

Discontinuous Variational-Hemivariational Inequalities Involving the p-Laplacian

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Abstract

In this paper we deal with discontinuous quasilinear elliptic variational-hemivariational inequalities. By using the method of sub- and supersolution and based on the results of S. Carl we extend the theory for discontinuous problems. The proof of the existence of extremal solutions within a given order interval of sub- and supersolutions is the main goal of this paper. In the last part we give an example of the construction of sub- and supersolutions.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. By $V = W^{1,p}(\Omega)$ and $V_0 = W_0^{1,p}(\Omega)$, $1 < p < \infty$, we denote the usual Sobolev spaces with their dual spaces $V^* = (W^{1,p}(\Omega))^*$ and $V_0^* = W^{-1,q}(\Omega)$, respectively, (q is the Hölder conjugate of p). In this paper, we consider the following elliptic variational-hemivariational inequality:

$$u \in K : \quad \langle -\Delta_p u + F(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx \geq 0, \quad \forall v \in K, \quad (1.1)$$

where $j^0(s; r)$ denotes the generalized directional derivative of the locally Lipschitz function $j : \mathbb{R} \rightarrow \mathbb{R}$ at s in the direction r given by

$$j^0(s; r) = \limsup_{y \rightarrow s, t \downarrow 0} \frac{j(y + tr) - j(y)}{t}, \quad (1.2)$$

(cf. [7, Chapter 2]), and $K \subset V_0$ is some closed and convex subset. The operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian, $1 < p < \infty$, and F denotes the

Nemytskij operator related to the function $f : \Omega \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ given by

$$F(u)(x) = f(x, u(x), u(x)).$$

In [3] the method of sub- and supersolution was developed for variational-hemivariational inequalities of the form (1.1) with $F(u) \equiv f \in V_0^*$. The aim of this paper is the generalization for discontinuous Nemytskij operators $F : L^p(\Omega) \rightarrow L^q(\Omega)$. Let us consider some special cases of problem (1.1):

- (i) For $f \in V_0^*$, (1.1) is also a variational-hemivariational inequality which is discussed in [3].
- (ii) If $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfying some growth condition and $j = 0$, then (1.1) is a classical variational inequality of the form

$$u \in K : \langle -\Delta_p u + F(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

for which the method of sub- and supersolution has been developed in [5, Chapter 5].

- (iii) For $K = V_0$, $f \in V_0^*$ and $j : \mathbb{R} \rightarrow \mathbb{R}$ smooth, (1.1) becomes a variational equality of the form

$$u \in V_0 : \langle -\Delta_p u + f + j'(u), \varphi \rangle = 0, \quad \forall \varphi \in V_0,$$

for which the sub-supersolution method is well known.

2 Notations and hypotheses

For functions $u, v : \Omega \rightarrow \mathbb{R}$ we use the notation $u \wedge v = \min(u, v)$, $u \vee v = \max(u, v)$, $K \wedge K = \{u \wedge v : u, v \in K\}$, $K \vee K = \{u \vee v : u, v \in K\}$, and $u \wedge K = \{u\} \wedge K$, $u \vee K = \{u\} \vee K$ and introduce the following Definitions:

Definition 2.1 *A function $\underline{u} \in V$ is called a subsolution of (1.1) if the following holds:*

1. $\underline{u} \leq 0$ on $\partial\Omega$ and $F(\underline{u}) \in L^q(\Omega)$
2. $\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j^0(\underline{u}; w - \underline{u}) dx \geq 0, \quad \forall w \in \underline{u} \wedge K$

Definition 2.2 A function $\bar{u} \in V$ is called a supersolution of (1.1) if the following holds:

1. $\bar{u} \geq 0$ on $\partial\Omega$ and $F(\bar{u}) \in L^q(\Omega)$
2. $\langle -\Delta_p \bar{u} + F(\bar{u}), w - \bar{u} \rangle + \int_{\Omega} j^0(\bar{u}; w - \bar{u}) dx \geq 0, \quad \forall w \in \bar{u} \vee K$

Definition 2.3 The multivalued operator $\partial j : \mathbb{R} \rightarrow 2^{\mathbb{R}} \setminus \{\emptyset\}$ is called Clarke's generalized gradient of j defined by

$$\partial j(s) := \{\xi \in \mathbb{R} : j^0(s; r) \geq \xi r, \forall r \in \mathbb{R}\}. \quad (2.1)$$

We impose the following hypotheses for j and the nonlinearity f in problem (1.1):

(A) There exists a constant $c_1 \geq 0$ such that

$$\xi_1 \leq \xi_2 + c_1(s_2 - s_1)^{p-1} \quad (2.2)$$

for all $\xi_i \in \partial j(s_i), i = 1, 2$, and for all s_1, s_2 with $s_1 < s_2$.

(B) There is a constant $c_2 \geq 0$ such that

$$\xi \in \partial j(s) : |\xi| \leq c_2(1 + |s|^{p-1}), \quad \forall s \in \mathbb{R}. \quad (2.3)$$

(C) (i) $x \mapsto f(x, r, u(x))$ is measurable for all $r \in \mathbb{R}$ and for all measurable functions $u : \Omega \rightarrow \mathbb{R}$.

(ii) $r \mapsto f(x, r, s)$ is continuous for all $s \in \mathbb{R}$ and for almost all $x \in \Omega$.

(iii) $s \mapsto f(x, r, s)$ is decreasing for all $r \in \mathbb{R}$ and for almost all $x \in \Omega$.

(iv) For a given ordered pair of sub- and supersolutions \underline{u}, \bar{u} of problem (1.1) there exists a function $k_1 \in L^q_+(\Omega)$ such that $|f(x, r, s)| \leq k_1(x)$ for all $r, s \in [\underline{u}(x), \bar{u}(x)]$ and for almost all $x \in \Omega$.

By [2] the mapping $x \mapsto f(x, u(x), u(x))$ is measurable for $x \mapsto u(x)$ measurable, but the associated Nemytskij operator $F : L^p(\Omega) \rightarrow L^q(\Omega)$ need not necessarily be continuous. In this paper we assume K has lattice structure, i.e., K fulfills

$$K \vee K \subset K \quad \text{and} \quad K \wedge K \subset K. \quad (2.4)$$

We recall that the normed space $L^p(\Omega)$ is equipped with the natural partial ordering of functions defined by $u \leq v$ if and only if $v - u \in L^p_+(\Omega)$, where $L^p_+(\Omega)$ is the set of all nonnegative functions of $L^p(\Omega)$.

3 Preliminaries

Here we consider (1.1) for a Carathéodory function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ (i.e., $x \mapsto h(x, s)$ is measurable in Ω for all $s \in \mathbb{R}$ and $s \mapsto h(x, s)$ is continuous on \mathbb{R} for almost all $x \in \Omega$), which fulfills the following growth condition:

$$|h(x, s)| \leq k_2(x) \text{ for all } s \in [\underline{u}(x), \bar{u}(x)] \text{ and for a.e. } x \in \Omega, \quad (3.1)$$

where $k_2 \in L^q_+(\Omega)$ and $[\underline{u}, \bar{u}]$ is some ordered pair in $L^p(\Omega)$, specified later. Note that the associated Nemytskij operator H defined by $H(u)(x) = h(x, u(x))$ is continuous and bounded from $[\underline{u}, \bar{u}] \subset L^p(\Omega)$ to $L^q(\Omega)$ (cf. [9]). Next we introduce the indicator function $I_K : V_0 \rightarrow \mathbb{R} \cup \{+\infty\}$ related to the closed convex set $K \neq \emptyset$ given by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K \\ +\infty & \text{if } u \notin K, \end{cases} \quad (3.2)$$

which is known to be proper, convex and lower semicontinuous. The variational-hemivariational inequality (1.1) can be rewritten as follows:

$$\begin{cases} \langle -\Delta_p u + H(u), v - u \rangle + I_K(v) - I_K(u) + \int_{\Omega} j^0(u; v - u) dx \geq 0, \forall v \in V_0 \\ u \in V_0 \end{cases} \quad (3.3)$$

If $H(u) \equiv h \in V_0^*$, problem (3.3) is a special case of the elliptic variational-hemivariational inequality in [5, Corollary 7.15] for which the method of sub- and supersolution was developed. In the next result we show the existence of extremal solutions of (3.3) for a Carathéodory function $h = h(x, s)$.

Lemma 3.1 *Let hypotheses (A),(B) and (2.4) be satisfied and assume the existence of sub- and supersolutions \underline{u} and \bar{u} satisfying $\underline{u} \leq \bar{u}$, $\underline{u} \vee K \subset K$ and $\bar{u} \wedge K \subset K$. Furthermore we suppose that the Carathéodory function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies (3.1). Then, (3.3) has a greatest solution u^* and a smallest solution u_* such that*

$$\underline{u} \leq u_* \leq u^* \leq \bar{u}, \quad (3.4)$$

that is, u_ and u^* are solutions of (3.3) that satisfy (3.4), and if u is any solution of (3.3) such that $\underline{u} \leq u \leq \bar{u}$, then $u_* \leq u \leq u^*$.*

Proof: The proof follows the same ideas as in the proof for $H(u) \equiv h \in V_0^*$ with an additional modification. We only introduce a truncation operator related to

the functions \underline{u} and \bar{u} defined by

$$Tu(x) = \begin{cases} \bar{u}(x) & \text{if } u(x) > \bar{u}(x) \\ u(x) & \text{if } \underline{u}(x) \leq u(x) \leq \bar{u}(x) \\ \underline{u}(x) & \text{if } u(x) < \underline{u}(x). \end{cases}$$

The mapping T is continuous and bounded from V into V which follows from the fact that the functions $\min(\cdot, \cdot)$ and $\max(\cdot, \cdot)$ are continuous from V to itself and that T can be represented as $Tu = \max(u, \underline{u}) + \min(u, \bar{u}) - u$ (cf. [8]). In the auxiliary problems of the proof of Corollary 7.15 in [5] we replace $h \in V_0^*$ by $(H \circ T)(u)$ and argue in an analogous way. \square

An important tool in extending the previous result to discontinuous Nemytskij operators is the next fixed point result. The proof of this Lemma can be found in [4, Theorem 1.1.1].

Lemma 3.2 *Let P be a subset of an ordered normed space, $G : P \rightarrow P$ an increasing mapping and $G[P] = \{Gx \mid x \in P\}$.*

1. *If $G[P]$ has a lower bound in P and increasing sequences of $G[P]$ converge weakly in P , then G has the least fixed point x_* , and $x_* = \min\{x \mid Gx \leq x\}$.*
2. *If $G[P]$ has an upper bound in P and decreasing sequences of $G[P]$ converge weakly in P , then G has the greatest fixed point x^* , and $x^* = \max\{x \mid x \leq Gx\}$.*

4 Main results

One of our main results is the following theorem.

Theorem 4.1 *Let hypotheses (A)–(C), (2.4) be satisfied and assume the existence of sub- and supersolutions \underline{u} and \bar{u} satisfying $\underline{u} \leq \bar{u}$, $\underline{u} \vee K \subset K$ and $\bar{u} \wedge K \subset K$. If f is right-continuous (respectively, left-continuous) in the third argument, then there exists a greatest solution u^* (respectively, a smallest solution u_*) of (1.1) in the order interval $[\underline{u}, \bar{u}]$.*

Proof: We choose a fixed element $z \in [\underline{u}, \bar{u}]$ which is a supersolution of (1.1) satisfying $z \wedge K \subset K$ and consider the following auxiliary problem

$$u \in K : \quad \langle -\Delta_p u + F_z(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx \geq 0, \quad \forall v \in K, \quad (4.1)$$

where $F_z(u)(x) = f(x, u(x), z(x))$. It is readily seen that the mapping $(x, u) \mapsto f(x, u, z(x))$ is a Carathéodory function satisfying some growth condition as in

(3.1). Since $F_z(z) = F(z)$, z is also a supersolution of (4.1). By Definition 2.1 we have for a given subsolution \underline{u} of (1.1)

$$\langle -\Delta_p \underline{u} + F(\underline{u}), w - \underline{u} \rangle + \int_{\Omega} j^0(\underline{u}; w - \underline{u}) dx \geq 0, \quad \forall w \in \underline{u} \wedge K.$$

Setting $w = \underline{u} - (\underline{u} - v)^+$ for all $v \in K$ and using the monotonicity of f with respect to s we get

$$\begin{aligned} 0 &\geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\underline{u}; -(\underline{u} - v)^+) dx \\ &\geq \langle -\Delta_p \underline{u} + F_z(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\underline{u}; -(\underline{u} - v)^+) dx, \quad \forall v \in K, \end{aligned}$$

which shows that \underline{u} is also a subsolution of (4.1). Lemma 3.1 implies the existence of a greatest solution $u^* \in [\underline{u}, z]$ of (4.1). Now we introduce the set A given by $A := \{z \in V : z \in [\underline{u}, \bar{u}] \text{ and } z \text{ is a supersolution of (1.1) satisfying } z \wedge K \subset K\}$ and define the operator $L : A \rightarrow K$ by $z \mapsto u^* =: Lz$. This means that the operator L assigns to each $z \in A$ the greatest solution u^* of (4.1) in $[\underline{u}, z]$. In the next step we construct a decreasing sequence as follows:

$$\begin{aligned} u_0 &:= \bar{u} \\ u_1 &:= Lu_0 \quad \text{with } u_1 \in [\underline{u}, u_0] \\ u_2 &:= Lu_1 \quad \text{with } u_2 \in [\underline{u}, u_1] \\ &\vdots \\ u_n &:= Lu_{n-1} \quad \text{with } u_n \in [\underline{u}, u_{n-1}] \end{aligned} \tag{4.2}$$

As $u_n \in [\underline{u}, u_{n-1}]$, we get $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$. Furthermore, the sequence u_n is bounded in V_0 , that is, $\|u_n\|_{V_0} \leq C$ for all n and due to the monotony of u_n and the compact embedding $V_0 \hookrightarrow L^p(\Omega)$ we obtain

$$u_n \rightharpoonup u \quad \text{in } V_0, \quad u_n \rightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. pointwise in } \Omega. \tag{4.3}$$

The fact that u_n is a solution of (4.1) with $z = u_{n-1}$ and $v = u \in K$ results in

$$\langle -\Delta_p u_n, u_n - u \rangle \leq \langle F_{u_{n-1}}(u_n), u - u_n \rangle + \int_{\Omega} j^0(\underline{u}; u - u_n) dx.$$

Applying Fatou's Lemma, (4.3) and the upper semicontinuity of $j^0(\cdot, \cdot)$ yields

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle -\Delta_p u_n, u_n - u \rangle &\leq \limsup_{n \rightarrow \infty} \underbrace{\|k\|_{L^q(\Omega)} \|u - u_n\|_{L^p(\Omega)}}_{\rightarrow 0} \\ &\quad + \int_{\Omega} \limsup_{n \rightarrow \infty} \underbrace{j^0(\underline{u}; u - u_n)}_{\leq j^0(\underline{u}; 0) = 0} dx \\ &\leq 0, \end{aligned}$$

which by the S_+ -property of $-\Delta_p$ on V_0 along with (4.3) implies

$$u_n \rightarrow u \quad \text{in } V_0.$$

The right-continuity of f and the strong convergence of the decreasing sequence (u_n) along with the upper semicontinuity of $j^0(\cdot; \cdot)$ allows us to pass to the lim sup in (4.1) where u (respectively, z) is replaced by u_n (respectively, u_{n-1}). We have

$$\begin{aligned} 0 &\leq \limsup_{n \rightarrow \infty} \langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \limsup_{n \rightarrow \infty} \int_{\Omega} j^0(u_n; v - u_n) dx \\ &\leq \lim_{n \rightarrow \infty} \langle -\Delta_p u_n + F_{u_{n-1}}(u_n), v - u_n \rangle + \int_{\Omega} \limsup_{n \rightarrow \infty} j^0(u_n; v - u_n) dx \\ &\leq \langle -\Delta_p u + F_u(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx, \quad \forall v \in K. \end{aligned}$$

This shows that u is a solution of (1.1) in the order interval $[\underline{u}, \bar{u}]$. Now, we still have to prove that u is the greatest solution of (1.1) in $[\underline{u}, \bar{u}]$. Let \tilde{u} be any solution of (1.1) in $[\underline{u}, \bar{u}]$. Because of the fact that K has lattice structure, \tilde{u} is also a subsolution of (1.1), respectively, a subsolution of (4.1). By the same construction as in (4.2) we obtain:

$$\begin{aligned} \tilde{u}_0 &:= \bar{u} \\ \tilde{u}_1 &:= Lu_0 \quad \text{with } \tilde{u}_1 \in [\tilde{u}, u_0] \\ \tilde{u}_2 &:= Lu_1 \quad \text{with } \tilde{u}_2 \in [\tilde{u}, u_1] \\ &\vdots \\ \tilde{u}_n &:= Lu_{n-1} \quad \text{with } \tilde{u}_n \in [\tilde{u}, u_{n-1}]. \end{aligned} \tag{4.4}$$

Obviously, the sequences in (4.2) and (4.4) create the same extremal solutions u_n and \tilde{u}_n which implies that $\tilde{u} \leq \tilde{u}_n = u_n$ for all n . Passing to the limit delivers the assertion. The existence of a smallest solution can be shown in a similar way. \square

In the next Theorem we will prove that only the monotony of f in the third argument is sufficient for the existence of extremal solutions. The function f needs neither be right-continuous nor left-continuous.

Theorem 4.2 *Assume that hypotheses (A)–(C), (2.4) are valid and let \underline{u} and \bar{u} be sub- and supersolutions of (1.1) satisfying $\underline{u} \leq \bar{u}$, $\underline{u} \vee K \subset K$ and $\bar{u} \wedge K \subset K$. Then there exist extremal solutions u^* and u_* of (1.1) with $\underline{u} \leq u_* \leq u^* \leq \bar{u}$.*

Proof: As in the proof of Theorem 4.1 we consider the following auxiliary problem

$$u \in K : \quad \langle -\Delta_p u + F_z(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx \geq 0, \quad \forall v \in K, \tag{4.5}$$

where $F_z(u)(x) = f(x, u(x), z(x))$. We define again the set $A := \{z \in V : z \in [\underline{u}, \bar{u}]$ and z is a supersolution of (1.1) satisfying $z \wedge K \subset K\}$ and introduce the fixed point operator $L : A \rightarrow K$ by $z \mapsto u^* =: Lz$. For a given supersolution $z \in A$, the element Lz is the greatest solution of (4.5) in $[\underline{u}, z]$ and thus it holds that $\underline{u} \leq Lz \leq z$ for all $z \in A$ which implies $L : A \rightarrow [\underline{u}, \bar{u}]$. Because of (2.4), Lz is also a supersolution of (4.5) satisfying

$$\langle -\Delta_p Lz + F_z(Lz), w - Lz \rangle + \int_{\Omega} j^0(Lz; w - Lz) dx \geq 0, \quad \forall w \in Lz \vee K.$$

By the monotonicity of f with respect to $Lz \leq z$ and using the representation $w = Lz + (v - Lz)^+$ for any $v \in K$ we obtain

$$\begin{aligned} 0 &\leq \langle -\Delta_p Lz + F_z(Lz), (v - Lz)^+ \rangle + \int_{\Omega} j^0(Lz; (v - Lz)^+) dx \\ &\leq \langle -\Delta_p Lz + F_{Lz}(Lz), (v - Lz)^+ \rangle + \int_{\Omega} j^0(Lz; (v - Lz)^+) dx, \quad \forall v \in K. \end{aligned}$$

Consequently, Lz is a supersolution of (1.1). This shows $L : A \rightarrow A$.

Let $v_1, v_2 \in A$ and assume that $v_1 \leq v_2$. Then we have

$$\begin{aligned} Lv_1 \in [\underline{u}, v_1] \text{ is the greatest solution of} \\ \langle -\Delta_p u + F_{v_1}(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx \geq 0, \quad \forall v \in K, \end{aligned} \quad (4.6)$$

and

$$\begin{aligned} Lv_2 \in [\underline{u}, v_2] \text{ is the greatest solution of} \\ \langle -\Delta_p u + F_{v_2}(u), v - u \rangle + \int_{\Omega} j^0(u; v - u) dx \geq 0, \quad \forall v \in K. \end{aligned} \quad (4.7)$$

Since $v_1 \leq v_2$ it follows that $Lv_1 \leq Lv_2$ and due to (2.4), Lv_1 is also a subsolution of (4.6), that is, (4.6) holds, in particular, for $v \in Lv_1 \wedge K$, that is,

$$\langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j^0(Lv_1; -(Lv_1 - v)^+) dx \leq 0, \quad \forall v \in K.$$

Using the monotonicity of f with respect to s yields

$$\begin{aligned} 0 &\geq \langle -\Delta_p Lv_1 + F_{v_1}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j^0(Lv_1; -(Lv_1 - v)^+) dx \\ &\geq \langle -\Delta_p Lv_1 + F_{v_2}(Lv_1), (Lv_1 - v)^+ \rangle - \int_{\Omega} j^0(Lv_1; -(Lv_1 - v)^+) dx, \quad \forall v \in K, \end{aligned}$$

and hence Lv_1 is a subsolution of (4.7). By Lemma (3.1), we know there exists a greatest solution of (4.7) in $[Lv_1, v_2]$. But Lv_2 is the greatest solution of (4.7) in $[\underline{u}, v_2] \supseteq [Lv_1, v_2]$ and therefore, $Lv_1 \leq Lv_2$. This shows that L is increasing.

In the last step we have to prove that any decreasing sequence of $L(A)$ converges weakly in A . Let $(u_n) = (Lz_n) \subset L(A) \subset A$ be a decreasing sequence. The same argument as in the proof of Theorem 4.1 delivers $u_n(x) \searrow u(x)$ a.e. $x \in \Omega$. The boundedness of u_n in V_0 and the compact imbedding $V_0 \hookrightarrow L^p(\Omega)$ along with the monotony of u_n implies

$$u_n \rightharpoonup u \quad \text{in } V_0, \quad u_n \rightarrow u \quad \text{in } L^p(\Omega) \text{ and a.e. } x \in \Omega. \quad (4.8)$$

Since $u_n \in K$ solves (4.5), it follows $u \in K$. From (4.5) with u replaced by u_n and v by u and the fact that $(s, r) \mapsto j^0(s; r)$ is upper semicontinuous we obtain by applying Fatou's Lemma

$$\begin{aligned} \limsup_{n \rightarrow \infty} \langle -\Delta_p u_n, u_n - u \rangle &\leq \limsup_{n \rightarrow \infty} \langle F_{z_n}(u_n), u - u_n \rangle + \limsup_{n \rightarrow \infty} \int_{\Omega} j^0(u_n; u - u_n) dx \\ &\leq \underbrace{\limsup_{n \rightarrow \infty} \langle F_{z_n}(u_n), u - u_n \rangle}_{\rightarrow 0} + \int_{\Omega} \underbrace{\limsup_{n \rightarrow \infty} j^0(u_n; u - u_n)}_{\leq j^0(u; 0) = 0} dx \\ &\leq 0. \end{aligned}$$

The S_+ -property of $-\Delta_p$ provides the strong convergence of (u_n) in V_0 . As $Lz_n = u_n$ is also a supersolution of (4.5) Definition 2.2 yields

$$\langle -\Delta_p u_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j^0(u_n; (v - u_n)^+) dx \geq 0, \quad \forall v \in K.$$

Due to $z_n \geq u_n \geq u$ and the monotonicity of f we get

$$\begin{aligned} 0 &\leq \langle -\Delta_p u_n + F_{z_n}(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j^0(u_n; (v - u_n)^+) dx \\ &\leq \langle -\Delta_p u_n + F_u(u_n), (v - u_n)^+ \rangle + \int_{\Omega} j^0(u_n; (v - u_n)^+) dx, \quad \forall v \in K, \end{aligned}$$

and, since the mapping $u \mapsto u^+ = \max(u, 0)$ is continuous from V_0 to itself (cf. [8]), we can pass to the upper limit on the right hand side for $n \rightarrow \infty$. This yields

$$\langle -\Delta_p u + F_u(u), (v - u)^+ \rangle + \int_{\Omega} j^0(u; (v - u)^+) dx \geq 0, \quad \forall v \in K,$$

which shows that u is a supersolution of (1.1), that is, $u \in A$. As \bar{u} is an upper bound of $L(A)$, we can apply Lemma 3.2, which yields the existence of a greatest fixed point u^* of L in A . This implies that u^* must be the greatest solution of (1.1) in $[\underline{u}, \bar{u}]$. By analogous reasoning, one shows the existence of a smallest solution u_* of (1.1). This completes the proof of the theorem. \square

Application

In the last part we give an example of the construction of sub- and supersolutions of problem (1.1). We denote by $\lambda_1 > 0$ the first eigenvalue of $(-\Delta_p, V_0)$ and by φ_1 the eigenfunction of $(-\Delta_p, V_0)$ corresponding to λ_1 satisfying $\varphi_1 \in \text{int}(C_0^1(\overline{\Omega})_+)$ and $\|\varphi_1\|_p = 1$ (cf. [1]). Here, $\text{int}(C_0^1(\overline{\Omega})_+)$ describes the interior of the positive cone $C_0^1(\overline{\Omega})_+$ given by

$$\text{int}(C_0^1(\overline{\Omega})_+) = \left\{ u \in C_0^1(\overline{\Omega}) : u(x) > 0, \forall x \in \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \forall x \in \partial\Omega \right\}.$$

We suppose the following conditions for f and Clarke's generalized gradient of j , where $\lambda > \lambda_1$ is any fixed constant:

- (D) (i) $\lim_{|s| \rightarrow \infty} \frac{f(x, s, s)}{|s|^{p-2}s} = +\infty$, uniformly with respect to a.a. $x \in \Omega$,
- (ii) $\lim_{s \rightarrow 0} \frac{f(x, s, s)}{|s|^{p-2}s} = -\lambda$, uniformly with respect to a.a. $x \in \Omega$,
- (iii) $\lim_{s \rightarrow 0} \frac{\xi}{|s|^{p-2}s} = 0$, uniformly with respect to a.a. $x \in \Omega$, for all $\xi \in \partial j(s)$,
- (iv) f is bounded on bounded sets.

Proposition 4.3 *Assume hypotheses (A), (B), (C)(i)-(iv), and (D). Then there exists a constant a_λ such that $a_\lambda e$ and $-a_\lambda e$ are supersolution and subsolution of problem (1.1), where $e \in \text{int}(C_0^1(\overline{\Omega})_+)$ is the unique solution of $-\Delta_p u = 1$ in V_0 . Moreover, $-\varepsilon \varphi_1$ is a supersolution and $\varepsilon \varphi_1$ is a subsolution of (1.1) provided that $\varepsilon > 0$ is sufficiently small.*

Proof: A sufficient condition for a subsolution $\underline{u} \in V$ of problem (1.1) is $\underline{u} \leq 0$ on $\partial\Omega$, $F(\underline{u}) \in L^q(\Omega)$ and

$$-\Delta_p \underline{u} + F(\underline{u}) + \xi \leq 0 \text{ in } V_0^*, \text{ for all } \xi \in \partial j(\underline{u}). \quad (4.9)$$

Multiplying (4.9) with $(\underline{u} - v)^+ \in V_0 \cap L_+^p(\Omega)$ and using the fact $j^0(\underline{u}; -1) \geq -\xi$, for all $\xi \in \partial j(\underline{u})$, yield

$$\begin{aligned} 0 &\geq \langle -\Delta_p \underline{u} + F(\underline{u}) + \xi, (\underline{u} - v)^+ \rangle \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle + \int_{\Omega} \xi (\underline{u} - v)^+ dx \\ &\geq \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\underline{u}; -1) (\underline{u} - v)^+ dx \\ &= \langle -\Delta_p \underline{u} + F(\underline{u}), (\underline{u} - v)^+ \rangle - \int_{\Omega} j^0(\underline{u}; -(\underline{u} - v)^+) dx, \quad \forall v \in K, \end{aligned}$$

and thus, \underline{u} is a subsolution of (1.1). Analogously, $\bar{u} \in V$ is a supersolution of problem (1.1) if $\bar{u} \geq 0$ on $\partial\Omega$, $F(\bar{u}) \in L^q(\Omega)$ and if the following inequality is satisfied

$$-\Delta_p \bar{u} + F(\bar{u}) + \xi \geq 0 \text{ in } V_0^*, \text{ for all } \xi \in \partial j(\bar{u}).$$

The main idea of this proof is to show the applicability of [6, Lemmas 2.1–2.3]. We put $g(x, s) = f(x, s, s) + \xi + \lambda|s|^{p-2}s$ for $\xi \in \partial j(s)$ and notice that in our considerations the nonlinearity g needs not be a continuous function. In view of assumption (B) we see at once that

$$\frac{|\xi|}{|s|^{p-1}} \leq c, \quad \text{for } |s| \geq k > 0, \text{ for all } \xi \in \partial j(s),$$

where c is a positive constant. This fact and the condition (D) yield the following limit values

$$\lim_{|s| \rightarrow \infty} \frac{g(x, s)}{|s|^{p-2}s} = +\infty, \quad \text{and} \quad \lim_{s \rightarrow 0} \frac{g(x, s)}{|s|^{p-2}s} = 0.$$

By [6, Lemmas 2.1–2.3] we obtain a pair of positive sub- and supersolutions given by $\underline{u} = \varepsilon\varphi_1$ and $\bar{u} = a_\lambda e$, respectively, a pair of negative sub- and supersolutions given by $\underline{u} = -a_\lambda e$ and $\bar{u} = -\varepsilon\varphi_1$. \square

In order to apply Theorem 4.2 we need to satisfy the assumptions

$$\underline{u} \vee K \subset K, \quad \bar{u} \wedge K \subset K, \quad K \vee K \subset K, \quad K \wedge K \subset K, \quad (4.10)$$

which depends on the specific K . For example we consider an obstacle problem given by

$$K = \{v \in V_0 : v(x) \leq \psi(x) \text{ for a.e. } x \in \Omega\}, \quad \psi \in L^\infty(\Omega), \psi \geq C > 0, \quad (4.11)$$

where C is a positive constant. One can show that for the positive pair of sub- and supersolutions in Proposition 4.3 all these conditions in (4.10) with respect to the closed convex set K defined in (4.11) can be satisfied.

Example 4.4 *The function $f : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ defined by*

$$f(r, s) = \begin{cases} -(\lambda + 1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s < -1 \\ -\lambda|s|^{p-2}s + |r|^{p-1}r & \text{for } -1 \leq s \leq 1 \\ -(\lambda + 1)|s|^{p-2}s + |r|^{p-1}r & \text{for } s > 1, \end{cases}$$

fulfills the assumption (C)(i)–(iv) with respect to \underline{u}, \bar{u} defined in Proposition 4.3. Moreover f satisfies the conditions (D)(i)–(ii), (D)(iv), where $\lambda > \lambda_1$ is fixed.

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