpha}{4}$. On the other hand, in view of the Neumann problem the above theorem yields the existence of strong solutions, only if $H > \frac{\alpha}{4}$. The following corollaries concern a result on mixed regularity classes with either full spatial regularity or full temporal regularity. In the Dirichlet case, i.e. $\mathcal{D} = I$, this results reads as

Corollary 3.1. *The Dirichlet problem (2.1) admits a unique solution u and*

(i) $u \in {}_0W_2^{\vartheta\alpha}(J; {}_0W_2^2(G; L_2(\Omega)))$, $\quad \vartheta \ge 0$,
(ii) $u \in {}_0W_2^{\alpha}(J; {}_0W_2^{2\vartheta}(G; L_2(\Omega)))$, $\quad \vartheta \in [0,1]$,

provided that

(a) $\vartheta < \frac{H}{\alpha} - \frac{3}{4}$*;*
(b) $\vartheta < \frac{1}{4}$*, in case* $G \neq \mathbb{R}^N_+$*;*
(c) $b \in U_{4\vartheta+3,H}$*.*

Proof. Set $\gamma = 4\vartheta + 3$, then clearly $3 \le \gamma < \frac{4}{\alpha}H$ and, in addition, $\gamma < 4$ if $G \neq \mathbb{R}^N_+$, thus Theorem 3.1 yields $u \in Z_{\gamma+1}$. By the mixed derivative theorem we obtain

$$Z_{\gamma+1} \hookrightarrow {}_0W_2^{\frac{\alpha(\gamma+1)}{4}\theta}\left(J; {}_0W_2^{\frac{\gamma+1}{2}(1-\theta)}(G; L_2(\Omega))\right), \qquad \theta \in [0,1]$$

and the choice $\theta = \frac{\gamma-3}{\gamma+1}$ proves assertion (i), while $\theta = \frac{4}{\gamma+1}$ gives (ii). □

In case of a boundary condition of Neumann type, i.e. $\mathcal{D} = \partial_\nu$, we deduce

Corollary 3.2. *The Neumann problem (2.1) admits a unique solution u and*

(i) $u \in {}_0W_2^{\vartheta\alpha}(J; {}_0W_2^2(G; L_2(\Omega)))$, $\quad \vartheta \geq 0$,
(ii) $u \in {}_0W_2^{\alpha}(J; {}_0W_2^{2\vartheta}(G; L_2(\Omega)))$, $\quad \vartheta \in [0, 1]$,

provided that

(a) $\vartheta < \frac{H}{\alpha} - \frac{1}{4}$;
(b) $\vartheta < \frac{3}{4}$, *in case* $G \neq \mathbb{R}_+^N$;
(c) $b \in U_{4\vartheta+1,H}$.

Proof. Repeat the arguments of the proof of Corollary 3.1 with $\gamma = 4\vartheta + 1$. □

Here we discuss mainly the $L_2(\Omega)$-valued case. The subsequent proposition – which is easy to prove but never the less useful – covers, in the presence of Theorem 3.1, a result in the pathwise sense.

Proposition 3.1. *Let u belong to the space* Z_δ *given by (2.3) and* $s \geq 0$ *a real number.*

(i) *If* $\delta > \frac{2}{\alpha}(2s + 1)$, *then* $u \in L_2(\Omega; C^s(J; L_2(G)))$.
(ii) *If* $2s + N < \delta \leq 4$, *then* $u \in L_2(\Omega; L_2(J; C^s(G)))$.
(iii) *If* $\delta > 4$ *and* $s < \frac{4-N}{2}$, *then* $u \in L_2(\Omega; L_2(J; C^s(G)))$.
(iv) *If* $\delta > \max\left\{4; \frac{8(2s+N+1)}{\alpha(3-2s-N)}\right\}$ *and* $s < \frac{3-N}{2}$, *then* $u \in L_2(\Omega; C^s(J \times G))$.

Proof. Fubini's Theorem yields

$$Z_\delta = L_2(\Omega; {}_0W_2^{\frac{\alpha\delta}{4}}(J; L_2(G))) \cap L_2(\Omega; L_2(J; {}_0W_2^{\min\{\frac{\delta}{2};2\}}(G)))$$

and it is due to Sobolev imbedding that ${}_0W_2^{\theta}(V) \hookrightarrow C^s(V)$ if $s < \theta - \frac{\dim V}{2}$. Then a simple computation confirms (i), (ii) and (iii). Turning to (iv) we allude to

$$Z_\delta \hookrightarrow L_2(\Omega; {}_0W_2^{\frac{2\alpha\delta}{\alpha\delta+8}}(J \times G)) \quad \text{if} \quad \delta > 4,$$

which is due to the mixed derivative theorem and again the claim follows again via Sobolev imbedding. □

It is worthwhile to mention that in view of Proposition 3.1, there exists a feasible δ for (ii) and (iii) only if $N \leq 3$, and for (iv) only if $N \leq 2$. The remaining part of this paper is devoted to the proof of Theorem 3.1. It is organized as follows.

Next, in Section 4.1, we provide the notion of a weak solution associated to problem (2.1). Then, in Section 4.2, we are going to tailor necessary and useful auxiliaries which make it possible to proof Theorem 3.1 in the half space setting. By means of spatial localization, which is the aim of Section 4.4, we will carry over the half space results to domains in Section 4.5.

4. Proof of the main results

4.1. **Weak solutions.** Let us denote by $g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}$, $\kappa > 0$ the standard kernel of fractional integration. We call a function $u \in X$ weak solution of the Dirichlet problem (2.1), i.e. $\mathcal{D} = I$, if it satisfies the integral equation

$$\int_G \int_J (\partial_t^2 \phi_D)(g_{2-\alpha} * u)\mathrm{d}t\mathrm{d}x + \int_G \int_J (\Delta\phi_D)u\mathrm{d}t\mathrm{d}x = \int_{\partial G} \int_J (\partial_\nu \phi_D)\psi\mathrm{d}t\mathrm{d}x \tag{4.1}$$

for all test functions ϕ_D in the class

$$\{\phi_D \in W_2^2(G; L_2(J)) : \phi_D \mid_{\partial G} = 0\} \\ \cap \{\phi_D \in W_2^2(J; L_2(G)) : \phi_D(T) = \partial_t\phi_D(T) = 0\}. \tag{4.2}$$

Similarly, we call a function $u \in X$ weak solution of the Neumann problem (2.1), i.e. $\mathcal{D} = \partial_\nu$, if is satisfies the integral equation

$$\int_G \int_J (\partial_t^2 \phi_N)(g_{2-\alpha} * u)\mathrm{d}t\mathrm{d}x + \int_G \int_J (\Delta\phi_N)u\mathrm{d}t\mathrm{d}x = \int_{\partial G} \int_J \phi_N\psi\mathrm{d}t\mathrm{d}x \tag{4.3}$$

for all test functions ϕ_N in the class

$$\{\phi_N \in W_2^2(G; L_2(J)) : \partial_\nu\phi_N \mid_{\partial G} = 0\} \\ \cap \{\phi_N \in W_2^2(J; L_2(G)) : \phi_N(T) = \partial_t\phi_N(T) = 0\}. \tag{4.4}$$

Equations (4.1) resp. (4.3) can be obtained by multiplying problem (2.1) with ϕ_D resp. ϕ_N and integrating over J and G. Note that by construction every strong solution is also a weak solution. The converse is not true in general. Observe that the classes (4.2) and (4.4) are nontrivial and dense in X, since they contain the C^∞-functions with compact support in $(0, T) \times G$.

In the half space setting, that is if $G = \mathbb{R}^N_+$, one achives a more explicit representation of a weak solution of problem (2.1). To this purpose we define the operator

$$F := (\partial_t^\alpha - \Delta_{x'})^{1/2} \tag{4.5}$$

acting on the basic space Y with domain

$$D(F) = {}_0W_2^{\frac{\alpha}{2}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^1(\mathbb{R}^{N-1}; L_2(\Omega))). \tag{4.6}$$

Since the operator ∂_t^α is sectorial of angle $\frac{\alpha\pi}{2}$ and, moreover, commutes with the negative Laplacian $-\Delta_{x'}$ it is due to the Kalton-Weis-Theorem [9, Theorem 6.3], that the operator F is sectorial of angle $\frac{\alpha\pi}{4}$, hence is the negative generator of an analytic C_0-semigroup, provided $0 < \alpha < 2$.

Let $\Lambda : \{\partial_\nu, I\} \to \{0,1\}$ be the function which indicates the Neumann problem; precisely $\Lambda_{\mathcal{D}} := \Lambda(\mathcal{D}) = 1$ if and only if $\mathcal{D} = \partial_\nu$. We are now in the position to rewrite problem (2.1) in coordinates according to (3.1) as the ordinary differential equation

$$\begin{cases} -\partial_y^2 u(y) + F^2 u(y) = 0, & y > 0, \\ (1-\Lambda_{\mathcal{D}})u(0) - \Lambda_{\mathcal{D}}\partial_y u(0) = \psi. \end{cases} \tag{4.7}$$

The deterministic case (cf. [11, Section 3]) gives raise to call a function u a (weak) solution of (4.7), if it satisfies

$$u(y) = e^{-Fy}F^{-\Lambda_{\mathcal{D}}}\psi, \quad t > 0, \tag{4.8}$$

where as usual $F^0 := I$. Here e^{tA} denotes the analytic C_0-semigroup generated by the operator A. In particular, this formula depicts the well-posedness of problem (2.1) in the sense of Hadamard, i.e. the problem admits a unique solution which depends continuously on the data, in some reasonable topology.

In order to show, that a weak solution of the form (4.8) satisfies the representation formula (4.1) resp. (4.3), we make use of an approximation argument. We exemplify this argument for the case of a boundary condition of Dirichlet type. To this end let ψ_n belonging to

$$D(F^{\frac{3}{2}}) = {}_0W_2^{\frac{3\alpha}{4}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^{\frac{3}{2}}(\mathbb{R}^{N-1}; L_2(\Omega)))$$

for all $n \in \mathbb{N}$ so that $\psi_n \to \psi \in Y$ as n tends to infinity. Theorem 3.1 yields that the function $u_n(y) = e^{-Fy}\psi_n$ affiliates to the class Z_4, hence is a strong solution of the Dirichlet problem

$$\begin{cases} \partial_t^\alpha u_n(t,x) - \Delta u_n(t,x) = 0, & t \in J, \quad x \in \mathbb{R}_+^N, \\ u_n(t,x) = \psi_n(t,x), & t \in J, \quad x \in \mathbb{R}^{N-1}, \\ u_n(0,x) = 0, & x \in \mathbb{R}_+^N \end{cases}$$

for every $n \in \mathbb{N}$. It is due to the C_0-property of the semigroup e^{-Fy} and Theorem 3.1 that $u_n \to u \in Z_1$ as $n \to \infty$ and in particular by maximal regularity (the functions u_n are strong solutions for all $n \in \mathbb{N}$) and representation (4.1) we have the validity of the integral equation

$$\int_{\mathbb{R}_+^N}\int_J (\partial_t^2\phi_D)(g_{2-\alpha} * u_n)\mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}_+^N}\int_J (\Delta\phi_D)u_n\mathrm{d}t\mathrm{d}x = \int_{\mathbb{R}^{N-1}}\int_J (\partial_\nu\phi_D)\psi_n\mathrm{d}t\mathrm{d}x$$

for all $n \in \mathbb{N}$. Passing n to the limit we see that in the half space setting, a weak solution of the form (4.8) satisfies equation (4.1). In this sense formulae (4.8) and (4.1) resp. (4.3) are connected.

4.2. **Auxiliaries.** Let us stress that the boundary condition does not involve a derivative of B^H in the distributional sense, which in particular means that a stochastic integration calculus is not required. For the reader's convenience we recall the definition of a scalar fractional Brownian motion.

Definition 4.1. *A real valued Gaussian process* $B^H := \{B^H(t)\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *is called a* fractional Brownian motion *with Hurst parameter* $H \in (0,1)$ *if for all* $s, t \in \mathbb{R}_+$ *it is*

(i) $\mathbb{P}\{B^H(0) = 0\} = 1,$
(ii) $\mathbb{E}[B^H(t)] = 0,$
(iii) $\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$

Note that $B^{\frac{1}{2}}$ satisfies the definition of an ordinary Wiener process. Note further that with probability 1 the graph of B^H, i.e. $(t, B^H(t))_{t\in J} \subset \mathbb{R}^2$, is a fractal of Hausdorff dimension $2 - H$ (cf. [7, Theorem 16.7]), which might be interesting for several applications. A detailed survey on fractional Brownian motions can be found in Mandelbrot & Van Ness [10], Anh & Grecksch [2] or Decreusefond & Üstünel [4], which is only a tiny fracture of a manifold range of papers related to this topic. Turning to spatial regularity, we furnish a necessary and sufficient condition on the pointwise multiplier $b := (b_i)_{i\in\mathbb{N}}$, so that the boundary disturbance ψ affiliates to the space

$$Y_s := L_2(J; {}_0W_2^s(\partial G; L_2(\Omega))), \qquad s \geq 0. \tag{4.9}$$

Note that Y_0 is isometrically isomorphic to the basic space Y.

Proposition 4.1. *Let* $G \subset \mathbb{R}^N$ *be a domain with boundary of class* C^1, ψ *given by (2.2),* $s \geq 0$, b *subject to Hypothesis (**b**) and the class* Y_s *given by (4.9). Then the following are equivalent.*

(i) $\psi \in Y_s$.
(ii) $b \in L_{2,H}(J; {}_0W_2^s(\partial G; \ell_2))$.

We have the isometry $\|\psi\|_{Y_s} = \|b\|_{L_{2,H}(J;{}_0W_2^s(\partial G;\ell_2))}$.

Proof. A straight forward computation gives

$$
\begin{aligned}
\|\psi\|_{Y_s} &= \left\| \left(\mathbb{E} \left| \sum_{k=1}^{\infty} b_k(t,x) B_k^H(t) \right|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} \\
&= \left\| \left(\mathbb{E} \left[\sum_{k=1}^{\infty} |b_k(t,x) B_k^H(t)|^2 + \sum_{k\neq l} b_k(t,x) b_l(t,x) B_k^H(t) B_l^H(t) \right] \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))}
\end{aligned}
$$

and with the aid of the independence of B_i^H and B_j^H for $i \neq j$ we proceed with

$$
\begin{aligned}
\|\psi\|_{Y_s} &= \left\| \left(\sum_{k=1}^{\infty} |b_k(t,x)|^2 \mathbb{E} |B_k^H(t)|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} \\
&= \left\| \left(\sum_{k=1}^{\infty} |t^H b_k(t,x)|^2 \right)^{\frac{1}{2}} \right\|_{L_2(J;{}_0W_2^s(\partial G))} \\
&= \|b\|_{L_{2,H}(J;{}_0W_2^s(\partial G;\ell_2))}
\end{aligned}
$$

and the proof is complete. □

Our next aim is to deduce an optimal condition on b, so that the boundary disturbance ψ admits some time regularity. To this purpose we state a useful imbedding result which is immediate by [8, Theorem 329]. This reads as

Lemma 4.1. *Let V be a Banach space, $H \in (0,1)$, and $0 < \sigma < H$. Then*

$$
{}_0W_{2,H}^{\sigma}(\mathbb{R}_+;V) \hookrightarrow L_{2,H-\sigma}(\mathbb{R}_+;V). \tag{4.10}
$$

It is worthwhile to mention that for $\gamma \in [0, \frac{4}{\alpha}H)$ Lemma 4.1 yields

$$
U_{\gamma,H} \hookrightarrow L_{2,H-\frac{\alpha\gamma}{4}}(J;L_2(\partial G;\ell_2)),
$$

where the class $U_{\delta,H}$ is given by (2.4). Before proving the announced regularity result for the boundary process ψ we may stress that, in general, one cannot expect that $\psi \in {}_0W_2^H(J;L_2(\partial G;L_2(\Omega)))$. This is due to the fact that we have $B^H \in L_2(\Omega;C^{\alpha}(J))$, if and only if $\alpha < H$ (cf. [13, Proposition 3.1]). By imbedding, it is immediate that

$$
B^H \in \bigcap_{0<\delta<H} W_2^{\delta}(J;L_2(\Omega)) \tag{4.11}
$$

and it is an easy thought that nontrivial pointwise multipliers can only preserve, but not improve regularity.

Proposition 4.2. *Let $G \subset \mathbb{R}^N$ be a domain with boundary of class C^1 and ψ given by (2.2). Suppose $H \in (0,1)$, $0 < \sigma < H$ and b satisfies Hypothesis (**b**). Then the following are equivalent*

(i) $b \in {}_0W^{\sigma}_{2,H}(J; L_2(\partial G; \ell_2))$;
(ii) $\psi \in {}_0W^{\sigma}_{2}(J; L_2(\partial G; L_2(\Omega)))$.

Proof. First observe that $\psi \in Y$ if b is subject to (i). This is due to Lemma 4.1, $L_{2,H-\sigma} \hookrightarrow L_{2,H}$, and Proposition 4.1 with $s = 0$. In the sequel we denote by $[\cdot]_\sigma$ the semi-norm of the Slobodeckij space $W^{\sigma}_2(J; L_2(\partial G; L_2(\Omega)))$. Then we have by Definition 4.1 (iii)

$$\begin{aligned}
[\psi]^2_\sigma &= \int_J\int_J \left\| \left(\frac{\mathbb{E}|\psi(t,\cdot) - \psi(s,\cdot)|^2}{|t-s|^{1+2\sigma}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t \\
&= \int_J\int_J \left\| \left(\frac{\mathbb{E}\left[\sum_{k=1}^\infty (b_k(t,\cdot)B^H_k(t) - b_k(s,\cdot)B^H_k(s))^2\right]}{|t-s|^{1+2\sigma}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t \\
&= \int_J\int_J \left\| \left(\frac{\sum_{k=1}^\infty (b_k(t,\cdot)t^H - b_k(s,\cdot)s^H)^2}{|t-s|^{1+2\sigma}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t \\
&\qquad + \int_J\int_J \left\| \left(\frac{\sum_{k=1}^\infty b_k(t,\cdot)b_k(s,\cdot)[|t-s|^{2H} - (t^H - s^H)^2]}{|t-s|^{1+2\sigma}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t
\end{aligned}$$

which proves that (ii) indeed implies (i). Turning to the converse, we may estimate

$$\begin{aligned}
[\psi]^2_\sigma &\le [b]^2_{\sigma,H} + \int_J\int_J \left\| \left(\frac{\sum_{k=1}^\infty b_k(t,\cdot)b_k(s,\cdot)|t-s|^{2H}}{|t-s|^{1+2\sigma}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t \\
&\le [b]^2_{\sigma,H} + \frac{1}{2}\int_J\int_J \left\| \left(\frac{\sum_{k=1}^\infty (b^2_k(t,\cdot) + b^2_k(s,\cdot))}{|t-s|^{1+2\sigma-2H}} \right)^{\frac{1}{2}} \right\|^2_{L_2(\partial G)} \mathrm{d}s\mathrm{d}t \\
&= [b]^2_{\sigma,H} + 2\int_0^T\int_0^t \frac{\|b(t,\cdot)\|^2_{L_2(\partial G;\ell_2)}}{(t-s)^{1+2\sigma-2H}} \mathrm{d}s\mathrm{d}t \\
&= [b]^2_{\sigma,H} + \frac{1}{H-\sigma}\int_J \|t^{H-\sigma} b(t,\cdot)\|^2_{L_2(\partial G;\ell_2)} \mathrm{d}t \\
&= [b]^2_{\sigma,H} + \frac{1}{H-\sigma}\|b\|^2_{L_{2,H-\sigma}(J;L_2(\partial G;\ell_2))},
\end{aligned}$$

which completes the proof. □

We are now in the position to proof the main result in the half space setting.

4.3. **Proof of Theorem 3.1: Half space setting.** Let $G = \mathbb{R}^N_+$, given by (3.1). The unique existence of a solution u of problem (2.1) is clear by (4.8). By Propositions 4.1 and 4.2 we have the equivalence

$$b \in U_{\gamma,H} \iff \psi \in D(F^{\frac{\gamma}{2}}),$$

where the operator F is given by (4.5) and

$$D(F^\theta) = {}_0W_2^{\frac{\alpha\theta}{2}}(J; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(J; W_2^\theta(\mathbb{R}^{N-1}; L_2(\Omega))), \tag{4.12}$$

for $\theta \geq 0$. Assertion (i) is proven, if we can show that $\psi \in D(F^{\frac{\gamma}{2}})$ is equivalent to $u \in Z_{\gamma+1}$. To this end we denote by $z \in \widetilde{X} := L_2(\mathbb{R} \times \mathbb{R}^N_+ \times \Omega)$ the solution of the problem

$$\begin{cases} -\partial_y^2 z(y) + \widetilde{F}^2 z(y) = 0, & y > 0, \\ z(0) = \Psi, \end{cases}$$

where the process Ψ belongs to $\widetilde{Y} := L_2(\mathbb{R} \times \mathbb{R}^{N-1} \times \Omega)$ so that $\Psi\mid_{t\in J} = \psi$ holds and we define

$$\widetilde{F} := \sqrt{\partial_t^\alpha - \Delta_{x'} + I}$$

with domain

$$D(\widetilde{F}) = {}_0W_2^{\frac{\alpha}{2}}(\mathbb{R}; L_2(\mathbb{R}^{N-1}; L_2(\Omega))) \cap L_2(\mathbb{R}; W_2^1(\mathbb{R}^{N-1}; L_2(\Omega))).$$

Recall that by (4.8) z is of the form $z(y) := e^{-\widetilde{F}y}\Psi$ with $y \geq 0$.

In what follows $\mathcal{F}$ means the Fourier transform with respect to time t and tangential variable x'. Let $m = m(\lambda, \xi) = \sqrt{\lambda^\alpha + |\xi|^2 + 1}$ with $\lambda = i\rho$, $\rho \in \mathbb{R}$, $\xi \in \mathbb{R}^{N-1}$, denote the Fourier symbol of $\widetilde{F}(t, x')$. Suppressing the argument $\omega \in \Omega$, Plancherel's Theorem yields

$$\begin{aligned} \|\widetilde{F}^{\frac{\gamma+1}{2}} z\|^2_{\widetilde{X}} = \|\mathcal{F}\{\widetilde{F}^{\frac{\gamma+1}{2}} z\}\|^2_{\widetilde{X}} &= \int_0^\infty \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |m^{\frac{1}{2}} \mathcal{F}\{\widetilde{F}^{\frac{\gamma}{2}} z(y)\}(\lambda, \xi)|^2 \mathrm{d}\xi \mathrm{d}\rho \mathrm{d}y \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \int_0^\infty |m| e^{-2\operatorname{Re} m y} |\mathcal{F}\{\widetilde{F}^{\frac{\gamma}{2}}\Psi\}(\lambda, \xi)|^2 \mathrm{d}y \mathrm{d}\xi \mathrm{d}\rho \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|m|}{2\operatorname{Re} m} |\mathcal{F}\{\widetilde{F}^{\frac{\gamma}{2}}\Psi\}(\lambda, \xi)|^2 \mathrm{d}\xi \mathrm{d}\rho. \end{aligned}$$

Observe now, that due to $\alpha \in (0, 2)$ the symbol m takes values in an open sector of the complex plane, symmetric with respect to the positive real half axis $\mathbb{R}_+$, with vertex 0 and opening angle $\vartheta < \pi$. This captures the existence of constants $c_1, c_2 > 0$, such that

$$c_1|m| \leq \operatorname{Re} m \leq c_2|m|$$

holds. Therefrom we obtain for $\gamma \in [0, \frac{4}{\alpha}H) \cap [0, 3]$

$$\|\widetilde{F}^{\frac{\gamma}{2}}\Psi\|^2_{\widetilde{Y}} \leq c \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} \frac{|m|}{2\operatorname{Re} m} |\mathcal{F}\{\widetilde{F}^{\frac{\gamma}{2}}\Psi\}(\lambda, \xi)|^2 \mathrm{d}\xi \mathrm{d}\rho = \|\widetilde{F}^{\frac{\gamma+1}{2}} z\|^2_{\widetilde{X}}$$

which is the key to necessity. Turning to sufficiency we deduce for $\gamma \in [0, \frac{4}{\alpha}H)$

$$\|\widetilde{F}^{\frac{\gamma+1}{2}} z\|_{\widetilde{X}}^2 \leq c \int_{\mathbb{R}} \int_{\mathbb{R}^{N-1}} |\mathcal{F}\{\widetilde{F}^{\frac{\gamma}{2}} \Psi\}(\lambda, \xi)|^2 \mathrm{d}\xi \mathrm{d}\rho = c\|\widetilde{F}^{\frac{\gamma}{2}} \Psi\|_{\widetilde{Y}}^2,$$

hence $\bar{z} := z \mid_{t \in J} \in Z_{\gamma+1}$, where $\bar{z} = z \mid_{t \in J}$ denotes the restriction of z to J. Observe now, that $u = \bar{z} + w$, where $w \in X$ is the solution of the problem

$$\begin{cases} -\partial_y^2 w(y) + F^2 w(y) = \bar{z}, & y > 0, \\ w(0) = 0, \end{cases}$$

with F given by (4.5). It is due to [15, Theorem 3.1] that

$$w \in {}_0W_2^{\alpha + \frac{\alpha(\gamma+1)}{4}}(J; L_2(G; L_2(\Omega))) \cap L_2\left(J; {}_0W_2^2(G; L_2(\Omega))\right) = Z_{\gamma+5},$$

which in turn yields $u \in Z_{\gamma+1}$ and assertion (i) is proven.

Turning to (ii) let us denote by v the solution of the Dirichlet problem (2.1). Recall that by (4.8) it is $u = F^{-1}v$, and so in particular we have

$$v \in Z_{\gamma+1} \iff u \in Z_{\gamma+3}$$

for all $0 \leq \gamma < \frac{4}{\alpha}H$. Employing (i) completes the proof.

4.4. **Spatial localization.** Let now $G \subset \mathbb{R}^N$ be a domain with compact boundary ∂G of class C^2. In case G is unbounded one has to think of an exterior domain. Since the problem under investigation is fully known in the full space $\mathbb{R}^N$ (e.g. [16, Theorem 3.1]) and, by the above, in the half space $\mathbb{R}_+^N$ the apparent strategy is to localize problem (2.1) and to apply the known results.

In order to prevent the localization with respect to time we do consider the following two auxiliary problems

$$\begin{cases} \partial_t^\alpha v(t,x) + (\lambda - \Delta)v(t,x) = 0, & t \in J, \quad x \in G, \\ \mathcal{D}v(t,x) = \psi(t,x), & t \in J, \quad x \in \partial G, \\ v(0,x) = 0, & x \in G, \end{cases} \tag{4.13}$$

where $\lambda > 0$ is chosen later and

$$\begin{cases} \partial_t^\alpha w(t,x) - \Delta w(t,x) = \lambda v(t,x), & t \in J, \quad x \in G, \\ \mathcal{D}w(t,x) = 0, & t \in J, \quad x \in \partial G, \\ w(0,x) = 0, & x \in G. \end{cases} \tag{4.14}$$

Note, that the function $u = v + w$ clearly solves the initial problem (2.1). Our strategy is as follows. We are going to localize problem (4.13) with respect to space and obtain a solution on the whole of $[0,T]$ by choosing λ sufficiently large. Then, depending on the resulting regularity of v, the regularity of w is known by [16, Theorem 3.4].

Since the technique of localization is well known (e.g. [5, Section 8]) we just go briefly through the prearrangements. Let $x_0 \in \partial G$ be an arbitrary element of the boundary. Without loss of generality, we may assume that $x_0 = 0$ and the outer normal at x_0 satisfies $n(x_0) = (0, \dots, 0, -1)$. This can always be achieved by a composition of a translation and a rotation in $\mathbb{R}^N$. Such affine mappings of $\mathbb{R}^N$ onto itself clearly leave all function spaces under consideration invariant. By definition of a C^2-boundary there is an open neighborhood $U = U_1 \times U_2 \subset \mathbb{R}^N$ of x_0 with $U_1 \subset \mathbb{R}^{N-1}$ and $U_2 \subset \mathbb{R}$ as well as a function $\zeta \in C^2(\overline{U_1})$, such that

$$\begin{aligned} \partial G \cap U &= \{x = (x', y) \in U : y = \zeta(x')\}\,, \\ G \cap U &= \{x = (x', y) \in U : y > \zeta(x')\}\,. \end{aligned}$$

Using now the notation $x = (x_1, \dots, x_N)$ we define $\vartheta : \overline{U} \to \mathbb{R}^N$ in virtue of

$$\vartheta_k(x) = \begin{cases} x'_k & : k = 1, \dots, N-1 \\ y - \zeta(x') & : k = N \end{cases}. \tag{4.15}$$

It is easy to see, that $\vartheta \in C^2(\overline{U}; \mathbb{R}^N)$ is one-to-one and satisfies $G \cap U = \{x \in U : \vartheta_N(x) > 0\}$ as well as $\partial G \cap U = \{x \in U : \vartheta_N(x) = 0\}$. Observe, that the function ζ can be extended to a function in $C^2(\mathbb{R}^{N-1})$ with compact support. For brevity we denote the extension of ζ again by ζ.

Regarding spatial localization, by the boundedness of ∂G, there exists a radius $r_0 > 0$ such that ∂G is entirely contained in the open ball $B_{r_0}(0)$. If G is unbounded we set $U_0 = \{x \in \mathbb{R}^N; |x| > r_0\}$, otherwise we may assume that $\overline{G} \subset B_{r_0}(0)$ and put $U_0 = \emptyset$. Now, we cover $\overline{B_{r_0}(0)}$ by finitely many open sets U_j, $j = 1, \dots, n$, which are subject to

(U1) $U_j \cap \partial G = \emptyset$ and $U_j = B_{r_j}(x_j)$ for all $j = 1, \dots n_1$.
(U2) $U_j \cap \partial G \neq \emptyset$ for $j = n_1 + 1, \dots, n$ and there exists $x_j \in U_j \cap \partial G$ and $\zeta_j \in C^2(\mathbb{R}^{N-1})$ with compact support such that $U_j \cap \partial G = \{x = (x', y) \in U_j : y = \zeta_j(x')\}$ as well as $U_j \cap G = \{x = (x', y) \in U_j : y > \zeta_j(x')\}$, and $U_j = \vartheta_j^{-1}(B_{r_j}(x_j))$.

In what follows we denote by $\{\varphi_j\}_{j=0}^n \subset C^\infty(\mathbb{R}^N; [0, 1])$ a partition of the unity such that $\sum_{j=0}^n \varphi_j(x) \equiv 1$ on $\overline{G}$ and $\operatorname{supp} \varphi_j \subset U_j$. Observe now, that v is a solution of (4.13) if and only if

$$\begin{cases} \partial_t^\alpha(\varphi_j v) + (\lambda - \Delta)(\varphi_j v) = -[\Delta, \varphi_j]v, & \text{in } J \times G, \quad j = 0, \dots, n \\ \mathcal{D}(\varphi_j v) = \varphi_j \psi + [\mathcal{D}, \varphi_j]v, & \text{on } J \times \partial G,\ j = n_1 + 1, \dots, n, \\ \varphi_j v\,|_{t=0} = 0. & \end{cases}$$

In case $j = 0, \dots, n_1$ we have to consider full space problems for the functions $\varphi_j v$, for which the existence of the corresponding solution operators $\mathcal{S}_j^F$ are known. One obtains

$$\varphi_j v = \mathcal{S}_j^F(-[\Delta, \varphi_j]v) =: h_j^F(v), \quad j = 1, \dots, n_1. \tag{4.16}$$

For $j = n_1 + 1, \dots, n$, we get problems on crooked half spaces with inhomogeneous Neumann or Dirichlet boundary condition. Using the common affine mappings that, in particular, transform x_j to the origin combined with an appropriate variable transformation and denoting by Γ_y the trace operator at $y = 0$ leads to

$$\begin{cases} \partial_t^\alpha \Theta_j^{-1}(\varphi_j v) + (\lambda - \Delta)^{\vartheta_j} \Theta_j^{-1}(\varphi_j v) = -\Theta_j^{-1}[\Delta, \varphi_j]v, & J \times \mathbb{R}_+^N, \\ \mathcal{D}^{\vartheta_j} \Theta_j^{-1}(\varphi_j v) = \Theta_j^{-1}(\varphi_j \psi) + \Theta_j^{-1}\Gamma_y[\mathcal{D}, \varphi_j]v, & J \times \mathbb{R}^{N-1}, \\ \Theta_j^{-1}(\varphi_j v)\mid_{t=0} = 0, & \end{cases}$$

that is, to half space problems for $\Theta_j^{-1}(\varphi_j v)$. Here the pull-back $\Theta_j v$ is defined on $\operatorname{int} G$ by $\Theta_j v(x) = v(\vartheta_j(x))$ and $(\lambda - \Delta)^{\vartheta_j} := \lambda - \Theta_j^{-1}\Delta\Theta_j$ as well as $\mathcal{D}^{\vartheta_j} := \Theta_j^{-1}\mathcal{D}\Theta_j$. Choosing the radii r_i, $i = 1, \dots, n$, sufficiently small, Theorem 3.1 in connection with a perturbation argument asserts the existence of solution operators $\mathcal{S}_j^B$ for the above problems. So we immediately get

$$\varphi_j v = \Theta_j \mathcal{S}_j^B \begin{pmatrix} -\Theta_j^{-1}[\Delta, \varphi_j]v \\ \Theta_j^{-1}(\varphi_j \psi) + \Theta_j^{-1}\Gamma_y[\mathcal{D}, \varphi_j]v \end{pmatrix} =: h_j^B(\psi, v), \tag{4.17}$$

for $j = n_1 + 1, \dots, n$. Summing now over all j yields the formula

$$v = \sum_{j=0}^{n_1} h_j^F(v) + \sum_{j=n_1+1}^{n} h_j^B(\psi, v) =: \mathcal{G}(v) + \mathcal{K}(\psi), \tag{4.18}$$

which is necessary for v to be a solution of (2.1). Summarizing, we deduced a fixed point equation (4.18) for v, where the first sum is determined by the data, and the second contains only terms of lower order. By means of the contraction principle, this fixed point equation can be solved on $J = [0, T]$ provided $\mathcal{G}$ is a strict contraction on J. But this can always be arranged by choosing λ sufficiently large.

Before focusing the concrete Neumann or Dirichlet case, we prove an extremely useful result. This reads as

Proposition 4.3. *Let $X = L_2(J \times \mathbb{R}_+^N \times \Omega)$ and $F = \sqrt{\partial_t^\alpha - \Delta_{x'}}$ with domain $D(F)$ given by (4.6) in coordinates according to (3.1). Then there is a constant $c > 0$, such that*

$$\left\| (F^2 + \lambda)^{\frac{\gamma+1}{4}} e^{-(F^2+\lambda)^{\frac{1}{2}} y} \right\|^2_{\mathcal{B}\left(D(F^{\frac{\gamma}{2}});X\right)} \leq c(1 + \lambda^{\frac{\gamma}{2}}) \tag{4.19}$$

holds for $\gamma \geq 0$ and $\lambda \geq 0$.

Proof. Let $g \in D(F^{\frac{\gamma}{2}})$ with $\gamma \geq 0$ and $Y = L_2(J \times \mathbb{R}^{N-1} \times \Omega)$. Then by means of Plancherel's Theorem and Fourier transform with respect to time and space it is

$$
\begin{aligned}
&\left\| (F^2+\lambda)^{\frac{\gamma+1}{4}} e^{-(F^2+\lambda)^{\frac{1}{2}} y} g \right\|_X^2 \\
&\leq \int\limits_{\mathbb{R}_+} \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}} \left| (i\tau)^\alpha + |\xi|^2 + \lambda \right|^{\frac{\gamma+1}{2}} e^{-2\operatorname{Re}\sqrt{(i\tau)^\alpha + |\xi|^2+\lambda} y} |\widetilde{g}(i\tau,\xi)|^2 \mathrm{d}\tau \mathrm{d}\xi \mathrm{d}y \\
&= \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{N-1}} \Big(\int\limits_{\mathbb{R}_+} e^{-2\operatorname{Re}\sqrt{(i\tau)^\alpha + |\xi|^2+\lambda} y} \mathrm{d}y \Big) \left| (i\tau)^\alpha + |\xi|^2 + \lambda \right|^{\frac{\gamma+1}{2}} |\widetilde{g}(i\tau,\xi)|^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq c \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{N-1}} \Big[\left| (i\tau)^\alpha + |\xi|^2 + \lambda \right|^{\frac{\gamma}{4}} |\widetilde{g}(i\tau,\xi)| \Big]^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq c \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}^{N-1}} \Big[[\left| (i\tau)^\alpha + |\xi|^2 \right|^{\frac{\gamma}{4}} + \lambda^{\frac{\gamma}{4}}] |\widetilde{g}(i\tau,\xi)| \Big]^2 \mathrm{d}\xi \mathrm{d}\tau \\
&\leq c \|g\|^2_{D(F^{\frac{\gamma}{2}})} + \lambda^{\frac{\gamma}{2}} \|g\|_Y^2 \\
&\leq c(1+\lambda^{\frac{\gamma}{2}}) \|g\|^2_{D(F^{\frac{\gamma}{2}})}
\end{aligned}
$$

with a generic constant $c > 0$ being independent of λ. □

4.5. **Proof of Theorem 3.1: Setting for domains.** This time we first facing assertion (ii). Let us shortly recall what we have done in the preview. We split up the initial problem (2.1) in the two auxiliary problems (4.13) and (4.14), so that it suffices to seek for the regularity of the solution v of the localized version of (4.13). This is what remains to do. Therefore we denote by $\mathcal{Z}_\gamma^\lambda$ the space Z_γ equipped with the norm

$$\|\cdot\|_{\mathcal{Z}_\gamma^\lambda} := \|\cdot\|_{Z_\gamma} + \lambda \|\cdot\|_X, \tag{4.20}$$

with λ from (4.13) and where the space Z_γ is given by (2.3) with an admissible γ. Thanks to our preview it remains to show, that $\mathcal{G}$ is a strict contraction on J for $v \in \mathcal{Z}_{\gamma+3}^\lambda$. Thus we proceed as follows. Let v_1 and v_2 belonging to $\mathcal{Z}_{\gamma+3}^\lambda$, then the linearity of the solution operators captures

$$
\begin{aligned}
\|\mathcal{G}(v_1) - \mathcal{G}(v_2)\|_{\mathcal{Z}_{\gamma+3}^\lambda} &\leq \sum_{j=0}^{n_1} \left\| S_j^F([\Delta,\varphi_j](v_2 - v_1)) \right\|_{\mathcal{Z}_{\gamma+3}^\lambda} \\
&\quad + \sum_{j=n_1+1}^{n} \left\| \mathcal{S}_j^B \begin{pmatrix} \Theta_j^{-1}[\Delta,\varphi_j](v_2-v_1) \\ \Theta_j^{-1}\Gamma_y[\mathcal{D},\varphi_j](v_1-v_2) \end{pmatrix} \right\|_{\mathcal{Z}_{\gamma+3}^\lambda}.
\end{aligned} \tag{4.21}
$$

Turning to the first sum we observe, that for every $\varepsilon_1 > 0$ there is a constant c_{ε_1} depending on ε_1, such that by interpolation and Young's inequality

$$\|[\Delta,\varphi_j]u\|_X \leq \varepsilon \|u\|_{Z_{\gamma+3}} + c_{\varepsilon_1}\|u\|_X \leq \|u\|_{\mathcal{Z}_{\gamma+3}^\lambda} \Big(\varepsilon_1 + \frac{c_{\varepsilon_1}}{\lambda} \Big) \tag{4.22}$$

holds for all $j = 0, 1, \dots, n$ and $u \in \mathcal{Z}^\lambda_{\gamma+3}$. Then by the boundedness of the solution operators and (4.22) we end up with

$$\begin{aligned}\left\|S^F_j([\Delta, \varphi_j](v_2 - v_1))\right\|_{\mathcal{Z}^\lambda_{\gamma+3}} &\le c_1 \|[\Delta, \varphi_j](v_2 - v_1)\|_X \\ &\le c_1 \left(\varepsilon_1 + \frac{c_{\varepsilon_1}}{\lambda}\right) \|v_2 - v_1\|_{\mathcal{Z}^\lambda_{\gamma+3}}.\end{aligned} \tag{4.23}$$

Facing the second sum from (4.21) we get

$$\begin{aligned}&\sum_{j=n_1+1}^{n} \left\| \mathcal{S}^B_j \begin{pmatrix} \Theta_j^{-1}[\Delta, \varphi_j](v_2 - v_1) \\ \Theta_j^{-1}\Gamma_y[\mathcal{D}, \varphi_j](v_1 - v_2) \end{pmatrix} \right\|_{\mathcal{Z}^\lambda_{\gamma+3}} \\ &\quad\le c_2 \sum_{j=n_1+1}^{n} \|[\Delta, \varphi_j](v_2 - v_1)\|_X + C(\lambda) \sum_{j=n_1+1}^{n} \|\Gamma_y[\mathcal{D}, \varphi_j](v_1 - v_2)\|_{D(F^{\frac{\gamma}{2}})},\end{aligned}$$

where $C(\lambda) \sim \lambda^{\frac{\gamma}{4}}$, which is due to Proposition 4.3. The first term is fine by (4.23). For the second we may stress that $u \in Z_{\gamma+1}$ implies $\Gamma_y u \in D(F^{\frac{\gamma}{2}})$. Now we estimate with the aid of Young's inequality

$$\begin{aligned}\|\Gamma_y[\mathcal{D}, \varphi_j](v_1 - v_2)\|_{D(F^{\frac{\gamma}{2}})} &\le c_3 \|v_1 - v_2\|_{Z_{\gamma+1}} \\ &\le c_4(\varepsilon_2 \|v_1 - v_2\|_{Z_{\gamma+3}} + c_{\varepsilon_2}\|v_1 - v_2\|_X) \\ &\le c_4(\varepsilon_2 + \frac{c_{\varepsilon_2}}{\lambda})\|v_1 - v_2\|_{\mathcal{Z}^\lambda_{\gamma+3}}.\end{aligned} \tag{4.24}$$

Hence, we deduced

$$\begin{aligned}&\|\mathcal{G}(v_1) - \mathcal{G}(v_2)\|_{\mathcal{Z}^\lambda_{\gamma+3}} \\ &\qquad\le c_5 \left[(n+1)\left(\varepsilon_1 + \frac{c_{\varepsilon_1}}{\lambda}\right) + (n - n_1)C(\lambda)\left(\varepsilon_2 + \frac{c_{\varepsilon_2}}{\lambda}\right)\right] \|v_1 - v_2\|_{\mathcal{Z}^\lambda_{\gamma+3}},\end{aligned}$$

where the Lipschitz constant can be made arbitrary small by choosing first ε_1 and ε_2 sufficiently small and than selecting λ appropriately large (recall that $C(\lambda) \sim \lambda^{\frac{\gamma}{4}}$ by Proposition 4.3 and $\gamma < 4$ by assumption).

Summarizing we have shown that the solution v of (4.13) belongs to $Z_{\gamma+3}$. Lastly, it is due to [15, Theorem 3.1] that the solution w of the auxiliary problem (4.14) in particular belongs to $Z_{\gamma+7}$, which immediately results in the fact that the function $u = v + w$ is a solution of the initial problem (2.1) and, moreover, affiliates to the space $Z_{\gamma+3}$. This completes the proof of (ii). The proof of the corresponding Dirichlet problem (i) can be obtained by following the arguments of the proof of (ii). Therefore we omit it.

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