ON PARABOLIC VOLTERRA EQUATIONS DISTURBED BY FRACTIONAL BROWNIAN MOTIONS

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Abstract. Aim of this paper is to study the parabolic Volterra equation

$$u(t) + (b * Au)(t) = (Q^{1/2}B^{H})(t), \quad t \geq 0,$$

on a separable Hilbert space. Throughout this work the operator $-A$ is assumed to be a differential operator like the Laplacian, the elasticity operator, or the Stokes operator. The random disturbance $Q^{1/2}B^{H}$ is modeled to be a system independent vector valued fractional Brownian motion with Hurst parameter $H \in (0,1)$. We derive optimal conditions for the existence of a unique mild solution and the Hölderianity of its trajectories. For this purpose we do the analysis on stochastic integrals of the form

$$\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^{H})(t), \quad t \geq 0,$$

where the integrand R is a deterministic, operator valued function.

1. Introduction

Random effects on mainly deterministic systems occur in many areas, for instance in flow mechanics or interest rate models. Usually *Wiener processes* are used to describe these random effects, but this is no longer adequate if the perturbations possess any chronological dependence. For this purpose Mandelbrot and van Ness proposed in their famous work [10] the notion of a *fractional Brownian motion.* Basically this concept formulates Gaussian processes indexed by a parameter $H \in (0,1)$. This parameter was named after the hydrologist Hurst who, together with some collaborators, demonstrated in their pioneering work [7] that this approach is appropriate to describe statistic time series in a hydrological framework. To the authors knowledge, the first work on stochastic differential equations with a vector valued fractional Brownian motion disturbance is Grecksch & Anh [6].

In *Section 2* we start to collect some basic tools needed for the investigation of the equations to be studied. After fixing some notations we review some results on

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the theory of the abstract Volterra equations. Concerning parabolic problems of Volterra type we use the monograph of Prüss [13] as a general reference.

The aim of *Section 3* is to introduce stochastic integration with respect to a vector valued fractional Brownian motion. We consider deterministic, operator valued integrands and study the stochastic integral

$$\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^{\boldsymbol{H}})(t), \quad t \geq 0.$$

Therefore we review the definition of a real valued fractional Brownian motion, repeat its property of Hölderianity and define a Q-covariance fractional Brownian motion which takes values in a separable Hilbert space $\mathcal{H}$. Unlike most publications concerning stochastic integration (e.g. Duncan, Jakubowski & Pasik-Duncan [5], van Neerven & Veraar [17], Dettweiler, Weis & van Neerven [4]) we adhere to the Q-covariance type instead of considering a cylindrical process. We obtain a mean square isometry of Itô type and deduce estimates of the first and second moment. Even though this result is not completely new, we connect it closely to the homogenous Bessel potential space of a certain order, which appears to be quite satisfactory.

Finally, in *Section 4*, we turn our attention to equations of the type

$$u(t) + (b * Au)(t) = (Q^{1/2}B^{\boldsymbol{H}})(t), \quad t \geq 0,$$

where A is assumed to be an unbounded, selfadjoint, positive definite operator on $\mathcal{H}$ with compact resolvent and the kernel b should be thought of as a stress relaxation kernel, or a material function of fluid behavior. We prove certain existence and regularity results for a system independent right hand side $Q^{1/2}B^{\boldsymbol{H}}$. With system independence we mean, that the eigensystems of the operators A and Q do not necessarily coincide.

2. Preliminaries

2.1. **Some notations, function spaces, Laplace transform.** In this section we fix some of the notations used throughout this paper, recall some basic definitions and give references concerning function spaces and the Laplace transform.

By $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of natural numbers, real and complex numbers, respectively, and let further $\mathbb{R}_+ = [0, \infty)$, $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0\}$. $\mathcal{H}$ will denote a separable Hilbert space with norm $|\cdot|_{\mathcal{H}}$ and inner product $(\cdot|\cdot)_{\mathcal{H}}$. X and Y will usually be Banach spaces; $|\cdot|_X$ designates the norm of X. When there are no ambiguities we will skip this index. The symbol $\mathcal{B}(X;Y)$ means the space of all bounded linear operators from X to Y, we write $\mathcal{B}(X) = \mathcal{B}(X;X)$ for short.

Furthermore the symbols $\mathscr{L}_1(\mathcal{H})$ and $\mathscr{L}_2(\mathcal{H})$ denotes the spaces of *nuclear operators* and *Hilbert-Schmidt operators* on $\mathcal{H}$, respectively; we have the embeddings

$$\mathscr{L}_1(\mathcal{H}) \hookrightarrow \mathscr{L}_2(\mathcal{H}) \hookrightarrow \mathcal{B}(\mathcal{H}).$$

In case the operator $T : \mathcal{H} \to \mathcal{H}$ is selfadjoint with eigenvalues $\lambda = (\lambda_n)_{n\in\mathbb{N}}$, the norms in these spaces can be written as

$$\begin{aligned} \|T\|_{\mathscr{L}_1(\mathcal{H})} &= \|\lambda\|_{\ell_1}, \\ \|T\|_{\mathscr{L}_2(\mathcal{H})} &= \|\lambda\|_{\ell_2}. \end{aligned}$$

For nuclear operators T on $\mathcal{H}$ one can define the trace of T in virtue of

$$\operatorname{Tr}[T] = \sum_{n=1}^{\infty} (Tg_n \mid g_n)_{\mathcal{H}},$$

where $(g_n)_{n\in\mathbb{N}}$ is an orthonormal basis in $\mathcal{H}$. Note that the series of the right hand side converges if T is nuclear and that its value is independent from the choice of the orthonormal basis $(g_n)_{n\in\mathbb{N}}$. Due to this property nuclear operators are sometimes called operators of *trace class*. One can show, that $|\operatorname{Tr}[T]| \leq \|T\|_{\mathscr{L}_1(\mathcal{H})}$ holds and, moreover, that $\operatorname{Tr}[T] = \|T\|_{\mathscr{L}_1(\mathcal{H})}$ if T is positive semidefinite and selfadjoint.

By $\dot{\mathrm{H}}_2^s$ we mean the homogenous Bessel potential space of order s which is defined as

$$\dot{\mathrm{H}}_2^s(\mathbb{R}) := \left\{ f \in \mathcal{S}^*(\mathbb{R}) : \int_{\mathbb{R}} |\mathscr{F} f(\tau)|^2 |\tau|^{2s} \mathrm{d}\tau < \infty \right\},$$

where $\mathcal{S}^*$ is the space of tempered distributions and $\mathscr{F} f$ denotes the Fourier transform of a function f. For a comprehensive account of the theory of these function spaces we refer to Triebel [16].

If A is a linear operator in X, $\mathcal{D}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$ stand for domain, range, and null space of A, respectively, while $\rho(A)$, $\sigma(A)$ designate the resolvent set and the spectrum of A.

In what follows, let X be a Banach space. For $J \subset \mathbb{R}$ open or closed, $C(J;X)$ and $C_b(J;X)$ stand for the spaces of continuous resp. bounded continuous functions $f : J \to X$. Further, if $\alpha \in (0,1)$, $C^\alpha(J;X)$ designates the space of all Hölder-continuous functions $f : J \to X$ of order α.

If not indicated otherwise, by $f * g$ we mean the *convolution* defined by

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau, \qquad t \geq 0,$$

of two functions f, g supported on the halfline.

For $u \in L_{1,\mathrm{loc}}(\mathbb{R}_+; X)$ of exponential growth, i.e. $\int_0^\infty e^{-\omega t}|u(t)|\mathrm{d}t < \infty$ with some $\omega \in \mathbb{R}$, the *Laplace transform* of u is defined by

$$\widehat{u}(\lambda) = \int_0^\infty e^{-\lambda t} u(t)\mathrm{d}t, \quad \operatorname{Re}\lambda \geq \omega.$$

The next section addresses our attention to the notion of parabolic problems.

2.2. **Evolutionary integral equations.** The theory of parabolic problems used in this study is taken from Prüss [13].

Let $\mathcal{H}$ be a separable Hilbert space, A a closed linear, but in general unbounded operator in $\mathcal{H}$ with dense domain $\mathcal{D}(A)$, and let $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ be of subexponential growth, which means $\int_0^\infty e^{-\varepsilon t}|a(t)|\mathrm{d}t < \infty$ for all $\varepsilon > 0$. Then it is readily seen that the Laplace transform $\widehat{a}(\lambda)$ of the kernel a exists for $\operatorname{Re}\lambda > 0$. We consider the Volterra equation

$$u(t) + (a * Au)(t) = f(t), \qquad t \geq 0, \tag{2.1}$$

where $f : \mathbb{R}_+ \to \mathcal{H}$ is a given function, strongly measurable and locally integrable, at least.

In the sequel we denote by $\mathcal{H}_A$ the domain of A equipped with the graph norm $|\cdot|_A$ of A, i.e. $|x|_A = |x| + |Ax|$; $\mathcal{H}_A$ is a Banach space since A is closed, and it is continuously embedded into $\mathcal{H}$. The following notations of solutions of (2.1) are natural. We abbreviate by J the time interval $[0,T]$, $T > 0$, or the whole halfline $\mathbb{R}_+$.

Definition 2.1 ([13, Definition 1.1])**.** *A function $u \in C(J;\mathcal{H})$ is called*

(a) strong solution *of (2.1) on J if $u \in C(J;\mathcal{H}_A)$ and (2.1) holds on J;*
(b) mild solution *of (2.1) on J if $a * u \in C(J;\mathcal{H}_A)$ and $u(t) = f(t) - A(a*u)(t)$ on J.*

Obviously, every strong solution of (2.1) is a mild one. The converse is not true, in general.

Now we may use the concept of parabolicity which was introduced in [13]:

Definition 2.2 ([13, Definition 3.1])**.** *Equation (2.1) is called* parabolic*, if the following conditions hold.*

(P1) $\widehat{a}(\lambda) \neq 0$, $1/\widehat{a}(\lambda) \in \rho(A)$ *for all* $\operatorname{Re}\lambda > 0$.

(P2) There is a constant $M \geq 1$ such that
$$\left|\frac{1}{\lambda}(I+\widehat{a}(\lambda)A)^{-1}\right| \leq \frac{M}{|\lambda|} \quad \textit{for all } \operatorname{Re}\lambda > 0.$$

The notion of sectorial kernels is given by

Definition 2.3 ([13, Definition 3.2])**.** *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth and suppose $\widehat{a}(\lambda) \neq 0$ for all $\operatorname{Re}\lambda > 0$. a is called* sectorial *with angle $\theta > 0$ (or merely θ-sectorial) if*

$$|\arg \widehat{a}(\lambda)| \leq \theta \quad \textit{for all } \operatorname{Re}\lambda > 0. \tag{2.2}$$

Here, $\arg \widehat{a}(\lambda)$ is defined as the imaginary part of a fixed branch of $\log \widehat{a}(\lambda)$, and θ in (2.2) is allowed to be greater than π. In case a is sectorial, we always choose that branch of $\log \widehat{a}(\lambda)$ which yields the smallest angle θ; in particular, if $\widehat{a}(\lambda)$ is real for real λ we choose the principal branch.
A standard situation leading to parabolic equations is described in

Proposition 2.1 ([13, Proposition 3.1])**.** *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be θ-sectorial for some $\theta < \pi$, suppose A is closed linear densely defined, such that $\rho(A) \supset \Sigma(0,\theta)$, and*
$$|(\mu + A)^{-1}| \leq \frac{M}{|\mu|} \quad \textit{for all } \mu \in \Sigma(0,\theta).$$
Then (2.1) is parabolic.

If A is sectorial with angle ϕ_A (for a detailed survey we refer to [3, Section 1]), and a is ϕ_a-sectorial, then (2.1) is parabolic provided that $\phi_A + \phi_a < \pi$, cf. [14, Proposition 3.1].
An important property of parabolic Volterra equations is the fact that they admit bounded *resolvents* whenever the kernel a is 1-regular, see [14, Theorem 3.1]. By a resolvent for (2.1) we mean a family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in $\mathcal{H}$ which satisfy the following conditions:

(S1) $S(t)$ is strongly continuous on $\mathbb{R}_+$ and $S(0) = I$;
(S2) $S(t)\mathcal{D}(A) \subset \mathcal{D}(A)$ and $AS(t)x = S(t)Ax$ for all $x \in \mathcal{D}(A)$, $t \geq 0$;
(S3) $S(t)x + A(a * Sx)(t) = x$, for all $x \in \mathcal{H}$, $t \geq 0$.

(S3) is called *resolvent equation*, cf. [13, Definition 1.3, Proposition 1.1]. One can show that (2.1) admits at most one resolvent, and if it exists, then (2.1) has a unique mild solution u represented by the variation of parameters formula

$$u(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-\tau)f(\tau)\mathrm{d}\tau, \qquad t \geq 0, \tag{2.3}$$

at least for such f for which (2.3) is meaningful, see [13, Sections 1.1 and 1.2].

The notion of k-regularity of a kernel $a(t)$ is discussed in the next section.

2.3. **Regular kernels.** The next definition introduces an appropriate notion of regularity of kernels.

Definition 2.4 ([13, Definition 3.3])**.** *Let $a \in L_{1,loc}(\mathbb{R}_+)$ be of subexponential growth and $k \in \mathbb{N}$. a is called* k-regular *if there is a constant $c > 0$ such that*

$$|\lambda^n \widehat{a}^{(n)}(\lambda)| \leq c|\widehat{a}(\lambda)|, \quad \textit{for all } \operatorname{Re}\lambda > 0,\ 0 \leq n \leq k. \tag{2.4}$$

It is not difficult to see that convolutions of k-regular kernels are again k-regular. Furthermore, k-regularity is preserved by integration and differentiation, while sums and differences of k-regular kernels need not be k-regular. In general, nonnegative, nonincreasing kernels are not 1-regular, but if the kernel is also convex, then it is 1-regular (cf. [13, Section I.3]). We call a kernel $a \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ 1-*monotone* if $a(t)$ is nonnegative and nonincreasing; for $k \geq 2$ we define

Definition 2.5 ([13, Definition 3.4])**.** *Let $a \in L_{1,loc}(\mathbb{R}_+)$ and $k \geq 2$. $a(t)$ is called* k-monotone *if $a \in C^{k-2}(0,\infty)$, $(-1)^n a^{(n)}(t) \geq 0$ for all $t > 0$, $0 \leq n \leq k-2$, and $(-1)^{k-2}a^{(k-2)}(t)$ is nonincreasing and convex.*

The aim of the subsequent proposition is to connect the notions of monotonicity and regularity.

Proposition 2.2 ([13, Proposition 3.3])**.** *Suppose $a \in L_{1,loc}$ is $(k+1)$-monotone, $k \geq 1$. Then $a(t)$ is k-regular and of positive type.*

3. Stochastic calculus for fractional Brownian motions

3.1. **Fractional Brownian motion.** Stochastic differential equations play an important role in studying random influences on deterministic systems. For this purpose usually Wiener processes are considered. However, this is not adequate if the chronological independence of the stochastic perturbations is not sufficiently warranted. Therefore we introduce the concept of a fractional Brownian motion.

Definition 3.1. *A real valued Gaussian process $\beta^{H} := \{\beta^{H}(t)\}_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is called a* fractional Brownian motion *with Hurst parameter $H \in (0,1)$ if for all $s, t \in \mathbb{R}_+$*

(i) $\beta^{H}(0) = 0$,
(ii) $\mathbb{E}\beta^{H}(t) = 0$,
(iii) $\operatorname{Cov}[\beta^{H}(t), \beta^{H}(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

And in deed, trajectories of a fractional Brownian motion admit the property of Hölder-continuity.

Proposition 3.1. *The trajectories of a real valued fractional Brownian motion* $\beta^H := \{\beta^H(t)\}_{t\geq 0}$ *with Hurst parameter* $H \in (0,1)$ *are Hölder continuous of order* $\theta \in (0, H)$ *for a.e.* $\omega \in \Omega$.

Proof. By the Kahane-Khinchine inequality (e.g. [9, Corollary 3.4.1]) it is

$$\mathbb{E}|\beta^H(t) - \beta^H(s)|^p \leq c_p \left(\mathbb{E}|\beta^H(t) - \beta^H(s)|^2\right)^{p/2} = c_p|t-s|^{pH}$$

for all $p \in [1, \infty)$. In particular

$$\mathbb{E}|\beta^H(t) - \beta^H(s)|^p \leq c_p|t-s|^{1+(pH-1)}$$

holds and the Kolmogorov-Čentsov-Theorem (e.g. [8, Theorem 2.8]) yields the claimed Hölderianity for every $\theta \in (0, H - \frac{1}{p})$ and for all $p \in [1, \infty)$. The proof is complete. □

A special case of a fractional Brownian motion is a Wiener process, which occurs in the case $H = \frac{1}{2}$. Unlike a Wiener process, a fractional Brownian motion with Hurst parameter $H \neq \frac{1}{2}$ is neither a martingale, nor a semi-martingale, nor Markovian. Throughout this work we want to deal with $\mathcal{H}$-valued fractional Brownian motions. This gives rise to

Hypothesis (B). *The operator* Q *belongs to* $\mathscr{L}_1(\mathcal{H})$ *is selfadjoint, positive semi-definite and is diagonal with respect to the orthonormal basis* $(e_n)_{n\in\mathbb{N}}$ *of* $\mathcal{H}$, *i.e.* $Qe_n = \gamma_n e_n$ *and* $\gamma_n > 0$ *for all* $n \in \mathbb{N}$. $B^H := \{B^H(t)\}_{t\geq 0}$ *is of the form*

$$(B^H(t)|x) = \sum_{n=0}^{\infty} \beta_n^H(t)(e_n|x), \quad t \in \mathbb{R}, \quad x \in \mathcal{H}, \tag{3.1}$$

where β_n^H *are mutually independent real valued fractional Brownian motions with fixed Hurst parameter* $H \in (0,1)$ *on the probability space* $(\Omega, \mathscr{F}, \mathbb{P})$.

It is well known that B^H (as in (3.1)) is not a well defined $\mathcal{H}$-valued random variable. However, due to $B^H(t) : \Omega \to \mathcal{H}_{Q^{-1/2}}$, where $\mathcal{H}_{Q^{-1/2}}$ is the completion of $\mathcal{H}$ with respect to the norm $|x|^2_{Q^{-1/2}} := |Q^{-1/2}x|_{\mathcal{H}}$, $x \in \mathcal{H}$, the process $Q^{1/2}B^H$ converges in $L_2(\Omega, \mathcal{H})$. Note that $Q^{1/2}$ is well defined and belongs to $\mathscr{L}_2(\mathcal{H})$.

Let us deduce the distributional properties of the process $Q^{1/2}B^H$, where the covariance operator Q and the process B^H satisfy Hypothesis **(B)**. It is obvious that $Q^{1/2}B^H$ is a Gaussian process, so it remains to calculate the mean value and the covariance.

$$\mathbb{E}\left(Q^{1/2}B^H(t) \mid x\right) = \mathbb{E}\left(\sum_{n=1}^{\infty} \sqrt{\gamma_n}\beta_n^H(t)e_n \mid x\right) = \sum_{n=1}^{\infty} \sqrt{\gamma_n}\mathbb{E}(\beta_n^H(t))(e_n \mid x) = 0,$$

for every $x \in \mathcal{H}$. Regarding the covariance we have for all $s, t \in \mathbb{R}_+$

$$\begin{aligned}\left(\mathrm{Cov}[Q^{1/2}B^{H}(t), Q^{1/2}B^{H}(s)]e_m \mid e_n\right) &= \mathbb{E}\left[\left(Q^{1/2}B^{H}(t) \mid e_m\right)\left(Q^{1/2}B^{H}(s) \mid e_n\right)\right] \\ &= \sqrt{\gamma_m\gamma_n}\mathbb{E}\left[\beta_m^{H}(t)\beta_n^{H}(s)\right] \\ &= \frac{1}{2}\delta_{mn}\gamma_n\left(t^{2H} + s^{H} - |t-s|^{2H}\right)\end{aligned}$$

and therefrom

$$\mathbb{E}\left|Q^{1/2}B^{H}(t) - Q^{1/2}B^{H}(s)\right|^2 = \|Q\|_{\mathscr{L}_1(\mathcal{H})}|t-s|^{2H}.$$

This leads to the subsequent

Definition 3.2. *A stochastic process $Q^{1/2}B^{H} = \{Q^{1/2}B^{H}(t)\}_{t\geq 0}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, which satisfies Hypothesis **(B)** is called $\mathcal{H}$-valued fractional Brownian motion with Hurst parameter $H \in (0,1)$.*

3.2. **Stochastic integration for deterministic integrands.** This section is devoted to stochastic integrals with respect to a $\mathcal{H}$-valued fractional Brownian motion

$$\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^{H})(t), \quad t \geq 0. \tag{3.2}$$

We want to deduce properties of the function $R : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ such that the integral (3.2) is well defined. For the theory of stochastic integration with respect to a vector valued Wiener process we refer to [18].

We may start to recall a result from [15, Section 1].

Proposition 3.2. *Let β^{H} be a scalar fractional Brownian motion with Hurst parameter $H \in (0,1)$, then the identity*

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+} f(\tau)\mathrm{d}\beta^{H}(\tau)\right)\left(\int_{\mathbb{R}_+} g(\tau)\mathrm{d}\beta^{H}(\tau)\right)\right] = (f \mid g)_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \tag{3.3}$$

holds for appropriate scalar functions f and g. Here $\dot{\mathrm{H}}_2^{\frac{1}{2}-H}$ denotes the homogeneous Bessel potential space of order $\frac{1}{2} - H$.

By the definition of a stochastic integral it is

$$\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^{H})(t) := \sum_{n=1}^{\infty}\int_0^\infty R(t)Q^{1/2}e_n\mathrm{d}\beta_n^{H}(t), \quad t \geq 0.$$

Our first goal is to calculate the covariance operator. For every $x, y \in \mathcal{H}$ it is

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t) \mid x\right)_{\mathcal{H}} \left(\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t) \mid y\right)_{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\sum_{k=1}^\infty \int_0^\infty \left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}} \mathrm{d}\beta_k^H(t) \sum_{l=1}^\infty \int_0^\infty \left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}} \mathrm{d}\beta_l^H(t)\right] \\
&= \mathbb{E}\left[\sum_{k=l} \int_0^\infty \left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}} \mathrm{d}\beta_k^H(t) \int_0^\infty \left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}} \mathrm{d}\beta_l^H(t)\right] \\
&\quad + 2\mathbb{E}\left[\sum_{k<l} \int_0^\infty \left(R(t)Q^{1/2}e_k \mid x\right)_{\mathcal{H}} \mathrm{d}\beta_k^H(t) \int_0^\infty \left(R(t)Q^{1/2}e_l \mid y\right)_{\mathcal{H}} \mathrm{d}\beta_l^H(t)\right] \\
&= \sum_{n=1}^\infty \mathbb{E}\left[\int_0^\infty \left(R(t)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \mathrm{d}\beta_n^H(t) \int_0^\infty \left(R(t)Q^{1/2}e_n \mid y\right)_{\mathcal{H}} \mathrm{d}\beta_n^H(t)\right] \\
&= \sum_{n=1}^\infty \left(\left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \mid \left(R(\cdot)Q^{1/2}e_n \mid y\right)_{\mathcal{H}}\right)_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}
\end{aligned}
$$

and the last equality holds if and only if $\left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$ for all $x \in \mathcal{H}$ and for all $n \in \mathbb{N}$ (cf. [15]). Now we may choose $x = y$ to obtain the variance

$$
\mathbb{E}\left(\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t) \mid x\right)_{\mathcal{H}}^2 = \sum_{n=1}^\infty \left\|\left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}}\right\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}
$$

for all $x \in \mathcal{H}$. Hence, letting $(h_n)_{n\in\mathbb{N}}$ an arbitrary orthonormal basis in $\mathcal{H}$, then Parseval's equation yields

$$
\begin{aligned}
\mathbb{E}\left|\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t)\right|^2_{\mathcal{H}} &= \sum_{k=1}^\infty \mathbb{E}\left(\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t) \mid h_k\right)^2_{\mathcal{H}} \\
&= \sum_{k=1}^\infty \sum_{n=1}^\infty \left\|\left(R(\cdot)Q^{1/2}e_n \mid h_k\right)_{\mathcal{H}}\right\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}.
\end{aligned}
$$

Note, that $R \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+; \mathcal{B}(\mathcal{H}))$ implies $R(\cdot)Q^{1/2} \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+; \mathscr{L}_2(\mathcal{H}))$ as well as $\left(R(\cdot)Q^{1/2}e_n \mid x\right)_{\mathcal{H}} \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$ for all $x \in \mathcal{H}$ and for all $n \in \mathbb{N}$. Resuming, we deduced the following identity of Itô-type.

Theorem 3.1. *Let $R : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ and $(h_n)_{n\in\mathbb{N}}$ an orthonormal basis in $\mathcal{H}$. Then the identity*

$$
\mathbb{E}\left|\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t)\right|^2_{\mathcal{H}} = \sum_{k=1}^\infty \sum_{n=1}^\infty \left\|\left(R(\cdot)Q^{1/2}e_n \mid h_k\right)_{\mathcal{H}}\right\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \tag{3.4}
$$

holds and the left hand side is independent from the choice of the basis $(h_n)_{n\in\mathbb{N}}$. In particular the stochastic integral (3.2) is well defined, if $R \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+; \mathcal{B}(\mathcal{H}))$.

Suppose now, that there is a second orthonormal system $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{H}$ and scalar functions $r(\cdot,\cdot):\mathbb{R}_+\times\mathbb{C}\to\mathbb{R}$ such that the operator $R(t)$ decomposes into

$$R(t)x=\sum_{n=1}^{\infty} r(t,\mu_n)(x\mid g_n)g_n,\quad t\geq 0,\quad x\in\mathcal{H}.$$

Note that the operator valued function $R:\mathbb{R}_+\to\mathcal{B}(\mathcal{H})$ admits this property if, for instance, $R(t)$ is selfadjoint and $\rho(R(t))$ is compact for all $t\geq 0$. We have

$$\begin{aligned}\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t) &= \sum_{n=1}^{\infty}\int_0^\infty R(t)Q^{1/2}e_n\mathrm{d}\beta_n^H(t)\\ &=\sum_{n=1}^{\infty}\sqrt{\gamma_n}\int_0^\infty R(t)e_n\mathrm{d}\beta_n^H(t)\\ &=\sum_{n=1}^{\infty}\sqrt{\gamma_n}\int_0^\infty \sum_{k=1}^{\infty}(e_n\mid g_k)r(t,\mu_k)g_k\mathrm{d}\beta_n^H(t)\\ &=\sum_{n=1}^{\infty}\sqrt{\gamma_n}\int_0^\infty \sum_{k,l}(e_n\mid g_k)r(t,\mu_k)(g_k\mid e_l)e_l\mathrm{d}\beta_n^H(t)\\ &=\sum_{n,k,l}\sqrt{\gamma_n}(e_n\mid g_k)(g_k\mid e_l)\int_0^\infty r(t,\mu_k)\mathrm{d}\beta_n^H(t)e_l\end{aligned}$$

and Proposition 3.2 gives

$$\begin{aligned}&\mathbb{E}\left|\int_0^\infty R(t)\mathrm{d}(Q^{1/2}B^H)(t)\right|_{\mathcal{H}}^2\\ &\quad=c_H\sum_{n,k,l,m}\gamma_n(e_n\mid g_k)(g_k\mid e_l)(e_n\mid g_m)(g_m\mid e_l)\left(r(\cdot,\mu_k)\mid r(\cdot,\mu_m)\right)_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}\\ &\quad=c_H\sum_{n,k,m}\gamma_n(e_n\mid g_k)(e_n\mid g_m)(g_m\mid g_k)\left(r(\cdot,\mu_k)\mid r(\cdot,\mu_m)\right)_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}\\ &\quad=c_H\sum_{n,k}\gamma_n(e_n\mid g_k)^2\|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}\\ &\quad=c_H\sum_{k,j}(Qg_j\mid g_k)(g_j\mid g_k)(r(\cdot,\mu_j)\mid r(\cdot,\mu_k))_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}\\ &\quad=c_H\sum_{k=1}^{\infty}|Q^{1/2}g_k|^2_{\mathcal{H}}\|r(\cdot,\mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}.\end{aligned}$$

Further let $0\leq s\leq t$ and define a notation of a shifted, time inverted and trivially extended function f supported on the half line $\mathbb{R}_+$ in virtue of

$$f^{\langle t\rangle}(\tau):=\begin{cases}f(t-\tau) & :-\infty<\tau\leq t;\\ 0 & :\tau>t.\end{cases}\tag{3.5}$$

For brevity we define $r_n(\tau) := r(\tau, \mu_n)$. With this aid the latter observations yield

$$\begin{aligned}\mathbb{E}\left|\int_0^\infty (R^{\langle t\rangle}(\tau) - R^{\langle s\rangle}(\tau))\mathrm{d}(Q^{1/2}B^H)(\tau)\right|_{\mathcal{H}}^2 \\ &= c_H \sum_{k=1}^\infty |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|^2_{\dot{\mathrm{H}}_2^{\frac12 - H}(\mathbb{R}_+)} \\ &\le c_H \sum_{k=1}^\infty |Q^{1/2}g_k|_{\mathcal{H}}^2 \|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|^2_{\dot{\mathrm{H}}_2^{\frac12 - H}(\mathbb{R})}.\end{aligned}$$

Now we may introduce the fractional integral of order α of a function ϕ by $\mathcal{I}^\alpha\phi$ in the extended form as

$$\mathcal{I}^\alpha\phi := \begin{cases} \mathcal{I}^{-\alpha}\phi & : \alpha < 0, \\ \phi & : \alpha = 0, \\ \mathcal{D}^\alpha\phi & : \alpha > 0. \end{cases} \tag{3.6}$$

Precisely, this means for $\alpha > 0$

$$(\mathcal{I}^\alpha\phi)(r) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}} \phi(\tau)(\tau - r)_+^{\alpha-1}\mathrm{d}\tau, \quad r \in \mathbb{R},$$

where $(x)_+$ denotes the positive part of a number x. $\mathcal{D}^\alpha$ is defined to be the right inverse of $\mathcal{I}^\alpha$, i.e. for appropriate functions ϕ it holds for $\alpha > 0$ that $\mathcal{D}^\alpha(\mathcal{I}^\alpha\phi) \equiv \phi$. We are now in the position to formulate an equivalent representation of the norm in $\dot{\mathrm{H}}_2^\sigma(\mathbb{R})$ for $-\frac12 < \sigma < \frac12$. This is goes back to Plancherel's Theorem and reads as

$$\|f\|_{\dot{\mathrm{H}}_2^\sigma(\mathbb{R})} = \|\mathcal{I}^\sigma f\|_{L_2(\mathbb{R})}. \tag{3.7}$$

With this norm representation it is obvious, that if $f \in \dot{\mathrm{H}}_2^\sigma(\mathbb{R}_+)$ then

$$\|f^{\langle t\rangle}\|_{\dot{\mathrm{H}}_2^\sigma(\mathbb{R}_+)} \le \|f^{\langle t\rangle}\|_{\dot{\mathrm{H}}_2^\sigma(\mathbb{R})} = \|f^{\langle 0\rangle}\|_{\dot{\mathrm{H}}_2^\sigma(\mathbb{R})} = \|f\|_{\dot{\mathrm{H}}_2^\sigma(\mathbb{R}_+)}$$

holds for $-\frac12 < \sigma < \frac12$ and $t \ge 0$.

Next, deduce that for $H \in (0,1)$ we have

$$\begin{aligned}\|r_k^{\langle t\rangle} - r_k^{\langle s\rangle}\|^2_{\dot{\mathrm{H}}_2^{\frac12 - H}(\mathbb{R})} &= \left\|\mathcal{I}^{\frac12 - H}(r_k^{\langle t\rangle} - r_k^{\langle s\rangle})\right\|^2_{L_2(\mathbb{R})} \\ &= \int_{\mathbb{R}} \left|(\mathcal{I}^{\frac12 - H}r_k^{\langle t\rangle})(\tau) - (\mathcal{I}^{\frac12 - H}r_k^{\langle s\rangle})(\tau)\right|^2 \mathrm{d}\tau \\ &= \int_{\mathbb{R}} \left|(\mathcal{I}^{\frac12 - H}r_k^{\langle 0\rangle})(\tau - t) - (\mathcal{I}^{\frac12 - H}r_k^{\langle 0\rangle})(\tau - s)\right|^2 \mathrm{d}\tau \\ &= \left\|(\mathcal{I}^{\frac12 - H}r_k^{\langle 0\rangle})(s - t + \cdot) - (\mathcal{I}^{\frac12 - H}r_k^{\langle 0\rangle})(\cdot)\right\|^2_{L_2(\mathbb{R})} \\ &\le \left\|\mathcal{I}^{\frac12 - H}r_k^{\langle 0\rangle}\right\|^2_{B^\theta_{2,\infty}(\mathbb{R})} |t - s|^{2\theta},\end{aligned} \tag{3.8}$$

where $B^{\theta}_{2,\infty}(\mathbb{R})$ denotes a Besov space; cf. [16, Section 2.3.2] to verify

$$\mathrm{H}^{\theta}_{2}(\mathbb{R}) \hookrightarrow B^{\theta}_{2,\infty}(\mathbb{R}).$$

Finally, with the apparent relation

$$\|f\|_{\dot{\mathrm{H}}^{\sigma}_{2}(\mathbb{R})} + \|f\|_{\dot{\mathrm{H}}^{\theta+\sigma}_{2}(\mathbb{R})} = \left\|\mathcal{I}^{\theta} f\right\|_{\mathrm{H}^{\theta}_{2}(\mathbb{R})}, \quad -\tfrac{1}{2} < \sigma < \tfrac{1}{2}, \tag{3.9}$$

we derived the estimate

$$\|r^{\langle t\rangle}_{k} - r^{\langle s\rangle}_{k}\|^{2}_{\dot{\mathrm{H}}^{\frac{1}{2}-H}_{2}(\mathbb{R})} \leq \left[\|r_k\|_{\dot{\mathrm{H}}^{\frac{1}{2}-H}_{2}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}^{\theta+\frac{1}{2}-H}_{2}(\mathbb{R}_+)}\right]^{2} |t-s|^{2\theta}. \tag{3.10}$$

Altogether, we proved the following result.

Theorem 3.2. *Let $R : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ and $(g_n)_{n\in\mathbb{N}}$ be an orthonormal basis in $\mathcal{H}$, such that $R(t)$ decomposes into*

$$R(t)x = \sum_{n=1}^{\infty} r(t,\mu_n)(x \mid g_n) g_n, \quad t \geq 0, \quad x \in \mathcal{H}.$$

Then it is

$$\mathbb{E}\left|\int_0^{\infty} R(t)\mathrm{d}(Q^{1/2}B^{H})(t)\right|^{2}_{\mathcal{H}} = c_H \sum_{k=1}^{\infty} |Q^{1/2} g_k|^{2}_{\mathcal{H}} \|r(\cdot,\mu_k)\|^{2}_{\dot{\mathrm{H}}^{\frac{1}{2}-H}_{2}(\mathbb{R}_+)} \tag{3.11}$$

and

$$\mathbb{E}\left|\int_0^{\infty} R^{\langle t\rangle}(\tau)\mathrm{d}(Q^{1/2}B^{H})(\tau)\right|^{2}_{\mathcal{H}} \leq c_H \sum_{k=1}^{\infty} |Q^{1/2} g_k|^{2}_{\mathcal{H}} \|r(\cdot,\mu_k)\|^{2}_{\dot{\mathrm{H}}^{\frac{1}{2}-H}_{2}(\mathbb{R}_+)} \tag{3.12}$$

as well as

$$\begin{aligned} \mathbb{E}&\left|\int_0^{\infty} (R^{\langle t\rangle}(\tau) - R^{\langle s\rangle}(\tau))\mathrm{d}(Q^{1/2}B^{H})(\tau)\right|^{2}_{\mathcal{H}} \\ &\leq c_H \sum_{k=1}^{\infty} |Q^{1/2} g_k|^{2}_{\mathcal{H}} \left[\|r_k\|_{\dot{\mathrm{H}}^{\frac{1}{2}-H}_{2}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}^{\theta+\frac{1}{2}-H}_{2}(\mathbb{R}_+)}\right]^{2} |t-s|^{2\theta}. \end{aligned} \tag{3.13}$$

4. Applications to parabolic Volterra equations

Let us consider the problem

$$u(t) + (b * Au)(t) = Q^{1/2}B^{H}(t), \quad t \geq 0 \tag{4.1}$$

in the Hilbert space $\mathcal{H}$, where the kernel b is supposed to be the antiderivative of a 3-monotone scalar function and the right hand side is due to Hypothesis **(B)**. In particular we recall that there is a sequence $(\gamma_n)_{n\in\mathbb{N}} \in \ell_1(\mathbb{R}_+)$ and an orthonormal basis $(e_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ of $\mathcal{H}$, such that $Qe_n = \gamma_n e_n$ for every $n \in \mathbb{N}$. Since in applications the operator $-A$ will usually be a differential operator like the Laplacian, the elasticity operator, or the Stokes operator, we formulate abstractly

Hypothesis (A): *A is an unbounded, selfadjoint, positive definite operator in $\mathcal{H}$ with compact resolvent. Consequently, the eigenvalues μ_n of A form a nondecreasing sequence with $\lim_{n\to\infty} \mu_n = \infty$, the corresponding eigenvectors $(a_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ form an orthonormal basis of $\mathcal{H}$.*

Observe, that Hypothesis **(A)** implies the sectoriality of the operator A with angle $\phi_A = 0$ (cf. [3, Section 1]). This observation allows us to define complex powers A^z, where $z \in \mathbb{C}$ is arbitrary; cf. [13, Section 8.1]. Note further, that the orthonormal bases of Hypotheses **(B)** and **(A)** do not necessarily have to coincide. In this sense the stochastic perturbation $Q^{1/2}B^H$ is independent of the left hand side of problem (4.1). This gives raise to call such a perturbation *system independent*. The case of an *A-synchronized* perturbation, i.e. the case of coinciding eigensystems of A and Q, is studied in [15]. A pioneering work of the problem under consideration in a differentiated form is [2], where the disturbance was modeled to be an A-synchronized Wiener process. We will show, that the results concerning existence and regularity of the mild solution u of (4.1) do not differ in both cases.

4.1. **Results for the standard kernel.** We start to study a special case of problem (4.1) that is

$$u(t) + (g_\alpha * Au)(t) = Q^{1/2}B^H(t), \quad t \geq 0, \tag{4.2}$$

where g_κ denotes the *Riemann-Liouville kernel* of fractional integration (for short *standard kernel*)

$$g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}, \quad t > 0, \quad \kappa > 0. \tag{4.3}$$

Observe, that (4.2) is the system independent version of the problem studied in [15, Section 4] if one sets $\beta = 1$. The existence and regularity result reads as

Theorem 4.1. *Let $\alpha \in (H, 2)$ and suppose that Hypotheses **(A)** and **(B)** are valid.*

(i) If $QA^{-2H/\alpha} \in \mathscr{L}_1(\mathcal{H})$, then the mild solution u of (4.2) exists and belongs to $C_b(\mathbb{R}_+, L_2(\Omega, \mathcal{H}))$, i.e. the trajectories of u are continuous and bounded on the half line for a.e. $\omega \in \Omega$.

(ii) If in addition there is $\theta \in (0, H)$ such that $QA^{2(\theta-H)/\alpha} \in \mathscr{L}_1(\mathcal{H})$, then $u \in C_b^\theta(\mathbb{R}_+; L_2(\Omega, \mathcal{H}))$, i.e. the trajectories of u are Hölder-continuous of order θ for a.e. $\omega \in \Omega$.

Proof. We denote by $R : \mathbb{R}_+ \to \mathcal{B}(\mathcal{H})$ the fundamental solution of (4.2). For every $n \in \mathbb{N}$ it is $R(t)a_n = r(t, \mu_n)a_n$ and the scalar fundamental solutions $r(t, \mu_n)$, $n \in \mathbb{N}$, are given by their Laplace transforms

$$\widehat{r}(\lambda, \mu_n) = \frac{\lambda^\alpha}{\lambda(\lambda^\alpha + \mu_n)}, \quad \operatorname{Re}\lambda > 0, \quad \lambda \neq 0, \quad \mu_n > 0.$$

It is proven in [15, Section 4] that for every $n \in \mathbb{N}$ the scalar kernel $r(\cdot, \mu_n)$ belongs to $\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-H}(\mathbb{R})$ whenever $\alpha \in (H, 2)$ and $\theta \in [0, H)$. The mild solution of problem (4.2) can be written as

$$u(t) = \int_0^t R(t-\tau)\mathrm{d}(Q^{1/2}B^H)(\tau), \quad t \geq 0$$

and Theorem 3.2 yields the estimates

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} \leq c_H \sum_{k=1}^\infty |Q^{1/2}a_k|^2_{\mathcal{H}} \|r(\cdot, \mu_k)\|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}$$

and for $0 \leq s \leq t$

$$\begin{aligned}&\mathbb{E}|u(t) - u(s)|^2_{\mathcal{H}} \\ &\quad \leq c_H \sum_{k=1}^\infty |Q^{1/2}a_k|^2_{\mathcal{H}} \left[\|r(\cdot, \mu_k)\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} + \|r(\cdot, \mu_k)\|_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \right]^2 |t-s|^{2\theta}.\end{aligned}$$

Now one may apply [15, Lemma 4.1] to justify the kernel estimates

$$\|r(\cdot, \mu_n)\|^2_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \leq c_{\alpha,\theta} \mu_n^{\frac{2(\theta-H)}{\alpha}}, \quad n \in \mathbb{N}$$

for $\alpha \in (H, 2)$ and $\theta \in [0, H)$. They give

$$\begin{aligned}\mathbb{E}|u(t)|^2_{\mathcal{H}} &\leq c_H \sum_{k=1}^\infty |Q^{1/2}a_k|^2_{\mathcal{H}} \mu_k^{\frac{-2H}{\alpha}} \\ &= c_H \sum_{k=1}^\infty (Qa_k \mid a_k)_{\mathcal{H}} \mu_k^{\frac{-2H}{\alpha}} \\ &= c_H \sum_{k=1}^\infty (Qa_k \mid A^{\frac{-2H}{\alpha}} a_k)_{\mathcal{H}} \\ &= c_H \operatorname{Tr}\left[QA^{-\frac{2H}{\alpha}} \right] \\ &= c_H \|QA^{-\frac{2H}{\alpha}}\|_{\mathscr{L}_1(\mathcal{H})}.\end{aligned}$$

To verify the last equality, observe that for arbitrary $z \in \mathbb{C}$ the operator QA^z is positive semidefinite and selfadjoint by Hypotheses **(B)** and **(A)**. With the aid of the latter algebraic manipulations

$$\mathbb{E}|u(t) - u(s)|^2_{\mathcal{H}} \leq c_H \|QA^{\frac{2(\theta-H)}{\alpha}}\|_{\mathscr{L}_1(\mathcal{H})} |t-s|^{2\theta}$$

holds. Now we may employ the Kahane-Khinchine inequality (e.g. [9, Corollary 3.4.1]) to obtain

$$\mathbb{E}\,|u(t)-u(s)|^p_{\mathcal{H}} \le c_p \left(\mathbb{E}\,|u(t)-u(s)|^2_{\mathcal{H}}\right)^{\frac{p}{2}} \le c_{H,p} \left\|QA^{\frac{2(\theta-H)}{\alpha}}\right\|^{\frac{p}{2}}_{\mathscr{L}_1(\mathcal{H})} |t-s|^{p\theta}$$

for all $1 \le p < \infty$. In particular

$$\mathbb{E}\,|u(t)-u(s)|^p_{\mathcal{H}} \le c_{H,p} \left\|QA^{\frac{2(\theta-H)}{\alpha}}\right\|^{\frac{p}{2}}_{\mathscr{L}_1(\mathcal{H})} |t-s|^{1+(p\theta-1)} \tag{4.4}$$

holds and the Kolmogorov-Čentsov-Theorem (e.g. [8, Theorem 2.8]) yields the claimed Hölderianity for every $\Theta \in (0, \theta - \frac{1}{p}) \subset (0, H - \frac{1}{p})$ and for all $p \in [1, \infty)$. Renaming Θ as θ completes the proof. □

4.2. **The main result.** Now we are going to study problem (4.1), where the right hand side and the operator A are subject to Hypotheses **(B)** and **(A)**, respectively. From now on we assume, that the kernel b is subject to

Hypothesis (b): *The kernel b is of the form*

$$b(t) = b_0 + \int_0^t b_1(\tau)\mathrm{d}\tau, \quad t > 0, \tag{4.5}$$

where $b_0 \ge 0$ and $b_1(t)$ is 3-monotone with $\lim_{t\to\infty} b_1(t) = 0$; in addition,

$$\lim_{t\downarrow 0} \frac{\frac{1}{t}\int_0^t \tau b_1(\tau)\mathrm{d}\tau}{b_0 + \int_0^t -\tau \dot{b}_1(\tau)\mathrm{d}\tau} < \infty. \tag{4.6}$$

In case **(A)** and **(b)** are valid the problem under consideration is well-posed and parabolic. Define

$$\phi_b := \sup\left\{|\arg \widehat{b}(\lambda)| : \operatorname{Re}\lambda > 0\right\};$$

then parabolicity means $\phi_b < \frac{\pi}{2}$. For kernels subject to (4.5), condition (4.6) is in fact equivalent to parabolicity. Since we deal with integrated kernels condition (4.5) entails $\phi_b \ge \frac{\pi}{2}$. Typical examples of kernels arising from the theory of linear viscoelasticity (cf. [13, Section I.5]), which satisfy Hypothesis **(b)** are the material functions of *Newtonian Fluids* ($b_0 > 0$, $b_1 \equiv 0$), *Maxwell Fluids* ($b_0 = 0$, $b_1(t) = \sigma\exp\{-\frac{\sigma t}{\nu}\}$) and of *Power Type Materials* ($b_0 = 0$, $b_1(t) = g_\alpha(t)$, $\alpha \in (1, 2)$). For brevity we scale the angle ϕ_b to

$$\rho := \frac{2}{\pi}\sup\left\{|\arg \widehat{b}(\lambda)| : \operatorname{Re}\lambda > 0\right\}. \tag{4.7}$$

Note that in case $b(t) = g_\alpha(t)$, where g_α denotes the standard kernel (cf. (4.3)), it is $\rho = \frac{2}{\pi}\phi_{g_\alpha} = \alpha$. Now, we are in the position to formulate the main result, which on the first view appears to be the analogon of Theorem 4.1 for more general kernels. However, Hypothesis **(b)** is too stringent as to countenance standard kernels g_α

with $\alpha < 1$. From this point of view Theorem 4.1 extends the main result for a special situation.

Theorem 4.2. *Assume that Hypotheses* **(A), (b), (B)** *are valid.*

(i) If $QA^{-2H/\rho} \in \mathscr{L}_1(\mathcal{H})$, then the mild solution u of (4.1) exists and belongs to $C_b(\mathbb{R}_+, L_2(\Omega, \mathcal{H}))$, i.e. the trajectories of u are continuous and bounded on the half line for a.e. $\omega \in \Omega$.

(ii) If in addition, there is $\theta \in (0, H)$ such that $QA^{2(\theta-H)/\rho} \in \mathscr{L}_1(\mathcal{H})$, then $u \in C_b^\theta(\mathbb{R}_+; L_2(\Omega, \mathcal{H}))$, i.e. the trajectories of u are Hölder-continuous in time of order θ for a.e. $\omega \in \Omega$.

Remark 4.1. All estimates used in the proof of Theorem 4.2 are sharp. In this sense the deduced sufficient conditions for existence and regularity are optimal.

Example 4.1. Let $\mathcal{H} = L_2(0, \pi)$ and consider

$$u(t) + (g_\alpha * Au)(t) = Q^{1/2}B^H(t), \quad \alpha \in (H, 2), \quad t \geq 0, \tag{4.8}$$

where g_α denotes the standard kernel; see (4.3). Set $A = A_0^m$, where $m \in \mathbb{N}$ and $A_0 = -(\mathrm{d}/\mathrm{d}x)^2$ with domain $D(A_0) = \mathrm{H}_2^2(0, \pi) \cap \mathring{\mathrm{H}}_2^1(0, \pi)$ and let Q be subject to Hypothesis **(B)** with $Qe_n = n^{-\nu}e_n$, $\nu > 1$, $n \in \mathbb{N}$. Observe that A is due to Hypothesis **(A)** and possesses the eigenvalues $\mu_k = k^{2m}$ for $k \in \mathbb{N}$. The standard kernel g_α satisfies Hypothesis **(b)**, whenever $\alpha \in [1, 2) \subset (H, 2)$ and Theorem 4.2 yields the existence of the mild solution for every $H \in (0, 1)$ and, in addition, its trajectories' Hölder-continuity of its trajectories of order $\theta \in (0, H)$. To preserve these properties also for $\alpha \in (H, 2)$ one may apply Theorem 4.1.

An interesting occurs if one assumes that $Q = I$, i.e. for all $x \in \mathcal{H}$ it is $Qx = x$, so that $Q^{1/2}B^H = B^H$ is only a cylindrical fractional Brownian motion.

Example 4.2. Assume the setting of Example 4.1, but let $Q = I$. Then the mild solution u of (4.8) exists if $H > \frac{\alpha}{4m}$. Moreover its trajectories are Hölder-continuous in time of order $\theta < H - \frac{\alpha}{4m}$. Note that in the cylindrical case θ depends on the parameter α and on the exponent m. Highly regular kernels g_α (this corresponds to α near 2) cause a loss in regularity, while an increasing exponent m improves Hölderianity.

The following section is dedicated to the proof of Theorem 4.2.

4.3. **Proof of the main result.** By Hypotheses **(A)** and **(b)** problem (4.1) admits a uniformly bounded resolvent $R(t)$ such that its unique mild solution is given by the variation of parameters formula

$$u(t) = \int_0^t R(t-\tau)\mathrm{d}(Q^{1/2}B^{H})(\tau), \quad t \geq 0. \tag{4.9}$$

By means of the spectral decomposition of A, the resolvent family $R(t)$ can be written explicitly as

$$R(t)x = \sum_{n=1}^{\infty} r(t,\mu_n)(x \mid a_n)a_n, \quad t \geq 0, \quad x \in \mathcal{H},$$

where the scalar functions $r(t,\mu_n)$ are the solutions of the scalar problems

$$r(t,\mu_n) + \mu_n \int_0^t b(t-\tau)r(\tau,\mu_n)\mathrm{d}\tau = 1, \quad t \geq 0, \quad n \in \mathbb{N}.$$

Since every information of the system is contained in the scalar kernels $r(t,\mu_n)$ we are interested in good estimates. To this end we prove the subsequent

Lemma 4.1. *Suppose the kernel $b(t)$ is subject to Hypothesis **(b)**, let $\rho \in [1,2)$. Then the scalar kernels $r(\cdot,\mu)$ satisfy the estimates*

(i) $|r(t,\mu)| \leq 1$ *for all* $t \geq 0$, $\mu > 0$;
(ii) $\|\dot{r}(\cdot,\mu)\|_{L_1(\mathbb{R}_+)} \leq c$ *for all* $\mu > 0$;
(iii) $\|r(\cdot,\mu)\|_{L_1(\mathbb{R}_+)} \leq c\mu^{-\frac{1}{\rho}}$ *for all* $\mu > 0$;
(iv) $\|r(\cdot,\mu)\|_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \leq c\mu^{\frac{\theta-H}{\rho}}$ *for* $H \in (0,1)$, $\theta \in [0,H)$ *and* $\mu \geq 1$,

where $c > 0$ is a generic constant which is independent of $\mu > 0$.

Proof. Assertion (i) follows from [13, Corollary 1.2], while (ii) is contained in [11, Proposition 6] (observe the relation $\dot{r}(t,\mu) = -\mu r_\mu(t)$ to connect the notations) and (iii) is proven in [2, Lemma 3.1]. To prove (iv) we first recall the facts that for all real numbers s and $1 \leq p < \infty$ we have the trivial embedding $\mathrm{H}_p^s(\mathbb{R}_+) \hookrightarrow \dot{\mathrm{H}}_p^s(\mathbb{R}_+)$ and, moreover, that the Bessel potential space $\mathrm{H}_p^0(\mathbb{R}_+)$ is isometrically isomorphic to the space $L_p(\mathbb{R}_+)$. We first consider the case $\theta = 0$.
Let $H = \frac{1}{2}$. Then by (i) and (iii) we obtain

$$\|r(\cdot,\mu)\|_{\dot{\mathrm{H}}_2^0(\mathbb{R}_+)} \leq \|r(\cdot,\mu)\|_{L_2(\mathbb{R}_+)} \leq \|r(\cdot,\mu)\|_{L_1(\mathbb{R}_+)}^{\frac{1}{2}} \leq c\mu^{-\frac{1}{2\rho}}. \tag{4.10}$$

Let $\frac{1}{2} < H < 1$. Since in this case we have $L_{\frac{1}{H}(\mathbb{R}_+)} \hookrightarrow \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$ (cf. [12, Proposition 3.2]; observe that by Plancherel's Theorem it is $\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+) \cong \Lambda_{H-\frac{1}{2}}$ to connect the results) assertions (i) and (iii) yield

$$\|r(\cdot,\mu)\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \leq c\|r(\cdot,\mu)\|_{L_{\frac{1}{H}}(\mathbb{R}_+)} \leq c\|r(\cdot,\mu)\|_{L_1(\mathbb{R}_+)}^{H} \leq c\mu^{-\frac{H}{\rho}}. \tag{4.11}$$

Let $0 < H < \frac{1}{2}$. Let us denote by $[X;Y]_\delta$ the complex interpolation space of the spaces X and Y with parameter $\delta \in (0,1)$. Then $[X;Y]_\delta = Z$ entails the interpolation inequality $|f|_Z \leq c|f|_X^{1-\delta}|f|_Y^{\delta}$ for all $f \in Z$. It follows from [16, Theorem 2.4.7] that

$$\big[L_1(\mathbb{R}_+); \mathrm{H}_q^{\tau}(\mathbb{R}_+)\big]_{\delta} = \mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$$

holds with

$$\delta = 1 - H, \quad \tau = \frac{1-2H}{2(1-H)}, \quad q = \frac{2(1-H)}{1-2H}. \tag{4.12}$$

This observation yields

$$\|r(\cdot,\mu)\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \leq \|r(\cdot,\mu)\|_{\mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \leq c\|r(\cdot,\mu)\|_{L_1(\mathbb{R}_+)}^{H}\|r(\cdot,\mu)\|_{\mathrm{H}_q^{\tau}(\mathbb{R}_+)}^{1-H}.$$

Now, one may apply [16, Theorem 2.7.1] to verify that the embedding

$$\mathrm{H}_1^1(\mathbb{R}_+) \hookrightarrow \mathrm{H}_q^{\tau}(\mathbb{R}_+)$$

holds with τ and q as in (4.12). This together with (iii) gives

$$\|r(\cdot,\mu)\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \leq c\|r(\cdot,\mu)\|_{L_1(\mathbb{R}_+)}^{H}\|r(\cdot,\mu)\|_{\mathrm{H}_1^1(\mathbb{R}_+)}^{1-H} \leq c\mu^{-\frac{H}{\rho}}, \tag{4.13}$$

because assertions (ii) and (iii) ensure the existence of a constant $M > 0$ such that $\|r(\cdot,\mu)\|_{\mathrm{H}_1^1(\mathbb{R}_+)} \leq M$ for all $\mu \geq 1$. Let now $\theta \in [0, H)$. Then $H - \theta \in (0, H] \subset (0, 1)$ and the desired result follows by repeating the prove of assertion (iv) with replacing H by $H - \theta$. $\square$

Theorem 3.2 and (iv) of Lemma 4.1 yield

$$\begin{aligned}
\mathbb{E}|u(t)|_{\mathcal{H}}^2 &\leq c_H \sum_{n=1}^{\infty} |Q^{1/2}a_n|_{\mathcal{H}}^2 \|r(\cdot,\mu_n)\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}^2 \\
&\leq c_H \sum_{n=1}^{\infty} (Qa_n \mid a_n)_{\mathcal{H}} \mu_n^{-\frac{2H}{\rho}} \\
&= c_H \sum_{n=1}^{\infty} (Qa_n \mid A^{-\frac{2H}{\rho}} a_n)_{\mathcal{H}} \\
&= c_H \sum_{n=1}^{\infty} (a_n \mid QA^{-\frac{2H}{\rho}} a_n)_{\mathcal{H}} \\
&= c_H \left\| QA^{-\frac{2H}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})}.
\end{aligned}$$

To verify the last equality, observe that due to Hypotheses **(B)** and **(A)** the operator QA^z is positive semidefinite and selfadjoint for arbitrary $z \in \mathbb{C}$. This, in particular, yields $\mathrm{Tr}[QA^z] = \|QA^z\|_{\mathscr{L}_1(\mathcal{H})}$ for all $z \in \mathbb{C}$. Concerning Hölder-continuity let $0 \le s \le t$ and $\theta \in (0, H)$. Theorem 3.2 and Lemma 4.1(iv) give

$$\begin{aligned}
\mathbb{E}\,|u(t)-u(s)|_{\mathcal{H}}^2 &\le c_H \sum_{k=1}^{\infty} |Q^{1/2} a_k|_{\mathcal{H}}^2 \left[\|r_k\|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} + \|r_k\|_{\dot{\mathrm{H}}_2^{\theta+\frac{1}{2}-H}(\mathbb{R}_+)} \right]^2 |t-s|^{2\theta} \\
&\le c_H \sum_{k=1}^{\infty} (Qa_k \mid a_k)_{\mathcal{H}} \mu_k^{\frac{2(\theta-H)}{\rho}} |t-s|^{2\theta} \\
&= c_H \left\| Q A^{\frac{2(\theta-H)}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})} |t-s|^{2\theta}.
\end{aligned}$$

Now we may employ the Kahane-Khinchine inequality (e.g. [9, Corollary 3.4.1]) to obtain

$$\mathbb{E}\,|u(t)-u(s)|_{\mathcal{H}}^p \le c_p \left(\mathbb{E}\,|u(t)-u(s)|_{\mathcal{H}}^2 \right)^{\frac{p}{2}} \le c_{H,p} \left\| Q A^{\frac{2(\theta-H)}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})}^{\frac{p}{2}} |t-s|^{p\theta}$$

for all $1 \le p < \infty$. In particular

$$\mathbb{E}\,|u(t)-u(s)|_{\mathcal{H}}^p \le c_{H,p} \left\| Q A^{\frac{2(\theta-H)}{\rho}} \right\|_{\mathscr{L}_1(\mathcal{H})}^{\frac{p}{2}} |t-s|^{1+(p\theta-1)} \tag{4.14}$$

holds and the Kolmogorov-Čentsov-Theorem (e.g. [8, Theorem 2.8]) yields the claimed Hölderianity for every $\Theta \in (0, \theta - \frac{1}{p}) \subset (0, H - \frac{1}{p})$ and for all $p \in [1, \infty)$. Renaming Θ as θ completes the proof.

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