FRACTIONAL WHITE NOISE PERTURBATIONS OF PARABOLIC VOLTERRA EQUATIONS

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Abstract. Aim of this work is to extend the results of Clément, Da Prato & Prüss [5] on the fractional white noise perturbation with Hurst parameter $H \in (0,1)$. We will obtain similar results and it will turn out that the regularity of the solution $u(t)$ increases with Hurst parameter H.

1. Introduction and Notations

Stochastic differential equations play an important role in studying random influences on deterministic systems. For this purpose usually Wiener processes are considered. However, this is not adequate if the chronological independence of the stochastic perturbations is not sufficiently warranted. Therefore we make use of the concept of a fractional Brownian motion, which was introduced by Mandelbrot & van Ness [9].

We are given a separable Hilbert space $\mathcal{H}$ with norm $|\cdot|_{\mathcal{H}}$ and inner product $(\cdot|\cdot)_{\mathcal{H}}$. Let A be a closed linear densely defined operator in $\mathcal{H}$, and $b \in L_1(\mathbb{R}_+)$ a scalar kernel. As in [5] we consider the integro-differential equation

$$\begin{cases} \dot{u}(t) + \displaystyle\int_0^t b(t-\tau)Au(\tau)\mathrm{d}\tau = Q^{1/2}\dot{B}^H(t), \quad t \geq 0, \\ u(0) = u_0. \end{cases} \tag{1.1}$$

Here the initial value u_0 is assumed to be an element of $\mathcal{H}$. Moreover, B^H is fractional Brownian motion in $\mathcal{H}$ with Hurst parameter $H \in (0,1)$, with corresponding fractional white noise $\dot{B}^H$ and the operator Q is of trace class (see Hypothesis **(B)** below).

Because problem (1.1) is motivated from applications of linear viscoelastic material behavior, we consider $G \subset \mathbb{R}^N$ to be an open and bounded domain and the

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operator $-A$ to be an elliptic differential operator like the Laplacian, the elasticity operator, or the Stokes operator, together with appropriate boundary conditions (e.g. Prüss [13, Section I.5]). In the following we are particulary interested in the case $\mathcal{H} = L_2(G)$.

Hypothesis (A). *A is an unbounded, selfadjoint, positive definite operator in $\mathcal{H}$ with compact resolvent. Consequently, the eigenvalues $\mu_n > 0$ of A form a nondecreasing sequence with $\lim_{n\to\infty} \mu_n = \infty$, the corresponding eigenvectors $(e_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ form an orthonormal basis of $\mathcal{H}$.*

Hypothesis (e). *There is a constant $C > 0$ such that*

$$|e_n(\xi)| \le C \quad \text{and} \quad |\nabla e_n(\xi)| \le C\mu_n^{1/2},$$

for all $n \in \mathbb{N}$ and all $\xi \in G$, where ∇ denotes the gradient with respect to the variable ξ.

Hypothesis (b). *$b \in L_1(\mathbb{R}_+)$ is 3-monotone, i.e. b and $-\dot{b}$ are nonnegative, nonincreasing, convex; in addition,*

$$\lim_{t\to 0} \frac{\frac{1}{t}\int_0^t \tau b(\tau)\mathrm{d}\tau}{\int_0^t -\tau\dot{b}(\tau)\mathrm{d}\tau} < \infty. \tag{1.2}$$

Prüss proved in [13, Section I.1] that if **(A)** and **(b)** are valid, the integrated version of problem (1.1) admits a resolvent $S(t)$ (which is strongly continuous, uniformly bounded by 1, with $\lim_{t\to\infty} |S(t)|_{\mathcal{B}(\mathcal{H})} = 0$ and $S \in L_1(\mathbb{R}_+; \mathcal{B}(\mathcal{H}))$) such that the unique mild solution of (1.1) is given by the variation of parameters formula

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)f(\tau)\mathrm{d}\tau, \quad t \ge 0, \tag{1.3}$$

whenever $u_0 \in \mathcal{H}$ and $f \in L_{1,\mathrm{loc}}(\mathbb{R}_+; \mathcal{H})$.

By means of the spectral decomposition of A, the resolvent family $S(t)$ can be written explicitly as

$$S(t)x = \sum_{n=1}^{\infty} s_n(t)(x|e_n)e_n, \quad t \ge 0, \quad x \in \mathcal{H}, \tag{1.4}$$

where the scalar functions $s_n(t)$ are the solutions of the scalar problems

$$\dot{s}_n(t) + \mu_n \int_0^t b(t-\tau)s_n(\tau)\mathrm{d}\tau = 0, \quad t \ge 0, \quad s_n(0) = 1. \tag{1.5}$$

For the reader's convenience we repeat the definition of a scalar fractional Brownian motion.

Definition 1.1. *A real valued Gaussian process* $\beta^H := \{\beta^H(t)\}_{t\geq 0}$ *on a probability space* $(\Omega, \mathscr{F}, \mathbb{P})$ *is called a fractional Brownian motion with Hurst parameter* $H \in (0,1)$ *if for all* $s, t \in \mathbb{R}_+$

(i) $\beta^H(0) = 0$,
(ii) $\mathbb{E}\beta^H(t) = 0$,
(iii) $\mathrm{Cov}[\beta^H(t), \beta^H(s)] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.

Next we want to give an abstract formulation of the assumptions on the covariance Q and the fractional white noise $\dot{B}^H$.

Hypothesis (B). $Q \in \mathscr{L}_1(\mathcal{H})$ *is selfadjoint, positive semi-definite and commutes with the operator* A*, i.e. there is a sequence* $(\gamma_n) \in \ell_1(\mathbb{R}_+)$*, such that* $Qe_n = \gamma_n e_n$ *for all* $n \in \mathbb{N}$*.* $B^H(t)$ *is of the form*

$$(B^H(t)|x) = \sum_{n=0}^{\infty} \beta_n^H(t)(x|e_n), \quad t \in \mathbb{R}, \quad x \in \mathcal{H}, \tag{1.6}$$

where β_n^H *are mutually independent real valued fractional Brownian motions with Hurst parameter* $H \in (0,1)$ *on the probability space* $(\Omega, \mathscr{F}, \mathbb{P})$*.*

Here, the symbol $\mathscr{L}_1(\mathcal{H})$ denotes the space of nuclear operators on $\mathcal{H}$. Note, that due to Hypothesis **(B)** the operator $Q^{1/2}$ is well defined and belongs to $\mathcal{L}_2(\mathcal{H})$; the space of Hilbert-Schmidt operators on $\mathcal{H}$. It is well known that $B^H(t)$ (as in (1.6)) is not a well defined $\mathcal{H}$-valued random variable. However, due to $B^H(t) : \Omega \to \mathcal{H}_{Q^{-1/2}}$, where $\mathcal{H}_{Q^{-1/2}}$ is the completion of $\mathcal{H}$ with respect to the norm $|x|^2_{Q^{-1/2}} := |Q^{-1/2}x|_{\mathcal{H}}$, $x \in \mathcal{H}$, the forcing function f is well defined since $Q^{1/2}B^H(t)$ is a mapping with values in $\mathcal{H}$. This observation yields an alternative strategy how to avoid the appearance of the trace class operator Q, namely if in considering the space $\mathcal{H}_{Q^{-1/2}}$ instead of $\mathcal{H}$.

In the sequel an upper index $\langle t\rangle$, $t > 0$ at a function $f : \mathbb{R}_+ \to \mathbb{R}$ means

$$f^{\langle t\rangle}(\tau) := \begin{cases} f(t-\tau) & : \tau \leq t; \\ 0 & : \tau > t. \end{cases}$$

Moreover, we will make use of the theory of integration with respect to fractional Brownian motions, which is provided by Pipiras and Taqqu [12]. Hence we denote the fractional integral of order $\alpha > 0$ of a function ϕ by $\mathcal{I}^\alpha\phi$, precisely this means

$$(\mathcal{I}^\alpha\phi)(r) = \frac{1}{\Gamma(\alpha)}\int_{\mathbb{R}} \phi(\tau)(\tau - r)_+^{\alpha-1}\mathrm{d}\tau, \quad r \in \mathbb{R},$$

where $(x)_+ := \max\{0, x\}$. Recall that the Marchaud fractional derivative $\mathcal{D}^\alpha$ is defined as the left inverse of $\mathcal{I}^\alpha$ for $\alpha > 0$ (see [14, Page 111]), i.e. for appropriate functions ϕ it holds that

$$\mathcal{D}^{\alpha}(\mathcal{I}^{\alpha}\phi) \equiv \phi. \tag{1.7}$$

Next we want to characterize the class of integrands f with respect to a fractional Brownian motion, such that the integral $\int_{\mathbb{R}} f(\tau)\mathrm{d}B^{H}(\tau)$ is well defined. In order to study the most general case, we consider the space Λ_H for integrands in the time domain which arises as

$$\Lambda_H := \left\{ f : \int_{\mathbb{R}} \left[(\mathcal{D}^{\frac{1}{2}-H} f)(r) \right]^2 \mathrm{d}r < \infty \right\} \quad \text{for} \quad 0 < H < \tfrac{1}{2}, \tag{1.8}$$

or alternatively as

$$\Lambda_H := \left\{ f : \int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} f)(r) \right]^2 \mathrm{d}r < \infty \right\} \quad \text{for} \quad \tfrac{1}{2} < H < 1. \tag{1.9}$$

In both cases Λ_H is a linear space with inner product

$$(f|g)_{\Lambda_H} = \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})} \int_{\mathbb{R}} (\mathcal{D}^{\frac{1}{2}-H} f)(r)(\mathcal{D}^{\frac{1}{2}-H} g)(r)\mathrm{d}r,$$

or accordingly

$$(f|g)_{\Lambda_H} = \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})} \int_{\mathbb{R}} (\mathcal{I}^{H-\frac{1}{2}} f)(r)(\mathcal{I}^{H-\frac{1}{2}} g)(r)\mathrm{d}r,$$

where

$$\zeta(H) = \left[\int_0^{\infty} \left[(1+\tau)^H - \tau^H \right]^2 \mathrm{d}\tau + \frac{1}{2H+1} \right]^{1/2}.$$

Remark. In the literature (e.g. [1, 11]) the space Λ_H is often defined by a scalar product of the form

$$(f|g)_{\Lambda_H} = c_H \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t)|s-t|^{2H-2}\mathrm{d}s\mathrm{d}t,$$

where c_H is an appropriate constant depending on H. Recall that f and g might be distributions.

Pipiras and Taqqu proved in [12, Proposition 3.2], that the embeddings

$$L_1(\mathbb{R}) \cap L_2(\mathbb{R}) \hookrightarrow L_{1/H}(\mathbb{R}) \hookrightarrow \Lambda_H, \tag{1.10}$$

hold true for $H \in (\frac{1}{2}, 1)$.
In the spectral domain we are interested in integrands being a member of the homogeneous Bessel potential space of order $\frac{1}{2} - H$,

$$\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}) = \left\{ f \in \mathcal{S}^*(\mathbb{R}) : \int_{\mathbb{R}} |\mathcal{F}f(\tau)|^2 |\tau|^{-2H+1} \mathrm{d}\tau < \infty \right\},$$

where $\mathcal{S}^*$ is the space of tempered distributions. It is well known, that for $f \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})$ the Fourier transform of $\mathcal{I}^{H-\frac{1}{2}} f$ or $\mathcal{D}^{\frac{1}{2}-H} f$ is

$$\psi_{H-\frac{1}{2}}(x)(\mathcal{F}f)(x)|x|^{\frac{1}{2}-H} = (\mathcal{F}f)(x)(ix)^{\frac{1}{2}-H}, \tag{1.11}$$

where

$$\psi_\alpha(x) = e^{-i\pi\alpha/2}\chi_{\{x>0\}} + e^{i\pi\alpha/2}\chi_{\{x<0\}}, \quad x \in \mathbb{R}.$$

Here χ_M denotes the indicator function of the set M. Hence by Plancherel's Theorem it holds that

$$\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}) \cong \Lambda_H. \tag{1.12}$$

An easy calculus shows that the identity

$$(f|g)_{\Lambda_H} = \mathbb{E}\left[\left(\int_{\mathbb{R}} f(\tau)\mathrm{d}\beta^H(\tau)\right)\left(\int_{\mathbb{R}} g(\tau)\mathrm{d}\beta^H(\tau)\right)\right] \tag{1.13}$$

holds for all $f, g \in \mathcal{E}$, where $\mathcal{E}$ denotes the set of all elementary functions. For a definition of the Wiener integral with respect to a fractional Brownian motion we refer to [12]. Since $\mathcal{E}$ is dense in Λ_H (see [12, Theorems 3.2 resp. 3.3]) equation (1.13) holds for all $f, g \in \Lambda_H$.

Remark.

(i) Since by (1.11) $\mathcal{F}(\mathcal{I}^{-\kappa} f) \equiv \mathcal{F}(\mathcal{D}^\kappa f)$ and by Plancherel's Theorem the norm in $\dot{\mathrm{H}}_2^\kappa(\mathbb{R})$ can be rewritten as

$$|f|_{\dot{\mathrm{H}}_2^\kappa(\mathbb{R})} = \begin{cases} |\mathcal{D}^\kappa f|_{L_2(\mathbb{R})} & : \kappa \geq 0; \\ |\mathcal{I}^{-\kappa} f|_{L_2(\mathbb{R})} & : \kappa < 0. \end{cases} \tag{1.14}$$

(ii) Observe that equation (1.13) also holds on an arbitrary set $M \subset \mathbb{R}$. This can be seen by replacing f and g by $f\chi_M$ and $g\chi_M$, respectively.

The plan of our paper is as follows. In Section 2 we state the main results about fractional white noise perturbations of equations in linear viscoelasticity, i.e. equation (1.1), assuming the Hypotheses **(A)**, **(b)**, and **(B)** explained above. These results are proved in Section 3 by means of the methods introduced in the monograph by Da Prato and Zabczyk [6], adapted to evolutionary integral equations in Clément and Da Prato [3], [4]. The required estimates were already available and taken from Monniaux and Prüss [10] and Clément, Da Prato & Prüss [5].
Section 4 is devoted to a study of the equation

$$u + g_\alpha * Au = g_\beta * Q^{1/2}\dot{B}^H$$

on the halfline, where $g_\kappa(t) = t^{\kappa-1}/\Gamma(\kappa)$, $t > 0$ for $\kappa > 0$ denotes the Riemann-Liouville kernel of fractional integration.

2. Main results

Concentrating on the stochastic case we let $h(t) = 0$, i.e. $f(t) = Q^{1/2}\dot{B}^{H}(t)$; w.l.o.g. we set $u_0 = 0$. This means that we have to investigate the stochastic convolution

$$u(t) = \int_0^t S(t-\tau)\mathrm{d}(Q^{1/2}B^{H})(\tau), \quad t \geq 0. \tag{2.1}$$

In virtue of the spectral decompositions of A and Q we may rewrite

$$u(t) = \sum_{n=1}^{\infty} \sqrt{\gamma_n} \int_0^t s_n(t-\tau)e_n \mathrm{d}\beta_n^{H}(\tau), \quad t \geq 0. \tag{2.2}$$

Our main result on problem (1.1) reads as follows.

Theorem 1. *Let $H \in (0,1)$. Assume that Hypotheses **(A)**, **(b)**, **(B)** are valid and suppose*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{-\frac{2H}{\rho}} < \infty, \tag{2.3}$$

where

$$\rho := 1 + \frac{2}{\pi} \sup\{|\arg \widehat{b}(\lambda)| : \mathrm{Re}\,\lambda > 0\}. \tag{2.4}$$

Then the series (2.2) converges in $L_2(\Omega;\mathcal{H})$, uniformly in t on bounded subsets of $\mathbb{R}_+$ and $u \in C_b(\mathbb{R}_+; L_2(\Omega;\mathcal{H}))$. $u(t)$ is a Gaussian random variable with mean zero and covariance operator Q_t, defined by

$$Q_t x = \sum_{n=1}^{\infty} \left\| s_n^{\langle t \rangle} \sqrt{\gamma_n} \right\|_{\Lambda_H}^2 (x|e_n)e_n, \quad x \in \mathcal{H}, \tag{2.5}$$

and we have $\mathrm{Tr}[Q_t] \leq c_H \,\mathrm{Tr}[QA^{-2H/\rho}]$.
If in addition, there is $\theta \in (0,1)$ such that

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2H(\theta-1)}{\rho}} < \infty, \tag{2.6}$$

then for each $\alpha \in (0, \theta H)$, the trajectories of $u(t)$ are almost everywhere α-Hölder-continuous, i.e. $u \in C_b^{\alpha}(\mathbb{R}_+; L_2(\Omega;\mathcal{H}))$.
*In case $\mathcal{H} = L_2(G)$ and Hypothesis **(e)** as well as*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\theta - \frac{2H}{\rho}} < \infty, \tag{2.7}$$

are met, the trajectories of $u(t,\xi)$ are almost surely α-Hölder-continuous in ξ, for each exponent $\alpha \in (0,\theta)$, i.e. $u \in C_b(\mathbb{R}_+; C^{\alpha}(G; L_2(\Omega)))$.

Here and in the sequel we denote by $c_H > 0$ a generic constant depending on H.

Remark.

(i) $\Lambda_{\frac{1}{2}}$ is isometrically isomorphic to $L_2(\mathbb{R})$. In this sense Theorem 1 is a generalization of [5, Theorem 2.1].

(ii) Note that by Hypothesis **(A)** and **(b)** the problem under consideration is *parabolic*, i.e. $\rho \in [1,2)$.

3. Proof of the main results

The idea of the proof is, of course, similar to Clément et al. [5] and follows the arguments for the Cauchy problem presented in Da Prato and Zabczyk [6].

Let us cite a useful lemma which was proven in [5, Lemma 3.1]:

Lemma 3.1. *Suppose the kernel $b(t)$ is subject to Hypothesis **(b)**, and let $\rho \in (1,2)$ be defined by (2.4). Then for every $n \in \mathbb{N}$ it is*

(i) $|s_n(t)| \le 1$ *for all* $t, \mu_n > 0$*;*

(ii) $|\dot{s}_n|_{L_1(\mathbb{R}_+)} \le C$ *for all* $\mu_n > 0$*;*

(iii) $|t\dot{s}_n|_{L_1(\mathbb{R}_+)} \le C\mu_n^{-1/\rho}$ *for all* $\mu_n > 0$*;*

(iv) $|s_n|_{L_1(\mathbb{R}_+)} \le C\mu_n^{-1/\rho}$ *for all* $\mu_n > 0$*,*

where $C > 0$ denotes a constant which is independent of $\mu_n > 0$.

Now, let the hypotheses of Theorem 1 be fulfilled. Observe that for $H \in (\frac{1}{2}, 1)$ by (iv) and (i) of Lemma 3.1 the functions $s_n^{\langle t\rangle}$ belongs to $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ and hence by embedding (1.10) to Λ_H. So by identity (1.13) we obtain

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n \int_{\mathbb{R}} \left[(\mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle})(r)\right]^2 \mathrm{d}r = c_H \sum_{n=1}^{\infty} \gamma_n \left|\mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle}\right|^2_{L_2(\mathbb{R})}. \tag{3.1}$$

As a result of [14, Theorem 5.3] the operator $\mathcal{I}^{H-\frac{1}{2}}$ is bounded from $L_{1/H}(\mathbb{R})$ into $L_2(\mathbb{R})$. Thus we have by (i) and (iv) of Lemma 3.1

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|^2_{L_{1/H}(\mathbb{R})} \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|^{2H}_{L_1(\mathbb{R})} \le c_H \sum_{n=1}^{\infty} \gamma_n \mu_n^{-2H/\rho} \tag{3.2}$$

which is finite by assumption. In the case $H \in (0, \frac{1}{2})$ one may argue as in the latter situation to obtain with the aid of (1.14) and the continuous embedding $\mathrm{H}_2^{\kappa}(\mathbb{R}) \hookrightarrow \dot{\mathrm{H}}_2^{\kappa}(\mathbb{R})$, $\kappa > 0$,

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n \left|\mathcal{D}^{\frac{1}{2}-H} s_n^{\langle t\rangle}\right|^2_{L_2(\mathbb{R})} \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|^2_{\mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R})}. \tag{3.3}$$

Hence by interpolation and Lemma 3.1

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle}|^{2H}_{L_1(\mathbb{R})} \cdot |s_n^{\langle t\rangle}|^{2(1-H)}_{\mathrm{H}_1^1(\mathbb{R})} \le c_H \sum_{n=1}^{\infty} \gamma_n \mu_n^{-2H/\rho} \tag{3.4}$$

holds. This can be seen as follows. Let us denote by $[X;Y]_\delta$ the complex interpolation space of the spaces X and Y with parameter $\delta \in (0,1)$. Then $[X;Y]_\delta = Z$ entails the interpolation inequality $|f|_Z \le c|f|_X^{1-\delta}|f|_Y^{\delta}$ for all $f \in Z$. It follows from [15, Theorem 2.4.7] that

$$\big[L_1(\mathbb{R}_+); \mathrm{H}_q^{\tau}(\mathbb{R}_+)\big]_\delta = \mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$$

holds with

$$\delta = 1-H, \quad \tau = \frac{1-2H}{2(1-H)}, \quad q = \frac{2(1-H)}{1-2H}. \tag{3.5}$$

This observation yields

$$|s_n^{\langle t\rangle}|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \le |s_n^{\langle t\rangle}|_{\mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \le c|s_n^{\langle t\rangle}|_{L_1(\mathbb{R}_+)}^{H}|s_n^{\langle t\rangle}|_{\mathrm{H}_q^{\tau}(\mathbb{R}_+)}^{1-H}.$$

Now, one may apply [15, Theorem 2.7.1] to verify that the embedding

$$\mathrm{H}_1^1(\mathbb{R}_+) \hookrightarrow \mathrm{H}_q^{\tau}(\mathbb{R}_+)$$

holds with τ and q as in (3.5).

Thus $u(t)$ is a zero mean $\mathcal{H}$-valued Gaussian random variable. Let Q_t be its covariance operator, then for $H \in (\frac{1}{2},1)$

$$\begin{aligned}
(Q_t x|y)_{\mathcal{H}} &= \mathbb{E}\left[(u(t)|x)(u(t)|y)\right] \\
&= \sum_{n=1}^{\infty} \gamma_n (e_n|x)(e_n|y)\mathbb{E}\left|\int_{\mathbb{R}} s_n^t(\tau)\mathrm{d}\beta_n^H(\tau)\right|^2 \\
&= \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})}\sum_{n=1}^{\infty}(e_n|x)(e_n|y)\int_{\mathbb{R}}\left[(\mathcal{I}^{H-\frac{1}{2}}s_n^{\langle t\rangle}\sqrt{\gamma_n})(\tau)\right]^2\mathrm{d}\tau \\
&= \frac{\Gamma^2(H+\frac{1}{2})}{\zeta^2(H-\frac{1}{2})}\sum_{n=1}^{\infty}\left(\int_{\mathbb{R}}\left[(\mathcal{I}^{H-\frac{1}{2}}s_n^{\langle t\rangle}\sqrt{\gamma_n})\right]^2(\tau)\mathrm{d}\tau(e_n|x)e_n\Big|y\right),
\end{aligned}$$

and with help of (3.1) and (3.2) $\mathrm{Tr}[Q_t] \le c_H \mathrm{Tr}[QA^{-2H/\rho}]$ follows. Replacing $\mathcal{I}^{H-\frac{1}{2}}$ by $\mathcal{D}^{\frac{1}{2}-H}$ yields the claim for $H \in (0,\frac{1}{2})$.

Concerning Hölder-continuity we will use the following two estimates with the convention $s_n(\tau) = 0$ for $\tau < 0$.

Lemma 3.2. *Suppose that the kernel $b(t)$ is subject to Hypothesis **(b)** and let $\kappa \in (1,2)$. Then for each $\theta \in (0,1)$ there is a constant $C_\theta > 0$ such that*

$$\int_x^t |s_n(t-\tau)|^\kappa \mathrm{d}\tau \le C_\theta \mu_n^{(\theta-1)/\rho}|t-x|^\theta, \quad 0<x<t, \tag{3.6}$$

and

$$\int_{-\infty}^x |s_n(t-\tau)-s_n(x-\tau)|^\kappa \mathrm{d}\tau \le C_\theta \mu_n^{(\theta-1)/\rho}|t-x|^\theta, \quad x<t. \tag{3.7}$$

The proof of Lemma 3.2 follows exactly the lines of [5, Proof of Lemma 3.1]. Therefore we omit it. For $H \in (\frac{1}{2}, 1)$ we use the identity (1.13) to obtain

$$\begin{aligned}\mathbb{E}|u(t)-u(x)|_{\mathcal{H}}^2 &= \mathbb{E}(u(t)-u(x)|u(t)-u(x))_{\mathcal{H}} = \sum_{n=1}^{\infty} \gamma_n (s_n^{\langle t\rangle} - s_n^{\langle x\rangle} | s_n^{\langle t\rangle} - s_n^{\langle x\rangle})_{\Lambda_H} \\ &= c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{I}^{H-\frac{1}{2}} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|_{L_2(\mathbb{R})}^2 \le c_H \sum_{n=1}^{\infty} \gamma_n |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|_{L_{1/H}(\mathbb{R})}^2\end{aligned}$$

and we have

$$|s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|_{L_{1/H}} = \left[\int_{-\infty}^{x} |s_n(t-\tau) - s_n(x-\tau)|^{1/H} \mathrm{d}\tau + \int_{x}^{t} |s_n(t-\tau)|^{1/H} \mathrm{d}\tau \right]^H .$$

For $H \in (0, \frac{1}{2})$ it is

$$\mathbb{E}|u(t)-u(x)|_{\mathcal{H}}^2 = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{D}^{\frac{1}{2}-H} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|_{L_2(\mathbb{R})}^2$$

and the estimate

$$\left| \mathcal{D}^{\frac{1}{2}-H} (s_n^{\langle t\rangle} - s_n^{\langle x\rangle}) \right|_{L_2(\mathbb{R})}^2 \le c |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|_{\mathrm{H}_2^{\frac{1}{2}-H}(\mathbb{R})}^2 \le \widetilde{c} |s_n^{\langle t\rangle} - s_n^{\langle x\rangle}|_{L_1(\mathbb{R})}^{2H} \tag{3.8}$$

holds for sufficient large $n \in \mathbb{N}$, by interpolation and Lemma 3.1. Thus by employing Lemmata 3.1 and 3.2 this yields

$$\mathbb{E}|u(t)-u(x)|_{\mathcal{H}}^2 \le c_H |t-x|^{2\theta H} \sum_{n=1}^{\infty} \gamma_n \mu_n^{2H(\theta-1)/\rho},$$

for $H \in (0,1)$ and with the aid of Kahane-Khinchine inequality (e.g. [8, Corollary 3.4.1]) and the Kolmogorov-Čentsov-Theorem (e.g. [7, Theorem 2.8]) we may conclude Hölder-continuity with respect to t of $u(t)$ as in the proofs given in Clément and Da Prato [3] or [4]. Similarly, in case **(e)** holds, we obtain spatial Hölder-continuity from the identities

$$\mathbb{E}|u(t,\xi) - u(t,\eta)|^2 = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{I}^{H-\frac{1}{2}} s_n^{\langle t\rangle} \right|_{L_2(\mathbb{R})}^2 |e_n(\xi) - e_n(\eta)|^2$$

for $H \in (\frac{1}{2}, 1)$ and

$$\mathbb{E}|u(t,\xi) - u(t,\eta)|^2 = c_H \sum_{n=1}^{\infty} \gamma_n \left| \mathcal{D}^{\frac{1}{2}-H} s_n^{\langle t\rangle} \right|_{L_2(\mathbb{R})}^2 |e_n(\xi) - e_n(\eta)|^2$$

for $H \in (0, \frac{1}{2})$ respectively.

4. Fractional derivatives and fractional white noise

In the remaining part of this paper we take up a different viewpoint to equations with fractional noise. We consider the problems

$$u + g_\alpha * Au = g_\beta * Q^{1/2}\dot{B}^H \tag{4.1}$$

in the Hilbert space $\mathcal{H}$, where the operator A is subject to Hypothesis **(A)** and also to **(e)** if appropriate, the covariance Q and the fractional Brownian motion B^H are subject to **(B)**, and g_κ denotes the fractional integration kernel

$$g_\kappa(t) = \frac{t^{\kappa-1}}{\Gamma(\kappa)}, \quad t > 0,$$

where $\kappa > 0$. Note that the kernel g_α is of subexponential growth, i.e.

$$\int_0^\infty e^{-\omega t}|g_\alpha(t)|\mathrm{d}t < \infty$$

for arbitrary small $\omega > 0$. This means that that Laplace transform $\widehat{g}_\alpha$ is well defined.

Remark. Problem (4.1) with $\beta = 1$, has also been studied in a recent paper of Bonaccorsi [2] in regard to existence of a mild solution.

For $\alpha \in (0,2)$, $\beta > 0$, define the scalar fundamental solution of (4.1) by

$$\widehat{r}_n(\lambda) = \frac{\widehat{g}_\beta(\lambda)}{1 + \mu_n \widehat{g}_\alpha(\lambda)} = \frac{\lambda^\alpha}{\lambda^\beta(\lambda^\alpha + \mu_n)}, \quad \mathrm{Re}\,\lambda > 0, \quad \mu_n > 0, \tag{4.2}$$

where $\widehat{r}_n$ denotes the Laplace transform of r_n. Furthermore with the convention $r_n(\tau) = 0$ for $\tau < 0$ we have by the Paley-Wiener Theorem

$$\begin{aligned} |r_n|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} &= \int_{\mathbb{R}} |(\mathcal{F}r_n)(\rho)|^2 |\rho|^{1-2H}\mathrm{d}\rho \\ &\le c_\alpha \int_{\mathbb{R}} \Big[\frac{|\rho|^\alpha}{|\rho|^\beta(|\rho|^\alpha + \mu_n)}\Big]^2 |\rho|^{1-2H}\mathrm{d}\rho \\ &= 2c_\alpha \int_0^\infty \Big[\frac{\rho^\alpha}{\rho^\beta(\rho^\alpha + \mu_n)}\Big]^2 \rho^{1-2H}\mathrm{d}\rho \\ &= 2c_\alpha \mu_n^{\frac{2(1-\beta-H)}{\alpha}} \int_0^\infty \Big[\frac{\tau^{\alpha-\beta-H+\frac{1}{2}}}{1+\tau^\alpha}\Big]^2 \mathrm{d}\tau, \end{aligned} \tag{4.3}$$

and the right integral is finite if and only if $1 - H < \beta < 1 - H + \alpha$. Thus by isomorphism (1.12) r_n belongs to Λ_H whenever $\alpha \in (0,2)$ and $\beta \in (1-H, 1-H+\alpha)$. The solution of (4.1) can be rewritten as

$$u(t) = \sum_{n=1}^\infty \sqrt{\gamma_n} \int_0^t r_n(t-\tau)\mathrm{d}\beta_n^H(\tau)e_n, \quad t > 0, \tag{4.4}$$

and therefore as in Section 3 it is by means of representation (1.14)

$$\mathbb{E}|u(t)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle}|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} \tag{4.5}$$

as well as

$$\mathbb{E}|u(t)-u(x)|^2_{\mathcal{H}} = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle} - r_n^{\langle x\rangle}|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} \tag{4.6}$$

and in case $\mathcal{H} = L_2(G)$ and **(e)** is valid

$$\mathbb{E}|u(t,\xi)-u(t,\eta)|^2 = c_H \sum_{n=1}^{\infty} \gamma_n |r_n^{\langle t\rangle}|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} |e_n(\xi)-e_n(\eta)|^2. \tag{4.7}$$

Moreover, it is due to

$$\begin{aligned}(\mathcal{F}f^{\langle t\rangle})(\xi) = \int_{\mathbb{R}} f(t-\tau)\chi_{(-\infty,t]}(\tau) e^{-i\xi\tau}\mathrm{d}\tau \\ = \int_{\mathbb{R}} f(-s)\chi_{(-\infty,0]}(s)e^{-i\xi(s+t)}\mathrm{d}s = e^{-i\xi t}(\mathcal{F}f^{\langle 0\rangle})(\xi)\end{aligned}$$

that for all $t\in\mathbb{R}$

$$\|f^{\langle t\rangle}\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R})} = \|f^{\langle 0\rangle}\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R})} = \|f\|_{\dot{\mathrm{H}}_2^{\sigma}(\mathbb{R}_+)}. \tag{4.8}$$

Thus

$$|r_n^{\langle t\rangle}|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} \le |r_n^{\langle t\rangle}|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} = |r_n^{\langle 0\rangle}|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R})} = |r_n|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)}$$

holds for $t\ge 0$, as soon as $r_n \in \dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)$. Identities (4.5) and (4.6) show that the solution $u(t)$ of (4.1) exists and is continuous in $L_2(\Omega;\mathcal{H})$ if and only if

$$\sigma_1 := \sum_{n=1}^{\infty} \gamma_n |r_n|^2_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(\mathbb{R}_+)} < \infty. \tag{4.9}$$

Next observe that we have for $H\in(\frac{1}{2},1)$

$$\begin{aligned}\left|\mathcal{I}^{H-\frac{1}{2}}(r_n^{\langle t\rangle}-r_n^{\langle x\rangle})\right|^2_{L_2(\mathbb{R})} &= \int_{\mathbb{R}} \left|(\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle t\rangle})(\tau) - (\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle x\rangle})(\tau)\right|^2 \mathrm{d}\tau \\ &= \int_{\mathbb{R}} \left|(\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle 0\rangle})(\tau-t) - (\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle 0\rangle})(\tau-x)\right|^2 \mathrm{d}\tau \\ &= \left|(\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle 0\rangle})(x-t+\cdot) - (\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle 0\rangle})(\cdot)\right|^2_{L_2(\mathbb{R})} \\ &\le \left|\mathcal{I}^{H-\frac{1}{2}}r_n^{\langle 0\rangle}\right|^2_{B^{\theta}_{2,\infty}(\mathbb{R})} |t-x|^{2\theta}\end{aligned}$$

and analogue for $H\in(0,\frac{1}{2})$

$$\left|\mathcal{D}^{\frac{1}{2}-H}(r_n^{\langle t\rangle}-r_n^{\langle x\rangle})\right|^2_{L_2(\mathbb{R})} \le \left|\mathcal{D}^{\frac{1}{2}-H}r_n^{\langle 0\rangle}\right|^2_{B^{\theta}_{2,\infty}(\mathbb{R})} |t-x|^{2\theta},$$

where $B^\theta_{2,\infty}(\mathbb{R})$ denotes a Besov space, with the equivalent norm

$$|f|_{B^\theta_{2,\infty}(\mathbb{R})} = \left[|f|^2_{L_2(\mathbb{R})} + \sup_{h\in\mathbb{R}}\int_{\mathbb{R}} \frac{|f(y+h)-f(y)|^2}{|h|^{2\theta}}\mathrm{d}y\right]^{1/2}.$$

Now we have the embedding

$$\mathrm{H}^\theta_2(\mathbb{R}) \hookrightarrow B^\theta_{2,\infty}(\mathbb{R}),$$

cf. [16, Theorem 2.3.2 (c)], and the apparent relation

$$|f|_{\dot{\mathrm{H}}^\kappa_2(\mathbb{R})} + |f|_{\dot{\mathrm{H}}^{\theta+\kappa}_2(\mathbb{R})} = \begin{cases} |\mathcal{D}^\kappa f|_{\mathrm{H}^\theta_2} & : \kappa \geq 0, \\ |\mathcal{I}^{-\kappa} f|_{\mathrm{H}^\theta_2} & : \kappa < 0. \end{cases} \tag{4.10}$$

So the condition

$$\sigma_2 := \sum_{n=1}^\infty \gamma_n \left[|r_n|_{\dot{\mathrm{H}}_2^{\frac12 - H}(\mathbb{R}_+)} + |r_n|_{\dot{\mathrm{H}}_2^{\theta+\frac12-H}(\mathbb{R}_+)}\right]^2 < \infty \tag{4.11}$$

implies Hölder continuity of $u(t)$ in time of order θ. Finally from **(e)** we obtain by interpolation

$$|e_n(\xi) - e_n(\eta)| \leq C|\xi-\eta|^\theta \mu_n^{\theta/2},$$

hence

$$\sigma_3 := \sum_{n=1}^\infty \gamma_n \mu_n^\theta |r_n|^2_{\dot{\mathrm{H}}_2^{\frac12-H}(\mathbb{R}_+)} < \infty \tag{4.12}$$

yields Hölder-continuity of $u(t,\xi)$ in space ξ of order θ. Therefore the goal is to estimate the $\dot{\mathrm{H}}_2^{\theta+\frac12-H}(\mathbb{R}_+)$-norms of r_n, where the functions $r_n(t)$ are the fundamental solutions of the scalar problems

$$r_n + \mu_n g_\alpha * r_n = g_\beta. \tag{4.13}$$

This will be done by the following Lemma.

Lemma 4.1. *Suppose $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$, and let $r_n(t)$ denote the solution of (4.13). Then*

$$|r_n|^2_{\dot{\mathrm{H}}_2^{\theta+\frac12-H}(\mathbb{R}_+)} \leq C_{\alpha,\beta,\theta}\mu_n^{\frac{2(1-\beta+\theta-H)}{\alpha}}, \quad \mu_n > 0,$$

whenever $\beta \in (1-H+\theta, 1-H+\alpha)$.

Proof. Again we extend the functions r_n trivially on negative halfline. Let $H \in (\frac12, 1)$. We first consider the case $\theta = 0$. Then by the Paley-Wiener theorem, $\mathcal{I}^{H-\frac12} r_n \in L_2(\mathbb{R})$ if and only if $\widehat{\mathcal{I}^{H-\frac12} r_n} \in \mathscr{H}_2(\mathbb{C}_+)$, the Hardy space of exponent 2 and $|\mathcal{I}^{H-\frac12} r_n|_{L_2(\mathbb{R})} = (1/\sqrt{2\pi})|\widehat{\mathcal{I}^{H-\frac12} r_n}|_{\mathscr{H}_2(\mathbb{C}_+)}$. Applying the Paley-Wiener theorem one more time, it suffices to show that $\mathcal{F}(\mathcal{I}^{H-\frac12} r_n) \in L_2(\mathbb{R})$. Now we may use (1.11) to compute

$$\int_{\mathbb{R}} \left|\mathcal{F}(\mathcal{I}^{H-\frac12} r_n)(\rho)\right|^2 \mathrm{d}\rho \leq \int_0^\infty \left[\frac{\rho^\alpha}{\rho^\beta(\rho^\alpha + \mu_n)}\right]^2 \rho^{-2H+1}\mathrm{d}\rho$$

and we have seen in (4.3) that the right integral converges if and only if $\beta \in (1-H, 1-H+\alpha)$. In case $\theta \neq 0$, observe that $|\cdot|_{L_2(\mathbb{R})} + |D^\theta \cdot|_{L_2(\mathbb{R})}$ defines an equivalent norm in $\mathrm{H}_2^\theta(\mathbb{R})$, hence replacing β by $\beta - \theta$ the result follows by Plancherel's Theorem. For $H \in (0, \frac{1}{2})$ one may proceed as above with replacing $\mathcal{I}^{H-\frac{1}{2}}$ by $\mathcal{D}^{\frac{1}{2}-H}$. □

Now we are in the position to state our result on (4.1).

Theorem 2. *Let $\alpha \in (0,2)$, $\beta > 0$, $\theta \in [0,1]$ such that $\beta \in (1-H+\theta, 1-H+\alpha)$. Assume that **(A)** and **(B)** are satisfied.*

(i) If

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta-H)}{\alpha}} < \infty,$$

then the solution u of (4.1) exists and belongs to $C_b(\mathbb{R}_+; L_2(\Omega; \mathcal{H}))$.

(ii) If

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta+\theta-H)}{\alpha}} < \infty,$$

then $u \in C_b^\theta(\mathbb{R}_+; L_2(\Omega; \mathcal{H}))$.

*(iii) If $\mathcal{H} = L_2(G)$, **(e)** holds, and*

$$\sum_{n=1}^{\infty} \gamma_n \mu_n^{\frac{2(1-\beta-H)+\alpha\theta}{\alpha}} < \infty,$$

then $u \in C_b(\mathbb{R}_+; C^\theta(G; L_2(\Omega)))$.

Proof. Use Lemma 4.1 to estimate the quantities σ_i, $i = 1, 2, 3$, arising in (4.9), (4.11) and (4.12), respectively. □

Remark.

(i) In case $\beta = 1$ and $\alpha > H$, Theorem 2 (i) coincides with the result of Bonaccorsi [2, Theorem 4.8 (iii)].

(ii) Sufficient conditions for the existence of a mild solution of (4.1) in case $\beta = 1$ and $\alpha \leq H$ can be found in [2, Theorem 4.8 (i),(ii)].

Example. Let $\mathcal{H} = L_2(0, \pi)$, $A = A_0^m$, where $A_0 = -(\mathrm{d}/\mathrm{d}x)^2$ with domain $D(A_0) = \mathrm{H}_2^2(0,\pi) \cap \mathring{\mathrm{H}}_2^1(0,\pi)$. It is obvious that A is subject to Hypothesis **(A)** and it is well known that eigenvalues of A are $\mu_k = k^{2m}$ for $k \in \mathbb{N}$. The covariance Q is given by its spectral decomposition

$$Qx = \sum_{k=1}^{\infty} \gamma_k (x|e_k) e_k,$$

with $(\gamma_k)_{k\in\mathbb{N}} \subset (0,1]$ such that $\sum_{k=1}^{\infty}\gamma_k < \infty$. For our example we choose $\gamma_k = k^{-l}$, $l > 1$, and we obtain

$$\sum_{k=1}^{\infty}\gamma_k\mu_k^{\frac{2(1-\beta-H)}{\alpha}} < \infty \iff \beta > 1 - H - \frac{\alpha(l-1)}{4m};$$

$$\sum_{k=1}^{\infty}\gamma_k\mu_k^{\frac{2(1-\beta+\theta-H)}{\alpha}} < \infty \iff \beta > 1 - H + \theta - \frac{\alpha(l-1)}{4m};$$

$$\sum_{k=1}^{\infty}\gamma_k\mu_k^{\frac{2(1-\beta-H)+\alpha\theta}{\alpha}} < \infty \iff \beta > 1 - H + \frac{\alpha\theta}{2} - \frac{\alpha(l-1)}{4m}.$$

Obviously the latter series converge for all $\beta \in (1-H+\theta, 1-H+\alpha)$, hence Theorem 2 applies independently from the choice of l and m. Observe that the spatial regularity is better than in time and that for $H \in (\frac{1}{2},1)$ the regularity in space and in time is better than in case $H = \frac{1}{2}$. On the other hand regularity degrade for $H \in (0,\frac{1}{2})$.

We conclude with a brief discussion of the case $\alpha = 2$. Then

$$\widehat{r}_n(\lambda) = \frac{\lambda^{2-\beta}}{\lambda^2+\mu_n}, \qquad n \in \mathbb{N},$$

hence there are poles $\pm i\sqrt{\mu_n}$ on the imaginary axis, and so Lemma 4.1 is not valid in this case. Therefore we proceed differently. It is shown in [5], that if $\frac{1}{2} < \beta < 3$ one obtains with the aid of the complex inversion formula for the Laplace transform

$$r_n(t) = \mu_n^{\frac{1-\beta}{2}}\left[\sin\left(\sqrt{\mu_n}t + \frac{(2-\beta)\pi}{2}\right) - \frac{1}{\pi}\sin((2-\beta)\pi)\int_0^{\infty} e^{-\sqrt{\mu_n}t\tau}\frac{\tau^{2-\beta}\mathrm{d}\tau}{1+\tau^2}\right],$$

where $t > 0$. This formula shows in particular, that for every $t \in (0,T)$ it is $|r_n(t)| \le c_T\mu_n^{\frac{1-\beta}{2}}$, where the constant c_T (in the sequel generic) may depend on T. Thanks to $L_{\frac{1}{H}} \hookrightarrow \dot{\mathrm{H}}_2^{\frac{1}{2}-H}$ we have in case $H > \frac{1}{2}$

$$|r_n|_{\dot{\mathrm{H}}_2^{\frac{1}{2}-H}(0,T)} \le c_T|r_n|_{L_{\frac{1}{H}}(0,T)} \le c_T\mu_n^{\frac{1-\beta}{2}},$$

for any fixed $T > 0$ and for all $n \in \mathbb{N}$. The condition for local existence in the case $\alpha = 2$ and $H > \frac{1}{2}$ is now immediate and reads as

$$\sum_{n=1}^{\infty}\gamma_n\mu_n^{\frac{1-\beta}{2}} < \infty.$$

Note, that this is not the limiting case of Theorem 2 (i) as $\alpha \to 2$.

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