Set–valued measures of risk

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Abstract

Extending the approach of Jouini et al. we define set–valued (convex) measures of risk and its acceptance sets. Using a new duality theory for set–valued convex functions we give dual representation theorems. A scalarization concept is introduced that has economical meaning in terms of prices of portfolios of reference instruments. Using primal and dual descriptions, we introduce new examples for set–valued measures of risk, e.g. set–valued expectations, Value at Risk, Average Value at Risk and entropic risk measure.

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1 Introduction

Recently, the concept of set–valued coherent measures of risk has been introduced by E. Jouini, M. Meddeb and N. Touzi in [11]. Only a few followers are known to us. While [15], [3], [2] mainly investigate special cases, [4] gives a new approach to risk measures with values in an abstract cone via depth–trimmed regions.

The theory for the scalar case, although less than 10 years old, is highly developed; we just mention the pioneering works [1], [5] and [6] with extensions and more related material, but Google Scholar gives more than 110.000 answers to "coherent risk measures" and ca. 1000 citations for [1]. A parallel theory for set–valued measures of risk is still missing which might be one main reason for the poor reception of [11]. Reference [8] and the diploma thesis [10], supervised by the other two authors, were first trials to collect the material and to link set–valued risk measures with new theories [9] for set–valued functions.

The goal of this work is to propose a theory of set–valued measures of risk including primal and dual representation theorems, systematic ways of constructing and investigating examples for set–valued risk measures and a scalarization procedure that links set–valued risk measures to more familiar objects, namely families of extended real–valued functions. The main idea is to find a formulation of the theory that gives expressions and formulas parallel to the scalar

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case. Knowing the "translation rules" of the present paper one should be able to produce set–valued versions of almost every scalar result about risk measures – including the construction of set–valued versions of known (or new) scalar risk measures. The translation mainly relies on substituting the mathematical expectation, a continuous linear functional on L^1, by a set–valued function that resembles as much as possible a linear functional. The tools for duality results are provided by a new duality theory for set–valued functions. As far as it is necessary, this theory can be found in the Appendix.

The economical motivation for set–valued risk measures is as follows. We assume that an investor acts in d different financial markets where "financial market" is a rather general concept including the case that a "market" just contains a single asset (like in [3] or [4]). The portfolio of the investor is modeled by a d-dimensional random vector where the ith component indicates the value of the holdings on market i measured in terms of some secure reference instrument on this market. We denote by E^i the position consisting of one unit of the reference instrument of market i and nothing in any other market. We assume that it is not possible or not desirable to transform everything into a position of one of the financial markets. The reasons may be transaction cost, liquidity bounds (even non-liquidity of certain positions), bad or oscillating current exchange rates etc. Further, we assume that the frictions between the markets are described by a closed convex cone $K \subseteq \mathbb{R}^d$. If $x \in K$ then the position $\sum_{i=1}^d x_i E^i$, with coefficients $x_i \in \mathbb{R}$, $i = 1, \ldots, d$ standing for a deterministic position $x_i E^i$ of x_i units of the reference instrument in the ith market can be converted into a position with non-negative values in each market. Loosely speaking, a deterministic position (in d markets) with $x \in K$ would be accepted by every rationally acting investor.

Therefore, it seems to be reasonable to assume $\mathbb{R}^d_+ \subseteq K \neq \mathbb{R}^d$ (see [11]). Below, we will give more precise assumptions. Following [11] and in contrast to [4] we do not assume that K is pointed ($K \cap -K = \{0\}$). This means that K may contain lines. Note that it is not a priori clear if K should be a convex cone – this assumptions allows the theory in this paper, but could be questionable and is not really necessary at several places.

An example involving proportional transaction costs can be found in [11], Section 2.2, another one is Example 4.4 below. Finally, we assume that m, $1 \leq m \leq d$, of the d markets are of particular importance to the investor: For example, they are accepted markets (by a supervising regulator or by the investor itself) for security deposits. The problem is to evaluate the risk of the portfolio in terms of a mixture (linear combination) of deterministic positions (positions of reference instruments) in the m accepted markets, i.e. the risk should be canceled by adding a linear combination of secure reference instruments in the m accepted markets to the portfolio.

Very likely, there is not only a single possibility for such a linear combination!

Trivially, if a certain linear combination with coefficients $u \in \mathbb{R}^m$ of secure assets does cancel the risk of the portfolio then every linear combination with coefficients in $u + \mathbb{R}^m_+$ does it as well. Moreover, it is reasonable to replace the cone $\mathbb{R}^m_+$ by a bigger cone K_m: This reflects the fact that some of the accepted markets can be replaced by some other. If K_m is just $\mathbb{R}^m_+$ then the accepted markets are mutually illiquid: The lack of a deterministic position in one of them can not be compensated by any deterministic position in another one. On the other hand, K_m should not contain an element with negative entries only; no rational agent would accept a negative deposit for a risky business. This gives the condition $K_m \cap -\operatorname{int} \mathbb{R}^m_+ = \emptyset$. The cone K_m is of course related to K: Since K reflects the frictions between all d markets and K_m the frictions between the m accepted markets, K_m should be a kind of restriction of K. An exact definition will be given in the next section.

The two "rationality conditions"

$$\mathbb{R}^m_+ \subseteq K_m, \qquad K_m \cap -\operatorname{int} \mathbb{R}^m_+ = \emptyset \tag{1.1}$$

do not exclude half spaces: If K_m is a half space then any deterministic position in one accepted market can be replaced by a proportional deterministic position in any other accepted market without e.g. transaction costs. This is also not very realistic, so the cone K_m should be something in between $\mathbb{R}^m_+$ and a half space not intersecting $-\operatorname{int} \mathbb{R}^m_+$. We will see that the collection of half spaces containing K_m plays a crucial role in dual representation theorems; in fact an additional dual variable reflects this collection.

Less trivial are cases with alternative linear combinations canceling the risk of the portfolio that are not comparable w.r.t. the order generated by K_m. Even more, it seems not to be favorable neither to be forced to use exactly one prescribed accepted market nor to exchange the whole linear combination into a deterministic position in one of the accepted markets for the reasons described above.

We shall formalize the situation by set–valued functions, i.e. functions with values in the set of all subsets of $\mathbb{R}^m$. It will turn out that these functions have a sample of properties being the affirmative analog to extended real–valued (convex) measures of risk as introduced by [1], [5], [6] and others. The handling of such set–valued functions can be done by a recent (duality) theory for set–valued (convex) functions due to one of the authors [8], [9]. It involves concepts like monotonicity, translativity, convexity, sublinearity, closedness, properness, Legendre–Fenchel conjugates for set–valued functions, a main result is a biconjugation theorem for set–valued convex function.

Our method is completely different from the more ad hoc approach in [11] which is restricted to the sublinear case (on L^∞_d) and does not relate dual representations to Legendre–Fenchel conjugates.

The paper is organized as follows. Section 2 contains a mathematical model of the situation including definitions for set–valued (convex) measures of risk and acceptance sets. In Section 3, the link between set–valued risk measures and its acceptance sets is given in terms of bijection theorems. Dual representations for convex and coherent risk measures can be found in Section 4 including "penalty function" representation formulas and a subsection about weak* lower semi-continuity ("Fatou property"). Section 5 is devoted to scalarizations of set–valued risk measures and its conjugates. Every section contains – as far as possible – an economical interpretation of the basic concepts and closes with a list of examples. As already mentioned, the Appendix contains the duality theory for set–valued convex functions.

2 The mathematical model and examples

2.1 Vector–valued random variables

Everywhere in this paper, $(\Omega, \mathcal{F}, P)$ is a probability space. Let $L^p_d := L^p_d(\Omega, \mathcal{F}, P)$, $1 \leq p \leq \infty$, be the usual Banach spaces of (equivalence classes of) d–vector-valued random variables. By $\left(L^p_d\right)_+$ we denote the members of L^p_d that are P a.s. non-negative. An element $X \in L^p_d$ has components $X_1, \dots, X_d$ in $L^p := L^p_1$. By E we denote the random variable $E : \Omega \to \mathbb{R}$ with $P(\{\omega \in \Omega : E(\omega) \neq 1\}) = 0$. We define d elements $E^1, \dots, E^d \in L^p_d$ by

$$E^i_j = E \in L^p \quad \text{for} \quad i = j \in \{1, \dots, d\} \quad \text{and} \quad E^i_j = 0 \quad \text{otherwise.}$$

We write $E^P[X] := \left(E^P[X_1], \ldots, E^P[X_d]\right)^T \in \mathbb{R}^d$ for the (componentwise) mathematical expectation of $X \in L^p_d$ under P with $E^P[X_i] = \int_\Omega X_i(\omega)\, dP$, $i = 1, \ldots, d$. Analog notation is used for expectations with respect to other measures.

Let $K \subseteq \mathbb{R}^d$ be a closed convex cone with $\mathbb{R}^d_+ \subseteq K$. Such a cone generates a reflexive transitive relation in $\mathbb{R}^d$ by setting $x^1 \leq_K x^2$ iff $x^2 - x^1 \in K$. This relation is not necessarily antisymmetric since K may contain straight lines. We assume

$$K \cap \left(-\operatorname{int} \mathbb{R}^m_+ \times \{0\}^{d-m}\right) = \emptyset. \tag{2.1}$$

The set

$$C := \left\{X \in L^p_d : \ P\left(\{\omega \in \Omega : \ X(\omega) \notin K\}\right) = 0\right\} \tag{2.2}$$

is a closed convex cone in L^p_d and generates the reflexive transitive relation $\leq_C$ for $\mathbb{R}^d$–valued random variables by setting $X^1 \leq_C X^2$ iff $X^2 - X^1 \in C$.

Since we distinguish the first m components of $\mathbb{R}^d$ variables as portions of secure assets, we need the $\mathbb{R}^m$ part of the cone K: Denote

$$K_m = \left\{(u_1, \ldots, u_m)^T : \ (u_1, \ldots, u_m, 0, \ldots, 0)^T \in K\right\}.$$

The cone $K_m \subseteq \mathbb{R}^m$ is also closed convex and from (2.1) we get $\mathbb{R}^m_+ \subseteq K_m$ and $K_m \cap -\operatorname{int} \mathbb{R}^m_+ = \emptyset$. Note that $u \in K_m$ if and only if $\sum_{i=1}^m u_i E^i \in C$.

A set–valued risk measure will be modeled as a function $\Phi : L^p_d \to \mathcal{F}_m$ or $\Phi : L^p_d \to \mathcal{C}_m$, a function from L^p_d into the set of lower closed and lower closed convex subsets of $\mathbb{R}^m$, respectively, see Appendix for definitions.

Dual variables are needed for $\mathbb{R}^m$ as well as for L^p_d. Define the positive dual or polar cone of K_m by $K^+_m := \left\{v \in \mathbb{R}^m : \ \forall u \in K_m : \ v^T u \geq 0\right\}$ and $B^+_m = \left\{v \in K^+_m : \ v^T e = 1\right\}$ with $e = (1, \ldots, 1)^T \in \operatorname{int} \mathbb{R}^m_+ \subseteq \operatorname{int} K_m$. The compact convex set B^+_m is a base for K^+_m. Similarly, $B^-_m = -B^+_m = \left\{v \in K^-_m : \ v^T e = -1\right\}$ is a base of the negative dual cone $K^-_m = -K^+_m$ of K_m. Note that $\mathbb{R}^m_+ \subseteq K_m$ implies $K^+_m \subseteq \mathbb{R}^m_+$.

For L^p_d, we shall introduce dual variables for the cases $p < \infty$ and $p = \infty$ separately. First, let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Since the topological dual of a direct sum of topological linear spaces is isomorphic to the direct product of the duals of those spaces, the topological dual space of L^p_d can be identified with L^q_d. We write $X \cdot Y = \sum_{i=1}^d X_i Y_i$ for the point-wise scalar product of two $\mathbb{R}^d$–valued random variables $X \in L^p_d$, $Y \in L^q_d$, and $\int_\Omega X \cdot Y\, dP$ is its duality pairing. The positive dual or polar cone of the cone $C \subseteq L^p_d$ is the set

$$C^+ = \left\{Z \in L^q_d : \ \forall X \in C : \ \int_\Omega X \cdot Z\, dP \geq 0\right\}$$

and its negative dual cone is $C^- = -C^+$. Since $\left(L^p_d\right)_+ \subseteq C$ we have $C^+ \subseteq \left(L^q_d\right)_+$. We assign an element $z \in \mathbb{R}^m$ to $Z \in L^q_d$ by $z_i := E^P[Z_i] = \int_\Omega Z_i dP$ for $i \in \{1, 2, \ldots, m\}$. In this paper, the symbol z will always denote such an element of $\mathbb{R}^m$ arising from a certain $Z \in L^q_d$. We define

$$\mathcal{Z}^q_m := \left\{Z \in L^q_d : \ Z \in C^+, \ z^T e = 1\right\}. \tag{2.3}$$

If $Z \in \mathcal{Z}^q_m$ then $z \in K^+_m$. Indeed, take $u \in K_m$ and set $X = \sum_{i=1}^m u_i E^i \in C$. Then

$$\int_\Omega X \cdot Z\, dP = z^T u \geq 0,$$

hence $z \in K^+_m$.

Remark 2.1 *Every $Z \in \mathcal{Z}_m^q$ generates a bounded IR^d–valued countably additive vector measure Q being absolutely continuous w.r.t. P by setting*

$$Q(\Omega') = \int_{\Omega'} Z(\omega)\, dP, \quad \Omega' \in \mathcal{F},$$

and, moreover, a probability measure $\widetilde{P}$ on $(\Omega, \mathcal{F})$ by

$$\widetilde{P}(\Omega') = \int_{\Omega'} \sum_{i=1}^{m} Z_i(\omega)\, dP, \quad \Omega' \in \mathcal{F}.$$

Since $Z \in C^+ \subseteq \left(L_d^q\right)_+$ and $z \in B_m^+$ this is indeed a probability measure that is absolutely continuous w.r.t. P and has the density $\frac{d\widetilde{P}}{dP} = \sum_{i=1}^m Z_i$.

The case $p = \infty$ is treated similarly: Let $L_d^\infty := L_d^\infty(\Omega, \mathcal{F}, P)$ be the Banach space of (equivalence classes of) IR^d–valued essentially bounded random variables. Its topological dual can be identified with $ba_d := ba_d(\Omega, \mathcal{F}, P)$, the space of additive set functions $Q : \mathcal{F} \to \mathrm{IR}^d$ with bounded variation and being absolutely continuous with respect to P, since $ba_d = (ba_1)^d$. The duality pairing between elements $X \in L_d^\infty$ and $Q \in ba_d$ is written as

$$\int_\Omega X \cdot dQ := \sum_{i=1}^{d} \int_\Omega X_i\, dQ_i$$

where $Q_i : \mathcal{F} \to \mathrm{IR}$ is the i-th component of the IR^d-valued additive set function Q. The positive dual cone to $C \subseteq L_d^\infty$ is

$$C^+ = \left\{ Q \in ba_d : \ \forall X \in C : \ \int_\Omega X \cdot dQ \geq 0 \right\}$$

and its negative dual is $C^- = -C^+$. We assign a vector $q \in \mathrm{IR}^m$ to an element $Q \in C^+$ by $q_i = \int_\Omega E^i\, dQ_i$, $i = 1, \ldots, m$, and we define the set

$$\mathcal{Q}_m = \left\{ Q \in ba_d : \ Q \in C^+, \ q^T e = 1 \right\}. \tag{2.4}$$

Again, if $Q \in \mathcal{Q}_m$ then $q \in K_m^+$.

2.2 Set–valued risk measures and acceptance sets

The definitions of this section are fundamental for this paper.

Definition 2.1 *A* set-valued measure of risk *is a function $\Phi : L_d^p \to \mathcal{F}_m$ that is*
(R0) normalized, i.e. $K_m \subseteq \Phi(0)$ and $\Phi(0) \cap -\mathrm{int}\, K_m = \emptyset$.
(R1) translative w.r.t. $E^1, \ldots, E^m \in \left(L_d^p\right)_+$, i.e.,

$$\forall X \in L_d^p, \ \forall u \in \mathrm{IR}^m : \ \Phi\left(X + \sum_{i=1}^m u_i E^i \right) = \Phi(X) + \{-u\}; \tag{2.5}$$

(R2) C–monotone, i.e., $X^2 - X^1 \in C$ implies $\Phi\left(X^2\right) \supseteq \Phi\left(X^1\right)$.

If Φ satisfies (R0), (R1), (R2) and is convex then it is called a (set-valued) convex measure of risk.

If Φ satisfies (R0), (R1), (R2) and is sublinear then it is called a (set-valued) coherent measure of risk.

If Φ is convex and maps into $\mathcal{F}_m$ then $\Phi(X)$ is convex for each $X \in L_d^p$, i.e. Φ actually maps into $\mathcal{C}_m$ (compare (6.1) in Appendix). If Φ is convex (or, more general, $\Phi(0)$ is a convex set) then the second condition in (R0) admits a separation of $\Phi(0)$ and $-K_m$ by means of $v \in K_m^+ \setminus \{0\}$:

$$\forall u \in \Phi(0), \; \forall k \in -K_m : \; v^T k \leq 0 \leq v^T u,$$

hence $\Phi(0) \subseteq H(-v) := \left\{u \in \mathrm{I\!R}^m : v^T u \geq 0\right\}$ and moreover, by the first condition of (R0), $\Phi(0) + H(-v) = H(-v)$. This shall make clear that (R0) is a replacement of $\rho(0) = 0$ for a risk measure $\rho : L_d^p \to \mathrm{I\!R} \cup \{+\infty\}$. Conversely, $\Phi(0) + H(-v) = H(-v)$ for some $v \in K_m^+ \setminus \{0\}$ implies $\Phi(0) \cap -\operatorname{int} K_m = \emptyset$.

We give an alternative condition for (R0).

(R0S) We say that Φ is strongly normalized iff $\Phi(0) = K_m$.

(R0S) implies (R0), while the converses is not true in general. If $m = 1$, $K_m = \mathrm{I\!R}_+$ the two normalization conditions do coincide and give $\Phi(0) = \mathrm{I\!R}_+$.

Of course, the above definition is motivated by the scalar case which is a special case for $d = m = 1$, $K_m = \mathrm{I\!R}_+$ (the only possibility). The values of Φ are of the form $\varrho(X) + \mathrm{I\!R}_+$ with $\varrho : L^p \to \mathrm{I\!R} \cup \{\pm\infty\}$ (and appropriate definitions for dealing with $\pm\infty$).

Interpretation. (R0) means: Any linear combination of deterministic positions in accepted markets with coefficient $u \in K_m$ cancels the risk of "doing nothing". Conversely, the risk of "doing nothing" can not be canceled by a linear combination of deterministic positions with "negative" (in $-\operatorname{int} K_m$) coefficients. There might be a deterministic portfolio in accepted markets with coefficient in $u \in -K_m$ that cancels the risk of "doing nothing" according to the risk measure Φ: perhaps with small loss in one and big gain in other positions.

We turn to properties of acceptance sets for set–valued risk measures.

Definition 2.2 *We call a set $A \subseteq L_d^p$* radially closed *with respect to $E^1, \ldots, E^m$ iff $X \in L_d^p$, $\left\{u^k\right\}_{k \in \mathrm{I\!N}} \subset \mathrm{I\!R}^m$, $\lim_{k\to\infty} u^k = 0$ and $X + \sum_{i=1}^m u_i^k E^i \in A$ for all $k \in \mathrm{I\!N}$ implies $X \in A$.*

This definition is due to [8]. Note that only topological properties of $\mathrm{I\!R}^m$ enter into this definition, not of L_d^p. Moreover, radial closedness with respect to $E^1, \ldots, E^m$ is weaker in general than the property of being algebraically closed. For example, the set $A = \left\{X \in \mathrm{I\!R}^3 : \; X_3 > 0\right\}$ is radially closed with respect to $E^1 = (1,0)^T$, $E^2 = (0,1)^T$, but not algebraically closed.

Definition 2.3 *An* acceptance set *is a subset $A \subseteq L_d^p$ that satisfies*
(A0) $u \in K_m$ implies $\sum_{i=1}^m u_i E^i \in A$, $u \in -\operatorname{int} K_m$ implies $\sum_{i=1}^m u_i E^i \notin A$;
(A1) A is radially closed with $A + \left\{\sum_{i=1}^m u_i E^i\right\} \subseteq A$ whenever $u \in K_m$;
(A2) $A + C \subseteq A$.

If A satisfies (A0), (A1), (A2) and is convex then it is called a convex acceptance set.

If A satisfies (A0), (A1), (A2) and is a convex cone then it is called a coherent acceptance set.

Note that (A2) and the definitions of C (see (2.2)) and K_m already imply the second part of (A1), i.e. $A + \left\{\sum_{i=1}^m u_i E^i\right\} \subseteq A$ whenever $u \in K_m$, but we will see in the next section that this property plays a particular role for the one-to-one-correspondence between risk measures and acceptance sets. Moreover, by (A0), $0 \in A$. This and (A2) imply $C \subseteq A$.

An alternative condition for (A0) corresponding to (R0S) reads as follows:

(A0S) $\sum_{i=1}^m u_i E^i \in A$ if and only if $u \in K_m$.

Again, (A0S) implies (A0), while the converse is not true in general.

The pairs of conditions (R0), (A0) and (R0S), (A0S) will correspond to different possibilities for generating a set–valued risk measure starting with a scalar one. One possibility will lead to a more, the other one to a less risk averse set-valued generalization of the same scalar risk measure.

Remark 2.2 *Jouini et al. define a (coherent) acceptance set (see [11], Definition 2.2) to be a closed convex cone containing C such that there is $u \in \mathrm{IR}^m$ with $\sum_{i=1}^m u_i E^i \notin A$. Of course, the latter condition is ensured by the second part of (A0). We prefer the condition in (A0) since it has an immediate economic interpretation (see Introduction) and all of our examples do satisfy this seemingly stronger condition.*

Remark 2.3 *In [4], the authors give a definition of risk measures with values in an abstract convex cone $\mathbb{G}$ which is essentially a conlinear space according to the terminology in [7]. The space $\mathcal{C}_m$ in this paper is a particular case if in Definition 2.1 of [4] the topological assumptions are dropped since nowhere in that paper topologies on $\mathbb{G}$ are introduced. Moreover, they establish a one-to-one correspondence between risk measures and depth–trimmed regions and come along with several operations producing new ($\mathbb{G}$–, set–valued) risk measures from existing ones (e.g. re-centering, translative and homogeneous hull).*

Before passing to the study of relationships between set–valued risk measures and acceptance sets in the next section we shall give a few examples.

2.3 Examples

Example 2.1 *Set–valued expectation: Let $1 \leq p < \infty$ and take $Z \in L^q_d$ with $\frac{1}{p} + \frac{1}{q} = 1$ such that $Z_1, \dots, Z_m \in L^q_+$ and $z^T e = 1$. Define*

$$E^Z_m[X] = \left\{ u \in \mathrm{IR}^m : \int_\Omega \left(X - \sum_{i=1}^m u_i E^i \right) \cdot Z \, dP = 0 \right\}.$$

We call this set–valued m-expectation of the d–vector–valued random variable X with respect to Z because in case $m = d = 1$ one re-discovers the expectation of X w.r.t. the probability measure Q with density $\frac{dQ}{dP} = Z$. The function $X \to E^Z_m[-X]$ maps into the power set of IR^m, but not into $\mathcal{F}_m$ (or $\mathcal{C}_m$). Therefore, we define the related function F^Z_m by

$$F^Z_m[X] = \left\{ u \in \mathrm{IR}^m : \int_\Omega \left(X - \sum_{i=1}^m u_i E^i \right) \cdot Z \, dP \leq 0 \right\} = \left\{ u \in \mathrm{IR}^m : \int_\Omega X \cdot Z \, dP \leq z^T u \right\}.$$

Claim. *If $Z \in \mathcal{Z}^q_m$ then $\Phi(X) := F^Z_m[-X]$ is a coherent measure of risk on L^p_d.*

Proof of the claim. *First, we shall show that Φ maps into $\mathcal{C}_m$. Since $\Phi(X)$ is a shifted closed half space and therefore closed and convex it is enough to show $\Phi(X) + K_m \subseteq \Phi(X)$. Indeed, for $u \in \Phi(X)$, $w \in K_m$ we have*

$$\int_\Omega \left(X + \sum_{i=1}^m u_i E^i \right) \cdot Z \, dP \geq 0,$$

and, since $z \in K^+_m$,

$$z^T w = \int_\Omega \left(\sum_{i=1}^m w_i E^i \right) \cdot Z \, dP \geq 0,$$

hence

$$\int_\Omega \left(X + \sum_{i=1}^{m} (u_i + w_i) E^i \right) \cdot Z \, dP \geq 0,$$

which is $u + w \in \Phi(X)$.

Sublinearity follows from Lemma 6.1 in Appendix with $\mathcal{X} = L^p_d$, $\mathcal{X}^* = L^q_d$; *observe* $F^Z_m[-X] = S_{-Z,-z}(X)$ *for this case.*

Let us consider (R0). We have $K_m \subseteq \Phi(0)$ *since* $z \in K^+_m$. *This proves the first part of (R0). We also have* $\Phi(0) = \left\{ u \in \mathrm{IR}^m : \ z^T u \geq 0 \right\}$. *If* $u \in \Phi(0) \cap -\mathrm{int}\, K_m$ *then* $z^T u \leq 0$ *since* $z \in K^+_m$, *hence* $z^T u = 0$. *Since* $u \in -\mathrm{int}\, K_m$ *there is* $\varepsilon > 0$ *such that* $u + \varepsilon \frac{z}{\|z\|} \in -K_m$. *Together, this gives the contradiction* $\varepsilon \leq 0$.

Axioms (R1) and (R2) follow from Lemma 6.3 in the Appendix, and this proves the claim.

Note that $\Phi(X) = F^Z_m[-X]$ *is not strongly normalized, in general.*

Remark 2.4 *Using elements* $Q \in (ba_d)_+$ *(elements of* ba_d *with non-negative values) with* $q^T e = 1$ *we may define set–valued expectations for random variables in* L^∞_d *by*

$$F^Q_m[X] = \left\{ u \in \mathrm{IR}^m : \ \int_\Omega \left(X - \sum_{i=1}^{m} u_i E^i \right) \cdot dQ \leq 0 \right\}.$$

One easily proves along the lines of the above example that $X \mapsto F^Q_m[-X]$ *is a coherent risk measure on* L^∞_d *whenever* $Q \in \mathcal{Q}_m$.

Example 2.2 *Let* $1 \leq p \leq \infty$. *The set*

$$A := \left\{ X \in L^p_d : \ E^P[X] \in K \right\}$$

is a coherent acceptance set. Indeed, it is obviously a convex cone and we have $\sum_{i=1}^m u_i E^i \in A$ *if* $u \in K_m$ *since* $(u_1, \ldots, u_m, 0, \ldots, 0)^T \in K$ *whenever* $u \in K_m$. *Moreover,* A *satisfies (A0S):* $u \notin K_m$ *implies* $(u_1, \ldots, u_m, 0, \ldots, 0)^T = E^P\left[\sum_{i=1}^m u_i E^i\right] \notin K$ *by definition of* K_m, *hence* $\sum_{i=1}^m u_i E^i \notin A$.

This example will be used later on and is also related to WCE *as defined in Section 2.5 of [11] (with a supercone* $J \supseteq K$ *instead of* K *itself).*

Example 2.3 *We shall give two set–valued variants of the negative essential infimum being a scalar coherent risk measure. The first one reads as*

$$-EI^S(X) = \left\{ u \in \mathrm{IR}^m : \ X + \sum_{i=1}^{m} u_i E^i \in C \right\} =$$

$$\left\{ u \in \mathrm{IR}^m : \ P\left(\left\{ \omega \in \Omega : \ X(\omega) + \sum_{i=1}^{m} u_i E^i(\omega) \notin K \right\} \right) = 0 \right\}.$$

The acceptance set belonging to this risk measure is the closed convex cone C *satisfying (A0S), (A1), (A2). Hence, it is a coherent measure of risk on* L^p_d *for* $1 \leq p \leq \infty$ *by Theorem 3.2 below (a direct proof is also possible). Note that* $-EI^S$ *satisfies (R0S).*

The second variant is

$$-EI^W(X) = \left\{ u \in \mathrm{IR}^m : \ P\left(\left\{ \omega \in \Omega : \ X(\omega) + \sum_{i=1}^{m} u_i E^i(\omega) \in -\mathrm{int}\, K \right\} \right) = 0 \right\}.$$

This risk measure (as well as its acceptance set) is not convex in general (for $m > 1$) and does satisfy (R0), but not (R0S).

Note $-EI^W(X) \supseteq -EI^S(X)$ for all $X \in L^1_d$, i.e. $-EI^S$ is more risk averse than $-EI^W$.

Both $-EI^S$ and $-EI^W$ do not have only "finite values" (see [8] for precise definitions) on L^p_d for $1 \le p < \infty$. A detailed study of related questions for this and other set–valued risk measures is postponed.

Example 2.4 *Value at Risk also has two (non convex) set–valued counterparts: Take $0 \le \lambda$. For the first one, define*

$$V@R^W_\lambda(X) = \left\{ u \in \mathbb{R}^m : \ P\left(\left\{ \omega \in \Omega : \ X(\omega) + \sum_{i=1}^m u_i E^i(\omega) \in -\operatorname{int} K \right\} \right) \le \lambda \right\}.$$

One may check that $V@R^W_\lambda$ satisfies (R0) whenever $0 \le \lambda < 1$ and (R1), (R2) (direct calculations), but not (R0S). For the second one, define

$$V@R^S_\lambda(X) = \left\{ u \in \mathbb{R}^m : \ P\left(\left\{ \omega \in \Omega : \ X(\omega) + \sum_{i=1}^m u_i E^i(\omega) \notin K \right\} \right) \le \lambda \right\}.$$

$V@R^S_\lambda$ satisfies (R0S) whenever $0 \le \lambda < 1$ and (R1), (R2).

Observe $V@R^W_\lambda(X) \supseteq V@R^S_\lambda(X)$ for all $X \in L^1_d$ as well as $V@R^W_{\lambda_1}(X) \supseteq V@R^W_{\lambda_2}(X)$ and $V@R^S_{\lambda_1}(X) \supseteq V@R^S_{\lambda_2}(X)$ for all $X \in L^1_d$ and $0 \le \lambda_1 \le \lambda_2$.

3 Primal representation of risk measures

In this section we shall establish bijection theorems for set–valued risk measures and acceptance sets. Let $\Phi : L^p_d \to \mathcal{F}_m$ be a function. By means of

$$A_\Phi := \left\{ X \in L^p_d : \ 0 \in \Phi(X) \right\} = \left\{ X \in L^p_d : \ K_m \subseteq \Phi(X) \right\} \tag{3.1}$$

we assign to Φ its zero sublevel set. Let $A \subseteq L^p_d$ be a set. By means of

$$\Phi_A(X) := \left\{ u \in \mathbb{R}^m : \ X + \sum_{i=1}^m u_i E^i \in A \right\} \tag{3.2}$$

we assign to A a function $\Phi_A : L^p_d \to \mathcal{P}(\mathbb{R}^m)$. It will turn out that these definitions yield one-to-one correspondences between acceptance sets and set–valued risk measures.

Theorem 3.1 *(i) Let $\Phi : L^p_d \to \mathcal{F}_m$ be translative w.r.t. $E^1, \dots, E^m$, i.e. it satisfies (R1). Then A_Φ satisfies (A1) and it holds $\Phi = \Phi_{A_\Phi}$. (ii) Let $A \subseteq L^p_d$ be a set satisfying (A1). Then Φ_A maps into $\mathcal{F}_m$, is translative and it holds $A = A_{\Phi_A}$.*

Proof. (i) First, we show (A1). Take $X \in L^p_d$, $\{u^k\}_{k\in\mathbb{N}} \subset \mathbb{R}^m$ with $\lim_{k\to\infty} u^k = 0$ and $X + \sum_{i=1}^m u^k_i E^i \in A$. Then (3.1) and (R1) imply $0 \in \Phi\left(X + \sum_{i=1}^m u^k_i E^i\right) = \Phi(X) + \{-u^k\}$, i.e. $u_k \in \Phi(X)$, for all $k \in \mathbb{N}$. Since Φ maps into $\mathcal{F}_m$, $\Phi(X)$ is closed, hence $0 \in \Phi(X)$ implying $X \in A_\Phi$ by (3.1). For the second part of (A1) take $X \in A_\Phi$ and $u \in K_m$. Since K_m is a convex cone we get $K_m \subseteq K_m + \{-u\}$. Since $X \in A_\Phi$ we have $K_m \subseteq \Phi(X)$, hence by (R1)

$$K_m \subseteq K_m + \{-u\} \subseteq \Phi(X) + \{-u\} = \Phi\left(X + \sum_{i=1}^m u_i E^i\right)$$

which gives $X+\sum_{i=1}^m u_iE^i \in A_\Phi$.

Direct calculations using (3.2), (3.1) and (R1) give

$$\begin{aligned}\Phi_{A_\Phi}(X) &= \left\{u\in \mathbb{R}^m:\ X+\sum_{i=1}^m u_iE^i \in A_\Phi\right\} \\ &= \left\{u\in \mathbb{R}^m:\ 0\in\Phi\left(X+\sum_{i=1}^m u_iE^i\right)=\Phi(X)+\{-u\}\right\}=\Phi(X).\end{aligned}$$

(ii) First, we show that Φ_A maps into $\mathcal{F}_m$. For this it is enough to show $\Phi_A(X)+K_m\subseteq\Phi_A(X)$ and $\Phi_A(X)$ is closed for each $X\in L_d^p$. Take $u\in\Phi_A(X)$ and $v\in K_m$. Then $X+\sum_{i=1}^m u_iE^i\in A$. (A1) implies $X+\sum_{i=1}^m(u_i+v_i)E^i\in A$, hence $u+v\in\Phi_A(X)$ by (3.2). Take a sequence $\{u^k\}_{k\in\mathbb{N}}\subset\Phi_A(X)$ with $\lim_{k\to\infty}u^k=u$. Then by (3.2) $X+\sum_{i=1}^m u_i^kE^i=\left(X+\sum_{i=1}^m u_iE^i\right)+\sum_{i=1}^m\left(u_i^k-u_i\right)E^i\in A$ for all $k\in\mathbb{N}$. Since A is radially closed this implies $X+\sum_{i=1}^m u_iE^i\in A$ which gives $u\in\Phi_A(X)$.

We turn to (R1). Take $X\in L_d^p$, $u\in\mathbb{R}^m$. Then by (3.2)

$$\begin{aligned}\Phi_A\left(X+\sum_{i=1}^m u_iE^i\right) &= \left\{v\in\mathbb{R}^m:\ X+\sum_{i=1}^m(u_i+v_i)E^i\in A\right\} \\ &= \left\{u+v\in\mathbb{R}^m:\ X+\sum_{i=1}^m(u_i+v_i)E^i\in A\right\}+\{-u\} \\ &= \Phi_A(X)+\{-u\}.\end{aligned}$$

From (3.2), (3.1) we get

$$A_{\Phi_A}=\left\{X\in L_d^p:\ 0\in\Phi_A(X)\right\}=\left\{X\in L_d^p:\ 0\in\left\{u\in\mathbb{R}^m:\ X+\sum_{i=1}^m u_iE^i\in A\right\}\right\}$$

which proves the desired equality. □

Theorem 3.2 *(i) Let $\Phi: L_d^p\to\mathcal{F}_m$ be a measure of risk. Then A_Φ is an acceptance set. If Φ satisfies (R0S) then A_Φ satisfies (A0S). If Φ is convex, so is A. If Φ is coherent then A is a coherent acceptance set.*

(ii) Let $A\subseteq L_d^p$ be an acceptance set. Then Φ_A is a measure of risk. If A satisfies (A0S) then Φ_A satisfies (R0S). If A is convex, so is Φ_A. If A is a coherent acceptance set then Φ_A is a coherent measure of risk.

Proof. (i) First, we show (A0): Take $u\in K_m$. Then $u\in\Phi(0)$ by (R0) and $0\in\Phi\left(\sum_{i=1}^m u_iE^i\right)$ by (R1). The construction (3.1) implies $\sum_{i=1}^m u_iE^i\in A_\Phi$. Now, take $u\in-\operatorname{int}K_m$. Then by (R0) $u\notin\Phi(0)$, hence by (R1) $0\notin\Phi\left(\sum_{i=1}^m u_iE^i\right)$ and $\sum_{i=1}^m u_iE^i\notin A_\Phi$.

(A1) follows from Theorem 3.1.

In order to check (A2) take $X_1\in A_\Phi$ and $X_2\in C$. Then $(X_1+X_2)-X_1\in C$ and by (R2) $0\in\Phi(X_1)\subseteq\Phi(X_1+X_2)$, hence $X_1+X_2\in A_\Phi$ as desired.

If Φ satisfies (R0S) then all the more (R0), hence A_Φ satisfies (A0) which gives $\sum_{i=1}^m u_iE^i\in A_\Phi$ for $u\in K_m$. Conversely, take $u\in\mathbb{R}^m$ with $\sum_{i=1}^m u_iE^i\in A_\Phi$. The definition of A_Φ gives $0\in\Phi\left(\sum_{i=1}^m u_iE^i\right)$ and (R1) implies $0\in\Phi(0)+\{-u\}$, hence $u\in K_m$ by (R0S). Together, A_Φ satisfies (A0S).

It is left to the reader to check that A_Φ is convex if Φ is convex and that A_Φ is a cone if Φ is positively homogeneous.

(ii) From Theorem 3.1 we already know that Φ_A maps into $\mathcal{F}_m$ and that (R1) holds true.

Next, we shall show (R0). By (A0) and (3.2) we get $u \in \Phi_A(0)$ whenever $u \in K_m$ and $u \notin \Phi_A(0)$ whenever $u \in -\operatorname{int} K_m$.

Finally, for a proof of (R2) take $X_1, X_2 \in L^p_d$ such that $X_2 - X_1 \in C$. Then

$$\begin{aligned}
\Phi_A(X_1) &= \left\{ u \in \mathrm{IR}^m : \; X_1 + \sum_{i=1}^m u_i E^i \in A \right\} \\
&= \left\{ u \in \mathrm{IR}^m : \; X_2 + \sum_{i=1}^m u_i E^i \in A + \{X_2 - X_1\} \right\} \\
&\subseteq \left\{ u \in \mathrm{IR}^m : \; X_2 + \sum_{i=1}^m u_i E^i \in A \right\}
\end{aligned}$$

where the inclusion holds true since $A + C \subseteq A$ by (A2).

If A satisfies (A0S) then all the more (A0), hence Φ_A satisfies (R0) which gives $K_m \subseteq \Phi(0)$. For the converse, take $u \in \Phi_A(0)$. Then, by (R1), $0 \in \Phi_A\left(\sum_{i=1}^m u_i E^i\right)$, hence $\sum_{i=1}^m u_i E^i \in A_{\Phi_A} = A$.

Again, it is a matter of exercise to show that Φ_A is convex if A is convex and that Φ_A is positively homogeneous if A is a cone. □

The proof of the above theorems makes it apparent what property of a translative function corresponds to a certain property of its zero sublevel set. In particular, the closedness of the values of the function corresponds to radial closedness of its zero sublevel set. This is a new observation due to [8] even for the case $d = m = 1$. The axioms (R1) and (A1) play a special role as Theorem 3.1 due to [8] shows.

Remark 3.1 *An analysis of the preceding theorems and its proofs shows that Theorem 3.2 yields a bijection between (1) functions $\Phi : L^p_d \to \mathcal{F}_m$ satisfying (R1) and sets $A \subseteq L^p_d$ satisfying (A1), (2) functions $\Phi : L^p_d \to \mathcal{F}_m$ satisfying (R0), (R1) and sets $A \subseteq L^p_d$ satisfying (A0), (A1), (3) functions $\Phi : L^p_d \to \mathcal{F}_m$ satisfying (R1), (R2) and sets $A \subseteq L^p_d$ satisfying (A1), (A2), (4) convex functions $\Phi : L^p_d \to \mathcal{C}_m$ satisfying (R1) and convex sets $A \subseteq L^p_d$ satisfying (A1) and, most important, (5) convex (coherent) risk measures and convex (coherent) acceptance sets.*

Remark 3.2 *Translativity of Φ is equivalent to*

$$\operatorname{epi}\Phi = \left\{ (X, u) \in L^p_d \times \mathrm{IR}^m : \; X + \sum_{i=1}^m u_i E^i \in A_\Phi \right\} \tag{3.3}$$

(see [8], Theorem 5). Using (3.3) we may observe that Φ is closed (by definition, iff $\operatorname{epi}\Phi$ is closed) if and only if A_Φ is closed. This is one more example for a bijection property, and closedness can be added in Theorem 3.2 and each item of the preceding remark.

Example 3.1 *As stated before, $-F^Z_m$, $Z \in \mathcal{Z}^q_m$, is a set–valued coherent risk measure on L^p_d, $1 \le p \le \infty$. Its acceptance set, actually a closed half space, is given by*

$$A = \left\{ X \in L^p_d : \; \int_\Omega X \cdot Z \, dP \ge 0 \right\}.$$

Example 3.2 *We consider Example 2.2 again. This gives a risk measure that could be considered as another set–valued counterpart of the scalar risk measure $E^P[-X]$. We call it* set–valued componentwise expectation. *Let $1 \leq p \leq \infty$. The set $A_{CE} = \left\{X \in L^p_d : \ E^P[X] \in K\right\}$ is a closed coherent acceptance set satisfying (A0S) (see Example 2.2) and defines (see Theorem 3.2) the closed coherent risk measure*

$$CE(X) := \Phi_{A_{CE}}(X) = \left\{u \in \mathrm{I\!R}^m : \ E^P\left[X + \sum_{i=1}^m u_i E^i\right] \in K\right\}$$

on L^p_d, $1 \leq p \leq \infty$, satisfying (R0S) (as one can also check with some effort directly). Observe that if $m = d$ then $-E^P[X] \in CE(X)$ for all $X \in L^p_d$.

Example 3.3 *$V@R^S_\lambda$ and $V@R^W_\lambda$ are non-convex set–valued risk measures. Its acceptance sets are given by*

$$A_{V@R^S_\lambda} = \left\{X \in L^p_d : \ P(\{\omega \in \Omega : \ X(\omega) \notin K\}) \leq \lambda\right\}$$

and

$$A_{V@R^W_\lambda} = \left\{X \in L^p_d : \ P(\{\omega \in \Omega : \ X(\omega) \in -\mathrm{int}\, K\}) \leq \lambda\right\},$$

respectively.

4 Dual representation

In this section, we shall state and prove dual representation formulas for convex risk measures in terms of set–valued expectations. The idea is to apply the convex duality theorems for set–valued functions (see Appendix) in combination with Example 2.1 and Remark 2.4. The results as well as the approach are entirely new.

4.1 The case $1 \leq p < \infty$

Let $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Recall the definitions $z = \left(E^P[Z_1], \dots, E^P[Z_m]\right)^T \in \mathrm{I\!R}^m$ for $Z \in L^q_d$ and of the sets $\mathcal{Z}^q_m = \left\{Z \in L^q_d : \ Z \in C^+, \ z^T e = 1\right\}$ in (2.3), $H(-z) = \left\{u \in \mathrm{I\!R}^m : \ z^T u \geq 0\right\}$ (see Appendix), $B^+_m = \left\{v \in K^+_m : \ v^T e = 1\right\}$.

Theorem 4.1 *Let $\Phi : L^p_d \to \mathcal{C}_m$ be a proper closed convex measure of risk. Then*

$$\forall X \in L^p_d : \ \Phi(X) = \bigcap_{Z \in \mathcal{Z}^q_m} \left(F^Z_m[-X] + \mathrm{cl} \bigcup_{X' \in A_\Phi} F^Z_m[X']\right) \tag{4.1}$$

If Φ is additionally positively homogeneous then

$$\forall X \in L^p_d : \ \Phi(X) = \bigcap_{Z \in \mathcal{Z}^q_m \cap A^+_\Phi} F^Z_m[-X] \tag{4.2}$$

Proof. In order to apply Theorem 6.2 we identify $\mathcal{X} = L^p_d$, $\mathcal{X}^* = L^q_d$. Observe that $(-Z, -v) \in \mathcal{Y}_m$ if and only if $v = z$ and $Z \in \mathcal{Z}^q_m$. Thus, we can replace (x^*, v) in (6.10) and (6.12) by $(-Z, -z)$ with $Z \in \mathcal{Z}^q_m$ taking into account $S_{(-Z,-z)}(X) = F^Z_m[-X]$ and $S_{(-Z,-z)}(-X) = F^Z_m[X]$. For the coherent case note that $-Z \in A^-_\Phi$ if and only if $Z \in A^+_\Phi$. □

The function $Z \mapsto -\alpha(Z) := \mathrm{cl} \bigcup_{X \in A_\Phi} F^Z_m[X]$ in (4.1) can be seen as the set–valued counterpart to a "penalty function" as defined in Section 4.2 of [6] for the scalar case, and we may say the convex risk measure Φ is *represented* by $-\alpha$. The other way round is also possible: We may start with a suitable function $-\alpha$ and get a convex risk measure. This procedure can be used to generate new convex risk measures as it is shown below for $AV@R$ and a set–valued variant of the entropy measure. The result reads as follows.

Theorem 4.2 *Let* $-\alpha : \mathcal{Z}_m^q \to \mathcal{C}_m$ *be a function such that*
(i) $-\alpha(Z) = \mathrm{cl}\,(-\alpha(Z) + H(-z))$ *for all* $Z \in \mathcal{Z}_m^q$*;*
(ii) $K_m \subseteq \bigcap_{Z \in \mathcal{Z}_m^q} -\alpha(Z)$*;*
(iii) $\bigcap_{Z \in \mathcal{Z}_m^q} -\alpha(Z) \cap -\mathrm{int}\, K_m = \emptyset$*.*

Then,

$$\Phi_\alpha(X) := \bigcap_{Z \in \mathcal{Z}_m^q} \left(-\alpha(Z) + F_m^Z[-X]\right)$$

is a convex measure of risk.

If $-\alpha(Z) \in \{\mathrm{I\!R}^m, H(-z)\}$ *for all* $Z \in \mathcal{Z}_m^q$ *then* Φ_α *is a coherent measure of risk.*

Proof. Follows from Theorem 6.3 of the Appendix. □

Remark 4.1 *In the preceding theorem, the set* $\{Z \in \mathcal{Z}_m^q : -\alpha(Z) \neq \mathrm{I\!R}^m\}$ *corresponds to* $\mathcal{Y}$ *in Theorem 6.3 of Appendix. This set characterizes the readiness of the investor to assume risk: the larger the set, the more risk averse is the investor. It also corresponds to the set of elements where* $-\Phi^*$ *is not equal to* $\mathrm{I\!R}^m$*. In the scalar case, the Fenchel conjugate of a scalar risk measure is not* $+\infty$ *on this set.*

Remark 4.2 *The assumption* $-\alpha(Z) = \mathrm{cl}\,(-\alpha(Z) + H(-z))$ *for all* $Z \in \mathcal{Z}_m^q$ *is not a true restriction since if it is not satisfied,* $-\alpha(Z)$ *can be replaced by* $\mathrm{cl}\,(-\alpha(Z) + H(-z))$*. This new function does still satisfy (ii), (iii) and yields the same* Φ_α*.*

Remark 4.3 *If one replaces the set* $\mathcal{Z}_m^q$ *in (4.1) or (4.2) by the corresponding set of vector measures (see Remark 2.1), the complete analogy to the scalar case is even more striking.*

4.2 The case $p = \infty$

We have the following dual representation formulas for convex risk measures on L_d^∞.

Theorem 4.3 *Let* $\Phi : L_d^\infty \to \mathcal{C}_m$ *be a proper closed convex risk measure. Then*

$$\forall X \in L_d^\infty :\ \Phi(X) = \bigcap_{Q \in \mathcal{Q}_m} \left(F_m^Q[-X] + \mathrm{cl} \bigcup_{X' \in A_\Phi} F_m^Q\left[X'\right] \right). \tag{4.3}$$

If Φ *is additionally positively homogeneous then*

$$\forall X \in L_d^\infty :\ \Phi(X) = \bigcap_{Q \in \mathcal{Q}_m \cap A_\Phi^+} F_m^Q[-X] \tag{4.4}$$

holds true.

Proof. Apply Theorem 6.2 for $\mathcal{X} = L_d^\infty$ and $\mathcal{X}^* = ba_d$. □

Remark 4.4 *The dual representation (4.4) of a proper closed coherent measure of risk on* L_d^∞

admits the following reformulation (use definition of F_m^Q in Remark 2.4)

$$\begin{aligned}
\Phi(X) &= \bigcap_{Q\in\mathcal{Q}_m\cap A_\Phi^+} F_m^Q[-X] \\
&= \bigcap_{Q\in\mathcal{Q}_m\cap A_\Phi^+} \left\{u\in\mathrm{I\!R}^m:\ \int_\Omega\left(X+\sum_{i=1}^m u_iE^i\right)\cdot dQ\geq 0\right\} \\
&= \left\{u\in\mathrm{I\!R}^m:\ \forall Q\in\mathcal{Q}_m\cap A_\Phi^+:\ \int_\Omega\left(X+\sum_{i=1}^m u_iE^i\right)\cdot dQ\geq 0\right\} \\
&= \left\{u\in\mathrm{I\!R}^m:\ 0\leq\inf_{Q\in\mathcal{Q}_m\cap A_\Phi^+} E^Q\left[X+\sum_{i=1}^m u_iE^i\right]\right\}.
\end{aligned}$$

The last line is a transcription of the dual representation of [11], Theorem 4.1 (and of the duality formulas in Section 9 of [4]), which can easily be adapted to the case $1\leq p<\infty$. In both references, only the sublinear case is dealt with and conjugates of set–valued functions are not considered.

For the sake of completeness we state the penalty function representation theorem for set–valued risk measures on L_d^∞.

Theorem 4.4 *Let $-\alpha:\mathcal{Q}_m\to\mathcal{C}_m$ be a function such that*
(i) $-\alpha(Q)=\mathrm{cl}\,(-\alpha(Q)+H(-q))$ for all $Q\in\mathcal{Q}_m$;
(ii) $K_m\subseteq\bigcap_{Q\in\mathcal{Q}_m}-\alpha(Q)$;
(iii) $\bigcap_{Q\in\mathcal{Q}_m}-\alpha(Q)\cap-\mathrm{int}\,K_m=\emptyset$.
Then,

$$\Phi_\alpha(X):=\bigcap_{Q\in\mathcal{Q}_m}\left(-\alpha(Q)+F_m^Q[-X]\right)$$

is a convex measure of risk.

If $-\alpha(Q)\in\{\mathrm{I\!R}^m,H(-q)\}$ for all $Q\in\mathcal{Q}_m$ then Φ_α is a coherent measure of risk.

Proof. Follows from Theorem 6.3 of the Appendix. □

One main question concerning risk measures on a non-reflexive space like L^∞ is under what circumstances a dual representation is possible based on a dual pairing of L_d^∞ and L_d^1, i.e. when weak* lower semicontinuity is present. An answer can be given that is parallel to the scalar case: A so-called Fatou property for set–valued functions can be formulated and proven to be equivalent to weak* lower semicontinuity. The result reads as follows.

Theorem 4.5 *The following things are equivalent for a convex risk measure $\Phi:L_d^\infty\to\mathcal{C}_m$:*
(i) Φ has a dual representation (4.3) with $\mathcal{Q}_m$ replaced by $\mathcal{Q}$ satisfying

$$\mathcal{Q}\subseteq\widetilde{\mathcal{Q}}_m:=\left\{Q\in\mathcal{Q}_m:\ \exists\frac{dQ_i}{dP}=Z_i\in L^1, i=1,\dots,m\right\};$$

(ii) Φ has the Fatou property, i.e. if $\left\{X^k\right\}_{k\in\mathrm{I\!N}}\subseteq L_d^\infty$ is a bounded sequence with $X^k\to X$ P-a.s. then

$$\Phi(X)\supseteq\liminf_{k\to\infty}\Phi\left(X^k\right)=\left\{u\in\mathrm{I\!R}^m:\ \forall k\in\mathrm{I\!N}:\ \exists u^k\in\Phi\left(X^k\right):\ \lim_{k\to\infty}u^k=u\right\};$$

(iii) $\mathrm{epi}\,\Phi$ is $\sigma\left(L_d^\infty,L_d^1\right)\times\mathrm{I\!R}^m$–closed;
(iv) A_Φ is $\sigma\left(L_d^\infty,L_d^1\right)$–closed.

PROOF. (iii) implies (i): Since Φ is proper convex and weakly* closed we can apply Theorem 6.2 (Appendix) for $\mathcal{X} = L_d^\infty$ with weak* topology and $\mathcal{X}^* = L_d^1$ getting the desired representation.

(i) implies (ii): Denote $M(Q) := \operatorname{cl} \bigcup_{X \in A_\Phi} F_m^Q [X]$. Take a bounded sequence $\left\{X^k\right\}_{k \in \mathrm{IN}} \subseteq L_d^\infty$ with $X^k \to X$ P-a.s. and $u \in \liminf_{k \to \infty} \Phi\left(X^k\right)$. Then there is a sequence $u^k \in \Phi\left(X^k\right)$ such that $u^k \to u$ in IR^m. Because of (i) it holds

$$\forall k \in \mathrm{IN},\ \forall Q \in \mathcal{Q}:\ u^k \in F_m^Q \left[-X^k\right] + M(Q).$$

We have to show $u \in \Phi(X)$, i.e. $u \in F_m^Q [-X] + M(Q)$ for all $Q \in \mathcal{Q}$. If $M(Q) = \mathrm{IR}^m$ then there is nothing to prove. If $M(Q) \neq \mathrm{IR}^m$ then there is $\widehat{u}$ such that $M(Q) = \widehat{u} + H(-q)$ with q belonging to Q as defined above, hence

$$\forall k \in \mathrm{IN},\ \forall Q \in \mathcal{Q}:\ u^k \in F_m^Q \left[-X^k\right] + \widehat{u} + H(-q)\,, i.e.$$

$$\forall k \in \mathrm{IN},\ \forall Q \in \mathcal{Q}:\ -q^T\left(u^k - \widehat{u}\right) \leq \int_\Omega X^k \cdot Z\, dP.$$

Since X^k is bounded and converges P a.s. to X we have $\int_\Omega X^k \cdot Z\, dP \to \int_\Omega X \cdot Z\, dP$ for each $Z \in L_d^1$ by Lebesgue's dominated convergence theorem, hence taking the limit $k \to \infty$ gives the desired result.

(ii) implies (iv): According to Lemma A.64 in [6] it suffices to show that the sets

$$A_\Phi^r := A_\Phi \cap \left\{X \in L_d^\infty :\ \|X\|_{L_d^\infty} \leq r\right\}, \quad r > 0$$

are closed in L_d^1. Take a sequence $\left\{X^k\right\}_{k \in \mathrm{IN}} \subseteq A_\Phi^r$ with $X^k \to X$ in L_d^1. Then there is a subsequence, again denoted by $\left\{X^k\right\}_{k \in \mathrm{IN}}$ that converges P-a.s. to X. The Fatou property ensures that $X \in A_\Phi$. The P a.s. convergence and boundedness of X^k's ensure $X \in A_\Phi^r$. Hence A_Φ^r is closed in L^1 and A_ϕ weakly* closed.

(iv) implies (iii): Compare Remark 3.2. □

4.3 Examples

Example 4.1 *The first example is the negative essential infimum. One of its set–valued counterparts (Example 2.3) is closed and convex, namely,*

$$-EI^S(X) = \left\{u \in \mathrm{IR}^m :\ X + \sum_{i=1}^m u_i E^i \in C\right\}.$$

Since $A_{-EI^S} = C$ Theorem 3, (4.2) yields

$$-EI^S(X) = \bigcap_{Z \in \mathcal{Z}_m^q} F_m^Z [-X]\,,$$

i.e. the set of dual variables entering the intersection is the largest possible one which corresponds to the fact that $-EI^S$ is the most risk averse risk measure. Since $-EI^S$ satisfies the Fatou property the dual representation for $-EI^S$ as a risk measure on L_d^∞ is (see Theorem 4.5, (i))

$$-EI^S(X) = \bigcap_{Q \in \tilde{\mathcal{Q}}_m} F_m^Q [-X]\,.$$

Example 4.2 *We investigate componentwise expectation, (see Example 2.2, 3.2). In case* $1 \leq p < \infty$*, the polar cone of the acceptance set of can be written (see below)*

$$A_{CE}^{+} = \left\{ Z \in C^{+} : \ \exists y \in \mathbb{R}^{d} : \ \forall i = 1, ..., d : \ Z_i = y_i E \right\}. \tag{4.5}$$

Therefore, the dual representation of CE *is*

$$CE\left(X\right) = \bigcap_{Z \in \mathcal{Z}_m^c} F_m^Z\left[-X\right]$$

with

$$\begin{aligned} \mathcal{Z}_m^c &:= \left\{ Z \in \mathcal{Z}_m^q : \ \exists y \in \mathbb{R}^d : \ \forall i = 1, ..., d : \ Z_i = y_i E \right\} \\ &= \left\{ Z \in \mathcal{Z}_m^q : \ \exists y \in K^{+} : \ \sum_{i=1}^{m} y_i = 1 \ \forall i = 1, ..., d : \ Z_i = y_i E \right\}. \end{aligned}$$

In case $p = \infty$ *the same dual representation holds true since* CE *has a weak* closed acceptance set.*

We sketch the proof for (4.5). Define a continuous linear operator $T : \mathbb{R}^d \to L_d^q$ *(if* $p = 1$ *then* $q = \infty$ *and* L_d^∞ *is considered with the weak* topology) by* $Ty = \sum_{i=1}^d y_i E^i$*,* $y \in \mathbb{R}^d$*. The adjoint operator* T^* *maps* L_d^p *into* $\mathbb{R}^d$ *by* $T^* X = E^P[X]$*, and we have* $A_{CE} = T^{*-1}(K) = \left\{X \in L_d^p : \ T^* X \in K\right\} = \left\{X \in L_d^p : \forall y \in K^+ : \ y^T (T^* X) \geq 0\right\}$*. The latter set coincides with*

$$\begin{aligned} T\left(K^{+}\right)^{+} &= \left\{ X \in L_d^p : \ \forall Y \in T\left(K^{+}\right) : \ \int_{\Omega} X \cdot Y \, dP \geq 0 \right\} \\ &= \left\{ X \in L_d^p : \ \forall y \in K^{+} : \ y^T \left(T^* X\right) \geq 0 \right\}, \end{aligned}$$

hence $A_{CE} = T^{*-1}(K) = T(K^+)^+$ *and*

$$A_{CE}^{+} = T\left(K^{+}\right)^{++} = \operatorname{cl} T\left(K^{+}\right) = \operatorname{cl} \left\{ Y \in L_d^q : \ \exists y \in K^{+} : \ Y = \sum_{i=1}^{d} y_i E^i \right\}.$$

The closure may be dropped since $T(K^+)$ *is already closed: Indeed, every convergent sequence in* L_d^q*,* $q < \infty$*, contains a P-a.s. convergent subsequence with the same limit, this limit is therefore constant P-a.s. if the members of the sequence are constant P-a.s. In case of* L_d^∞ *the closedness with respect to weak* topology can be proven using Lemma A. 64 of [6].*

Example 4.3 *We give an extension of Average Value at Risk to the set–valued case. Let* $1 \leq p < \infty$ *and* $0 < \lambda \leq 1$*. We use the dual way of definition which gives via Theorem 4.47 of [6] the scalar special case. Define*

$$\mathcal{Z}_\lambda := \left\{ Z \in \mathcal{Z}_m^q : \ \exists v \in \mathbb{R}_+^m : \sum_{i=1}^{m} v_i = \frac{1}{\lambda}, \ \forall i = 1, ...m : Z_i \leq v_i E \right\}. \tag{4.6}$$

Since $\mathcal{Z}_\lambda \subseteq \mathcal{Z}_m^q$*,*

$$AV@R_\lambda\left(X\right) := \bigcap_{Z \in \mathcal{Z}_\lambda} F_m^Z\left[-X\right]$$

defines a coherent measure of risk on L_d^p *according to Theorem 4.2. A moments thought will end with* $\mathcal{Z}_m^c \subseteq \mathcal{Z}_\lambda$*, hence*

$$\forall \lambda \in (0, 1], \ \forall X \in L_d^p : \ CE\left(X\right) \supseteq AV@R_\lambda\left(X\right).$$

Since CE satisfies (R0S) we obtain

$$K_m \subseteq AV@R_\lambda(0) \subseteq CE(0) = K_m,$$

hence $AV@R_\lambda$ satisfies (R0S), too. The acceptance set of $AV@R_\lambda$ is

$$A_\lambda = \left\{ X \in L^p_d :\ \forall Z \in \mathcal{Z}_\lambda : 0 \in F^Z_m[-X] \right\}.$$

Moreover, since $\mathcal{Z}_{\lambda_1} \supseteq \mathcal{Z}_{\lambda_2}$, for all $X \in L^p_d$ it holds $AV@R_{\lambda_1}(X) \subseteq AV@R_{\lambda_2}(X)$ whenever $\lambda_1 \leq \lambda_2$. Finally, the limit $\lambda \to 0$ goes along the lines of the scalar case:

Claim. *It holds*

$$\forall X \in L^p_d :\ \bigcap_{\lambda \in (0,1]} AV@R_\lambda(X) = -EI^S(X).$$

Proof of the claim. *Obviously, $\bigcap_{\lambda \in (0,1]} AV@R_\lambda(X) \supseteq -EI^S(X)$. We shall show the converse. Take $u \in \bigcap_{\lambda \in (0,1]} AV@R_\lambda(X)$, i.e. $u \in F^Z_m[-X]$ for all $Z \in \mathcal{Z}_\lambda$, $\lambda \in (0,1]$. We have to show $u \in F^Z_m[-X]$ for all $Z \in \mathcal{Z}^q_m$. Fix an arbitrary $Z \in \mathcal{Z}^q_m$. Since each component Z_i of Z is a non-negative random variable there exist, for each $i \in \{1, \dots, d\}$, a sequence of non-negative simple functions $\left\{S^k_i\right\}_{k \in \mathbb{N}}$ with $S^k_i \to Z_i$ P-a.s. and $S^k_i \leq S^{k+1}_i \leq Z_i$. We have $\left|X_i S^k_i\right| = |X_i| S^k_i \leq |X_i| Z_i \in L^1$ since $X \in L^p_d$, $Z \in L^q_d$. From the dominated convergence theorem we get $\int_\Omega X_i S^k_i dP \to \int_\Omega X_i Z_i dP$. Of course $E^P\left[S^k_i\right] \to E^P[Z_i]$. Defining $\bar{S}^k_i = \frac{S^k_i}{\sum_{i=1}^m E^P\left[S^k_i\right]}$ we get $\bar{S}^k_i \in \bigcup_{\lambda \in (0,1]} \mathcal{Z}_\lambda$. Now, we have $u \in F^{\bar{S}^k_i}_m[-X]$ for all $k \in \mathbb{N}$. The definition of $F^{\bar{S}^k_i}_m$ gives $\int_\Omega X_i \bar{S}^k_i dP + \sum_{i=1}^m u_i E^P\left[\bar{S}^k_i\right] \geq 0$ being equivalent to $\int_\Omega X_i S^k_i dP + \sum_{i=1}^m u_i E^P\left[S^k_i\right] \geq 0$. Taking the limit $k \to \infty$ we get*

$$\int_\Omega X_i Z_i \, dP + \sum_{i=1}^m u_i E^P[Z_i] \geq 0 \ \Leftrightarrow\ u \in F^Z_m[-X].$$

Since $Z \in \mathcal{Z}^q_m$ is arbitrary, the claim is proven.

In contrast to the scalar case where $AV@R_1(X) = E^P[-X]$ holds for all $X \in L^p$ in the set–valued case $AV@R_1(X) = CE(X)$ is not true in general, see the next example. Finally, we mention the problem of representing $AV@R_\lambda$ as a true average over $V@R$'s (Definition 4.43 in [6] for the scalar case) and of finding a primal representation for $AV@R_\lambda$. This would relate it to WCE as defined in [11].

Example 4.4 *Assume there are risky positions in two currencies with an exchange rate of* $1:1$, *but* 100% *exchange fees. So, for* 2 *units of currency 1 one can get* 1 *unit of currency 2 and vice versa.*

This leads to a model with $d = 2$ and $K = \left\{x \in \mathbb{R}^2 :\ x_1 + 2x_2 \geq 0 \text{ and } 2x_1 + x_2 \geq 0\right\}$ where $x \in K$ means that the position x can be transferred by an exchange into a position with non-negative amounts in both currencies. On the other hand, $x \in -K$ means that $(0,0)^T$ can be transformed into a position with at least the amounts in both currencies determined by the position x.

We consider the case $m = 1$, i.e. the risk of an uncertain position should be canceled by adding a certain amount of currency 1.

In this situation the acceptance set for the componentwise expectation is given by

$$\begin{aligned} A_{CE} &= \left\{X \in L^p_2 : E^P[X] \in K\right\} \\ &= \left\{X \in L^p_2 : E^P[X_1] + 2E^P[X_2] \geq 0 \text{ and } E^P[X_1] + \frac{1}{2}E^P[X_2] \geq 0\right\}. \end{aligned}$$

The acceptance set corresponding to $AV@R_1$ is

$$A_1 = \left\{X \in L_2^p : \ \forall Z \in \mathcal{Z}_1 : \ 0 \in F_m^Z[-X]\right\}$$
$$= \left\{X \in L_2^p : \ \forall Z_2 \in L^q \ \textit{with} \ \frac{1}{2} \le Z_2 \le 2 \ P \ a.s.: E^P[X_1] + E^P[Z_2 X_2] \ge 0\right\}.$$

In a discrete model with $\Omega = \{\omega_1, \omega_2\}$ and $P(\{\omega_1\}) = P(\{\omega_2\}) = \frac{1}{2}$, e.g. the position X with $X(\omega_1) = (3,1)^T, X(\omega_2) = (1,-3)^T$ is in A_{CE} because $E^P[X] = (2,-1)^T \in K$ but $X \notin A_1$ (take $Z_2(\omega_1) = 1, Z_2(\omega_2) = 2$).

Example 4.5 *The following construction is one possibility for a set–valued analog of the scalar entropic risk measure. More variants and relationships to set–valued utility functions are subject of current research. Let $1 \le m \le d$. The m-entropy $H(Q \mid P)$ of $Q \in \widetilde{\mathcal{Q}}_m$ (see Theorem 4.5, (i)) with respect to P is defined to be*

$$H(Q \mid P) := E_m^Q\left[E^{(d)} \log\left(\sum_{i=1}^m \frac{dQ_i}{dP}\right)\right]$$

where $E^{(d)} = (E, \dots, E)^T \in L_d^\infty$. Replacing E_m^Q in this definition by F_m^Q we obtain a $\mathcal{C}_m$–valued function $Q \mapsto -G(Q \mid P)$. Take $\beta > 0$. A set–valued counterpart for the entropic measure of risk *may be defined by*

$$R_\beta(X) := \bigcap_{Q \in \widetilde{\mathcal{Q}}_m} \left[-\frac{1}{\beta} G(Q \mid P) + F_m^Q[-X]\right].$$

It is a convex measure of risk on L_d^∞ according to Theorem 4.5 above. Condition (R0) may be checked directly using the fact that the scalar entropies $H(Q_i \mid P)$, $i = 1, \dots, m$ are non-negative. In case $m = d = 1$ we have $R_\beta(X) = [\varrho_\beta(X), +\infty)$ with ϱ_β the scalar entropic risk measure as defined e.g. in [6], Chapter 4.3., i.e. R_β is a generalization of ϱ_β. This shows that R_β is not coherent in general.

5 Scalarization

Let $1 \le p < \infty$, $\Phi : L_d^p \to \mathcal{F}_m$ be a risk measure and $X \in L_d^p$. Which element of $\Phi(X)$ should be selected in order to cancel the risk of X? A natural answer to this question is: Choose a minimal element of the set $\Phi(X)$ w.r.t. the order relation generated by the convex cone K_m. As far as convexity is present such elements can be identified via linear scalarizations, the topic of this section. Let us mention that it seems to be possible to replace the values $\Phi(X)$ by its infimal sets w.r.t. the cone K_m and get close relationships between vector optimization and set–valued risk measures. We refer the reader to [12], [13], [14] for this approach.

In the scalar case, looking for minimal elements of $\Phi(X)$ w.r.t. the cone K_m ("efficient points") means looking for the infimum of $[\varrho, +\infty)$, ϱ being an extended real–valued risk measure. However, it is not a trivial procedure if $m > 1$.

Take $v \in B_m^+ = \left\{v \in K_m^+ : \ v^T e = 1\right\}$ (see Section 2) and define a function $\varphi_v : L_d^p \to \mathrm{I\!R} \cup \{\pm\infty\}$ by

$$\varphi_v(X) = \inf_{u \in \Phi(X)} v^T u \tag{5.1}$$

which is nothing else than the negative support function (in the sense of convex analysis, see e.g. [16], p. 28) of $\Phi(X)$ at $-v \in \mathbb{R}^m$. Since we shall consider φ_v as a function of X, not of v we use the above notation. Since for $v \in K_m^+$

$$\varphi_v(X) = \inf_{u \in \Phi(X)} v^T u = \inf_{u \in \operatorname{cl}\operatorname{co}(\Phi(X)+K_m)} v^T u$$

we may assume in this section that Φ maps into $\mathcal{C}_m$.

Interpretation. An element $u \in \Phi(X)$ gives amounts of deterministic positions in m accepted markets, namely u_i units of the reference instrument in market i, $i = 1, \ldots, m$. The risk of X can be canceled by $\sum_{i=1}^m u_i E^i \in L_d^p$. Let us assume that a unit in market i can be bought at price p_i (\$, say, or any other appropriate currency). The total price of $\sum_{i=1}^m u_i E^i$ is $p^T u$\$. Hence, $\inf_{u \in \Phi(X)} p^T u$ gives the minimal price in \$ that has to be paid (with a given fixed price structure p, at this moment in time, for given exchange rates including proportional transaction costs) to get a portfolio in the m markets that cancels the risk of the original portfolio X in all d markets.

Instead of using the price structures $p \in K_m^+$ directly we shall consider the collection of reduced price structures $v \in B_m^+$. That is, if $p^T e > 0$ (the case $= 0$ can be excluded for obvious reasons) gives the total price of a portfolio consisting of one unit of the deterministic position in each of the markets $1, \ldots, m$, then we consider $v = \frac{p}{p^T e} \in B_m^+$. Since $K_m^+ \subseteq \mathbb{R}_+^m$, $p \in K_m^+$ has always non-negative entries. It is adequate to consider only price structures $p \in K_m^+$, since $p \notin K_m^+$ implies the existence of some $u \in K_m$ with $p^T u < 0$ meaning that it is possible to buy a deterministic position $\sum_{i=1}^m u_i E^i$, canceling the risk of "doing nothing" at a negative price. This is not reasonable because in this way some kind of "free lunch without risk" can be created. Mathematically, this fact is expressed by $\varphi_v(X) = -\infty$ if $v \notin K_m^+$.

A set–valued function $\Phi : L_d^p \to \mathcal{C}_m$ can be seen as a mapping from L_d^p into the collection of families $\{\varphi_v\}_{v \in B_m^+}$ via (5.1), but we do not emphasize this point of view in this paper. Several important properties of Φ can be expressed as properties of the family $\{\varphi_v\}_{v \in B_m^+}$. We shall give a selection in the following theorem.

Theorem 5.1 *Let $\Phi : L_d^p \to \mathcal{C}_m$ be a function. Then*

(i) if $K_m \subseteq \Phi(0)$ then $\varphi_v(0) \leq 0$ for each $v \in B_m^+$;

(ii) if $\Phi(0) \cap -\operatorname{int} K_m = \emptyset$ then there is $v \in B_m^+$ such that $\varphi_v(0) \geq 0$;

(iii) if Φ satisfies (R1) then φ_v is translative w.r.t. $\sum_{i=1}^m E^i$ for each $v \in B_m^+$;

(iv) if Φ satisfies (R2) then φ_v is monotone w.r.t. C for each $v \in B_m^+$;

(v) if Φ is convex then φ_v is convex for each $v \in B_m^+$;

(vi) if Φ is positively homogeneous the so is φ_v for each $v \in B_m^+$;

(vii) if $\Phi(X) \neq \emptyset$ for at least one $X \in L_d^p$ then φ_v is not identical $+\infty$ for each $v \in B_m^+$, if $\Phi(X) \neq \mathbb{R}^m$ for each $X \in L_d^p$ then for each $X \in L_d^p$ there is $v \in B_m^+$ such that $-\infty < \varphi_v(X)$;

The proof of this theorem is omitted since the facts are reformulations of well-known results or straightforward to prove using the corresponding definitions and elementary arguments. Observe that closedness of Φ does not imply closedness of φ_v in general. A simple counterexample in two dimensions is the function $\Phi : \mathbb{R} \to \mathcal{C}_2$ with $K_2 = \mathbb{R}_+^2$ defined by $\Phi(x) = \left\{ \left(\frac{1}{x}, 0\right)^T \right\} + \mathbb{R}_+^2$ for $x > 0$ and $\Phi(x) = \emptyset$ for $x \leq 0$. This function is closed and convex, but φ_v for $v = (0,1)^T$ is convex, but not closed.

It is worth noting that (vii) of the above theorem yields the following necessary conditions for properness of a convex valued function: If $\Phi : L_d^p \to \mathcal{C}_m$ is proper then (a) $\forall X \in L_d^p$, $\exists v \in B_m^+$:

$-\infty < \varphi_v(X)$ and (b) $\forall v \in B_m^+$, $\exists X \in L_d^p$: $\varphi_v(X) < +\infty$. Even more, the converse is also true: if (a) and (b) are satisfied then $\Phi : L_d^p \to \mathcal{C}_m$ is proper. The last remark also applies to (v), for example. We do not make use of these facts, the interested reader may consult the forthcoming [9] for more details. Instead, we turn to "dual scalarization".

The question is if the Fenchel conjugates (in the usual sense) of the extended real–valued functions φ_v have anything to do with the set–valued conjugate of Φ, i.e. $-\Phi^*$. The answer is affirmatively positive.

Define the extended real–valued functions $\varphi_v^* : L_d^q \to \mathbb{R} \cup \{\pm\infty\}$ by

$$\varphi_v^*(Y) = \sup_{u \in -\Phi^*(Y,-v)} -v^T u,$$

being nothing else than the support functions of the sets $-\Phi^*(Y,-v)$ at $-v \in \mathbb{R}^m$. The following result holds true.

Theorem 5.2 *Let $\Phi : L_d^p \to \mathcal{F}_m$ be a function and $v \in K_m^+$. Then $(\varphi_v)^* = \varphi_v^*$, i.e.,*

$$\forall Y \in L_d^q : \ \varphi_v^*(Y) = \sup_{X \in L_d^p} \left\{ \int_\Omega X \cdot Y \, dP - \varphi_v(X) \right\}$$

PROOF. First, the definitions of $-\Phi^*(Y,-v)$ and $S_{(Y,-v)}(-X)$ yield

$$\sup_{u \in -\Phi^*(Y,-v)} -v^T u \le \sup_{X \in L_d^p} \left[\sup_{u \in \Phi(X)} -v^T u + \int_\Omega X \cdot Y \, dP \right] = (\varphi_v)^*(Y)$$

Assume that strict inequality holds in this relationship. Then there are $X \in \operatorname{dom}\Phi$ and $u \in \Phi(X)$ such that

$$\sup_{u' \in -\Phi^*(Y,-v)} -v^T u' < \int_\Omega X \cdot Y \, dP - v^T u = -v^T \left(-e \int_\Omega X \cdot Y \, dP + u \right).$$

Since $-e \int_\Omega X \cdot Y \, dP + u \in -\Phi^*(Y,-v)$ (observe $-e \int_\Omega X \cdot Y \, dP \in S_{(Y,-v)}(-X)$), this is a contradiction. $\square$

Of course, the case $p = \infty$ can be dealt with analogously replacing $Y \in L_d^q$ by $Q \in ba_d$.

Theorem 5.2 is not only of theoretical interest (it yields a commuting diagram for Φ, $-\Phi^*$ and the φ_v's, φ_v^*'s), it can also be useful for detecting scalarizations of set–valued risk measures. We sketch the procedure. Let $\Phi : L_d^p \to \mathcal{C}_m$ be a proper closed coherent risk measure and $v \in B_m^+$. Theorem 5.2 (use the definition of $(\varphi_v)^{**}$, see e.g. [16], and the formula (6.11) from the Appendix for $-\Phi^*$) yields the following representation formula for the Fenchel biconjugate of the scalarization

$$(\varphi_v)^{**}(X) = \sup \left\{ -\int_\Omega X \cdot Z \, dP : \ Z \in A_\Phi^+, \ \forall i = 1, ..., m : \ v_i = E^P[Z_i] \right\}. \tag{5.2}$$

Since φ_v is convex, but not closed in general (see discussion after Theorem 5.1) (5.2) does not give φ_v itself, but its closed hull $(\varphi_v)^{**}$.

Example 5.1 *We look for scalarizations of the coherent risk measure given by $\Phi(X) = F_m^Z[-X]$ (see Example 2.1). Since for $Z \in \mathcal{Z}_m^q$ (recall $z = \left(E^P[Z_1], \dots, E^P[Z_m]\right)^T \in B_m^+$)*

$$\begin{aligned} F_m^Z[-X] &= \left\{ u \in \mathbb{R}^m : \int_\Omega \left(X + \sum_{i=1}^m u_i E^i \right) \cdot Z \, dP \ge 0 \right\} \\ &= \left\{ u \in \mathbb{R}^m : -\int_\Omega X \cdot Z \, dP \le z^T u \right\} \end{aligned}$$

one may see that

$$\varphi_v(X) = \begin{cases} -\infty & : \ z \neq v \in B_m^+ \\ -\int_\Omega X \cdot Z \, dP & : \ z = v \in B_m^+. \end{cases}$$

which means that φ_z is a linear function w.r.t. Z and coincides with $E^Q[-X]$ in case $m = d = 1$ and Q being the probability measure with density Z w.r.t. P. Any other case does not make sense, there is only one proper scalarization!

Example 5.2 *The biconjugate of the scalarization for set–valued componentwise expectation (see Example 2.2, 3.2, 4.2) can be derived using (5.2) to be*

$$(\varphi_v)^{**}(X) = \sup\left\{-\sum_{i=1}^{d} y_i E^P[X_i] : \ y \in K^+, \forall i = 1, ..., m : \ v_i = y_i\right\}.$$

Using the Fenchel-Rockafellar duality theorem for scalar optimization problems (see [16]) one can show that $(\varphi_v)^{**}$ coincides with φ_v if the cone K satisfies the following condition:

$$\operatorname{int} K + \left(\mathbb{R}^m \times \{0\}^{d-m}\right) = \mathbb{R}^d. \tag{5.3}$$

Indeed, if we define $f : \mathbb{R}^m \to \mathbb{R}$ by $f(u) := v^T u$, $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$ by

$$g(x) := I_{K+\{E^P[-X]\}}(x) = \begin{cases} 0 & : \ x \in K + \left\{E^P[-X]\right\} \\ +\infty & : \ x \notin K + \left\{E^P[-X]\right\} \end{cases}$$

and $T : \mathbb{R}^m \to \mathbb{R}^d$ by $Tu := (u_1, ..., u_m, 0, ..., 0)^T$ then

$$\varphi_v(X) = \inf_{u \in \mathbb{R}^m} \{f(u) + g(Tu)\}.$$

Since the conjugates of f and g are $f^* : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$, $f^*(w) = I_{\{v\}}(w)$ and $g : \mathbb{R}^d \to \mathbb{R} \cup \{+\infty\}$, $g^*(y) = I_{K^-}(y) - y^T E^P[X]$, respectively, and the adjoint operator of T is $T^* : \mathbb{R}^d \to \mathbb{R}^m$ with $T^* y = (y_1, ..., y_m)^T$ one can discover

$$(\varphi_v)^{**}(X) = \sup_{y \in \mathbb{R}^d} \{-f^*(T^* y) - g^*(-y)\}.$$

Since condition (5.3) for each $X \in L_d^1$ implies the existence of some $u \in \mathbb{R}^m$ such that $Tu \in \operatorname{int}\left(K + \left\{E^P[-X]\right\}\right)$ a regularity condition is satisfied and the Fenchel-Rockafellar duality theorem provides $\varphi_v(X) = (\varphi_v)^{**}(X)$.

Note that in the case $m = d$ condition (5.3) is always satisfied and one obtains $\varphi_v(X) = -\sum_{i=1}^{d} v_i E^P[X_i]$. Moreover, if $m = d = 1$ then $\varphi_v(X) = -E^P[X]$ for $X \in L^1$, i.e. (not very surprising, but comforting) the scalarization coincides with the original risk measure.

Interpretation. The regularity condition (5.3) means that each deterministic position can be transferred into a "positive" deterministic position (a position $\sum_{i=1}^{d} x_i E^i$ with $x \in \operatorname{int} K$) by adding certain amounts of the reference instruments in the accepted markets, in particular, this signifies that negative values in the last $d - m$ markets may be compensated by positive ones in (at least one of) the first m. This assumption is reasonable in our framework.

Example 5.3 *According to (5.2), the biconjugate of the scalarization for $-EI^S : L_d^\infty \to \mathcal{C}_m$ reads as*

$$(\varphi_v)_v^{**}(X) = \sup\left\{-\int_\Omega X \cdot dQ : \ Q \in \mathcal{Q}_m, \ v = q\right\}$$

agreeing on $\sup \emptyset = -\infty$*. A set of* $X^0 \in L^\infty_d$ *with* $\varphi_v^{**}\left(X^0\right) = \varphi_v\left(X^0\right)$ *can be calculated as in the previous example. Using the definition of* $-EI^S$ *we get*

$$\varphi_v\left(X^0\right) = \inf\left\{v^T u :\ X^0 + \sum_{i=1}^{m} u_i E^i \in C\right\},\ X^0 \in L^\infty_d.$$

Defining a continuous linear operator $T : \mathbb{R}^m \to L^\infty_d$ *by* $Tu := \sum_{i=1}^m u_i E^i$ *and functions* $f : \mathbb{R}^m \to \mathbb{R}$ *and* $g : L^\infty_d \to \mathbb{R} \cup \{+\infty\}$ *by* $f(u) := v^T u$, $g(X) := I_C\left(X + X^0\right)$*, respectively, with* $I_C(X) = 0$ *if* $X \in C$ *and* $I_C(X) = +\infty$ *if* $X \notin C$ *we obtain*

$$\varphi_v\left(X^0\right) = \inf_{u \in \mathbb{R}^m}\left\{f(u) + g(Tu)\right\}.$$

The Fenchel conjugates (see [16]) of f *and* g *are* $f^*(w) = I_{\{v\}}(w)$*,* $x \in \mathbb{R}^m$ *and* $g^*(Q) = I_{C^-}(Q) - E^Q\left[X^0\right]$*,* $Q \in ba_d$*, thus the dual problem for the one above defining* $\varphi_v\left(X^0\right)$ *is (observe* $T^*Q = q$*)*

$$\sup_{Q \in ba_d}\left\{-f^*\left(T^*Q\right) - g^*(-Q)\right\} = \sup_{Q \in C^+,\, q = v} E^Q\left[-X^0\right].$$

If strong duality holds for the two optimization problems then

$$\varphi_v\left(X^0\right) = \sup_{Q \in \mathcal{Q}_m, q = v} E^Q\left[-X^0\right] \tag{5.4}$$

since $v \in B^+_m$ *and* $q = v$ *implies* $Q \in \mathcal{Q}_m$*. A sufficient condition for strong duality is that there is* $u^0 \in \mathbb{R}^m$ *such that* g *is continuous at* $Tu^0 = \sum_{i=1}^m u_i^0 E^i$ *which is the case if* $\sum_{i=1}^m u_i^0 E^i + X^0 \in \operatorname{int} C$*. In particular, this is satisfied if* $X_i^0 > 0$ P *a.s. for* $i = m+1, \ldots, d$*. Again, for* $m = d$ *this is automatically satisfied. In case* $m = d = 1$ *we re-discover the dual representation of the negative essential infimum from* (5.4)*.*

6 Appendix

In order to make the paper as self-consistent as possible we give a few results on conjugates of set-valued functions. In particular, we shall give a proof of a biconjugation theorem which might be of independent interest. The reader is referred to [9] for further details on this new theory.

Let $\mathcal{X}$ be a separated locally convex space with topological dual $\mathcal{X}^*$ and $\mathcal{P}(\mathbb{R}^m)$ the set of all subsets of $\mathbb{R}^m$ including the empty set. For several reasons it does make sense to select certain subsets out of $\mathcal{P}(\mathbb{R}^m)$. This will be done as follows: We say that a set $M \subseteq \mathbb{R}^m$ is *lower closed* iff $M = \operatorname{cl}(M + K_m)$ and *lower closed convex* if $M = \operatorname{cl}\operatorname{co}(M + K_m)$. Denote by

$$\mathcal{F}_m := \left\{M \subseteq \mathbb{R}^m :\ M = \operatorname{cl}(M + K_m)\right\},\ \mathcal{C}_m := \left\{M \subseteq \mathbb{R}^m :\ M = \operatorname{cl}\operatorname{co}(M + K_m)\right\}$$

the set of all lower closed and lower closed convex subsets of $\mathbb{R}^m$, respectively. Here, $+$ denotes the Minkowski sum with conventions $\emptyset + M = M + \emptyset = \emptyset$ for all $M \in \mathcal{P}(\mathbb{R}^m)$ and $\emptyset$ is considered to be closed and convex[1].

The order relation in $\mathcal{F}_m$ and $\mathcal{C}_m$ that replaces the usual $\leq$–relation for real numbers shall be $\supseteq$. Since the Minkowski sum of two closed sets in $\mathbb{R}^m$ is not closed in general we use the addition $M_1 \oplus M_2 := \operatorname{cl}(M_1 + M_2)$. Note that $\oplus$ is commutative, associative and it holds

[1] One of the authors (A.H.) wrote "clonvex" having closed convex in mind. This seems to be a useful abbreviation.

$M \oplus K_m = K_m \oplus M = M$ for all $M \in \mathcal{F}_m$. Multiplication with scalars is defined as usual for nonempty sets and nonzero scalars. We agree on $t\emptyset = \emptyset$ for all $t \neq 0$ and $0M = K_m$ for all $M \in \mathcal{F}_m$ (in particular, $0\emptyset = K_m$), thus the closed convex cone K_m serves as zero element in $(\mathcal{F}_m, \oplus, \supseteq)$ and $(\mathcal{C}_m, \oplus, \supseteq)$ being ordered conlinear spaces according to [7].

Let $F : \mathcal{X} \to \mathcal{F}_m$ be a function. We assign to F the sets

$$\operatorname{epi} F := \{(x, u) \in \mathcal{X} \times \mathbb{R}^m : u \in F(x)\}, \quad \operatorname{dom} F := \{x \in \mathcal{X} : \ F(x) \neq \emptyset\},$$

its *epigraph* and *effective domain*, respectively. We say that F is convex iff $\operatorname{epi} F$ is a convex and that F is closed iff $\operatorname{epi} F$ is a closed subset of $\mathcal{X} \times \mathbb{R}^m$. Note that F is convex if and only if

$$\forall x^1, x^2 \in \mathcal{X}, \ \forall t \in (0,1): \ F\left(tx^1 + (1-t)x^2\right) \supseteq tF\left(x^1\right) \oplus (1-t)F\left(x^2\right). \tag{6.1}$$

Using this inequality for $x^1 = x^2$ one may observe that a convex function $F : \mathcal{X} \to \mathcal{F}_m$ actually maps into $\mathcal{C}_m$.

We call F positively homogeneous iff

$$\forall x \in \mathcal{X}, \ \forall t > 0: \ F(tx) = tF(x) \tag{6.2}$$

and sublinear iff it is convex and positively homogeneous. The function F is sublinear if and only if $\operatorname{epi} F$ is a convex cone.

A function $F : \mathcal{X} \to \mathcal{F}_m$ is called *proper* iff $\operatorname{dom} F \neq \emptyset$ and F never attains the value $\mathbb{R}^m$. One may observe that $\emptyset$ is the largest and $\mathbb{R}^m$ the smallest element with respect to the order relation $\supseteq$ in $\mathcal{F}_m$.

Take $x^* \in \mathcal{X}^*$ and $v \in \mathbb{R}^m \setminus \{0\}$. By

$$S_{(x^*,v)}(x) := \left\{u \in \mathbb{R}^m : \ x^*(x) + v^T u \leq 0\right\}, \ x \in \mathcal{X} \tag{6.3}$$

a function $S_{(x^*,v)} : \mathcal{X} \to \mathcal{P}(\mathbb{R}^m)$ is defined. Each value is a shifted half space in $\mathbb{R}^m$ since $S_{(x^*,v)}(x)$ is the sublevel set of the function $u \mapsto v^T u$ at level $-x^*(x)$. We shall use such functions as substitutes for real–valued linear functions in our set–valued framework. Indeed, the functions $x \mapsto S_{(x^*,v)}(x)$ share several properties with linear functions as the following lemma shows.

Lemma 6.1 *For each $x^* \in \mathcal{X}^*$ and $v \in \mathbb{R}^m \setminus \{0\}$ it holds*

$$\forall x^1, x^2 \in \mathcal{X}: \ S_{(x^*,v)}\left(x^1 + x^2\right) = S_{(x^*,v)}\left(x^1\right) + S_{(x^*,v)}\left(x^2\right)$$
$$\forall x \in \mathcal{X}, \ \forall t \neq 0: \ S_{(x^*,v)}(tx) = tS_{(x^*,v)}(x).$$

Moreover, $S_{(x^*,v)}(0) = \left\{u \in \mathbb{R}^m : \ v^T u \leq 0\right\} =: H(v)$. If $v \in K_m^- \setminus \{0\}$ then $S_{(x^*,v)} : \mathcal{X} \to \mathcal{C}_m$.

Proof. Almost everything is a direct consequence of (6.3). The only thing one may wonder about is additivity. For one inclusion, take $u^1 \in S_{(x^*,v)}\left(x^1\right)$, $u^2 \in S_{(x^*,v)}\left(x^2\right)$. Then $x^*\left(x^1\right) + v^T u^1 \leq 0$ and $x^*\left(x^2\right) + v^T u^2 \leq 0$, hence $x^*\left(x^1 + x^2\right) + v^T\left(u^1 + u^2\right) \leq 0$ which is $u^1 + u^2 \in S_{(x^*,v)}\left(x^1 + x^2\right)$. For the converse inclusion, take $u \in S_{(x^*,v)}\left(x^1 + x^2\right)$ and $u^1 \in \mathbb{R}^m$ such that $x^*\left(x^1\right) + v^T u^1 = 0$. Such u^1 always exists since $v \neq 0$, and certainly $u^1 \in S_{(x^*,v)}\left(x^1\right)$. Set $u^2 = u - u^1$. Then

$$x^*\left(x^2\right) + v^T u^2 = x^*\left(x^1 + x^2\right) + v^T\left(u - u^1\right) - x^*\left(x^1\right) = x^*\left(x^1 + x^2\right) + v^T(u) \leq 0$$

since $u \in S_{(x^*,v)}\left(x^1 + x^2\right)$. Hence $u^2 \in S_{(x^*,v)}\left(x^2\right)$ which proves the desired inclusion. □

The very definition of $S_{(x^*,v)}$ ensures

$$\left\{S_{(x^*,v)}:\ x^*\in\mathcal{X}^*,\ v\in K_m^-\backslash\{0\}\right\}=\left\{S_{(x^*,v)}:\ x^*\in\mathcal{X}^*,\ v\in B_m^-\right\},$$

therefore we restrict v to $B_m^-=\left\{v\in K_m^-:\ v^Te=-1\right\}$ in the following.

The conjugate of a function $F:\mathcal{X}\to\mathcal{F}_m$ is a function $-F^*:\mathcal{X}^*\times B_m^-\to\mathcal{F}_m$ defined by

$$-F^*\left(x^*,v\right):=\operatorname{cl}\bigcup_{x\in\mathcal{X}}\left[F\left(x\right)+S_{(x^*,v)}\left(-x\right)\right]. \tag{6.4}$$

The biconjugate $F^{**}:\mathcal{X}\to\mathcal{F}_m$ of F is defined to be

$$F^{**}\left(x\right):=\bigcap_{(x^*,v)\in\mathcal{X}^*\times B_m^-}\left[-F^*\left(x^*,v\right)+S_{(x^*,v)}\left(x\right)\right].$$

We prefer using $-F^*$ instead of F^* since F^* has the collection of upper closed subsets as image set, so one would have two different image spaces for F and F^*. The following result shows that the conjugate and the biconjugate are well-defined.

Proposition 1 *(i) $-F^*:\mathcal{X}^*\times B_m^-\to\mathcal{F}_m$. Moreover, $-F^*\left(x^*,v\right)$ is of the form $\{u\}+H\left(v\right)$ for some $u\in\mathrm{IR}^m$ or an element of $\{\mathrm{IR}^m,\emptyset\}$. (ii) $F^{**}:\mathcal{X}\to\mathcal{C}_m$.*

Proof. (i) We have to show $\operatorname{cl}\left(-F^*\left(x^*,v\right)+K_m\right)=-F^*\left(x^*,v\right)$ which is a consequence of the definition of $-F^*$ and $S_{(x^*,v)}$. For the second part, observe that $S_{(x^*,v)}\left(-x\right)$ is a shifted closed half space with normal vector v, a set $F\left(x\right)+S_{(x^*,v)}\left(-x\right)$ is the union of such half spaces and hence $-F^*\left(x^*,v\right)$ is by definition the closure of the union of such half spaces or empty.

(ii) We have to show $F^{**}\left(x\right)=\operatorname{cl}\operatorname{co}\left(F^{**}\left(x\right)+K_m\right)$. This follows from the definitions of F^{**} and $S_{(x^*,v)}$. □

The following result is the set–valued analog to Young–Fenchel's inequalities.

Proposition 2 *For each $x\in\mathcal{X}$, $x^*\in\mathcal{X}^*$, $v\in B_m^-$ it holds*

$$-F^*\left(x^*,v\right)\supseteq F\left(x\right)\oplus S_{(x^*,v)}\left(-x\right),\quad S_{(x^*,v)}\left(x\right)\oplus -F^*\left(x^*,v\right)\supseteq F^{**}\left(x\right). \tag{6.5}$$

Proof. Immediate from the definitions of $-F^*$ and F^{**}. □

Furthermore, we have the following elementary properties.

Proposition 3 *Let $F,F_1,F_2:\mathcal{X}\to\mathcal{F}_m$. Then*
(i) $F_1\supseteq F_2\Rightarrow -F_1^*\supseteq -F_2^*\Rightarrow F_1^{**}\supseteq F_2^{**}$;
(ii) $F^{**}\supseteq F$;
(iii) $-\left(F^{**}\right)^*=-F^*$
where the superset relation is understood pointwise.

Proof. (i), (ii) are immediate consequences of the definitions of (bi)conjugates. The inclusion $-\left(F^{**}\right)^*\supseteq -F^*$ is a consequence of (i) and (ii). The converse inclusion follows from

$$\begin{aligned}-\left(F^{**}\right)^*\left(y^*,w\right)\quad&=\quad\operatorname{cl}\bigcup_{x\in\mathcal{X}}\left[\bigcap_{(x^*,v)\in\mathcal{X}^*\times B_m^-}\left[-F^*\left(x^*,v\right)+S_{(x^*,v)}\left(x\right)\right]+S_{(y^*,w)}\left(-x\right)\right]\\ &\overset{x^*=y^*,v=w}{\subseteq}\quad\operatorname{cl}\left(-F^*\left(y^*,w\right)+H\left(w\right)\right)=-F^*\left(y^*,w\right).\end{aligned}$$

The last equality follows from Proposition 1, (i). □

Before establishing the announced biconjugation theorem we state a technical result that corresponds to the fact that the conjugate of a proper closed convex (extended real-valued) function is proper. This is needed in the proof of the biconjugation theorem.

Lemma 6.2 *Let $F : \mathcal{X} \to \mathcal{F}_m$ be proper closed convex. Then there are $y^* \in \mathcal{X}^*$ and $w \in B_m^-$ such that* $\sup_{u \in -F^*(y^*,w)} w^T u \in \mathrm{I\!R}$.

Proof. Since F is proper, $-F^*(x^*, v) \neq \emptyset$, hence $-\infty < \sup_{u \in -F^*(x^*,v)} v^T u$ for all $(x^*, v) \in \mathcal{X}^* \times B^-$.

Moreover, there are $\bar{x} \in \operatorname{dom} F$ and $\bar{u} \in \mathrm{I\!R}^m$ with $\bar{u} \notin F(\bar{x})$, i.e. $(\bar{x}, \bar{u}) \notin \operatorname{epi} F$. Since $\operatorname{epi} F$ is non-empty closed convex we can separate $(\bar{x}, \bar{u})$ and $\operatorname{epi} F$ by $(y^*, w) \in \mathcal{X}^* \times \mathrm{I\!R}^m$ obtaining

$$\sup_{(x,u) \in \operatorname{epi} F} \left\{ y^*(x) + w^T u \right\} < y^*(\bar{x}) + w^T \bar{u}. \tag{6.6}$$

By standard arguments using the existence of some $u^0 \in F(\bar{x})$ such that $\left(\bar{x}, u^0 + tk\right) \in \operatorname{epi} F$ for all $t \geq 0$ and $k \in K_m \backslash \{0\}$ we may see that $w \in K_m^- \backslash \{0\}$. Since B_m^- is a basis of K_m^- we can arrange by a suitable division that $w \in B_m^-$. The very definition of $-F^*(y^*, w)$ yields

$$\sup_{u \in -F^*(y^*,w)} w^T u \leq \sup_{x \in \operatorname{dom} F} \left\{ \sup_{u \in F(x)} w^T u + y^*(x) \right\} < w^T \bar{u}$$

which proves the lemma. □

Theorem 6.1 *Let $F : \mathcal{X} \to \mathcal{F}_m$ be proper closed convex. Then $F = F^{**}$.*

Proof. By Proposition 3, (ii) we have $F^{**}(x) \supseteq F(x)$ for all $x \in \mathcal{X}$. We shall show the converse inclusion by contradiction. Assume there are $x_0 \in \mathcal{X}$ and $u_0 \in F^{**}(x_0)$ with $u_0 \notin F(x_0)$. Then $(x_0, u_0) \notin \operatorname{epi} F$, the latter set being nonempty closed convex. We can separate (x_0, u_0) and $\operatorname{epi} F$ obtaining $x^* \in \mathcal{X}^*$, $v \in \mathrm{I\!R}^m$ such that

$$\sup_{(x,u) \in \operatorname{epi} F} \left\{ x^*(x) + v^T u \right\} < x^*(x_0) + v^T u_0. \tag{6.7}$$

By standard arguments (observe $F(x) = F(x) \oplus K_m$) one may see $v \in K_m^-$. We consider two cases.

First, let $v \neq 0$. Since B_m^- is a base of K_m^- we can arrange by a suitable division, if necessary, $v \in B_m^-$. We claim

$$u_0 \notin S_{(x^*,v)}(x_0) + \operatorname{cl} \bigcup_{x \in \mathcal{X}} \left[F(x) + S_{(x^*,v)}(-x) \right] = -F^*(x^*, v) + S_{(x^*,v)}(x_0)$$

contradicting $u_0 \in F^{**}(x_0)$. Indeed, if this would not be the case, there would be $\bar{u} \in S_{(x^*,v)}(x_0)$ and sequences $\left\{u^k\right\}$, $\left\{\tilde{u}^k\right\} \subseteq \mathrm{I\!R}^m$ and $\left\{x^k\right\} \subseteq \operatorname{dom} F$ such that $\bar{u} + \lim_{k\to\infty} \left(u^k + \tilde{u}^k\right) = u_0$ and $u^k \in F\left(x^k\right)$, $\tilde{u}^k \in S_{(x^*,v)}\left(-x^k\right)$, hence

$$\begin{aligned}
\forall k \in \mathrm{I\!N}: \; v^T \left(\bar{u} + u^k + \tilde{u}^k\right) &\leq -x^*(x_0) + x^*\left(x^k\right) + v^T u^k \\
&\leq -x^*(x_0) + \sup_{(x,u) \in \operatorname{epi} F} \left\{ x^*(x) + v^T u \right\} \\
&< v^T u_0.
\end{aligned}$$

The very left hand side of this inequality chain tends to $v^T u_0$ as $k \to \infty$ which gives a contradiction. This proves the claim.

Under the assumption $u_0 \notin F(x_0)$ we obtained $u_0 \notin -F^*(x^*, v) + S_{(x^*,v)}(x_0)$ and hence $u_0 \notin F^{**}(x_0)$, thus a contradiction.

Let us check the case $v = 0$. We get from (6.7)

$$\sup_{x \in \operatorname{dom} F} x^*(x) < x^*(x_0). \tag{6.8}$$

Take $(y^*, w) \in \mathcal{X}^* \times B_m^-$ such that $\sup_{u \in -F^*(y^*,w)} w^T u =: a \in \mathrm{I\!R}$. Such a couple exists due to Lemma 6.2. For an arbitrary $t > 0$ it holds

$$\begin{aligned} -F^*(y^* + tx^*, w) &= \operatorname{cl} \bigcup_{x \in \operatorname{dom} F} \left[F(x) + S_{(y^*+tx^*,w)}(-x)\right] \\ &= \operatorname{cl} \bigcup_{x \in \operatorname{dom} F} \left[F(x) + S_{(y^*,w)}(-x) + tS_{(x^*,w)}(-x)\right] \\ &\subseteq -F^*(y^*, w) \oplus t \bigcup_{x \in \operatorname{dom} F} S_{(x^*,w)}(-x). \end{aligned}$$

On the other hand, the second Young–Fenchel inequality gives

$$F^{**}(x_0) \subseteq S_{(y^*+tx^*,w)}(x_0) \oplus -F^*(y^* + tx^*, w)$$

and putting the last two inclusions together we get

$$\begin{aligned} u_0 \in F^{**}(x_0) &\subseteq -F^*(y^*, w) \oplus S_{(y^*+tx^*,w)}(x_0) \oplus t \bigcup_{x \in \operatorname{dom} F} S_{(x^*,w)}(-x) \\ &= -F^*(y^*, w) \oplus S_{(y^*,w)}(x_0) \oplus t \left[S_{(x^*,w)}(x_0) + \bigcup_{x \in \operatorname{dom} F} S_{(x^*,w)}(-x)\right]. \end{aligned}$$

Applying w to this inclusion we obtain

$$w^T u_0 \leq a - y^*(x_0) + t\left[-x^*(x_0) + \sup_{x \in \operatorname{dom} F} x^*(x)\right].$$

Since the term in square brackets is < 0 and $t > 0$ can be chosen arbitrarily, this can not be true. This contradiction is enough to finish the proof of the theorem. □

Example 6.1 *The set–valued indicator function of a set $A \subseteq \mathcal{X}$ is*

$$I_A(x) = \begin{cases} K_m & : \; x \in A \\ \emptyset & : \; x \notin A \end{cases}$$

The conjugate of I_A is

$$-P_A(x^*, v) := -(I_A)^*(x^*, v) = \operatorname{cl} \bigcup_{x \in A} S_{(x^*,v)}(-x).$$

We call this function the set–valued negative support function of A because it is the analog to $x \to -\sup_{x \in A} x^*(x) = \inf_{x \in A} x^*(-x)$ in the scalar case.

Next, we ask for the consequences of monotonicity and translativity of an $\mathcal{F}_m$–valued function F for $-F^*$ and F^{**}. Here comes a definition.

Definition 6.1 *(i) A function $F : \mathcal{X} \to \mathcal{F}_m$ is called* translative *with respect to linear independent elements $y^1, \dots, y^m \in \mathcal{X}$ iff*

$$\forall x \in \mathcal{X}, \forall u \in \mathrm{I\!R}^m : \; F\left(x + \sum_{i=1}^m u_i y^i\right) = F(x) + \{-u\}.$$

(ii) A function $F : \mathcal{X} \to \mathcal{F}_m$ is called monotone *with respect to a convex cone $C \subseteq \mathcal{X}$ iff*

$$x^1, x^2 \in \mathcal{X}, \; x^2 - x^1 \in C \quad \Longrightarrow \quad F(x^2) \supseteq F(x^1).$$

We shall consider the special case of the functions $S_{(x^*,v)}$.

Lemma 6.3 *Let $v \in B_m^-$. The function $S_{(x^*,v)} : \mathcal{X} \to \mathcal{C}_m$ is monotone w.r.t. C if $x^* \in C^-$. It is translative w.r.t. $y^1, \ldots, y^m \in \mathcal{X}$ if $v_i = x^*\left(y^i\right)$ for $i = 1, \ldots, m$.*

Proof. Firstly, take $x^1, x^2 \in \mathcal{X}$ such that $x^2 - x^1 \in C$. Then $0 \in S_{(x^*,v)}\left(x^2 - x^1\right)$ since $x^* \in C^-$ and (see Lemma 6.1)

$$S_{(x^*,v)}\left(x^2\right) = S_{(x^*,v)}\left(x^2 - x^1 + x^1\right) = S_{(x^*,v)}\left(x^2 - x^1\right) + S_{(x^*,v)}\left(x^1\right) \supseteq S_{(x^*,v)}\left(x^1\right).$$

Secondly, take $x \in \mathcal{X}$ and $w \in \mathbb{R}^m$. It holds

$$x^*\left(x + \sum_{i=1}^m w_i y^i\right) + v^T u = x^*(x) + v^T(u + w),$$

hence

$$S_{(x^*,v)}\left(x + \sum_{i=1}^m w_i y^i\right) = \left\{u \in \mathbb{R}^m : \; x^*(x) + v^T(u + w) \le 0\right\} = S_{(x^*,v)}(x) + \{-w\}$$

which is translativity of $S_{(x^*,v)}$. $\square$

Note that the converses of the assertions in Lemma 6.3 are also true, but we will not use this fact in this paper.

Define the zero sublevel set of $F : \mathcal{X} \to \mathcal{F}_m$ as

$$A_F := \{x \in \mathcal{X} : \; 0 \in F(x)\} = \{x \in \mathcal{X} : \; K_m \subseteq F(x)\}.$$

Observe that if F is monotone and $0 \in F(0)$ ($\Leftrightarrow K_m \subseteq F(0)$) then $C \subseteq A_F$. We denote by $C^- := \{x^* \in \mathcal{X}^* : \; \forall x \in C : \; x^*(x) \le 0\}$ the negative dual cone of C and set $C^+ = -C^-$. The same convention is used for A_F^- if F is sublinear and hence A_F is a convex cone. The announced result reads as follows.

Theorem 6.2 *Let $F : \mathcal{X} \to \mathcal{C}_m$ be proper closed convex translative w.r.t. $y^1, \ldots, y^m \in \mathcal{X}$ and monotone w.r.t. C such that $0 \in F(0)$. Then*

$$-F^*(x^*, v) = \begin{cases} -P_{A_F}(x^*, v) & : \; (x^*, v) \in \mathcal{Y}_m \\ \mathbb{R}^m & : \; (x^*, v) \notin \mathcal{Y}_m \end{cases} \tag{6.9}$$

where $\mathcal{Y}_m := \left\{(x^*, v) \in C^- \times B_m^- : \; v_i = x^*\left(y^i\right), i = 1, \ldots, m\right\}$. The dual representation of F is given by

$$\forall x \in \mathcal{X} : \; F(x) = \bigcap_{(x^*,v) \in \mathcal{Y}_m} \left[-P_{A_F}(x^*, v) + S_{(x^*,v)}(x)\right]. \tag{6.10}$$

Let $F : \mathcal{X} \to \mathcal{C}_m$ be additionally positively homogeneous. Then

$$-F^*(x^*, v) = \begin{cases} H(v) & : \; (x^*, v) \in \mathcal{Y}_m \text{ and } x^* \in A_F^- \\ \mathbb{R}^m & : \; (x^*, v) \notin \mathcal{Y}_m \text{ or } x^* \notin A_F^- \end{cases} \tag{6.11}$$

The dual representation of F is given by

$$\forall x \in \mathcal{X} : \; F(x) = \bigcap_{(x^*,v) \in \mathcal{Y}_m, \, x^* \in A_F^-} S_{(x^*,v)}(x). \tag{6.12}$$

PROOF. First, take $(x^*, v) \in \mathcal{X} \times B_m^-$. Then $x + \sum_{i=1}^m u_i y^i \in A_F$ and

$$-F^*(x^*, v) \supseteq \mathrm{cl} \bigcup_{x \in A_F} \left[F(x) + S_{(x^*,v)}(-x)\right] \supseteq \mathrm{cl} \bigcup_{x \in A_F} S_{(x^*,v)}(-x)$$

by definition of $-F^*$ and A_F. For the converse inclusion take $(x^*, v) \in \mathcal{Y}_m$, $x \in \mathrm{dom}\, F$ and $u \in F(x)$. Then

$$-P_{A_F}(x^*, v) \supseteq S_{(x^*,v)}\left(-x - \sum_{i=1}^m u_i y^i\right) = S_{(x^*,v)}(-x) + \{u\},$$

(the last equation is translativity of $S_{(x^*,v)}$ for $(x^*, v) \in \mathcal{Y}_m$, see Lemma 6.3) hence $-P_{A_F}(x^*, v) \supseteq S_{(x^*,v)}(-x) + F(x)$. This is all the more true if $F(x) = \emptyset$. Hence $-P_{A_F}(x^*, v) \supseteq -F^*(x^*, v)$ whenever $(x^*, v) \in \mathcal{Y}_m$.

It remains to show that $-F^*(x^*, v) \neq \mathrm{I\!R}^m$ implies $(x^*, v) \in \mathcal{Y}_m$. If $-F^*(x^*, v) \neq \mathrm{I\!R}^m$ then $-F^*(x^*, v) \subseteq u + H(v)$ for some $u \in \mathrm{I\!R}^m$ by Proposition 1, hence $-\infty < \sup_{u' \in -F^*(x^*,v)} v^T u' \leq v^T u < +\infty$ since the properness of F also ensures $-F^*(x^*, v) \neq \emptyset$.

Since F is translative, it holds for arbitrary $w \in \mathrm{I\!R}^m$

$$\begin{aligned} -F^*(x^*, v) &= \mathrm{cl} \bigcup_{x \in \mathcal{X}} \left[F(x) + S_{(x^*,v)}(-x)\right] \\ &= \mathrm{cl} \bigcup_{x \in \mathcal{X}} \left[F\left(x + \sum_{i=1}^m w_i y^i\right) + S_{(x^*,v)}\left(-x - \sum_{i=1}^m w_i y^i\right)\right] \\ &\subseteq -F^*(x^*, v) - w + S_{(x^*,v)}\left(-\sum_{i=1}^m w_i y^i\right). \end{aligned}$$

This gives (use definition of $S_{(x^*,v)}\left(-\sum_{i=1}^m w_i y^i\right)$)

$$\sup_{u' \in -F^*(x^*,v)} v^T u' \leq \sup_{u' \in -F^*(x^*,v)} v^T u' - v^T w + \widehat{v}^T w$$

with $\widehat{v} = \left(x^*\left(y^1\right), \ldots, x^*\left(y^m\right)\right)^T$. We obtain $0 \leq (\widehat{v} - v)^T w$ for all $w \in \mathrm{I\!R}^m$. This is only possible if $\widehat{v} = v$, i.e. $v_i = x^*\left(y^1\right)$, $i = 1, \ldots, m$.

On the other hand, Young–Fenchel inequality, the definition of A_F and $C \subseteq A_F$ (by monotonicity and $0 \in F(0)$) imply

$$\forall x \in C: \; -F^*(x^*, v) \supseteq F(x) + S_{(x^*,v)}(-x) \supseteq S_{(x^*,v)}(-x).$$

Multiplying this inclusion by v^T and observing that C is a cone we get (use definition of $S_{(x^*,v)}(-x)$)

$$\forall t > 0: \sup_{u' \in -F^*(x^*,v)} v^T u' \geq t x^*(x)$$

Since the left hand side of these inequalities is a real number (see above) they can only be satisfied if $x^*(x) \leq 0$. Since x can arbitrarily be chosen in C this proves $x^* \in C^-$.

Altogether, we get $(x^*, v) \in \mathcal{Y}_m$.

Finally, the dual representation formula (6.10) is a consequence of (6.9) and the biconjugation theorem (Theorem 6.1).

For the sublinear case, if $(x^*, v) \in \mathcal{Y}_m$ we have by the first part $-F^*(x^*, v) = -P_{A_F}(x^*, v) = \mathrm{cl} \bigcup_{x \in A_F} S_{(x^*,v)}(-x)$. Since $0 \in A_F$ we have $H(v) \subseteq -P_{A_F}(x^*, v)$. On the other hand, if

$x^* \in A_F^-$ then $x^*(x) \leq 0$, hence $u \in S_{(x^*,v)}(-x) = \left\{u' \in \mathrm{I\!R}^m : v^T u \leq x^*(x)\right\}$ implies $u \in H(v)$. This gives $-P_{A_\Phi}(x^*, v) \subseteq H(v)$.

Now, assume $x^* \notin A_F^-$. Then there is $x \in A_F$ such that $x^*(x) > 0$. Since A_F^- is a cone, $tx \in A_F^-$ whenever $t > 0$, hence

$$-P_{A_F}(x^*, v) \supseteq \bigcup_{t>0} \left\{u \in \mathrm{I\!R}^m : \ v^T u \leq t x^*(x)\right\} = \mathrm{I\!R}^m,$$

hence $-P_{A_F}(x^*, v) = \mathrm{I\!R}^m$ whenever $x^* \notin A_F^-$.

The dual representation formula (6.12) is a direct consequence of (6.11) and the first part. $\square$

Finally, we shall give an abstract version of the penalty function representation of convex risk measures. For the scalar case, compare [6], Chapter 4.

Theorem 6.3 *Let $\mathcal{Y} \subseteq \mathcal{Y}_m$ be nonempty and $-G : \mathcal{Y} \to \mathcal{C}_m$ be a function such that*
(i) $-G(x^*, v) = -G(x^*, v) \oplus H(v)$ for all $(x^*, v) \in \mathcal{Y}$;
(ii) $K_m \subseteq \bigcap_{(x^*,v)\in\mathcal{Y}} -G(x^*, v)$;
(iii) $\bigcap_{(x^*,v)\in\mathcal{Y}} -G(x^*, v) \cap -\mathrm{int}\, K_m = \emptyset$.
Then,

$$F(x) := \bigcap_{(x^*,v)\in\mathcal{Y}} \left[-G(x^*, v) + S_{(x^*,v)}(x)\right] \tag{6.13}$$

maps into $\mathcal{C}_m$, is closed, convex, translative w.r.t. $y^1, \ldots, y^m \in \mathcal{X}$, C–monotone and normalized. If, additionally, $-G(x^*, v) = H(v)$ for all $(x^*, v) \in \mathcal{Y}$ then F is additionally positively homogeneous.

Proof. Fix $x \in \mathcal{X}$. Every set $-G(x^*, v) + S_{(x^*,v)}(x)$ is closed, convex with $-G(x^*, v) + S_{(x^*,v)}(x) + K_m \subseteq -G(x^*, v) + S_{(x^*,v)}(x)$ by assumption (i). Hence F maps into $\mathcal{C}_m$.

To prove closedness, take a net $\left(x^i, u^i\right)_{i\in I} \subseteq \mathrm{epi}\, F$ such that $\left(x^i, u^i\right) \to (x, u) \in \mathcal{X} \times \mathrm{I\!R}^m$. Then $u^i \in -G(x^*, v) + S_{(x^*,v)}\left(x^i\right)$ for all $i \in I$ and for all $(x^*, v) \in \mathcal{Y}$. This implies $u \in -G(x^*, v) + S_{(x^*,v)}(x)$ for all $(x^*, v) \in \mathcal{Y}$, hence $(x, u) \in \mathrm{epi}\, F$.

The convexity of F follows from additivity of $S_{(x^*,v)}$ and properties of the intersection.

The translativity of F follows from the translativity of $S_{(x^*,v)}$ and the fact that the intersection of a collection of subsets of $\mathrm{I\!R}^m$ shifted by an element $u \in \mathrm{I\!R}^m$ is equal to the intersection of the by u shifted subsets.

The C–monotonicity follows from C–monotonicity of $S_{(x^*,v)}$ (observe $x^* \in C^-$) and the construction of F.

Finally, we have $K_m \subseteq F(0)$ and $F(0) \cap -\mathrm{int}\, K_m = \emptyset$ by assumptions (ii), (iii) and the construction of F.

The positive homogeneity in case $-G = H(v)$ in $\mathcal{Y}$ is easy to check. $\square$

Remark 6.1 *The function $-G$ can be extended to all of $\mathcal{X}^* \times B_m^-$ by setting $-G(x^*, v) = \mathrm{I\!R}^m$ whenever $(x^*, v) \notin \mathcal{Y}$. Thus, there is a one-to-one correspondence between subsets $\mathcal{Y} \subseteq \mathcal{Y}_m$ and sublinear, translative, C–monotone, normalized functions.*

Remark 6.2 *The function $-F^*$ is the smallest (w.r.t. pointwise inclusion) function that can be used as a "penalty function" $-G$ in the above theorem in order to generate a closed, convex, translative w.r.t. $y^1, \ldots, y^m \in \mathcal{X}$, C–monotone and normalized F. Indeed, from (6.13) we get*

$$\forall x \in \mathcal{X}, \ (x^*, v) \in \mathcal{Y} : \ F(x) \subseteq -G(x^*, v) + S_{(x^*,v)}(x),$$

hence $F(x)+S_{(x^*,v)}(-x) \subseteq -G(x^*,v) \oplus H(v) = -G(x^*,v)$ *for all* $x \in \mathcal{X}$ *and for all* $(x^*,v) \in \mathcal{Y}$. *This gives* $-F^*(x^*,v) \subseteq -G(x^*,v)$ *for all* $(x^*,v) \in \mathcal{Y}$. *See [8], Remark 4 for the scalar case and compare the discussion in Chapter 4 of [6] for convex risk measures on spaces of measurable functions.*

7 Conclusions

A new theory for set–valued risk measures has been developed that is more systematic and complete then previous approaches. Several new constructions turned out to be unavoidable. In particular, a new "convex analysis" for vector– and set–valued convex functions has been presented in order to derive dual representation results for set–valued convex risk measures. As a byproduct, systematic ways of constructing set–valued counterparts of scalar risk measures come along. The substitutes for linear functions on linear spaces of random variables are natural generalizations of mathematical expectations leading to "dual ways" of introducing set–valued risk measures. We believe that these concepts will be of some importance beyond its use in risk measure theory.

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