# Global smooth solutions to a fourth order quasilinear fractional evolution equation

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*Dedicated to the memory of Günter Lumer*

**Abstract.** We study a quasilinear fractional evolution equation, which is of order four in space and $1+\beta$ in time, where $\beta \in (0,1)$. Under the restriction $\beta < 3/5$ we are able to prove existence and uniqueness of global smooth solutions. This result can be seen as the analogue of a result obtained by Engler for a related problem of second order.

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## 1. Introduction

In this paper we investigate the existence and uniqueness of global smooth solutions to the problem

$$\begin{cases} \partial_t^\beta(u_t - u_1) + \sigma(u_{xx})_{xx} = f(t,x), & t \in (0,T],\ x \in [0,L] \\ u(t,0) = u(t,L) = 0, & t \in [0,T] \\ u_{xx}(t,0) = u_{xx}(t,L) = 0, & t \in [0,T] \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in [0,L]. \end{cases} \tag{1.1}$$

Here $\partial_t^\beta$ denotes the Riemann-Liouville fractional derivation operator of order $\beta \in (0,1)$ defined by

$$\partial_t^\beta u(t) = \partial_t \int_0^t g_{1-\beta}(t-\tau)u(\tau)\, d\tau, \tag{1.2}$$

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where

$$g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \quad t > 0, \quad \alpha > 0.$$

The nonlinearity $\sigma$ is a smooth real-valued function satisfying the condition

$$0 < \kappa_1 \le \sigma'(s) \le \kappa_2, \quad s \in \mathbb{R}. \tag{1.3}$$

The functions $f$, $u_0$, and $u_1$ are given data.

The corresponding second order problem, that is,

$$\begin{cases} \partial_t^\beta (u_t - u_1) - \sigma(u_x)_x = f(t,x), & t \in (0,T],\ x \in [0,L] \\ u(t,0) = u(t,L) = 0, & t \in [0,T] \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in [0,L], \end{cases} \tag{1.4}$$

as well as variants of it have been studied by many authors. Existence of global weak solutions but not uniqueness has been obtained in [10] for all $\beta \in (0,1)$. Existence of global strong solutions for all $\beta \in (0,1)$ has been established in [8] and [3] by means of a perturbation argument requiring the smallness of the number

$$\frac{\kappa_2 - \kappa_1}{\kappa_1}. \tag{1.5}$$

We further mention Gripenberg's result [9], where global weak solutions $u$ with $u_{xx}$ square integrable but no uniqueness were obtained under the condition $\beta < 1/2$. Restricting further $\beta$ to be less than $1/3$, Engler [7] was able to show existence and uniqueness of global smooth solutions for a variant of (1.4) without smallness condition on the number in (1.5).

In this paper we prove an analogue of Engler's result [7] in the 'fourth order case', see Theorem 4.2. We also need to impose a restriction on $\beta$, which is $\beta < 3/5$. Assuming this condition together with (1.3) and suitable smoothness and compatibility conditions on the data and the nonlinearity (see (H1)-(H4) below), we establish global existence and uniqueness of smooth solutions of (1.1) (with $u_1 = 0$, see below).

Our proof consists of two parts. In the first step we obtain the local well-posedness of (1.1) for all $\beta \in (0,1)$ in the framework of continuous interpolation spaces, see Theorem 3.2. Here we make use of a recent result on abstract quasilinear fractional evolution equations, [4, Theorem 13]. This result requires $u_1 = 0$, which will be assumed throughout this paper. We remark that by using the results in [11], it is also possible to treat the case $u_1 \neq 0$. We recall that the method of continuous interpolation spaces has been introduced by Da Prato and Grisvard in [5] and extended by Angenent [2], Lunardi [12], and Simonett [13].

In the second part of our proof we derive a priori estimates which imply the global well-posedness of (1.1). A crucial step here is to obtain an a priori bound for $u_{xx}$ in a Hölder space, which is achieved by using Engler's method, see [7]. In order to justify the corresponding computations, we are forced to work with higher regularity. So the parameters in our setting are chosen in such a way that a solution on a time-interval $[0,T]$, say, necessarily belongs to the space $C^1([0,T]; C^4([0,L]))$.

We remark that short-time existence and uniqueness of smooth solutions can be shown under weaker assumptions on the function $\sigma$ and the data.

Taking the a priori Hölder estimate for $u_{xx}$ as a starting point, we then carry out a bootstrap argument, which eventually yields the global well-posedness of (1.1). Note that in contrast to the second order case, problem (1.4), here one is confronted with an extra nonlinear term, which is of third order, as can be seen by writing the first equation in (1.1) as

$$\partial_t^\beta(u_t - u_1) + \sigma'(u_{xx})u_{xxxx} = -\sigma''(u_{xx})u_{xxx}^2 + f(t,x), \;\; t > 0, \; x \in [0, L].$$

The paper is organized as follows. In Section 2 we fix some notation. Section 3 is devoted to the local well-posedness, while Section 4 is on a priori estimates and global existence. Finally, in Section 5 we prove an auxiliary result, which is needed in Section 3.

## 2. Preliminaries

By $f * g$ we mean the convolution defined by $(f*g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$, $t \geq 0$, of two functions $f$, $g$ supported on the positive halfline.

Let $X$ be a Banach space and $T > 0$. We say that a function $u \in L_1((0,T);X)$ has a fractional derivative of order $\beta \in (0,1)$ if $u = g_\beta * f$ for some $f \in L_1((0,T);X)$. In this case we write $\partial_t^\beta u = f$.

We next consider functions defined on $J_0 := (0,T]$ which have (at most) a singularity of prescribed order at $t = 0$. Letting $J = [0,T]$ and $\mu \in (0,1)$ we define the space

$$\begin{aligned} BUC_{1-\mu}(J;X) = \{u \in & C(J_0;X) : \; t^{1-\mu}u(t) \in BUC(J_0;X) \\ & \text{and} \lim_{t\to 0+} t^{1-\mu}|u(t)|_X = 0\}, \end{aligned}$$

which becomes a Banach space when endowed with the norm

$$|u|_{BUC_{1-\mu}(J;X)} = \sup_{t\in J_0} t^{1-\mu}|u(t)|_X.$$

We further introduce the following subspace of $BUC_{1-\mu}(J;X)$. For $\beta \in (0,1)$ we set (cf. [4, p. 423])

$$\begin{aligned} BUC_{1-\mu}^{1+\beta}(J;X) = \{u \in BUC_{1-\mu}(J;X) : \; & u = x + g_{1+\beta} * f, \\ & \text{for some } x \in X \text{ and } f \in BUC_{1-\mu}(J;X)\}. \end{aligned}$$

## 3. Local well-posedness

In order to prove existence and uniqueness of local (in time) smooth solutions of (1.1) with $u_1 = 0$, we will apply [4, Theorem 13]. In what follows we explain the underlying setting and verify the assumptions needed in this result.

Let $L > 0$ and set $I = [0, L]$. Let

$$F_0 = \{v \in C(I) : \; v(0) = v(L) = 0\}$$

and

$$F_1 = \{v \in C^4(I) : \; v^{(i)}(0) = v^{(i)}(L) = 0, \; i = 0, 2, 4\},$$

endowed with the canonical norms. For $s \in (0, 8)$ with $s \notin \mathbb{N}$ we define

$$h^s_{bc}(I) = \{v \in h^s(I) : \; v^{(i)}(0) = v^{(i)}(L) = 0 \text{ for all } i \in \{0, 2, 4, 6\} \text{ with } i < s\},$$

where $h^s(I)$ stands for the little Hölder space with exponent $s$. It is well known, see [12], that the continuous interpolation space

$$F_\theta := (F_0, F_1)^0_{\theta,\infty} = h^{4\theta}_{bc}(I), \quad \theta \in (0, 1), \; 4\theta \notin \mathbb{N}.$$

Putting

$$E_0 = h^{4\theta}_{bc}(I), \quad E_1 = h^{4+4\theta}_{bc}(I), \quad \theta \in (0, 1), \; 4\theta \notin \mathbb{N},$$

the following embeddings hold true:

$$E_1 \hookrightarrow F_1 \hookrightarrow E_0 \hookrightarrow F_0.$$

We further set

$$E_\eta = (E_0, E_1)^0_{\eta,\infty}, \quad \eta \in (0, 1).$$

Then

$$E_\eta = h^{4\eta+4\theta}_{bc}(I), \quad \theta \in (0, 1), \; 4\theta \notin \mathbb{N}, \; \eta \in (0, 1), \; 4(\eta + \theta) \notin \mathbb{N}. \tag{3.1}$$

We next put

$$\hat{\mu} = \frac{\mu + \beta}{1 + \beta},$$

and assuming that $E_{\hat{\mu}} \hookrightarrow C^3(I)$ (cp. (3.8) below) we may define

$$\mathcal{A}(v)w = \sigma'(v_{xx})w_{xxxx}, \quad v \in E_{\hat{\mu}}, \, w \in E_1,$$

and

$$\mathcal{F}(v) = -\sigma''(v_{xx})v^2_{xxx}, \quad v \in E_{\hat{\mu}}.$$

Then (1.1) with $u_1 = 0$ can be written as an abstract quasilinear problem of the form

$$\begin{cases} \partial^\beta_t u_t + \mathcal{A}(u)u = \mathcal{F}(u) + f(t), \quad t > 0 \\ \qquad\qquad u(0) = u_0, \quad u_t(0) = 0. \end{cases} \tag{3.2}$$

Letting $\mu$, $\beta \in (0, 1)$ and $J = [0, T]$, we choose

$$\tilde{E}_0(J) := BUC_{1-\mu}(J; E_0)$$

as the base space for the fractional differential equation in (3.2) and seek solutions in the corresponding maximal regularity class

$$\tilde{E}_1(J) := BUC^{1+\beta}_{1-\mu}(J; E_0) \cap BUC_{1-\mu}(J; E_1).$$

It has been shown in [4, Theorem 10], cf. also [11], that

$$\tilde{E}_1(J) \hookrightarrow BUC^{(1+\beta)(1-\eta)-(1-\mu)}(J; E_\eta), \quad 0 \le \eta \le \hat{\mu}. \tag{3.3}$$

In particular, we have

$$\tilde{E}_1(J) \hookrightarrow BUC(J; E_{\hat{\mu}}). \tag{3.4}$$

Note that if $\mu+\beta > 1$, then the Hölder exponent in (3.3) exceeds 1, provided $\eta > 0$ is sufficiently small.

We will next fix the parameters $\mu$, $\theta \in (0,1)$ appropriately, ensuring among other things that $E_{\hat\mu} \hookrightarrow C^3(I)$ and $\tilde E_1(J) \hookrightarrow C^1(J; C^4(I))$.

Let $\varepsilon \in (0, \frac{\beta}{4(1+\beta)})$ and set

$$\mu = 1 - \varepsilon(1+\beta), \quad \theta = \frac{1}{1+\beta} + 3\varepsilon, \quad \eta = \frac{\beta}{1+\beta} - 2\varepsilon. \tag{3.5}$$

Here we exclude those values of $\varepsilon$, for which the condition

$$4\theta \notin \mathbb{N}, \quad 4(\eta+\theta) \notin \mathbb{N}, \quad \text{and} \quad 4(\hat\mu+\theta) \notin \mathbb{N}$$

is violated. Then

$$\hat\mu = \frac{\mu+\beta}{1+\beta} = 1 - \varepsilon,$$

and it is readily checked that $\eta \in (0, \hat\mu)$. We further have $\theta + \eta = 1 + \varepsilon$, and

$$\begin{aligned}(1+\beta)(1-\eta) - (1-\mu) &= (1+\beta)(\frac{1}{1+\beta} + 2\varepsilon) - \varepsilon(1+\beta) \\ &= 1 + \varepsilon(1+\beta).\end{aligned}$$

Using (3.1) and (3.3), we thus see that

$$\begin{aligned}\tilde E_1(J) &\hookrightarrow BUC^{(1+\beta)(1-\eta)-(1-\mu)}(J; h_{bc}^{4\eta+4\theta}(I)) \\ &= BUC^{1+\varepsilon(1+\beta)}(J; h_{bc}^{4(1+\varepsilon)}(I)) \hookrightarrow C^1(J; C^4(I)).\end{aligned} \tag{3.6}$$

Notice as well that

$$\hat\mu + \theta = \frac{2+\beta}{1+\beta} + 3\varepsilon - \varepsilon \in \Big(\frac{3}{2} + 2\varepsilon, 2 - \varepsilon\Big). \tag{3.7}$$

In particular, we have

$$E_{\hat\mu} = h_{bc}^{4\hat\mu+4\theta}(I) \hookrightarrow C^{6+8\varepsilon}(I). \tag{3.8}$$

We will assume that the data and the nonlinearity in (1.1) are subject to the following conditions:

**(H1)** $\sigma \in C^7(\mathbb{R})$, $\sigma^{(k)}(0) = 0$, $k = 0, 2, 4$;
**(H2)** $0 < \kappa_1 \le \sigma'(s) \le \kappa_2$, $\quad s \in \mathbb{R}$;
**(H3)** $f \in C^1(\mathbb{R}_+; C(I)) \cap C(\mathbb{R}_+; C^4(I))$,
$f(t,0) = f(t,L) = f_{xx}(t,0) = f_{xx}(t,L) = 0$, $t \ge 0$;
**(H4)** $u_0 \in C^8(I)$, $u_0^{(k)}(0) = u_0^{(k)}(L) = 0$, $k = 0, 2, 4, 6$; $u_1 = 0$.

Observe that (H3) implies that $f \in BUC_{1-\mu}([0,T]; E_0)$, for any $T > 0$, while (H4) and (3.7) ensure that $u_0 \in E_{\hat\mu} = h_{bc}^{4\hat\mu+4\theta}(I)$. Therefore condition (47) in [4] is satisfied.

We remark that for the Theorems 3.2 and 4.2 below we do not need the full regularity of $u_0$ required in (H4). In fact, $u_0 \in h_{bc}^{4(\hat\mu+\theta+\varepsilon)}(I)$ would be sufficient.

For Banach spaces $X$, $Y$, and a mapping $\mathcal{G}$ of $X$ into $Y$, we write $\mathcal{G} \in C^{1-}_{loc}(X;Y)$, if every point $x \in X$ has a neighbourhood $U$ such that $\mathcal{G}$ restricted to $U$ is globally Lipschitz continuous. By $\mathcal{B}(X,Y)$ we mean the space of bounded linear operators from $X$ into $Y$. We write $\mathcal{B}(X) = \mathcal{B}(X,X)$ for short.

In order to be able to apply [4, Theorem 13], it remains to verify that (cf. [4, condition (46)])

$$(\mathcal{A}, \mathcal{F}) \in C^{1-}_{loc}(E_{\hat{\mu}}; \mathcal{M}_{\beta,\,\mu}(E_1, E_0) \times E_0). \tag{3.9}$$

Here $\mathcal{M}_{\beta,\,\mu}(E_1, E_0)$ denotes the space of all operators $A \in \mathcal{B}(E_1, E_0)$ satisfying the following two conditions: (i) $\exists \omega \geq 0$ such that $A_\omega := A + \omega I$ is a nonnegative closed operator in $E_0$ with spectral angle $\varphi_{A_\omega} < \frac{\pi}{2}(1-\beta)$; (ii) $\partial_t^\beta u_t + Au = h(t)$, $u(0) = 0$, $u_t(0) = 0$, has maximal regularity in $\tilde{E}_0(J)$, i.e. there exists $C > 0$ such that for any $h \in \tilde{E}_0(J)$,

$$|u|_{\tilde{E}_1(J)} \leq C|h|_{\tilde{E}_0(J)},$$

where $u$ solves $\partial_t^\beta u_t + Au = h(t)$, $u(0) = 0$, $u_t(0) = 0$. $\mathcal{M}_{\beta,\,\mu}(E_1, E_0)$ is equipped with the topology of $\mathcal{B}(E_1, E_0)$.

Let $v \in E_{\hat{\mu}}$ and $w \in E_1$. Then, obviously, $w_{xxxx} \in E_0 = h^{4\theta}_{bc}(I)$ and $v_{xx} \in h^{4\hat{\mu}+4\theta-2}_{bc}(I)$. Note that $2 < 4\theta < 4 < 4\hat{\mu} + 4\theta - 2$, due to (3.5) and (3.7). Since

$$\Big(\sigma'(v_{xx})w_{xxxx}\Big)_{xx} = \sigma'(v_{xx})_{xx}w_{xxxx} + 2\sigma''(v_{xx})v_{xxx}w_{xxxxx} \\ + \sigma'(v_{xx})w_{xxxxxx},$$

and $\sigma''(0) = 0$, by (H1), we see that $(\sigma'(v_{xx})w_{xxxx})_{xx}$ vanishes at $x = 0$ and $x = L$. In view of (H1) ($\sigma \in C^6$ is enough) it is then clear that $\mathcal{A} \in C^{1-}_{loc}(E_{\hat{\mu}}; \mathcal{B}(E_1, E_0))$. Similarly, one checks that $\mathcal{F} \in C^{1-}_{loc}(E_{\hat{\mu}}, E_0)$. Note that here one needs one derivative more for $\sigma$; $\sigma \in C^6$ does not suffice. Notice also that

$$\Big(\sigma''(v_{xx})v_{xxx}^2\Big)_{xx} = \sigma''''(v_{xx})v_{xxx}^4 + 5\sigma'''(v_{xx})v_{xxx}^2 v_{xxxx} \\ + 2\sigma''(v_{xx})[v_{xxxx}^2 + v_{xxx}v_{xxxxx}],$$

which shows that $(\sigma''(v_{xx})v_{xxx}^2)_{xx}$ vanishes at $x = 0$ and $x = L$, by (H1).

Finally, let $v \in E_{\hat{\mu}}$ be fixed and define the operators

$$\tilde{A}w = \sigma'(v_{xx})w_{xxxx}, \quad w \in F_1,$$

and

$$Aw = \mathcal{A}(v)w = \sigma'(v_{xx})w_{xxxx}, \quad w \in E_1.$$

Then it follows from (H2) and the preceding considerations that $\tilde{A}$ and $A$ are isomorphisms mapping $F_1$ into $F_0$ and $E_1$ into $E_0$, respectively. Note that $Av = \tilde{A}v$ for all $v \in E_1$. Furthermore, $\tilde{A}$ as an operator in $F_0$ is nonnegative with spectral angle $\phi_{\tilde{A}} = 0$. The latter property is a consequence of the following result.

**Lemma 3.1.** *Let* $L > 0$ *and* $F_0 = \{v \in C([0,L];\mathbb{C}) : \ v(0) = v(L) = 0\}$ *equipped with the supremum norm. Suppose further that* $m \in C([0,L])$ *is strictly positive. Then the operator* $\tilde{A} : D(\tilde{A}) \subset F_0 \to F_0$ *defined by*

$$D(\tilde{A}) = F_1 = \{v \in C^4([0,L];\mathbb{C}) : \ v^{(i)}(0) = v^{(i)}(L) = 0, \ i = 0,2,4\},$$

*and*

$$\tilde{A}w = mw'''', \quad w \in D(\tilde{A}),$$

*is invertible and sectorial with spectral angle* $\phi_{\tilde{A}} = 0$*. We have* $\mathbb{C} \setminus (0,\infty) \subset \rho(\tilde{A})$ *and for any* $\vartheta \in [0,\pi)$ *there exists* $M_1(\vartheta) > 0$ *such that*

$$|(\lambda + \tilde{A})^{-1}|_{\mathcal{B}(F_0)} \le \frac{M_1(\vartheta)}{1+|\lambda|}, \quad \lambda \in \mathbb{C} \setminus \{0\}, \ |\arg \lambda| \le \vartheta. \tag{3.10}$$

A proof of Lemma 3.1 is given in Section 5.

It follows now from [4, Theorem 11] applied to the operators $\tilde{A}$ and $A$, that $A \in \mathcal{M}_{\beta,\mu}(E_1,E_0)$. This shows that $\mathcal{A}(v) \in \mathcal{M}_{\beta,\mu}(E_1,E_0)$ for all $v \in E_{\hat{\mu}}$. Hence condition (3.9) is satisfied.

We are now in position to apply [4, Theorem 13]. This establishes the local well-posedness of (1.1) in the described setting.

**Theorem 3.2.** *Let the assumptions (H1)-(H4) be satisfied. Let* $\beta \in (0,1)$ *and assume that the parameters* $\mu$, $\theta \in (0,1)$ *are chosen as in (*3.5*). Then there exists a unique maximal solution* $u$ *defined on the maximal interval of existence* $[0,T_0)$*, where* $T_0 \in (0,\infty]$*, and such that for any* $T \in (0,T_0)$ *one has*

$$u \in Z^T := BUC^{1+\beta}_{1-\mu}([0,T];h^{4\theta}_{bc}([0,L])) \cap BUC_{1-\mu}([0,T];h^{4+4\theta}_{bc}([0,L]))$$

*and* $u$ *solves (*1.1*) on* $[0,T]$*. Further, for any* $T \in (0,T_0)$, $u \in C^1([0,T];C^4([0,L]))$*. If* $T_0 < \infty$ *then*

$$\limsup_{t\uparrow T_0} |u(t)|_{h^{4\delta+4\theta}_{bc}([0,L])} = \infty, \quad \text{for any } \delta \in (\hat{\mu},1), \text{ where } \hat{\mu} = \frac{\mu+\beta}{1+\beta}.$$

## 4. A priori estimates and global well-posedness

In this section we will prove that the solution $u$ of (1.1) constructed in Theorem 3.2 exists globally, i.e. $T_0 = \infty$. We will make use of the following simple lemma.

**Lemma 4.1.** *Let* $T > 0$, $\beta \in (0,1)$*, and* $v \in Lip(-\infty,T]$ *with* $v(t) = v(0)$, $t < 0$*. Then*

$$\int_{-\infty}^t g_\beta(t-s)[v(s)-v(t)]_s \, ds = \int_{-\infty}^t \dot{g}_\beta(t-s)[v(s)-v(t)] \, ds, \quad 0 < t \le T. \tag{4.1}$$

*Proof.* We split the integral on the right-hand side of (4.1) and integrate by parts. This gives for $t \in (0,T]$,

$$\begin{aligned}\int_{-\infty}^t \dot g_\beta(t-s)[v(s)-v(t)]\,ds &= \int_{-\infty}^0 \ldots\, ds \quad + \int_0^t \ldots\, ds \\ &= -\,[v(0)-v(t)]\,g_\beta(t) + \Big[(v(t)-v(s))\,g_\beta(t-s)\Big]_{s=0}^{s=t} \\ &\quad + \int_0^t g_\beta(t-s)[v(s)-v(t)]_s\,ds \\ &= \int_0^t g_\beta(t-s)[v(s)-v(t)]_s\,ds = \int_{-\infty}^t g_\beta(t-s)[v(s)-v(t)]_s\,ds.\end{aligned}$$

Note that the first line shows that the integral on the right-hand side of (4.1) is well-defined. In the step before last we used the Lipschitz continuity of $v$ to conclude that $\lim_{s\uparrow t}(v(t)-v(s))\,g_\beta(t-s)=0$. □

The main result of the present paper is now the following.

**Theorem 4.2.** *Let the assumptions (H1)-(H4) be satisfied. Assume that*

$$0<\beta<\frac{3}{5},$$

*and suppose that the parameters* $\mu,\,\theta\in(0,1)$ *are chosen as in (*3.5*). Then the unique maximal solution* $u$ *of (*1.1*) constructed in Theorem* 3.2 *exists globally, that is,* $T_0=\infty$*: For any* $T>0$ *one has*

$$u\in Z^T = BUC^{1+\beta}_{1-\mu}([0,T];h^{4\theta}_{bc}([0,L]))\cap BUC_{1-\mu}([0,T];h^{4+4\theta}_{bc}([0,L]))$$

*and* $u$ *solves (*1.1*) on* $[0,T]$.

*Proof.* Suppose that $T_0<\infty$ and let $T\in[T_0/2,T_0)$. By means of a series of estimates for $u$ on $[0,T]\times[0,L]$ (uniform with respect to $T$), we will show that

$$\limsup_{t\uparrow T_0} |u(t)|_{h^{4(\bar\mu+\theta+\varepsilon)}_{bc}([0,L])}<\infty, \tag{4.2}$$

where $\varepsilon$ is the positive number that was used in the definition of $\theta$ and $\mu$ in (3.5). By the blow up criterion given in Theorem 3.2, (4.2) leads to a contradiction, which will imply that $T_0=\infty$.

The proof of (4.2) proceeds in four steps. In the first step we will obtain the basic a priori bound for $u_{xx}$ in a space of Hölder continuous functions. In the Steps 2–4 we will carry out a bootstrap argument which eventually yields (4.2).

**Step 1: An estimate for $u_{xx}$ in $C^\delta([0,T];C^\delta(I))$ with some $\delta>0$.** Since $u\in C^1([0,T];C^4([0,L]))$, by Theorem 3.2, and $f\in C^1([0,T];C([0,L]))$, due to (H3), we may convolve the first equation in (1.1) with $g_\beta$ and differentiate with respect to time, thereby obtaining

$$u_{tt}+g_\beta*[\sigma(u_{xx})_{xxt}] = g_\beta*(f_t)+g_\beta(t)\varphi(x), \tag{4.3}$$

where $\varphi(x) = f(0,x) - \sigma(u_0''(x))_{xx}$. Setting $u(t,x) = u_0(x)$ for $t < 0$, (4.3) can be written as

$$\begin{aligned} u_{tt}(t,x) + \int_{-\infty}^{t} g_\beta(t-s)[\sigma(u_{xx}(s,x)) - \sigma(u_{xx}(t,x))]_{xxs}\, ds \\ = (g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x), \end{aligned}$$

which after an integration by parts, cf. Lemma 4.1, appears in the form

$$\begin{aligned} u_{tt}(t,x) + \int_{-\infty}^{t} \dot{g}_\beta(t-s)[\sigma(u_{xx}(s,x)) - \sigma(u_{xx}(t,x))]_{xx}\, ds \\ = (g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x). \end{aligned} \tag{4.4}$$

We multiply (4.4) by $u_t$, integrate over $[0, L]$, and integrate by parts. This gives ($\sigma(0) = 0$, by (H1))

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_0^L u_t(t,x)^2\, dx - \int_{-\infty}^{t} \dot{g}_\beta(t-s)\,\frac{\partial}{\partial t} H(t,s)\, ds \\ = \int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]u_t(t,x)\, dx, \quad t \in (0,T] \end{aligned} \tag{4.5}$$

where

$$H(t,s) = \int_0^L \int_{u_{xx}(s,x)}^{u_{xx}(t,x)} [\sigma(y) - \sigma(u_{xx}(s,x))]\, dy\, dx.$$

Since $\sigma$ is strictly increasing, we see that $H(t,s) \geq 0$ for all $0 \leq s,\, t \leq T$. Also, by continuity of $u_{xxt}$ on $[0,T] \times [0,L]$, there exists a constant $M > 0$ such that

$$H(t,s) \leq M|t-s|^2, \qquad |\frac{\partial}{\partial t} H(t,s)| \leq M|t-s|, \quad 0 \leq s,\, t \leq T.$$

Therefore, setting

$$W(t) = \frac{1}{2}\int_0^L u_t(t,x)^2\, dx - \int_{-\infty}^{t} \dot{g}_\beta(t-s)H(t,s)\, ds, \quad t \in [0,T],$$

we may rewrite (4.5) as

$$\begin{aligned} \dot{W}(t) = \int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]u_t(t,x)\, dx \\ - \int_{-\infty}^{t} \ddot{g}_\beta(t-s)\, H(t,s)\, ds, \quad t \in (0,T]. \end{aligned}$$

Since $\ddot{g}_\beta$ and $H$ are nonnegative and by using Young's inequality, we then obtain

$$\dot{W}(t) \leq W(t) + \frac{1}{2}\int_0^L [(g_\beta * f_t)(t,x) + g_\beta(t)\varphi(x)]^2\, dx, \quad t \in (0,T],$$

which yields the estimate

$$W(t) = \frac{1}{2}\int_0^L u_t(t,x)^2\, dx - \int_{-\infty}^{t} \dot{g}_\beta(t-s)H(t,s)\, ds \leq C, \quad t \in [T_0/2, T], \tag{4.6}$$

where the constant $C$ depends only on $W(T_0/2)$ and the data.

It is not difficult to see (cf. [7, Lemma 3.3]) that

$$\frac{1}{2\kappa_2}\,|\sigma(u_{xx}(t,x))-\sigma(u_{xx}(s,x))|^2\le\int_{u_{xx}(s,x)}^{u_{xx}(t,x)}[\sigma(y)-\sigma(u_{xx}(s,x))]\,dy. \qquad (4.7)$$

Proceeding as in Engler [7] (cf. Lemma 3.4), it follows from (4.6) and (4.7) that

$$|\sigma(u_{xx})(t,\cdot)-\sigma(u_{xx})(s,\cdot)|_{L_2(I)}\le C_1|t-s|^{(1-\beta)/2},\quad t,\,s\in[T_0/2,T], \qquad (4.8)$$

where the constant $C_1$ depends only on $W(T_0/2)$ and the data. In fact, writing $\xi(t)=\sigma(u_{xx})(t,\cdot)\in L_2(I)$ for $t\in[T_0/2,T]$, (4.6) and (4.7) imply that

$$\int_{t-h}^{t}(t-\tau)^{\beta-2}|\xi(t)-\xi(\tau)|^2_{L_2(I)}\,d\tau\le\tilde{C},\quad t\in[T_0/2,T],\,h\in(0,T_0/2], \qquad (4.9)$$

where the constant $\tilde{C}$ depends only on $W(T_0/2)$ and the data. From (4.9) and Hölder's inequality we then obtain

$$\int_{t-h}^{t}|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\le\sqrt{\tilde{C}}\,\Big(\int_{t-h}^{t}(t-\tau)^{2-\beta}\,d\tau\Big)^{\frac{1}{2}}\le\tilde{C}_1h^{\frac{3-\beta}{2}}$$

for all $t\in[T_0/2,T]$ and $h\in(0,T_0/2]$, where $\tilde{C}_1=\tilde{C}_1(\tilde{C},\beta)$. Setting

$$\xi_h(t)=\frac{2}{h^2}\int_{t-h}^{t}(\tau-t+h)\xi(\tau)\,d\tau,\quad t\in[T_0/2,T],\,h\in(0,T_0/2],$$

this yields

$$\begin{aligned}|\xi(t)-\xi_h(t)|_{L_2(I)}&=\frac{2}{h^2}\,|\int_{t-h}^{t}(\tau-t+h)(\xi(t)-\xi(\tau))\,d\tau|_{L_2(I)}\\&\le\frac{2}{h}\int_{t-h}^{t}|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\;\le2\tilde{C}_1h^{\frac{1-\beta}{2}}\end{aligned}$$

as well as

$$|\dot{\xi}_h(t)|_{L_2(I)}\le\frac{2}{h^2}\int_{t-h}^{t}|\xi(t)-\xi(\tau)|_{L_2(I)}\,d\tau\le2\tilde{C}_1h^{-\frac{1+\beta}{2}}.$$

Hence, for $T_0/2\le s<t\le T$ and $h:=t-s\in(0,T_0/2)$ we have

$$\begin{aligned}|\xi(t)-\xi(s)|_{L_2(I)}&\le|\xi(t)-\xi_h(t)|_{L_2(I)}+|\xi_h(t)-\xi_h(s)|_{L_2(I)}+|\xi_h(s)-\xi(s)|_{L_2(I)}\\&\le4\tilde{C}_1h^{\frac{1-\beta}{2}}+2\tilde{C}_1(t-s)h^{-\frac{1+\beta}{2}}\le6\tilde{C}_1(t-s)^{\frac{1-\beta}{2}},\end{aligned}$$

which proves (4.8).

Thanks to (4.6), (4.8), and the smoothness of $\sigma$ we obtain bounds for

$$u_t\in L_\infty([0,T];L_2(I))\quad\text{and}\quad\sigma(u_{xx})\in C^{\frac{1-\beta}{2}}([0,T];L_2(I)), \qquad (4.10)$$

which depend only on the data, $W(T_0/2)$, and the corresponding bounds on the interval $[0,T_0/2]$.

We now put $v=1*\sigma(u_{xx})$. Then we have a bound for $v$ in the space $C^{1+(1-\beta)/2}([0,T];L_2(I))$ in terms of the bound for $\sigma(u_{xx})\in C^{\frac{1-\beta}{2}}([0,T];L_2(I))$.

On the other hand, we may integrate the first equation in (1.1) with respect to time to the result

$$v_{xx} = -g_{1-\beta} * u_t + 1 * f,$$

which yields a bound for $v$ in the space $C^{1-\beta}([0,T]; H_2^2(I))$ in terms of the bound for $u_t \in L_\infty([0,T]; L_2(I))$ and the data. By means of interpolation (cp. [7, pp. 283-284]) and Sobolev embedding, we have the embeddings

$$\begin{aligned} C^{\frac{3-\beta}{2}}([0,T]; L_2(I)) \cap C^{1-\beta}([0,T]; H_2^2(I)) &\hookrightarrow C^{(1-\tau)\,\frac{3-\beta}{2}+\tau(1-\beta)}([0,T]; H_2^{2\tau}(I)) \\ &\hookrightarrow C^{1+\delta}([0,T]; C^\delta(I)) \end{aligned}$$

for some $\delta > 0$ and some $\tau \in (\frac{1}{4}, \frac{1-\beta}{1+\beta})$, the latter being possible, since $\beta < 3/5$. Hence we obtain an a priori estimate for $\sigma(u_{xx})$ in $C^\delta([0,T]; C^\delta(I))$. Since $\sigma$ is strictly increasing and smooth, we get also an a priori bound for $u_{xx}$ itself in the space $C^\delta([0,T]; C^\delta(I))$. Note that this bound is uniform with respect to $T \in [T_0/2, T_0)$.

**Step 2: An estimate for $u$ in $BUC([0,T]; C^{4+\delta_1}(I))$ with $\delta_1 \in (0,\delta)$.** We write the first equation in (1.1) as

$$\partial_t^\beta u_t + \sigma'(u_{xx})u_{xxxx} = f - \sigma''(u_{xx})u_{xxx}^2, \quad t \in (0,T],\ x \in [0,L], \tag{4.11}$$

and view it as a linear equation for $u$ of the form

$$\partial_t^\beta u_t + m(t,x)u_{xxxx} = \tilde{f}, \tag{4.12}$$

where $m = \sigma'(u_{xx})$ and $\tilde{f} = f - \sigma''(u_{xx})u_{xxx}^2$. Note that $\tilde{f}(t,0) = \tilde{f}(t,L) = 0$, $t \in [0,T]$, since, by assumptions, $f$ enjoys the same property and $\sigma''(0) = 0$. We use then maximal regularity of (4.12) together with the boundary and initial conditions as in (1.1), in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T]; h_{bc}^{4\theta}(I))$ with $\mu \in (0,1)$ and $\theta \in (0, \delta/4)$. Letting

$$\tilde{E}_1([0,T]) = BUC_{1-\mu}^{1+\beta}([0,T]; h_{bc}^{4\theta}(I)) \cap BUC_{1-\mu}([0,T]; h_{bc}^{4+4\theta}(I)),$$

this gives the estimate

$$|u|_{\tilde{E}_1([0,T])} \le C\Big(|\tilde{f}|_{\tilde{E}_0([0,T])} + |u_0|_{h_{bc}^{4\theta+4\hat{\mu}}(I)}\Big), \tag{4.13}$$

where $C$ is a positive constant which depends only on the parameters and the bound for $u_{xx}$ in $C^\delta([0,T]; C^\delta(I))$. The space $C^{4\theta}(I)$ forms an algebra with respect to pointwise multiplication, and we have

$$|\sigma''(u_{xx}(t,\cdot))u_{xxx}(t,\cdot)^2|_{C^{4\theta}(I)} \le |\sigma''(u_{xx}(t,\cdot))|_{C^{4\theta}(I)} |u_{xxx}(t,\cdot)|_{C^{4\theta}(I)}^2, \quad t \in [0,T].$$

Since $4\theta < \delta$, there exists $\eta \in (0, 1/2)$ such that

$$|u_{xxx}(t,\cdot)|_{C^{4\theta}(I)} \le C_1 |u_{xx}(t,\cdot)|_{C^{2+4\theta}(I)}^\eta |u_{xx}(t,\cdot)|_{C^\delta(I)}^{1-\eta}, \quad t \in [0,T],$$

where $C_1 > 0$ is a positive constant. Using these inequalities we may estimate

$$\begin{aligned} |\tilde{f}|_{\tilde{E}_0([0,T])} &\le |\sigma''(u_{xx})u_{xxx}^2|_{\tilde{E}_0([0,T])} + |f|_{\tilde{E}_0([0,T])} \\ &\le C_2 \sup_{t\in(0,T]} t^{1-\mu}|u_{xx}(t,\cdot)|^{2\eta}_{C^{2+4\theta}(I)}|u_{xx}(t,\cdot)|^{2(1-\eta)}_{C^\delta(I)} + |f|_{\tilde{E}_0([0,T])} \\ &\le C_3|u|^{2\eta}_{BUC_{1-\mu}([0,T];h_{bc}^{4+4\theta}(I))} + |f|_{\tilde{E}_0([0,T])}, \end{aligned} \tag{4.14}$$

where the constants $C_2, C_3$ depend on $T_0$ and the bound for $u_{xx}$ in $C^\delta([0,T];C^\delta(I))$. It follows then from (4.13) and (4.14) that

$$|u|_{\tilde{E}_1([0,T])} \le C_2C|u|^{2\eta}_{\tilde{E}_1([0,T])} + C\Big(|f|_{\tilde{E}_0([0,T])} + |u_0|_{h_{bc}^{4\theta+4\hat{\mu}}(I)}\Big).$$

Thanks to $2\eta < 1$ and by Young's inequality, this yields a bound for $u$ in $\tilde{E}_1([0,T])$ in terms of $T_0$, the parameters, the data, and $|u|_{C^\delta([0,T];C^\delta(I))}$. In view of (3.4), (3.8), and $\theta < \delta/4$ the space $Z^T$ embeds into $BUC([0,T];C^{4+4\theta}(I))$. We thus obtain a bound for $u$ in the latter space in terms of the preceding set of quantities and the corresponding bound on the interval $[0,T_0/2]$. Putting $\delta_1 = 4\theta$, this is the desired bound of Step 2.

**Step 3: An estimate for $u$ in $BUC([0,T];C^{6+\delta_2}(I))$ with some $\delta_2 \in (0,\delta_1)$.** We differentiate the first equation in (1.1) twice with respect to $x$, which is possible since $f$ and $\sigma(u_{xx})_{xx}$ belong to the space $C([0,T];C^2(I))$ (by (3.4), (3.8), (H1), (H3)) and thus $\partial_t^\beta u_t$ does so, by (1.1). Letting $w = u_{xx}$ we obtain

$$\begin{aligned} \partial_t^\beta w_t + \sigma'(u_{xx})w_{xxxx} =& f_{xx} - 4\sigma''(u_{xx})u_{xxx}w_{xxx} - 3\sigma''(u_{xx})u_{xxxx}^2 \\ & - 6\sigma'''(u_{xx})u_{xxx}^2u_{xxxx} - \sigma''''(u_{xx})u_{xxx}^4. \end{aligned} \tag{4.15}$$

Furthermore

$$w(t,0) = w(t,L) = w_{xx}(t,0) = w_{xx}(t,L) = 0, \ \ t \in (0,T],$$

and

$$w(0,x) = u_0''(x), \quad w_t(0,x) = 0, \ \ x \in [0,L].$$

Denoting the right-hand side of (4.15) by $\tilde{f}$, we have $\tilde{f}(t,0) = \tilde{f}(t,L) = 0$, $t \in (0,T]$, as $f_{xx}$, $u_{xx}$, and $u_{xxxx}$ enjoy this property, and $\sigma''(0) = \sigma''''(0) = 0$, by assumption. By means of maximal regularity in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T];h_{bc}^{\delta_2}(I))$ with $\mu \in (0,1)$ as in (3.5) and $\delta_2 \in (0,\min\{\delta_1, 8\varepsilon\})$, there is a positive constant $C$ depending only on the parameters and the a priori bound for $u_{xx}$ in $C^\delta([0,T];C^\delta(I))$ such that with

$$\tilde{E}_1([0,T]) = BUC^{1+\beta}_{1-\mu}([0,T];h_{bc}^{\delta_2}(I)) \cap BUC_{1-\mu}([0,T];h_{bc}^{4+\delta_2}(I))$$

there holds the estimate

$$|w|_{\tilde{E}_1([0,T])} \le C\Big(|\tilde{f}|_{\tilde{E}_0([0,T])} + |u_0''|_{h_{bc}^{\delta_2+4\hat{\mu}}(I)}\Big). \tag{4.16}$$

Since

$$|w_{xxx}(t,\cdot)|_{C^{\delta_2}(I)} \le C_1|w_{xx}(t,\cdot)|^{1/2}_{C^{2+\delta_2}(I)}|u_{xxxx}(t,\cdot)|^{1/2}_{C^{\delta_2}(I)},$$

we obtain, similarly to the argument in Step 2,

$$|\tilde{f}|_{\tilde{E}_0([0,T])} \le C_2\Big(|w|^{1/2}_{\tilde{E}_1([0,T])} + |f_{xx}|_{\tilde{E}_0([0,T])} + 1\Big), \tag{4.17}$$

where the constant $C_2$ depends on $T_0$, the parameters, and the a priori bounds for $u$ from Step 1 and Step 2. Combining (4.16) and (4.17), and employing Young's inequality, we find a bound for $u_{xx}$ in $\tilde{E}_1([0,T])$ in terms of $T_0$, the parameters, the data, and the a priori estimates for $u$ from Step 1 and Step 2. In view of (3.4), (3.8), and $\delta_2 < 8\varepsilon$, the space $Z^T$ embeds into $BUC([0,T];C^{6+\delta_2}(I))$. Therefore we obtain a bound for $u$ in the latter space in terms of the preceding set of quantities and the corresponding bound on the interval $[0,T_0/2]$. This completes Step 3.

**Step 4: An estimate for $u$ in $BUC([0,T];C^{4(\theta+\hat{\mu}+\varepsilon)}(I))$ with $\theta$, $\mu$ as in (3.5).** Letting again $w = u_{xx}$, it follows from (4.15) and Step 3 that

$$\partial_t^\beta w_t + \sigma'(u_{xx})w_{xxxx} + 4\sigma''(u_{xx})u_{xxx}w_{xxx} = \tilde{g}, \tag{4.18}$$

where $\tilde{g}$ is a function which is a priori bounded in $C([0,T];C^2(I))$, uniform with respect to $T \in [T_0/2, T_0)$, and satisfies $\tilde{g}(t,0) = \tilde{g}(t,L) = 0$, $t \in (0,T]$.

Let $\theta$ and $\mu$ as in (3.5) and define

$$\gamma = 4\theta + 4\varepsilon - 2. \tag{4.19}$$

Note that $\gamma \in (0,2)$, because $\beta \in (0,\, \frac{3}{5})$ and $\varepsilon \in (0,\, \frac{\beta}{4(1+\beta)})$.

We then consider (4.18) as an equation for $w$ in the space $\tilde{E}_0([0,T]) = BUC_{1-\mu}([0,T];h^\gamma_{bc}(I))$. The coefficients are a priori bounded in $C([0,T];C^2(I))$, uniform with respect to $T \in [T_0/2, T_0)$. The linear term $4\sigma''(u_{xx})u_{xxx}w_{xxx}$ is of lower order, and hence by a perturbation argument, one has maximal regularity in $\tilde{E}_0([0,T])$, that is, with

$$\tilde{E}_1([0,T]) = BUC^{1+\beta}_{1-\mu}([0,T];h^\gamma_{bc}(I)) \cap BUC_{1-\mu}([0,T];h^{4+\gamma}_{bc}(I))$$

we get the estimate

$$|w|_{\tilde{E}_1([0,T])} \le C\Big(|\tilde{g}|_{\tilde{E}_0([0,T])} + |u_0''|_{h^{\gamma+4\hat{\mu}}_{bc}(I)}\Big),$$

where $C$ depends only on $T_0$, the parameters, the data, and the bound for $u$ from Step 3. By the embedding

$$\tilde{E}_1([0,T]) \hookrightarrow BUC([0,T];h^{\gamma+4\hat{\mu}}_{bc}(I)),$$

we thus obtain a uniform bound for $u$ in $BUC([0,T];h^{2+\gamma+4\hat{\mu}}_{bc}(I))$. In view of (4.19) this means we have a bound for $u$ in $BUC([0,T];h^{4(\theta+\hat{\mu}+\varepsilon)}_{bc}(I))$, uniform with respect to $T \in [T_0/2, T_0)$. This contradicts the hypothesis that $T_0 < \infty$. Hence we have global existence. □

## 5. Proof of Lemma 3.1

*Proof.* Define the operator $B$ by means of

$$D(B) = \{v \in C^2([0,L];\mathbb{C}) :\ v^{(i)}(0) = v^{(i)}(L) = 0,\ i = 0, 2\},$$

$$(Bu)(x) = -u''(x), \quad u \in D(B).$$

Then $B : D(B) \subset F_0 \to F_0$ is invertible, i.e. $0 \in \rho(B)$, and it is sectorial with spectral angle $\phi_B = 0$. The same then holds for $B^2 : D(B^2) \subset F_0 \to F_0$. We have $\mathbb{C} \setminus (0, \infty) \subset \rho(B^2)$ and for any $\vartheta \in [0, \pi)$ there exists $C_0(\vartheta) > 0$ such that

$$|(\lambda + B^2)^{-1}|_{\mathcal{B}(F_0)} \le \frac{C_0(\vartheta)}{1 + |\lambda|}, \quad \lambda \in \mathbb{C} \setminus \{0\},\ |\arg \lambda| \le \vartheta.$$

Moreover, $D(B^2) = F_1 = \{v \in C^4([0,L];\mathbb{C}) :\ v^{(i)}(0) = v^{(i)}(L) = 0,\ i = 0, 2, 4\}$, and by using standard interpolation inequalities we see that the graph norm of $B^2$ on $D(B^2)$ is equivalent to the usual norm of $C^4([0,L])$. Again from standard interpolation inequalities and from the identity

$$B^2(\lambda + B^2)^{-1} = I - \lambda(\lambda + B^2)^{-1}$$

we obtain for any $\vartheta \in [0, \pi)$ and $0 \le k \le 4$,

$$|D^k(\lambda + B^2)^{-1}|_{\mathcal{B}(F_0)} \le C_k(\vartheta)\Big(\frac{1}{1 + |\lambda|}\Big)^{1-\frac{k}{4}},\ \lambda \in \mathbb{C} \setminus \{0\},\ |\arg \lambda| \le \vartheta, \tag{5.1}$$

where $D = \frac{d}{dx}$, and $C_k(\vartheta) > 0$ are constants depending only on $k$ and $\vartheta$.

Let now $m \in C([0,L])$ be strictly positive and set

$$m_1 := \min_{x \in [0,L]} m(x), \qquad m_2 := \max_{x \in [0,L]} m(x).$$

Define the operator $M : F_0 \to F_0$ by means of

$$(Mu)(x) = m(x)u(x), \quad x \in [0,L],\ u \in F_0.$$

Then $M \in \mathrm{Isom}(F_0)$, $|M|_{\mathcal{B}(F_0)} \le m_2$, and $|M^{-1}|_{\mathcal{B}(F_0)} \le 1/m_1$. Furthermore we have $\tilde{A} = MB^2$ with $D(\tilde{A}) = D(B^2)$, and so clearly $0 \in \rho(\tilde{A})$ and $\tilde{A}^{-1} = (B^2)^{-1}M^{-1}$.

In order to show that $\mathbb{C} \setminus (0, \infty) \subset \rho(\tilde{A})$, it is sufficient to prove that $\lambda + \tilde{A} : D(\tilde{A}) \to F_0$ is bijective for any $\lambda \in \mathbb{C} \setminus (-\infty, 0)$. To this end, for such a $\lambda$, we define the operator $K : D(\tilde{A}) \to F_0$ ($D(\tilde{A})$ equipped with the graph norm) as follows:

$$(Ku)(x) = \lambda u(x), \quad u \in D(\tilde{A}).$$

Then $K$ is compact, and since $\tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$, it follows then from the stability of the index under compact perturbations that the index of $K + \tilde{A}$ is zero.

The null space of $K + \tilde{A}$ is $\{0\}$. Indeed, let $u \in D(\tilde{A})$ be such that $Ku + \tilde{A}u = 0$. We divide this equation by $m$, multiply by $\bar{u}$, and integrate over $[0,L]$; this yields

$$\lambda \int_0^L \frac{1}{m(x)}\, |u(x)|^2\, dx + \int_0^L |u''(x)|^2\, dx = 0.$$

In view of the positivity of $m$ and due to $\lambda \in \mathbb{C} \setminus (-\infty, 0)$, it follows that $u = 0$. We conclude that $K + \tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$. Hence $\lambda + \tilde{A} \in \mathrm{Isom}(D(\tilde{A}), F_0)$ for all $\lambda \in \mathbb{C} \setminus (-\infty, 0)$.

As to (3.10), observe that by continuity of the resolvent and as a consequence of what we have just proved, (3.10) holds provided $|\lambda| \le \rho$ for $\rho > 0$, with $M_1(\vartheta)$ replaced with some $M_1(\vartheta, \rho)$. Therefore it remains to show the following:

$$\left\{ \begin{array}{l} \quad \forall \vartheta \in [0, \pi)\ \exists \rho > 0\ \exists M(\vartheta, \rho) > 0 \text{ such that} \\ |(\lambda + \tilde{A})^{-1}|_{\mathcal{B}(F_0)} \le \frac{M(\vartheta,\rho)}{1+|\lambda|}\,,\ |\lambda| \ge \rho,\ |\arg \lambda| \le \vartheta. \end{array} \right. \tag{5.2}$$

Employing the resolvent estimates (5.1) for the operator $B^2$ and using the continuity of $m$, (5.2) can be proved by the method of localization and perturbation arguments, see e.g. [1, pp. 479–480] or [6]. Due to limitations of space, we do not carry out the details. □

## References


[1] Angenent, S.: Local existence and regularity for a class of degenerate parabolic equations. Math. Ann. **280** (1988), pp. 465–482.

[2] Angenent, S.: Nonlinear analytic semiflows. Proc. Roy. Soc. Edinburgh Sect. A **115** (1990), pp. 91–107.

[3] Bazhlekova, E.; Clément, Ph.: Global smooth solutions for a quasilinear fractional evolution equation. J. Evol. Equ. **3** (2003), pp. 237–246.

[4] Clément, Ph.; Londen, S.-O.; Simonett, G.: Quasilinear evolutionary equations and continuous interpolation spaces. J. Differential Equations **196** (2004), pp. 418–447.

[5] Da Prato, G.; Grisvard, P.: Equations d'évolution abstraites non linéaires de type parabolique. Ann. Mat. Pura Appl. **120** (1979), pp. 329–396.

[6] Denk, R.; Hieber, M.; Prüss, J.: *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type.* Mem. Amer. Math. Soc. 166 (2003), no. 788.

[7] Engler, H.: Global smooth solutions for a class of parabolic integrodifferential equations. Trans. Amer. Math. Soc. **348** (1996), pp. 267–290.

[8] Gripenberg, G.: Global existence of solutions of Volterra integrodifferential equations of parabolic type. J. Differential Equations **102** (1993), pp. 382–390.

[9] Gripenberg, G.: Nonlinear Volterra equations of parabolic type due to singular kernels. J. Differential Equations **112** (1994), pp. 154–169.

[10] Heikonen, J.: On the existence of a global mild solution for a nonlinear parabolic integro-differential equation. Licenciate Thesis, Helsinki University of Technology, Espoo, Finland, 1993.

[11] Londen, S.-O.: Interpolation spaces for initial values of abstract fractional differential equations. In Operator Theory: Advances and Applications, Vol. **108**, pp. 153–168, Birkhäuser Verlag, Basel, 2006.

[12] Lunardi, A.: *Analytic semigroups and optimal regularity in parabolic problems.* Birkhäuser Verlag, Basel, 1995.

[13] Simonett, G.: Quasilinear parabolic equations and semiflows. Evolution equations, control theory, and biomathematics (Han sur Lesse, 1991), pp. 523–536, Lecture Notes in Pure and Appl. Math., 155, Dekker, New York, 1994.

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