An Approximation of the Minimal Point Set

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Abstract

In this paper, we consider an approximation of the ordering cone by a sequence of convex well-based cones that are contained inside the original cone. We derive related results on approximating minimal point sequences and propose a density result being evocative of the well-known ABB theorem.

Keywords. vector optimization, partially ordered linear spaces, approximating cones, density, ABB theorem

1 Introduction

Determining minimal points of a subset Y of a partially ordered linear space $(\mathfrak{Y}, \mathfrak{Y}_+)$ usually calls for scalarization. I.e., the real task consists in solving auxiliary problems of the type

$$z(y) \to \min, \qquad y \in Y,$$

where $z : \mathfrak{Y} \to \mathbb{R}$ denotes an appropriate monotone functional. The most popular scalarization functional z is the positive linear one, $z \in \mathfrak{Y}'_+$. The related auxiliary problems are easy to handle and reproduce in most cases at least the practically rewarding minimal points.

The main question arising in the context of linear scalarization is that of density: Can we replicate all or at least a dense subset of minimal points via linear scalarization? The eponymous theorem in this direction goes back to Arrow, Barankin and Blackwell [1]. Later on, the extension to more general partially ordered spaces became a fast-paced field of research: A rich spectrum of proof constructions have been developped, among them outer cone approximation methods like Henig's dilating cones (Henig [13], for a general treatment compare e.g. Gong [8] and Sterna-Karwat [17]) and D-cones (Gallagher and Saleh [7]), approximations of the dual cone (Chen [3]), and approximations of the minimal points via Daneš' drop theorem (Ferro [6]). For a deeper discussion of existing results and the tradeoff between assumptions on the set and the cone we refer the reader to Ferro [6], Truong [18], and Woo and Goodrich [20]. Göpfert, Tammer and Zălinescu [12] prove density results with respect to Henig proper minimal points.

In this paper, we consider an approximation of the ordering cone by a sequence of convex well-based cones that are contained inside the original cone ("inner approximation"). In contrast to the concept of Henig's dilating cones, the original ordering cone can be described as (the closure of) the union of a monotone sequence of approximating cones. Consequentially, the intersection of the minimal point sets with respect to the approximating cones is a superset of the original minimal point set and a subset of the set of weakly minimal points. Finally, we propose a density result being evocative of the well-known ABB theorem.

After some preliminaries in section 2, section 3 contains the construction and main properties of the approximating cones. In section 4, we state our main results concerning approximations of the set of minimal points.

2 Basic notions and preliminaries

Let $(Y, \|.\|)$ be a normed vector space. For a nonempty subset $Y \subset \mathfrak{Y}$ we denote by $\operatorname{int} Y$, $\operatorname{cl} Y$ and $\operatorname{conv} A$ the interior, closure and the convex hull of Y, respectively. $S(0, r)$ will denote the closed ball with radius r around 0, and

$$\operatorname{dist}(\bar{y}, Y) := \inf_{y \in Y} \|\bar{y} - y\|$$

the distance of a point $\bar{y}$ from the set Y. Let $\mathfrak{Y}'$ denote the topological dual of $\mathfrak{Y}$.

A nonempty subset $C \subset \mathfrak{Y}$ is a **cone** if $\lambda y \in C$ for all $\lambda \geq 0$ and all $y \in C$. The cone C is called **convex** if $C + C \subseteq C$, **proper** (or **nontrivial**) if $C \neq \{0\}$ and $C \neq \mathfrak{Y}$, and **pointed** if $C \cap -C = \{0\}$. Within this paper, C is assumed to be a proper, pointed, convex cone.

A set $B \subset \mathfrak{Y}$, $0 \notin \operatorname{cl} B$ such that

$$C = \bigcup_{\lambda \geq 0} \lambda B$$

is called a **base** of the cone C. If there exists a bounded convex base B for C, the cone C is called **well-based**. Note that for a well-based closed convex cone, the base can be chosen to be closed as well. The cone C is called **supernormal** if there exists some $v \in \mathfrak{Y}'$ such that

$$\|y\| \leq v(y) \qquad \forall\, y \in C.$$

Since we stay in normed spaces, supernormality implies well-basedness (compare e.g. Göpfert et al. [11, Proposition 2.2.15]).

We define by

$$\begin{aligned} C^+ &:= \{v \in \mathfrak{Y}' : \ v(y) \geq 0 \ \ \forall\, y \in C\} \\ \operatorname{qint} C^+ &:= \{v \in \mathfrak{Y}' : \ v(y) > 0 \ \ \forall\, y \in C \backslash \{0\}\} \end{aligned}$$

the **dual cone** for C and its **quasi-interior**. Remind $C^{++} = \operatorname{cl} \operatorname{cone} \operatorname{conv} C$, especially $C^{++} = C$ for closded convex cones (compare e.g. Zălinescu [21, Theorem 1.1.9]).

Let $Y \subset \mathfrak{Y}$. An element $\bar{y} \in Y$ is called **minimal in Y with respect to the cone C** if $Y \cap (\bar{y} - C \backslash \{0\}) = \emptyset$; the set of all efficient elements is denoted by $\mathcal{E}(Y, C)$. Assuming $\operatorname{int} C \neq \emptyset$, $\bar{y} \in Y$ is said to be **weakly minimal** if $Y \cap (\bar{y} - \operatorname{int} C) = \emptyset$; the set of all weakly efficient elements is denoted by $\mathcal{E}_w(Y, C)$. Finally, an element $\bar{y} \in Y$ is called **proper minimal (in the sense of linear scalarization)**, if there exists some $v \in \operatorname{qint} C^+$ such that

$$v(\bar{y}) \leq v(y) \qquad \forall\, y \in Y.$$

We denote the set of those elements by $\mathcal{E}_l(Y)$.

Later on, we need the fact that the sum of a weakly compact and a closed, convex set is closed again. It seems to be a very natural result, we found no proof in literature.

Lemma 2.1 *Let A, B be two nonempty subsets of the space $\mathfrak{Y}$, A weakly compact, and B convex, closed and bounded. Then the sum $A + B = \{a + b : \ a \in A, \ b \in B\}$ is closed.*

Proof. Consider nets $(a_i)_{i \in I} \subseteq A$ and $(b_i)_{i \in I} \subseteq B$ such that $a_i + b_i$ strongly converges to some y. Then, $a_i + b_i$ weakly converges to y, too. By the weak compactness of A there exists a subnet $(a_j)_{j \in J}$, $J \subset I$, weakly converging to some $a \in A$. Define $b := y - a$. Obviously, b is a weak limit of the net $b_j = (a_j + b_j) - a_j$. B is convex, so by the Mazur theorem, b_j also strongly converges to b. Closedness of B implies $b \in B$, hence $y \in A + B$. ■

Finally, for the convenience of the reader, we recall the well-known drop theorem of Daneš [4] (compare also [5]).

Proposition 2.1 *Let $\mathfrak{Y}$ be a Banach space, $Y \subset \mathfrak{Y}$ a nonempty closed convex subset, $y \in \mathfrak{Y} \backslash Y$, $\varepsilon > 0$ and $0 < r < \operatorname{dist}(y, Y)$. Then there exists a point $\bar{y} \in Y$ such that*

$$\|\bar{y} - y\| < \operatorname{dist}(y, Y) + \varepsilon \qquad \text{and} \qquad Y \cap \operatorname{conv}(S(y, r) \cup \{\bar{y}) = \{\bar{y}\}.$$

3 Approximating Cones

We fix some $c \in C \backslash \{0\}$ and define sets

$$C_n := \{y \in \mathfrak{Y} : \; y - \frac{1}{n} \|y\| \, c \in C\}.$$

Cones of this type – with $c \in \operatorname{int} C$ – have still been investigated by Göpfert and Tammer [9, 10]. So, let us collect some elementary properties of the sets C_n.

Lemma 3.1 *The sets C_n, $n \in \mathbb{N}$, are nontrivial convex cones. If the cone C ist closed, then the cones C_n are closed, too.*

Proof. Obviously, $0 \in C_n$ for all $n \in \mathbb{N}$. Further, for $y, y_1, y_2 \in C_n$ and $\lambda > 0$ we verify the cone property and convexity by

$$\begin{aligned} \lambda y - \frac{1}{n} \|\lambda y\| \, c = \lambda(y - \frac{1}{n} \|y\| \, c) \quad &\in \quad C, \\ y_1 + y_2 - \frac{1}{n} \|y_1 + y_2\| \, c \in y_1 + y_2 - \frac{1}{n} (\|y_1\| + \|y_2\|) \, c + C \quad &\subseteq \quad C. \end{aligned}$$

Now, let C be closed. Consider $y \notin C_n$, i.e. $y - n^{-1}\|y\|c \notin C$. Hence, by closedness of C, there exists $r > 0$ such that $\tilde{y} \notin C$ for all $\tilde{y} \in \mathfrak{Y}$ with

$$\left\| \tilde{y} - \left(y - \frac{1}{n} \|y\| \, c \right) \right\| < r.$$

Define

$$r' := \frac{n}{n+1} \, \frac{1}{\|c\|} \, r > 0.$$

For $u \in \mathfrak{Y}$ with $\|u\| < r'$ we verify

$$\begin{aligned} \left\| y + u - \frac{1}{n} \|y + u\| \, c - \left(y - \frac{1}{n} \|y\| \, c \right) \right\| \quad &\leq \quad \|u\| + \frac{1}{n} (\|y + u\| - \|y\|) \, \|c\| \\ &\leq \quad \|u\| + \frac{1}{n} \|u\| \, \|c\| \\ &< \quad r \end{aligned}$$

and therefore $y + u - n^{-1}\|y + u\|c \notin C$, i.e. $y + u \notin C_n$. This implies closedness of C_n. ■

Lemma 3.2 *The cones C_n, $n \in \mathbb{N}$, are supernormal, hence well-based.*

Proof. Fix $n \in \mathbb{N}$ and choose $v \in C_n^+$ such that $v(c) = n$ for $c \in C \backslash \{0\}$ from the definition of C_n. Let $y \in C_n$. Then there exists some $\tilde{c} \in C_n$ with $y - n^{-1} \|y\| \, c = \tilde{c}$. Therefore

$$-\|y\| = v(-\frac{1}{n} \|y\| \, c) = v(\tilde{c}) - v(y) \geq -v(y)$$

and finally $\|y\| \leq v(y)$ for all $y \in C_n$. ■

The proof shows that the cones C_n are contained in a so-called Phelps cone, i.e. for each $n \in \mathbb{N}$ there exists $v \in \mathfrak{Y}'$ such that $C_n \subseteq \{y \in \mathfrak{Y} : \; \|y\| \leq v(y)\}$.

Note that the bases B_n of the cones C_n can be chosen to be bounded by 1: Take $v_n \in C_n^+$ as in the proof above, then the sets $B_n := \{y \in C_n : \ v_n(y) = 1\}$ define bases for the cones C_n with $\|b\| \leq v_n(b_n) = 1$ for all $b \in B_n$.

Lemma 3.3 *For all $n \in \mathbb{N}$, the inclusion chain $C_n \subseteq C_{n+1} \subseteq C$ holds. Moreover, if C is closed and has nonempty interior, we have*

$$\operatorname{int} C \subseteq \bigcup_{n\in\mathbb{N}} C_n \subseteq \operatorname{cl} \bigcup_{n\in\mathbb{N}} C_n = C.$$

Proof. The inclusion $\operatorname{int} C \subseteq \bigcup_{n\in\mathbb{N}} C_n$ is obvious. $C_n \subseteq C_{n+1} \subseteq C$ can be verified straightforward, too, so we complete

$$\operatorname{int} C \subseteq \bigcup_{n\in\mathbb{N}} C_n \subseteq C.$$

Taking the closure on this inclusion chain (remember $\operatorname{cl}\operatorname{int} C = \operatorname{cl} C$, compare e.g. van Tiel [19, Theorem 2.27]), we derive the first result in our lemma. ■

Note that the first part of the inclusion chain may be strict: For $C = \mathbb{R}^2_+$ and $c = (1,0)^T$ we verify $C_n \not\subseteq \operatorname{int} C$.

As a simple consequence of Lemma 3.3 we state

Corollary 3.1 *For all $n \in \mathbb{N}$, we have $C^+ \subseteq C^+_{n+1} \subseteq C^+_n$ and $\operatorname{qint} C^+ \subseteq \operatorname{qint} C^+_{n+1} \subseteq \operatorname{qint} C^+_n$.*

Lemma 3.4 *Let $c \in \operatorname{int} C$. Then $C^+\backslash\{0\} \subseteq \operatorname{qint} C^+_n$ for all $n \in \mathbb{N}$.*

Proof. Consider $v \in C^+\backslash\{0\}$ and $y \in C_n\backslash\{0\}$. Thus, $y - n^{-1}\,\|y\|\,c \in C$ holds and therefore $v(y) \geq n^{-1}\,\|y\|\,v(c)$. The choice $c \in \operatorname{int} C$ implies $v(c) > 0$, hence $v(y) > 0$. ■

4 Approximation of the minimal point sets

We first derive some immediate consequences of the results in the section before.

Lemma 4.1 *For all $n \in \mathbb{N}$, the inclusion chain $\mathcal{E}(Y,C) \subseteq \mathcal{E}(Y,C_n) \subseteq \mathcal{E}(Y,C_{n+1})$ holds. Moreover, if C has nonempty interior, we have*

$$\mathcal{E}(Y,C) \subseteq \bigcap_{n\in\mathbb{N}} \mathcal{E}(Y,C_n) = \mathcal{E}(Y, \bigcup_{n\in\mathbb{N}} C_n) \subseteq \mathcal{E}_w(Y,C).$$

Proof. Apply Lemma 3.3. ■

For the results below, in accordance with the definitions in the preliminaries section we define the set of proper minimal points with respect to the cone C_n by

$$\mathcal{E}^n_l(Y) := \bigcup_{v\in\operatorname{qint} C^+_n} \{\bar{y} \in Y : \ v(\bar{y}) \leq v(y) \ \ \forall\, y \in Y\}.$$

Lemma 4.2 *For all $n \in \mathbb{N}$, the inclusion chain $\mathcal{E}_l(Y) \subseteq \mathcal{E}^{n+1}_l(Y) \subseteq \mathcal{E}^n_l(Y)$ holds.*

Proof. Apply Corollary 3.1. ■

The next result can be understood as an ABB theorem.

Theorem 4.1 *Assume $\mathfrak{Y}$ to be a Banach space. Let $Y \subset \mathfrak{Y}$ be a convex, weakly compact subset of $\mathfrak{Y}$ and $C \subset \mathfrak{Y}$ a closed, convex, pointed cone in $\mathfrak{Y}$. Then we have*

$$\mathcal{E}_l(Y) \subseteq \mathcal{E}(Y, C) \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{cl} \mathcal{E}_l^n(Y).$$

Proof. Let $\bar{y} \in \mathcal{E}(Y, C)$, hence $\bar{y} \in \mathcal{E}(Y, C_n)$ for all $n \in \mathbb{N}$. We can assume $\bar{y} = 0$. Let B_n be closed and (uniformly) bounded bases for the cones C_n. Then we have $0 \notin Y + n^{-1}B_n$ for all $n \in \mathbb{N}$. Denote by r the uniform bound of the bases B_n and by r_n the distance between 0 and $Y + n^{-1}B_n$. Then

$$0 < r_n \leq \frac{1}{n}\, r \qquad \forall\, n \in \mathbb{N}. \tag{1}$$

By the drop theorem, for every $n \in \mathbb{N}$ there exist elements $y_n \in Y$ and $b_n \in B_n$ such that

$$\|y_n + \frac{1}{n}\, b_n\| \leq 2r_n \tag{2}$$

$$\left(Y + \frac{1}{n}\, B_n\right) \cap \operatorname{conv}\left(S(0, \frac{r_n}{2}) \cup \{y_n + \frac{1}{n}\, b_n\}\right) = \{y_n + \frac{1}{n}\, b_n\} \tag{3}$$

The application of a classical separation theorem on equation (3) yields the existence of some $v_n \in \mathfrak{Y}'$, $\|v_n\| = 1$ with

$$v_n(z) \leq v_n(y + \frac{1}{n}\, b_n) \tag{4}$$

for $z \in \operatorname{conv}(S(0, r_n/2) \cup \{y_n + n^{-1}b_n\})$, $y \in Y$ and $b_n \in B_n$. Inequation (4) holds strictly for each $z \in \operatorname{int}\operatorname{conv}(S(0, r_n/2) \cup \{y_n + n^{-1}b_n\})$, hence the choice $z = y = 0$ yields

$$v_n(b_n) > 0 \qquad \forall\, b_n \in B_n,\ n \in \mathbb{N}, \tag{5}$$

i.e. $v_n \in \operatorname{qint} C_n^+$. On the other hand, for $z = y_n + n^{-1}b_n$ we derive

$$v_n(y_n) \leq v_n(y) \qquad \forall\, y \in Y,\ n \in \mathbb{N} \tag{6}$$

and therefore $y_n \in \mathcal{E}_l^n(Y)$. Finally, we conclude

$$\|y_n\| \leq \|y_n + \frac{1}{n}\, b_n\| + \frac{1}{n}\, \|b_n\| \leq 2\, r_n + \frac{1}{n}\, \|b_n\| \leq \frac{3}{n}\, r \to 0$$

for $n \to \infty$. Hence, $\bar{y} = 0$ is contained in the set of limits of sequences $y_n \in \mathcal{E}_l^n(Y)$. Taking into account Lemma 4.2 and the formula for limits of decreasing sequences of set (compare e.g. Aubin and Frankowska [2, p. 18]), we deduce $\bar{y} \in \bigcap_{n\in\mathbb{N}} \operatorname{cl} \mathcal{E}_l^n(Y)$. ■

The main steps of the proof are due to Ferro [6, Theorem 3.1. (ii)]. He applied the drop theorem on the original ordering cone – we did it with respect to the approximating cones.

Remark 4.1 Lemma 4.2 implies $\operatorname{cl} \mathcal{E}_l(Y) \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{cl} \mathcal{E}_l^n(Y)$, but the inverse inclusion may fail.

The proof of the above theorem offers another density assertion.

Theorem 4.2 *Assume $\mathfrak{Y}$ to be a Banach space. Let $Y \subset \mathfrak{Y}$ be a convex, weakly compact subset of $\mathfrak{Y}$ and $C \subset \mathfrak{Y}$ a closed, convex, pointed cone in $\mathfrak{Y}$. Then for each $\bar{y} \in \mathcal{E}(Y, C)$ there exists $v \in C^+$ such that*

(i) $v(\bar{y}) \leq v(y)$ for all $y \in Y$,

(ii) $\bar{y}$ is the limit of a sequence $(y_n)_{n\in\mathbb{N}} \subset Y$ and v is the $\sigma(\mathfrak{Y}', \mathfrak{Y})$-limit of a sequence $(v_n)_{n\in\mathbb{N}}$ such that $v_n \in \operatorname{qint} C_n^+$ and y_n is a minimizer of v_n over Y.

Proof. We revisit the sequences of $v_n \in \operatorname{qint} C_n^+$ and $y_n \in \mathcal{E}_l^n(Y)$ from the proof of theorem 4.1. Since $\|v_n\| = 1$, the Alaoglu theorem yields a subsequence (again denoted by $(v_n)_{n\in\mathbb{N}}$) which converges to some $v \in \mathfrak{Y}'$ in the $\sigma(\mathfrak{Y}', \mathfrak{Y})$-topology. Hence

$$\|v(\bar{y}) - v_n(y_n)\| \leq \|v(\bar{y}) - v_n(\bar{y})\| + \|v_n(\bar{y}) - v_n(y_n)\| \to 0.$$

Now inequality (5) can be extended to $v(y) \geq 0$ for all $y \in C$, i.e. $v \in C^+$. Inequality (6) states exactly minimality property proposed in part (ii); moreover, it can be extended to $v(\bar{y}) \leq v(y)$ for all $y \in Y$. ■

Majumdar [14], Peleg [15], and Radner [16] proved similar results for the l^∞-setting. Under the assumption of compactness for the set Y, they obtain a sequence $(y_n) \subset \mathcal{E}(Y, C)$ and strong convergence of the sequence (v_n).

5 Conclusions

We investigated an approximation of the ordering cone by a sequence of convex well-based cones that are contained inside the original cone ("inner approximation"). In contrast to the concept of Henig's dilating cones, the original ordering cone can be described as (the closure of) the union of a monotone sequence of approximating cones. Consequentially, the intersection of the minimal point sets with respect to the approximating cones is a superset of the original minimal point set and a subset of the set of weakly minimal points. Finally, we proposed a density result being evocative of the well-known ABB theorem.

In our opinion, up to now, inner approximation concepts have been inadequately investigated in literature. Our results provide an insight into the capability of this approach.

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