AN ANALYSIS OF ASIAN OPTIONS

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Dedicated to the memory of Gunter Lumer.

Abstract. The objective of this paper is to provide an analytic theory for pricing of Asian options of European type. We present a partial differential equation describing the fair price process of an Asian option. This appears as

$$(\partial_t - A - x \cdot \nabla_y)u = 0$$

and the associated payoff function as the end value. Here the operator A is the d-dimensional Black-Scholes operator, and $B = x \cdot \nabla_y$ represents the path dependence in terms of the price averaging in Asian options. The main result will be to prove, that a solution of this partial differential equation exists, is unique, and depends continuously on the data in appropriate function spaces, i.e. that the problem is well-posed. On our way we are going to employ semigroup methods, in particular the **Lumer-Phillips theorem**.

1. Introduction

Let $J := [0,T]$ and $(\Omega, \mathcal{F}, P)$ a complete probability space with filtration $\{\mathcal{F}_t\}_{t\in J}$; thus

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \cup \mathcal{N} \subset \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathcal{F} \qquad s,t \in J, \quad s < t,$$

and $\mathcal{N} = \{N \in \mathcal{F} : P(N) = 0\}$. We consider a market containing $d+1$ assets, the 0th being riskless, the remaining d being risky with prices $S_t^i : \Omega \to \mathbb{R}$, $i = 0, \dots, d$, in time $t \in J$. Let $\{S_t\} = \{(S_t^0, \dots, S_t^d)^T\} \in L_2(J \times \Omega; \mathbb{R}^{d+1})$ be the $\{\mathcal{F}_t\}$-adapted vectorial price process. We assume as usual that the asset price processes S_t are driven by stochastic differential equations

$$\begin{cases} \mathrm{d}S_t^0 = r(t)S_t^0\mathrm{d}t, & S_0^0 = 1 \\ \mathrm{d}S_t^i = \mu^i(t)S_t^i\mathrm{d}t + \sum_{j=1}^m \sigma_j^i(t)S_t^i\mathrm{d}B_t^j, & S_0^i > 0 \text{ given} \end{cases} \tag{1.1}$$

where $i = 1, \dots, d$ and $t \in J$.

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Here $r \in L_\infty(J)$ is the deterministic rate of interest of the riskless asset at time t, $\mu^i \in L_\infty(J \times \Omega)$ the growth rate of asset i, $\sigma^i_j \in L_\infty(J)$ the variances, also known as volatilities of the market which is assumed to be complete, i.e. the matrix (σ^i_j) is surjective. The process $\{B_t\}_{t\in J}$ denotes a m-dimensional Brownian motion with independent components B^j_t.
Henceforth $\mathcal{F}_t$ is assumed to be the complete σ-algebra, which is induced by the history of the Brownian motion $\{B_s\}_{s<t}$. Moreover we assume that $\{\mu^i(t)\}$ is an adapted process, i.e. $\mu^i(t)$ is $\mathcal{F}_t$-adapted. We define

$$\mu(t) := (\mu^1(t), \dots, \mu^d(t))^T \qquad \text{and} \qquad \sigma(t) := (\sigma^i_j(t))_{i,j}.$$

Our objective is to find a self-financing portfolio strategy θ_t and the fair price for an option at time t, if a function $g \in C((0,T] \times \mathbb{R}^d_+ \times \mathbb{R}^d_+)$ is appointed, which replicates g, i.e. $Z := g(T, S^*_T, I(S^*_T))$, with $S^*_t := \big(S^1_t, \dots, S^d_t\big)^T$ and $I(S^*_t) := \big(I(S^1_t), \dots, I(S^d_t)\big)^T$ is defined by $I(S^k_t) := \int_0^t S^k_\tau \mathrm{d}\tau$. Subject to these conditions $Y_t := \theta_t \cdot S_t$ is the fair price at time t with end constraint $Y_T = Z$. Our candidates of interest for g are

$$g(t, S^*_t, I(S^*_t)) = \left[\frac{1}{t} I(S^k_t) - K\right]_+ \quad \text{(average price call on asset } k\text{)},$$
$$g(t, S^*_t, I(S^*_t)) = \left[S^k_t - \frac{1}{t} I(S^k_t)\right]_+ \quad \text{(average strike call on asset } k\text{)},$$
$$g(t, S^*_t, I(S^*_t)) = \left[K - \frac{1}{t} I(S^k_t)\right]_+ \quad \text{(average price put on asset } k\text{)},$$
$$g(t, S^*_t, I(S^*_t)) = \left[\frac{1}{t} I(S^k_t) - S^k_t\right]_+ \quad \text{(average strike put on asset } k\text{)},$$

with $k = 1, \dots, d$. We restrict our attention to the "European case" which means that an option can only be executed at expiration time T. Therefore it will suffice to evaluate the payoff function g in $t = T$ and we will write $g(T, x, y) =: g(x, y)$.

In the following section we will construct a $2d+1$ dimensional partial differential equation of elliptic-hyperbolic type, describing the fair price process of an Asian option. Then, in Section 3, the Lumer-Phillips theorem is employed to obtain well-posedness of the Euler transformed problem on the spaces

$$X_\infty := C_0(\mathbb{R}^d \times \mathbb{R}^d_+) := \left\{ u \in C(\mathbb{R}^d \times \mathbb{R}^d_+) : \lim_{|x|+|y| \to \infty} u(x,y) = 0 \right\}$$

endowed with norm $\|u\|_\infty = \sup_{x\in\mathbb{R}^d, y\in\mathbb{R}^d_+} \{|u(x,y)|\}$, and

$$X_p := L_p(\mathbb{R}^d \times \mathbb{R}^d_+), \quad 1 \le p < \infty,$$

equipped with their natural norm. We will make use of Yosida approximation to verify the generator property of a sum of noncommuting generators. Finally in

Section 4 we present an appropriate scaling of the end conditions and we show that well-posedness is invariant under this scaling. Lastly the Lumer-Phillips theorem is the key to obtain the well-posedness results.

We introduce some notations. The dot between two vectors a and b denotes the inner product, i.e. $a \cdot b = \sum_{i=1}^{d} a_i b_i$. A double dot between two matrices A and B similarly denotes the double summation, i.e. $A : B = \sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}b_{ij}$. Moreover ∇_x^2 means $(\nabla_x^2)_{ij} = \partial_{x_i}\partial_{x_j}$ and we define $(xy)_i := x_i y_i$, $(xx\nabla_x^2)_{ij} := x_i x_j \partial_{x_i}\partial_{x_j}$ for $x, y \in \mathbb{R}^d$.

2. The Black-Scholes Approach

The basic idea consists in the approach following the fundamental work of Black-Scholes [BS73]. We use the ansatz $Y_t = u(t, S_t^*, I(S_t^*))$ and try to determine the function $u(t,x,y)$ with $t \in J$, $x = (x^1, \ldots, x^d) \in \mathbb{R}_+^d$, and $y = (y^1, \ldots, y^d) \in \mathbb{R}_+^d$. We already know that u satisfies the end constraint $u(T, S_T^*, I(S_T^*)) = Y_T = Z = g(S_T^*, I(S_T^*))$, i.e.

$$u(T,x,y) = g(x,y), \qquad x \in \mathbb{R}_+^d, \quad y \in \mathbb{R}_+^d. \tag{2.1}$$

Applying Itô's formula we have

$$\begin{aligned} \mathrm{d}Y_t &= \partial_t u \mathrm{d}t + \partial_x u \mathrm{d}S_t^* + \partial_y u \mathrm{d}I_t + \frac{1}{2}\partial_x^2 u \mathrm{d}S_t^* \mathrm{d}S_t^* + \partial_{xy}^2 u \mathrm{d}S_t^* \mathrm{d}I_t \\ &\quad + \frac{1}{2}\partial_y^2 u \mathrm{d}I_t \mathrm{d}I_t \\ &= \partial_t u \mathrm{d}t + \sum_{i=1}^{d} \partial_{x_i} u \mathrm{d}S_t^i + \sum_{i=1}^{d} \partial_{y_i} u \mathrm{d}I_t^i + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{x_i}\partial_{x_k} u \mathrm{d}S_t^i \mathrm{d}S_t^k \\ &\quad + \sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{x_i}\partial_{y_k} u \mathrm{d}S_t^i \mathrm{d}I_t^k + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{y_i}\partial_{y_k} u \mathrm{d}I_t^i \mathrm{d}I_t^k. \end{aligned} \tag{2.2}$$

The relation $\mathrm{d}I_t^k = S_t^k \mathrm{d}t$ and (1.1) yield

$$\begin{aligned} \mathrm{d}Y_t =& \partial_t u \mathrm{d}t + \sum_{i=1}^{d} \partial_{x_i} u \left[\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^{m} \sigma_j^i S_t^i \mathrm{d}B_t^j \right] + \sum_{i=1}^{d} \partial_{y_i} u \left[S_t^i \mathrm{d}t \right] \\ &+ \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{x_i}\partial_{x_k} u \left[\left(\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^{m} \sigma_j^i S_t^i \mathrm{d}B_t^j \right) \left(\mu_t^k S_t^k \mathrm{d}t + \sum_{j=1}^{m} \sigma_j^k S_t^k \mathrm{d}B_t^j \right) \right] \\ &+ \sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{x_i}\partial_{y_k} u \left[\left(\mu_t^i S_t^i \mathrm{d}t + \sum_{j=1}^{m} \sigma_j^i S_t^i \mathrm{d}B_t^j \right) \left(S_t^k \mathrm{d}t \right) \right] \\ &+ \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} \partial_{y_i}\partial_{y_k} u \left[\left(S_t^i \mathrm{d}t \right) \left(S_t^k \mathrm{d}t \right) \right]. \end{aligned}$$

With subject to the common conventions $(\mathrm{d}t)^2 = \mathrm{d}t \cdot \mathrm{d}B_t = 0$ and $(\mathrm{d}B_t)^2 = \mathrm{d}t$, the following equation follows

$$\mathrm{d}Y_t = \underbrace{\left[\partial_t u + \sum_{i=1}^{d} \mu_t^i S_t^i \partial_{x_i} u + \sum_{i=1}^{d} S_t^i \partial_{y_i} u + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d}\left(\sum_{j=1}^{m} \sigma_j^i \sigma_j^k\right) S_t^i S_t^k \partial_{x_i}\partial_{x_k} u\right] \mathrm{d}t}_{\text{deterministic part}} + \underbrace{\sum_{j=1}^{m}\left(\sum_{i=1}^{d} \partial_{x_i} u \sigma_j^i S_t^i\right)}_{\text{stochastic part}} \mathrm{d}B_t^j. \tag{2.3}$$

On the other hand $Y_t = \theta_t \cdot S_t = \theta_t^* \cdot S_t^* + \theta_t^0 \cdot S_t^0$, hence with the self-financing condition

$$\mathrm{d}\theta_t \cdot S_t + \mathrm{d}\theta_t \cdot \mathrm{d}S_t = 0 \tag{2.4}$$

another representation of $\mathrm{d}Y_t$ arises as

$$\begin{aligned} \mathrm{d}Y_t = \theta_t \cdot \mathrm{d}S_t &= \theta_t^0 r_t S_t^0 \mathrm{d}t + \sum_{i=1}^{d} \theta_t^i S_t^i \mu_t^i \mathrm{d}t + \sum_{i=1}^{d}\sum_{j=1}^{m} \theta_t^i S_t^i \sigma_j^i \mathrm{d}B_t^j \\ &= \underbrace{\left[\theta_t^0 r_t S_t^0 + \sum_{i=1}^{d} \theta_t^i S_t^i \mu_t^i\right]}_{\text{deterministic part}} \mathrm{d}t + \underbrace{\sum_{j=1}^{m}\sum_{i=1}^{d} \theta_t^i S_t^i \sigma_j^i}_{\text{stochastic part}} \mathrm{d}B_t^j. \end{aligned} \tag{2.5}$$

Thanks to the uniqueness of the Itô transform we are able to compare the deterministic and stochastic coefficients of both representations (2.3) and (2.5).

The comparison of the stochastic parts yields

$$\sigma_t^T \cdot (S_t^* \nabla_x u) = \sigma_t^T \cdot (\theta_t^* S_t^*)$$

and due to the injectivity of the matrix σ_t^T we obtain

$$\theta_t^* = \nabla_x u(t, S_t^*, I(S_t^*)). \tag{2.6}$$

The comparison of the deterministic coefficients provides

$$\partial_t u + (S_t^* \mu_t) \cdot \nabla_x u + S_t^* \cdot \nabla_y u + \frac{1}{2}\sigma_t \sigma_t^T : (S_t^* S_t^* \nabla_x^2 u) = \theta_t^0 r_t S_t^0 + \theta_t^* \cdot (S_t^* \mu_t). \tag{2.7}$$

After insertion of (2.6) the remaining equation reads as follows

$$\partial_t u + S_t^* \cdot \nabla_y u + \frac{1}{2}(\sigma_t \sigma_t^T) : (S_t^* S_t^* \nabla_x^2 u) = \theta_t^0 r_t S_t^0. \tag{2.8}$$

Since $u(t, S_t^*, I(S_t^*)) = Y_t = \theta_t^* \cdot S_t^* + \theta_t^0 S_t^0$ we have

$$\theta_t^0 S_t^0 = u - \theta_t^* \cdot S_t^* \overset{(2.6)}{=} u - S_t^* \cdot \nabla_x u$$

and therefore

$$\theta_t^0 = (S_t^0)^{-1}\left[u(t, S_t^*, I(S_t^*)) - S_t^* \cdot \nabla_x u(t, S_t^*, I(S_t^*))\right]. \tag{2.9}$$

As a last step we have to insert (2.9) into (2.8) and resubstitute S_t^* with x and $I(S_t^*)$ with y, so that the following partial differential equation arises

$$\partial_t u + x \cdot \nabla_y u + \frac{1}{2}\underbrace{(\sigma_t \sigma_t^T)}_{=:a_t} : xx\nabla_x^2 u = r_t(u - x \cdot \nabla_x u), \qquad x, y \in \mathbb{R}_+^d.$$

Hence we derived a $2d+1$ dimensional partial differential equation with inverse time direction

$$\begin{cases} \partial_t u + \sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d a_{ik}(t) x_i x_k \partial_{x_i}\partial_{x_k} u = r(t)\left[u - \sum_{i=1}^d x_i \partial_{x_i} u\right], \\ u(T, x, y) = g(T, x, y), \end{cases} \tag{2.10}$$

where $x, y \in \mathbb{R}_+^d$ and $t \in J$.

For simplicity, we assume, that $a(t) \equiv a$ and $r(t) \equiv r$. By doing so we can write (2.10) as

$$\begin{cases} \partial_t u + \sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2}\sum_{i=1}^d \sum_{k=1}^d a_{ik} x_i x_k \partial_{x_i}\partial_{x_k} u = r\left[u - \sum_{i=1}^d x_i \partial_{x_i} u\right], \\ u(T, x, y) = g(x, y), \end{cases} \tag{2.11}$$

with $x, y \in \mathbb{R}_+^d$ and $t \in [0, T]$.

This is the basic model for the pricing of an Asian option of European style. In short form it is written as

$$\partial_t u + x \cdot \nabla_y u + \frac{1}{2} a : xx\nabla_x^2 u = r(u - x \cdot \nabla_x u), \qquad t \in [0, T),\ x, y \in \mathbb{R}_+^d,$$
$$u(T, x, y) = g(x, y).$$

In order to eliminate the strong degeneracy of the coefficients in (2.11) we are going to run an Euler-Transformation. Therefore we substitute $x_i \rightsquigarrow e^{\xi_i}$ and $u \rightsquigarrow v$ with

$$v(t, \xi, y) = u(t, x, y) = u(t, e^\xi, y) = v(t, \log x, y),$$

where $e^\xi = (e^{\xi_1}, \ldots, e^{\xi_d})^T$ and $\log x = (\log x_1, \ldots, \log x_d)^T$. Hence the first partial derivative of v with respect to ξ is

$$\partial_{\xi_i} v(t, \xi, y) = x_i \partial_{x_i} u(t, x, y) \tag{2.12}$$

and the relevant second partial derivation results as

$$\begin{aligned} \partial_{\xi_i}\partial_{\xi_j} v(t, \xi, y) &= (x_j \partial_{x_j})(x_i \partial_{x_i} u)(t, x, y) \\ &= \begin{cases} x_i x_j \partial_{x_i}\partial_{x_j} u(t, x, y) & : i \neq j \\ (x_i)^2 \partial_{x_i}^2 u(t, x, y) + x_i \partial_{x_i} u(t, x, y) & : i = j \end{cases}. \end{aligned} \tag{2.13}$$

Thus the differential equation in v appears as

$$\partial_t v + \sum_{i=1}^{d} e^{\xi_i}\partial_{y_i} v + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}\partial_{\xi_i}\partial_{\xi_k} v = r\left[v - \sum_{i=1}^{d}\partial_{\xi_i} v\right] + \frac{1}{2}\sum_{i=1}^{d} a_{ii}\partial_{\xi_i} v.$$

Time direction can be inverted with a time reflection by substituting $t \rightsquigarrow T-t$ and $v \rightsquigarrow w$ with $w(t,\xi,y) = v(T-t,\xi,y)$. Thus the final problem can be written as

$$-\partial_t w + \sum_{i=1}^{d} e^{\xi_i}\partial_{y_i} w + \frac{1}{2}\sum_{i=1}^{d}\sum_{k=1}^{d} a_{ik}\partial_{\xi_i}\partial_{\xi_k} w = r\left[w - \sum_{i=1}^{d}\partial_{\xi_i} w\right] + \frac{1}{2}\sum_{i=1}^{d} a_{ii}\partial_{\xi_i} w,$$
$$w(0,\xi,y) = g\left(e^{\xi}, y\right),$$

with $\xi \in \mathbb{R}^d,\ y \in \mathbb{R}^d_+$ and $t \in [0,1]$. Introducing the vector b by $b_i := r - \frac{a_{ii}}{2}$ we obtain the following problem

(2.14)
$$\begin{cases} \partial_t w + rw - b\cdot\nabla_\xi w - \dfrac{1}{2}a : \nabla_\xi^2 w = e^{\xi}\cdot\nabla_y w,\ t \in (0,1],\ \xi \in \mathbb{R}^d,\ y \in \mathbb{R}^d_+, \\ \qquad w(0,\xi,y) = w_0(\xi,y) := g\left(e^{\xi}, y\right). \end{cases}$$

It is this problem we will study mathematically, the inverse Euler transform is left to the reader.

3. Well-Posedness of the Problem

Our objective is to prove that the problem (2.14) is well-posed in the spaces X_p, $1 < p \leq \infty$, introduced in Section 1, i.e. that its solution exists, is unique, and depends continuously on the data. We start providing two preliminary lemmata.

Lemma 3.1. *The family of operators* $\{T_B(t)\}_{t\geq 0} \subset \mathcal{B}(X_p)$, $1 \leq p \leq \infty$ *given by*

$$(T_B(t)u_0)(\xi,y) = u_0(\xi, y + te^{\xi}), \quad t \geq 0,\ \xi \in \mathbb{R}^d,\ y \in \mathbb{R}^d_+$$

defines a C_0*-semigroup of contractions in* X_p.

Proof. It is obvious that $\{T_B(t)\}_{t\geq 0}$ satisfies the semigroup property. The following equation assures the contractivity of semigroup $\{T_B(t)\}_{t\geq 0}$ for $1 \leq p < \infty$

$$\begin{aligned} \|T_B(t)u_0(\xi)\|_p^p &= \int\limits_{\mathbb{R}^d_+} \left|u_0\left(\xi, y + te^{\xi}\right)\right|^p \mathrm{d}y \\ &= \int\limits_{U\subset\mathbb{R}^d_+} |u_0(\xi,z)|^p\,\mathrm{d}z, \qquad (\text{with } z = y + te^{\xi}) \\ &\leq \|u_0(\xi)\|_p^p. \end{aligned}$$

Integrating over $\xi \in \mathbb{R}^d$ yields the claim. For $p = \infty$ we receive

$$\|T_B(t)u_0(\xi)\|_\infty = \sup_{y\in\mathbb{R}^d_+} \left|u_0\left(\xi, y + te^\xi\right)\right| \le \sup_{y\in\mathbb{R}^d_+} |u_0(\xi, y)| = \|u_0(\xi)\|_\infty .$$

This implies that $\|T_B(t)u_0\|_p \le \|u_0\|_p$ holds for $1 \le p \le \infty$.
The C_0 property for $1 \le p < \infty$ follows directly from the fact that the space of test functions is dense in X_p. □

In the sequel we denote with B the generator of $\{T_B(t)\}_{t\ge 0}$; note that $B = e^\xi \cdot \nabla_y$ holds at least on $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^d_+) \subset D(B)$.

Lemma 3.2. *The operator $A : D(A) \subset X_p \to X_p$, $1 < p \le \infty$, defined by*

$$Au := \frac{1}{2} a : \nabla^2_\xi u + b \cdot \nabla_\xi u - ru$$

with domain

$$D(A) = \begin{cases} W^2_p\left(\mathbb{R}^d; L_p(\mathbb{R}^d_+)\right) & \text{for } 1 < p < \infty, \\ \left\{u \in C_0(\mathbb{R}^d) \cap \bigcap_{q>1} W^2_{q,loc}(\mathbb{R}^d) : Au \in C_0(\mathbb{R}^d)\right\} & \text{for } p = \infty \end{cases}$$

generates an analytic C_0-semigroup of contractions in X_p.

Proof. Lunardi proved in [Lun95, Corollary 3.1.9] that A generates an analytic C_0-semigroup $\{T_A(t)\}_{t\ge 0}$. This semigroup is given by

$$(T_A(t)u_0)(\xi, y) = e^{-rt} \int_{\mathbb{R}^d} u_0(\xi - \eta, y)\gamma_{tb,ta}(\eta)\mathrm{d}\eta, \tag{3.1}$$

where $\gamma_{\mu,\sigma}(\eta)$ is the Gaussian distribution, i.e.

$$\gamma_{\mu,\sigma}(\eta) = \frac{1}{\sqrt{(2\pi)^d \det \sigma}} \exp\left\{-\frac{1}{2}(\sigma^{-1}(\eta - \mu) \mid \eta - \mu)\right\}.$$

Since $\gamma_{\mu,\sigma}(\eta) \ge 0$ and $\int_{\mathbb{R}^d} \gamma_{\mu,\sigma}(\eta)\mathrm{d}\eta = 1$ we obtain by Young's inequality

$$\|T_A(t)\|_p \le e^{-rt}$$

for all $1 \le p \le \infty$. □

In the following $B_\lambda := B(I - \lambda B)^{-1}$ denotes the Yosida-approximation of the operator B. It is well known that $\lim_{\lambda\to 0^+} B_\lambda x = Bx$ for $x \in D(B)$ and also that if B is a generator of a C_0-semigroup of contractions then so is B_λ.

Proposition 3.1. *Let X be a reflexive Banach space. Let A and B be dissipative generators and suppose that the solution u_λ of*

$$\omega u - Au - B_\lambda u = f \tag{3.2}$$

satisfies

$$\sup_{\lambda\in(0,1)} |B_\lambda u_\lambda| < \infty \tag{3.3}$$

for a dense set of right hand sides f. Then $\overline{A+B}$ is the generator of a C_0-semigroup of contractions in X.

Proof. We employ the theorem of Lumer and Phillips [LP61]. Obviously $A+B$ is dissipative. Due to the assumption that $\{B_\lambda u_\lambda\}$ is bounded it holds that $\{u_\lambda\}$ is bounded and hence $\{Au_\lambda\}$ is bounded as well. Since X is reflexive there is a sequence $(\lambda_n) \subset \mathbb{R}$ with $\lambda_n \to 0$ such that $u_n := u_{\lambda_n} \rightharpoonup u$, $Au_n \rightharpoonup v$, and $B_{\lambda_n} u_n \rightharpoonup w$. Because graph$(A)$ is closed and convex we have

$$\text{graph}(A) \ni (u_n, Au_n) \rightharpoonup (u,v) \in \text{graph}(A) \quad \text{in } X\times X \tag{3.4}$$

and thus $(u,v) = (u, Au)$ with $u \in D(A)$. Moreover, since $(I-\lambda_n B)^{-1}u_n \rightharpoonup u$ as well, we obtain also

$$B_{\lambda_n} u_n = B(I-\lambda_n B)^{-1} u_n \rightharpoonup w = Bu \tag{3.5}$$

with $u \in D(B)$. Summarizing, we have proven that u_n converges weakly to a solution $u \in D(A)\cap D(B)$ of

$$\omega u - Au - Bu = f \tag{3.6}$$

for a dense set of right hand sides f; hence $\overline{R(w - Au - Bu)} = X$ and the theorem of Lumer-Phillips applies. □

Theorem 1. *Let the operator $L : D(L) \subset X_p \to X_p$, $1 < p \leq \infty$, be defined as*

$$Lw := \frac{1}{2} a : \nabla_\xi^2 w + b\cdot\nabla_\xi w - rw + e^\xi \partial_y w,$$

with domain $D(L) = D(A)\cap D(B)$ and consider the abstract Cauchy problem

$$\begin{cases} \partial_t w - \overline{L}w = 0 & \text{in } (0,T]\times\mathbb{R}^d\times\mathbb{R}^d_+, \\ w = w_0 & \text{on } \{t=0\}\times\mathbb{R}^d\times\mathbb{R}^d_+. \end{cases} \tag{3.7}$$

Then problem (3.7) is well-posed in X_p.

Proof. Lemma 3.2 resp. Lemma 3.1 the Lumer-Phillips theorem [LP61] imply that the operator A given by $Au = \frac{1}{2}a : \nabla_\xi^2 u + b\cdot\nabla_\xi u - ru$ and the operator B

introduced after Lemma 3.1 are dissipative generators. We want to show that the range of $\omega + A + B$ is dense in X_p. So we consider the equation

$$\omega u_\lambda - Au_\lambda - B_\lambda u_\lambda = f \tag{3.8}$$

which admits a unique solution $u_\lambda \in D(A)$, $\lambda > 0$. Here we take $f \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d_+)$, $\operatorname{supp} f$ compact; this set is dense in X_p for each $1 \le p \le \infty$.
Since A and e^{ξ_j}, $j = 1, \dots, d$, do not commute we obtain

$$\begin{aligned} \omega(e^{\xi_j}u_\lambda) - A(e^{\xi_j}u_\lambda) - B_\lambda(e^{\xi_j}u_\lambda) &= e^{\xi_j}f - A(e^{\xi_j}u_\lambda) + e^{\xi_j}Au_\lambda \\ &= e^{\xi_j}f - [A, e^{\xi_j}]u_\lambda, \quad j = 1, \dots, d, \end{aligned} \tag{3.9}$$

where $[A, e^{\xi_j}]$ denotes the commutator of A and e^{ξ_j}. Employing the sum convention we have

$$\begin{aligned} [A, e^{\xi_j}]v &= \frac{a_{ik}}{2}\partial_i\partial_k(e^{\xi_j}v) + b_i\partial_i(e^{\xi_j}v) - e^{\xi_j}\frac{a_{ik}}{2}\partial_i\partial_k v - e^{\xi_j}b_i\partial_i v \\ &= \frac{a_{ik}}{2}\partial_i(e^{\xi_j}\partial_k v + \delta_{jk}e^{\xi_j}v) + e^{\xi_j}b_i(\partial_i v + \delta_{ji}v) - e^{\xi_j}\frac{a_{ik}}{2}\partial_i\partial_k v - e^{\xi_j}b_i\partial_i v \\ &= \frac{a_{ik}}{2}\left(\delta_{ji}e^{\xi_j}\partial_k v + \delta_{jk}e^{\xi_j}\partial_i v + \delta_{ji}\delta_{jk}e^{\xi_j}v\right) + \left(r - \frac{a_{jj}}{2}\right)e^{\xi_j}v \\ &= \frac{a_{jk}}{2}e^{\xi_j}\partial_k v + \frac{a_{ij}}{2}e^{\xi_j}\partial_i v + re^{\xi_j}v \\ &= a_{jk}e^{\xi_j}\partial_k v + re^{\xi_j}v \\ &= a_{jk}\partial_k(e^{\xi_j}v) - a_{jj}(e^{\xi_j}v) + r(e^{\xi_j}v), \end{aligned}$$

due to $b_j = r - a_{jj}/2$ and since (a_{ij}) is symmetric. Thus we obtain the equation

$$(\omega + r - a_{jj})(e^{\xi_j}u_\lambda) - A(e^{\xi_j}u_\lambda) - A_j(e^{\xi_j}u_\lambda) - B_\lambda(e^{\xi_j}u_\lambda) = e^{\xi_j}f \tag{3.10}$$

with $A_j v := -a_{jk}\partial_k v$. Because the operators A_j are also dissipative we choose $\omega > 2\max\{a_{jj} - r : j = 1, \dots, d\}$ and it results

$$\|e^{\xi_j}u_\lambda\|_p \le \frac{2}{\omega}\|e^{\xi_j}f\|_p \quad \text{as well as} \quad \|e^{\xi_j}\partial_{y_k}u_\lambda\|_p \le \frac{2}{\omega}\|e^{\xi_j}\partial_{y_k}f\|_p. \tag{3.11}$$

This implies that

$$\|B_\lambda u_\lambda\|_p = \left\|\frac{1}{\lambda}\left(\frac{1}{\lambda} - B\right)^{-1} Bu_\lambda\right\|_p \le c_1\|Bu_\lambda\|_p \le c$$

for $1 < p \le \infty$ and Proposition 3.1 applies for $1 < p < \infty$. For the case $p = \infty$ it follows that $\|Au_\lambda\|_\infty \le c$ and we obtain $\partial_{\xi_i}u_\lambda \in C^\alpha_\xi$ as well as $\partial_{\xi_i}\partial_{y_k}u_\lambda \in C^\alpha_\xi$ with $\alpha \in (0,1)$. Thus there is a sequence $(\lambda_n) \subset \mathbb{R}_+$ with $\lim_{n\to\infty}\lambda_n = 0$, such that $\nabla_y u_{\lambda_n} \to \nabla_y u$ and $u_{\lambda_n} \to u$ as $n \to \infty$ uniformly on compact sets. In particular this means that for every ball $B_r(0)$ with radius $r \in \mathbb{N}$ we find a sequence $(\lambda_{r,n})_n \subset \mathbb{R}_+$ with $\lim_{n\to\infty}\lambda_{r,n} = 0$ such that $B_{\lambda_{r,n}}u_{\lambda_{r,n}} \to Bu$ and $Au_{\lambda_{r,n}} \to Au$ as $n \to \infty$ uniformly on $B_r(0)$. By a diagonal-sequence argument we obtain the existence of a sequence $(\lambda_k) \subset \mathbb{R}_+$ with $\lim_{k\to\infty}\lambda_k = 0$ such that $B_{\lambda_k}u_{\lambda_k} \to Bu$

as well as $Au_{\lambda_k} \to Au$ as $k \to \infty$ on an arbitrary compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d_+$. This implies

$$\omega u + Au + Bu = f \tag{3.12}$$

also in the case $p = \infty$, and the theorem is proved. □

4. The Call-Put Parity

Suppose that u_c is a solution of problem (2.11) with $g(x,y) = [\frac{1}{T}y_k - K]_+$ resp. $g(x,y) = [x_k - \frac{1}{T}y_k]_+$. Accordingly suppose that u_p is a solution of (2.11) with $g(x,y) = [K - \frac{1}{T}y_k]_+$ resp. $g(x,y) = [\frac{1}{T}y_k - x_k]_+$. Since problem (2.11) is linear, we obtain that $u_c - u_p$ is a solution for $g(x,y) = \frac{1}{T}y_k - K$ resp. $g(x,y) = x_k - \frac{1}{T}y_k$.

In this section we will present an appropriate scaling $\tilde{g}$ of the end conditions g, such that $\tilde{g} \in X_\infty$ holds and that the scaled problem is still well-posed. Then the Lumer-Phillips theorem results that in particular the solutions of the scaled problem, hence the solutions of (2.11), are unique. Thus the following propositions provide the call-put-parities for the average price option resp. the average strike option.

Proposition 4.1. $u(t,x,y) = e^{r(t-T)}\left[T^{-1}\left(y_k + r^{-1}x_k\left(e^{r(T-t)} - 1\right)\right) - K\right]$ *is a solution of (2.11) with* $g(x,y) = \frac{1}{T}y_k - K$.

Proof. Obviously, the end condition with the postulated g holds. Thus it remains to prove, that u is a solution of the partial differential equation (2.11):

$$\begin{aligned}
\partial_t u &+ \sum_{i=1}^d x_i \partial_{y_i} u + \frac{1}{2}\sum_{i,j=1}^d a_{ij}x_i x_j \partial_{x_i}\partial_{x_j} u \\
&= e^{r(t-T)}\left[\frac{1}{T}\left(ry_k - x_k\right) - rK\right] + \frac{1}{T}e^{r(t-T)}x_k \\
&= \frac{r}{T}e^{r(t-T)}y_k - \frac{1}{T}e^{r(t-T)}x_k - re^{r(t-T)}K + \frac{1}{T}e^{r(t-T)}x_k \\
&= \frac{r}{T}e^{r(t-T)}y_k + \frac{1}{T}x_k - \frac{1}{T}e^{r(t-T)}x_k - re^{r(t-T)}K - \frac{1}{T}x_k + \frac{1}{T}e^{r(t-T)}x_k \\
&= r\left[u - \sum_{i=1}^d x_i \partial_{x_i} u\right].
\end{aligned}$$

□

Proposition 4.2. $u(t,x,y) = x_k - T^{-1}e^{r(t-T)}\left[y_k + x_k r^{-1}\left(e^{r(T-t)} - 1\right)\right]$ *is a solution of (2.11) with* $g(x,y) = x_k - \frac{1}{T}y_k$.

PROOF. The end condition with the postulated g holds obviously. Thus it remains to prove, that u is a solution of the partial differential equation (2.11):

$$\begin{aligned}
\partial_t u &+ \sum_{i=1}^{d} x_i \partial_{y_i} u + \frac{1}{2} \sum_{i,j=1}^{d} a_{ij} x_i x_j \partial_{x_i} \partial_{x_j} u \\
&= \frac{1}{T} e^{r(t-T)} (x_k - r y_k) - x_k \frac{1}{T} e^{r(t-T)} \\
&= -\frac{r}{T} e^{r(t-T)} y_k + \frac{1}{T} e^{r(t-T)} x_k - \frac{1}{T} e^{r(t-T)} x_k \\
&= r x_k - \frac{r}{T} e^{r(t-T)} y_k - \frac{1}{T} x_k + \frac{1}{T} e^{r(t-T)} x_k - r x_k + \frac{1}{T} x_k - \frac{1}{T} e^{r(t-T)} x_k \\
&= r \left[u - \sum_{i=1}^{d} x_i \partial_{x_i} u \right].
\end{aligned}$$

□

Consider the scaling of the end condition

$$v_0 = \frac{u_0}{1 + |x|^2 + |y|^2}, \quad u_0 \in \left\{ [x_k - \tfrac{1}{T} y_k]_+ ; [K - \tfrac{1}{T} y_k]_+ \right\} \tag{4.1}$$

for $x, y \in \mathbb{R}_+^d$, where $|x|^2 = \sum_{i=1}^{d} x_i^2$. Hence we have $u = (1 + |x|^2 + |y|^2) v$ and we compute the relevant derivatives, i.e.

$$\partial_t u = (1 + |x|^2 + |y|^2) \partial_t v \tag{4.2}$$

$$\partial_{x_i} u = (1 + |x|^2 + |y|^2) \partial_{x_i} v + 2 x_i v \tag{4.3}$$

$$\partial_{y_i} u = (1 + |x|^2 + |y|^2) \partial_{y_i} v + 2 y_i v \tag{4.4}$$

$$\partial_{x_j} \partial_{x_i} u = (1 + |x|^2 + |y|^2) \partial_{x_j} \partial_{x_i} v + 2 x_j \partial_{x_i} v + 2 x_i \partial_{x_j} v + \delta_{ij} 2 v. \tag{4.5}$$

By means of sum convention and equation (2.11) we have

$$\begin{aligned}
\partial_t v + x_i \partial_{y_i} v &+ \frac{1}{2} a_{ij} x_i x_j \partial_{x_i} \partial_{x_j} v - r (v - x_i \partial_{x_i} v) \\
&= -m(x,y) (2 a_{ij} x_i x_j^2 \partial_{x_i} + a_{ii} x_i^2 + r |x|^2 + 2 y_i) v,
\end{aligned} \tag{4.6}$$

with $m(x,y) := 1/(1 + |x|^2 + |y|^2)$. To run an Euler-transform in x, i.e. $x_i \rightsquigarrow e^{x_i}$, $i = 1, \ldots, d$ and $v \rightsquigarrow w$ with $w(t, \xi, y) = v(t, e^{\xi}, y)$, we use the calculated derivatives (2.12) and (2.13) and receive

$$\begin{aligned}
\partial_t w + e^{\xi_i} \partial_{y_i} w &+ b_i \partial_{\xi_i} w + \frac{1}{2} a_{ij} \partial_{\xi_i} \partial_{x_j} w - r w \\
&= -m(e^{\xi}, y) (2 a_{ij} e^{2\xi_j} \partial_{\xi_i} + a_{ii} e^{2\xi_i} + r |e^{\xi}|^2 + 2 y_i) v,
\end{aligned} \tag{4.7}$$

with vector b given by $b_i = r - \frac{a_{ii}}{2}$. As a last step we invert the time, i.e. $t \rightsquigarrow T-t$, and denote the inverted function again with w; hence

$$
\begin{aligned}
(4.8)\quad \partial_t w - e^{\xi_i}\partial_{y_i} w - b_i\partial_{\xi_i} w - \frac{1}{2}a_{ij}\partial_{\xi_i}\partial_{x_j} w + rw \\
= m(e^{\xi}, y)(2a_{ij}e^{2\xi_j}\partial_{\xi_i} + a_{ii}e^{2\xi_i} + r|e^{\xi}|^2 + 2y_i)w
\end{aligned}
$$

holds. Let us introduce the operators G and H by

$$
\begin{aligned}
Gw &:= 2m(e^{\xi}, y)\sum_{i=1}^{d}\sum_{j=1}^{d} a_{ij}e^{2\xi_j}\partial_{\xi_i} w \\
Hw &:= m(e^{\xi}, y)\left(\sum_{i=1}^{d} a_{ii}e^{2\xi_i} + r|e^{\xi}|^2 + 2y_i\right)
\end{aligned}
$$

and recall that the left hand side of equation (4.8) precisely is $(\partial_t - L)u$ with $L = A + B$ as defined in Theorem 1. With an easy calculus we obtain that

$$
A + G = \frac{1}{2}a : \nabla_{\xi}^2 + \widetilde{b}(\xi)\cdot\nabla_{\xi} - r, \tag{4.9}
$$

with $\widetilde{b}_i(\xi) := b_i + 2m(e^{\xi}, y)\sum_{j=1}^{d} a_{ij}e^{2\xi_j}$, $i = 1, \ldots, d$, is the ω-dissipative generator of an analytic C_0-semigroup with $\omega \le \sup\{|\operatorname{div}\widetilde{b}(\xi)| : \xi \in \mathbb{R}^d\}$, provided the right hand side of this inequality is finite. And indeed we have

$$
\begin{aligned}
|\operatorname{div}\widetilde{b}(\xi)| &= \left|\sum_{i=1}^{d}\partial_{\xi_i} m(e^{\xi}, y)\sum_{j=1}^{d} a_{ij}e^{2\xi_j}\right| \\
&\le 2\left(\sum_{i=1}^{d}\sum_{j=1}^{d}|a_{ij}|\left|\frac{e^{2\xi_i}e^{2\xi_j}}{(1+|e^{\xi}|^2)^2}\right| + \sum_{i=1}^{d}|a_{ii}|\left|\frac{e^{2\xi_i}}{1+|e^{\xi}|^2}\right|\right) \le 4d^2 M_a,
\end{aligned}
$$

where $M_a := \max\{|a_{ij}| : i, j = 1, \ldots, d\}$. Thus Theorem 1 applies for the shifted operator sum $(\eta + A + G)$ and B, η sufficiently large, i.e. the sum $A + G + B$ generates a C_0-semigroup. Since the operator H is linear and bounded the Bounded Perturbation Theorem (e.g. [EN00]) applies and thus $A + G + B + H$ generates a C_0-semigroup and we obtain the existence of a solution of the scaled problem (4.8) with initial value $w(0, \xi, y) = v_0(e^{\xi}, y)$.

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